Probability theory, stochastic processes econophysics, financial markets Stochastic analysis methods

Modeling wealth distribution in growing markets

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We introduce an auto-regressive model which captures the growing nature of realistic markets. In our model agents do not trade with other agents, they interact indirectly only through a market. Change of their wealth depends, linearly on how much they invest, and stochastically on how much they gain from the noisy market. The average wealth of the market could be fixed or growing. We show that in a market where investment capacity of agents differ, average wealth of agents generically follow the Pareto-law. In few cases, the individual distribution of wealth of every agent could also be obtained exactly. We also show that the underlying dynamics of other well studied kinetic models of markets can be mapped to the dynamics of our auto-regressive model.

PACS numbers: 02.50.-r,89.65.Gh,05.10.Gg

Kinetic models have been drawing substantial attention as model markets. In these models[1], markets are compared with systems of ideal gases, where agents and their wealth are considered analogous to the gas particles and their energy respectively. Trading between any pair of agent is similar to a collision process where energy or wealth is shared between agents. Several models of both conserved [2, 3, 4, 5, 6] and open [3, 7, 8, 9] economic systems have been studied recently, which differ mainly in their exchange rules, namely whether collision is elastic or inelastic, if a fraction or the whole energy of a pair is shared between agents, etc. A minimal model of a closed market is when a randomly chosen pair of particles(agents) collide (trade) elastically such that the total energy(wealth) of the pair is shared randomly between the particles (agents). This wealth conserving dynamics naturally predicts [4] a Gibb's distribution of wealth $P(x) \sim exp(-\beta x)$ in equilibrium, which has been observed in distribution of income-tax return of individuals in several countries[10]. However, the tails of the wealth distribution in [10] and other economic systems[11] follow a power-law distribution, as predicted by V. Pareto[12] (known as *Pareto-law*). In an attempt to get a scale free distribution within the frame-work of wealth conserving dynamics, a simple model was proposed by Chatterjee Chakrabarti and Manna (CCM)[6], where agents contribute only a fraction of their wealth for trading depending on their savings propensities which differ among agents. Numerical simulations of this model strongly suggest that the the distribution of *average* wealth $\{w_i = \langle x_i \rangle\}$ follow $P(w) \sim w^{-\gamma}$, with $\gamma = 2$. Later, exact results[13] show that the tail of the distribution is generically scale-free with $\gamma = 2$. For certain typical variations of the model one may get $\gamma \neq 2$. Note that in these models, it is *only* the average wealth which show a scale free distribution. Wealth of any individual agent is distributed about his average following a Gamma-*like*[14] distribution.

Although, kinetic models [1] are successful in describing basic features of wealth distribution, they do not capture the growing features of most realistic markets. Recently, growing markets are modeled by pouring an extra amount of wealth during each trading which is proportional to the wealth of one[3] or both[8] agents participating in trading. A power-law distribution for rich was observed only in [8] where $P(w) \sim w^{-1.7}$. To have finite average wealth, the tail of this distribution can not be scale-free; it must be cut off at some finite w.

In this article, we introduce a minimal model of growing markets and show that this class of models generically produce a power-law tail in the wealth distribution, independent of the details of the market and the trading rules. Both static and growing markets having conserving or non-conserving dynamics lead to Pareto-distribution of wealth, $P(w)$ $w^{-\gamma}$ with $\gamma \geq 2$. Kinetic models are just a sub-class satisfying conservation of wealth and their dynamics could be mapped to the dynamics of our model. This exact mapping suggests that wealth conservation is not necessary for the description of markets. It also provides a route to capture the exact distribution of the fluctuations of wealth of individual agents. Finally, these models, being auto-regressive(AR), bridge a connection between kinetic models studied recently in econophysics and other AR models[15] of markets studied by economists.

First, the model. Let us take a system of N independent agents $i = 1 \ldots N$, whose wealth at a given time t is $x_i(t)$. Each agent i, depending on his investment capacity $0 < \mu_i \leq 1$, invests a definite fraction of wealth $\mu_i x_i(t)$ in the market. The market stochastically returns a net gain (t). Thus, wealth of agent i at time t is

$$x_i(t) = (1 - \mu_i)x_i(t-1) + \xi(t). \quad (1)$$

In this minimal model $\xi(t)$ is taken as a uncorrelated positive stochastic variable with probability distribution function (PDF) $h(\xi)$; it does not depend on $\{x_i\}$. Thus, agents may gain or lose from the market. The auto-regressive nature of the model that $x(t)$ depends on $x(t-$

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1) is clear from (1).

First we will calculate the steady state of (1). Let us define an operator $\mathcal{B}$ which moves the variables one step backward in time, i.e., $x(t-1) = \mathcal{B}x(t)$. For convenience let us take $\lambda_i \equiv 1 - \mu_i$, which is similar to the savings propensity defined in kinetic models [1]. Now, (1) can be written as,

$$\begin{aligned} x(t) &= \frac{1}{1-\lambda\mathcal{B}}\xi(t) = \sum_{n=0}^{\infty} \lambda^n \xi(t-n) \\ &= \sum_{n=0}^{t} \lambda^n \xi(n), \end{aligned} \tag{2}$$

where we have dropped the index i as agents are independent. In the last step we have used the fact that $\xi(t)$ is a uncorrelated random variable and that $\xi(n<0) = 0$. Thus, the steady state distribution $P(x)$ which is reached as $t \to \infty$ is the PDF of the stochastic variable

$$x = \sum_{n=0}^{\infty} \lambda^n \xi(n) \tag{3}$$

which is just a weighted sum of $\{\xi(n)\}$ with weights$\{\lambda^n\}$. Let $x_m = \sum\limits_{n=0}^{m} \lambda^n \xi(n)$ be the first m terms of (3) and their distribution be $P_m(x)$. From (2) and (3) it is clear that $x_m = x(t=m)$. It implies, first that true steady state gets contributions from all orders of λ^n. Secondly, $P_m(x)$ can be considered as the distribution at $t = m$.

Since $x_m = \lambda^m \xi(m) + x_{m-1}$, $P_m(x)$ satisfies a recursion relation,

$$P_m(x) = \frac{1}{\lambda^m} \int_0^x P_{m-1}(y) h(\frac{x-y}{\lambda^m}) dy. \tag{4}$$

The steady state distribution is then $P(x) \equiv \lim\limits_{m\to\infty} P_m(x)$. Clearly, from (4) one can see that

$$P_m(0) = 0 \quad \text{for all} \quad m > 0. \tag{5}$$

Thus in steady state we must have $P(x=0) = 0$. Equation (5), being independent of the choice of $h(\xi)$, can be used as generic boundary conditions for (4). Secondly, it indicates that the steady state distribution is neither Gibb's nor Pareto like, where $P(x=0)$ is finite.

To proceed further, we need to be more specific, namely we need to know $h(\xi)$. Before considering the generic growing markets, we consider few examples of *static* markets where average wealth of the market $a \equiv \langle\xi\rangle$ is fixed.

- *Normal distribution of ξ :* The first example is when fluctuation of the market is normal, *i.e*, $h(\xi)$ is a Gaussian distribution denoted by $\mathcal{G}(\alpha_0, \sigma_0)$ with mean α_0 and standard deviation σ_0. In this case, the steady state distribution $P(x)$ is $\mathcal{G}(\alpha, \sigma)$ where

$$\alpha = \frac{\alpha_0}{1-\lambda} \quad \text{and} \quad \sigma = \frac{\sigma_0}{\sqrt{1-\lambda^2}}. \tag{6}$$

It is easy to check that $\mathcal{G}(\alpha, \sigma)$ satisfy Eq. (1) in steady state; i.e, if PDF of x and ξ are $\mathcal{G}(\mu_0, \sigma_0)$ and $\mathcal{G}(\mu, \sigma)$ respectively, then PDF of $\lambda x + \xi$ is same as PDF of x. Note that agents in this case can have negative wealth even though $\langle x\rangle > 0$. The negative wealth may be interpreted as *debt.*

- *Exponential distribution of ξ:* In the next example we take $h(\xi) = \exp(-\xi)$. This case is interesting, because for $\lambda = 0$ it gives same steady state distribution as that of the CC model[5], *i.e* $P(x) = \exp(-x)$. For non zero λ, we need to solve the integral equation (4). Instead we rewrite it as a differential equation (which is possible in this case),

$$\frac{d}{dx} P_m(x) = \frac{1}{\lambda^m} \left[P_{m-1}(x) - P_m(x)\right], \tag{7}$$

where $m > 0$, and the boundary conditions are given by Eq. (5). For $m = 0$, $P_0(x) \equiv h(x)$. In terms of $G_m(s)$, which is the Laplace transform (LT) of $P_m(x)$, Eq. (7) becomes a difference equation

$$G_m(s) = \frac{1}{1+\lambda^m s} G_{m-1}(s), \tag{8}$$

whose formal solution is

$$G_m(s) = \prod_{k=0}^{m-1} (1+\lambda^k s)^{-1} G_0(s).$$

Again, let us remind that $G_0(s)$ is the LT of $P_0(x) = h(x)$. Finally, $P(x)$ is the inverse LT of

$$G(s) = \prod_{k=1}^{\infty} \frac{1}{1+\lambda^k s} G_0(s), \tag{9}$$

which can be written as the following series :

$$\begin{aligned} P(x) &= \sum_{m=1}^{\infty} C_m \exp(-x/\lambda^m) \\ &\text{where} \quad C_m^{-1} = \lambda^m \prod_{0<n\neq m}^{\infty} (1-\lambda^{n-m}). \end{aligned} \tag{10}$$

Although Eq. (10) is an infinite series, first few terms are good enough for numerical evaluation of the distribution. Terms up to $m = n$ gives $P_n(x)$, which can be interpreted either as an approximation of true steady state distribution $P(x)$ to n^{th} order in λ or as the distribution at finite time $t = n$. In Fig. 1 we compare $P(x)$ which is obtained numerically with the first four terms of (10) for $\lambda = 0.4$. Note, that $P(x)$ is a Gamma-like distribution similar to what has been obtained in [3, 5, 14].

- *Pareto-law :* In our model, the wealth distributions of individual agents are not simple and depend

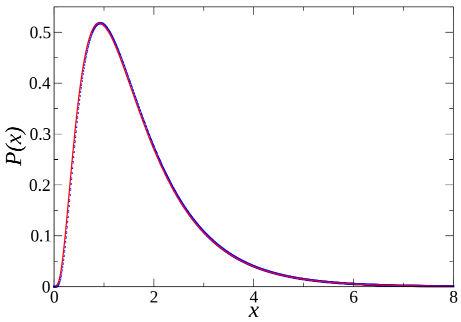

FIG. 1: $P(x)$ as a function of x, for $h(\xi) = exp(-\xi)$. The numerically obtained steady state distribution(for $\lambda = 0.4$) of wealth (symbols) is compared with the first four terms of $P(x)$ from Eq. (10), drawn as solid line.

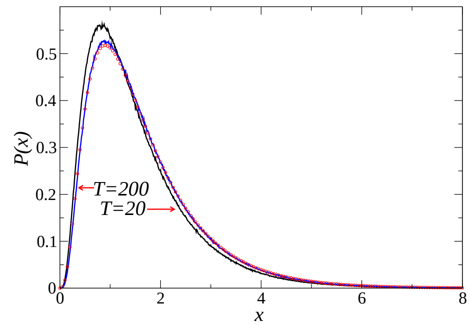

FIG. 2: The figure compares the wealth distribution $P(x, t = T)$ of growing markets where average wealth $a(t) = t/T$, with that of the *static* market with $a(t) = 1$ (line). As expected, For slowly growing market (large T), $P(x, t = T)$ compares well with the distribution in *static* market, suggesting that 'for adiabatically growing markets, instantaneous distribution of wealth is independent of the history'.

on their investment capacities μ_i. Their averages, however, follow a power-law. To prove this let us define $\langle x_i \rangle = w_i$. In steady state $\langle x(t)\rangle = \langle x(t-1)\rangle$. Thus, Eq. (1) gives

$$w_i = \frac{\langle \xi \rangle}{\mu_i}. \tag{11}$$

Agents in this model differ in their investment capacities. In a system of N agents the average number of agents having investment capacity μ is $Ng(\mu)$ where $g(\mu)$ is the distribution of μ. Thus, we can write $w(\mu) = \langle \xi \rangle / \mu$. Distribution of w is then

$$P(w) = g(\mu)|\frac{d\mu}{dw}| = \langle \xi \rangle \frac{g(\langle \xi \rangle / w)}{w^2}. \tag{12}$$

A similar argument was used in [13] for deriving the wealth distribution of CCM model. Although, distribution for the rich (large w) is generically $P(w) \sim w^{-2}$, one can obtain $\gamma > 2$ in typical cases. For example, if PDF of μ is $g(\mu) = \mu^\alpha/(\alpha - 1)$ with $0 \le \alpha < 1$, the asymptotic distribution of (12) results $P(w) \sim w^{-\gamma}$, where $\gamma = 2 + \alpha$.

- *Growing markets :* The kinetic models of markets [4, 5, 6] are defined with wealth conserving dynamics, which keeps the total wealth of the system constant. In our model, we can easily incorporate the growth feature of the market (say, stock-markets) by introducing explicit time dependence in the distribution of ξ. For example, the mean $\langle \xi \rangle \equiv a(t)$ may vary in time. The distribution of wealth $P(x, t)$ will then depend explicitly on t. However, in the adiabatic limit, when $a(t)$ varies slowly (such that $a(t-1) \approx a(t)$), we have $P(x, t-1) \approx P(x, t)$. In this limit, thus, $P(x, \tau)$ is identical to the steady state distribution of the time-independent model, where ξ has an average $\langle \xi \rangle = a(\tau)$.

  For demonstration, we take $h(\xi)$ to be an exponential distribution with varying average $a(t) = t/T$. In other words, $h(\xi, t) = \exp[-x/a(t)]/a(t)$. From the numerical simulations we calculate the distribution $P(x, T)$ at $t = T$ for different values of T. Since $a(T) = 1$, $P(x, T)$ is compared with the steady state distribution (10). Figure 2 compares the distributions for $T = 20$ and $T = 200$, which clearly suggests that in the quasi-static limit $T \to \infty$ the instantaneous distribution depends only on the instantaneous distribution of ξ.

- *Annealed λ:* Another interesting case is when savings propensity of agents change in time. This is modeled by taking λ as a stochastic variable distributed, say uniformly in $(0, 1)$. Let $h(\xi) = \exp(-\xi)$. The steady state distribution of wealth is then $P(x) = \Gamma_2(x) = x\exp(-x)$, which can be proved as follows. If $P(x) = \Gamma_2(x)$ then $P(u = \lambda x) = exp(-u)$[16]. Thus, PDF of right hand side of Eq. (1) is $\Gamma_2(x)$[17] which is same as the PDF of left hand side. This, completes the proof.

In rest of the article we discuss kinetic models studied in the context of wealth distribution and show that the dynamics of these models can be mapped to the AR model defined in Eq. (1). First let us consider the CCM model[6]. The main idea in this model is that the agents, labeled by $i = 1 \dots N$, are considered to have savings propensities $\{\lambda_i\}$ distributed as $g(\lambda)$. During trading, wealth x_i and x_j of two randomly chosen agents i and j changes to x'_i and x'_j respectively such that

$$\begin{aligned} x'_i &= \lambda_i x_i + rT_{ij} \\ x'_j &= \lambda_j x_j + (1-r)T_{ij}, \end{aligned} \tag{13}$$

where $T_{ij} = (1-\lambda_i)x_i + (1-\lambda_j)x_j$, and r is a stochastic variable with PDF $\mathcal{U}(r)$, uniform in $(0, 1)$. The wealth conserving dynamics (13) can be interpreted as follows. Both agents (i, j) save a fraction (λ_i, λ_j) of their wealth and contribute the rest for trading. The total trading

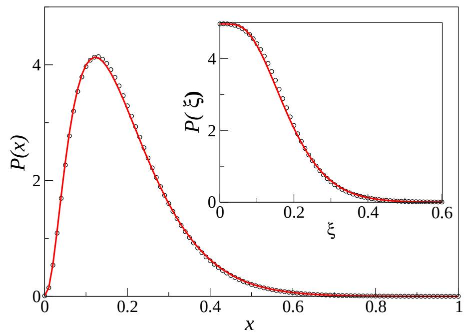

FIG. 3: Comparison of wealth distribution of a tagged agent k, having $\lambda_k = 0.4$ in CCM model (symbol) and the CC model (line) where savings propensity is identical ($\lambda = 0.4$) for every agent. Average wealth of CC model is taken as $w_k = 0.198$, which is the average wealth of the tagged agent. For both CCM and CC model, $N = 100$. In the inset, we compare the distribution of "noise" in these systems.

wealth T_{ij} is then randomly shared between agents i and j. A special case of the model $\{\lambda_i = \lambda\}$ was studied earlier by Chakrabarti and Chakraborti (CC)[5]. Explicitly, the dynamics of the model is

$$\begin{aligned} x_i' &= \lambda x_i + [r(1-\lambda)(x_i + x_j)] \\ x_j' &= \lambda x_j + [(1-r)(1-\lambda)(x_i + x_j)] \,, \end{aligned} \tag{14}$$

where a skew wealth distribution was found for $\lambda \neq 0$. Although, the exact steady state wealth distribution $P(x)$ of CC model is not known, it can be fitted well to a Gamma distribution $\Gamma_n(x) \equiv x^{n-1}\exp(-x)/(n-1)!$ with $n = (1+2\lambda)/(1-\lambda)$[18]. However, later analytical studies [19] disagree with the Gamma distribution. In this article we refer to it as *Gamma-like* distribution.

To see why the dynamics of CCM model is same as Eq. (1) we look at the steady state wealth distribution of a tagged agent k whose savings propensity is $\lambda_k = \lambda$. Wealth of this agent, which fluctuates in time about the mean $w_k \sim 1/(1-\lambda)$, is distributed as $P_\lambda^{CCM}(x)$. Now we take a system (namely the CC model) of N agents with total wealth $w_k N$ and all agents having the same savings propensity as that of the tagged agent in CCM model, *i.e*, $\{\lambda_i = \lambda\}$. Numerically calculated steady state distribution of wealth $P_\lambda^{CC}(x)$ in this case is found to be identical to $P_\lambda^{CCM}(x)$. As an example, we take a tagged agent in CCM having $\lambda = 0.4$ and compare $P_\lambda^{CC}(x)$ and $P_\lambda^{CCM}(x)$ in Fig. 3. An excellent agreement suggests that agents in CCM model are independent, unaware of savings propensity of other agents. Since each agent in CCM model trade with every other agent, it is not surprising that this model turns out to be mean-field system of non-interacting agents (it is also observed in [14]), which suggests and supports that one can replace (14) by a single equation

$$x_i' = \lambda x_i + \eta, \tag{15}$$

where $\eta = rT_{ij}$ is the noise. Note that x_j satisfies the same equation because PDF of r is same as that of $1-r$. Replacement of the conserving dynamics (14) by a single equation (15) which do not have conservation suggests that *conservation is not important* in these systems.

To emphasize this point further that 'the wealth conserving dynamics can be replaced by a non-conserving one similar to (1)', we consider other kinetic models. In a generic wealth conserving dynamics a pair of agents interact as follows,

$$\begin{aligned} x_i' &= \lambda x_i + \eta x_j \\ x_j' &= (1-\lambda)x_i + (1-\eta)x_j, \end{aligned} \tag{16}$$

where both η and λ are stochastic variables with PDF $\mathcal{U}(x)$. We will prove, by mapping wealth conserving dynamics (16) to a non-conserving one, that the steady state distribution of this model is in fact $P(x) = x\exp(-x) \equiv \Gamma_2(x)$ (here $\langle x\rangle = 2$). The non-conserving dynamics for this model is then

$$x' = \lambda x + \xi, \tag{17}$$

where the noise $\xi = \eta\tilde{x}$ and $\tilde{x}$ is the wealth of the other agent in the conserving dynamics. Both x and $\tilde{x}$ have the same distribution in the steady state. If that distribution is $\Gamma_2(x)$, the PDF of ξ is $P(\xi) = \exp(-\xi)$[16]. Thus, the dynamics of this model is effectively the same as that of the annealed λ case of the AR model with exponential noise studied earlier in this article, where the steady state distribution is $P(x) = \Gamma_2(x)$. We have done numerical simulation of the conserving dynamics (16) and calculated the distribution of $\xi = \eta\tilde{x}$, and the steady state distribution of wealth $P(x)$. The resulting distributions are found (see Fig. 4) to be $P(\xi) = \exp(-\xi)$ and $P(x) = x\exp(-x)$, as expected from the non-conserving dynamics. Finally, we take the special case of the model with $\eta = \lambda$, which is the kinetic model studied in [4], where the steady state distribution is $P(x) = \exp(-x)$. One may write a non-conserving dynamics in this case as

$$x' = \lambda(x + \tilde{x}). \tag{18}$$

Again, both x and $\tilde{x}$ have the same distribution in steady state. If $P(x) = exp(-x)$, then using [16, 17] one can show that PDF of right hand side of (18) is same as that of the left hand side. These generic examples thus strongly suggest that both conserving and non-conserving dynamics approaches the same steady state.

In conclusion, we introduce a simple model which captures the growing feature of realistic markets. Agents in these models do not involve in direct trading, they interact only through the market. Their net gain depend on how much they invest and how much they gain from the market. The market, naturally noisy, is modeled by a stochastic variable having a specific distribution with fixed or varying mean. In our model, return from the market is considered to be independent of individual agent's investment (a natural extension would be when $\xi(t)$ explicitly depends on $x(t)$). One of our main results is that, the average wealth of agents generically follow Pareto-distribution. We also argue that, when average wealth of the market grows adiabatically, the wealth

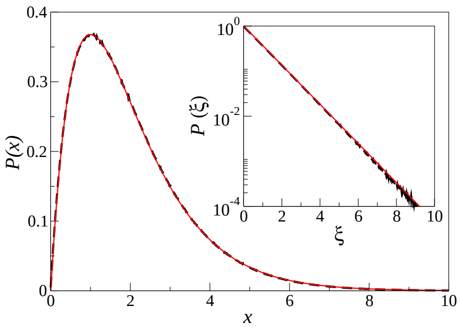

FIG. 4: Wealth distribution $P(x)$ of a generic kinetic model (16) (dashed line), obtained from simulations (with $N = 100$ and $\langle x \rangle = 2$), is compared with the steady state distribution $P(x) = x\exp(-x)$ (solid line) of corresponding non-conserving model (17). In the inset we compare PDF of noise ξ for conserving (dashed line) and non-conserving (solid-line) models. For the later one $P(\xi) = \exp(-\xi)$.

distribution of agents at any given time depends only on the instantaneous market. For *static* markets, exact steady state wealth distribution of agents was calculated for a few cases. More importantly, the dynamics of usual wealth conserving kinetic models studied in econophysics as model markets can be mapped to the dynamics of our model which does not have conservation.

Auto regression is a usual technique for economists, for study of financial time series. These new models, being auto-regressive in nature, build connections between standard auto-regressive models and other kinetic models of markets. Kinetic models which are believed to be the only model explaining Pareto-law for the tail of the wealth distribution is not quite correct. In particular both, conservation of wealth during each trading and global conservation of wealth are not necessary for obtaining Pareto-distribution. Auto-regression is an alternative which is more generic.

Acknowledgments : One of the authors (UB) would like to acknowledge thankfully the financial support of the Council of Scientific and Industrial Research, India (SPM-07/489(0034)/2007)

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