

# SU(5)-invariant decomposition of ten-dimensional Yang–Mills supersymmetry

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## Abstract

The N=1, d=10 superYang–Mills action is constructed in a twisted form, using  $SU(5)$  invariant decomposition of spinors in 10 dimensions. The action and its off-shell closed twisted scalar supersymmetry operator  $Q$  derive from a Chern–Simons term. The action can be decomposed as the sum of a term in the cohomology of  $Q$  and of a term that is  $Q$ -exact. The first term is a fermionic Chern-Simons term for a twisted component of the Majorana–Weyl gluino and it is related to the second one by a twisted vector supersymmetry with 5 parameters. The cohomology of  $Q$  and some topological observables are defined from descent equations. In this  $SU(5) \subset SO(10)$  invariant decomposition, the N=1, d=10 theory is determined by only 6 supersymmetry generators, as in the twisted N=4, d=4 theory. There is a superspace with 6 twisted fermionic directions, with solvable constraints.

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# 1 Introduction

There is a huge literature concerning the construction of supersymmetric theories in a twisted form. In twisted formulations one often gets great simplifications of supersymmetric transformations, and sets of auxiliary fields can be found for smaller supersymmetric subalgebra, which are however big enough to determine the theory. The twisted formulations are also the privileged framework to define topological observables.

The case of maximal supersymmetry has been studied quite extensively, in 4, 6 and 8 dimensions, but the twist of the  $N=1$ ,  $d=10$  theory has not been studied in depth. In fact, twisted aspects of theories with  $N=1$  supersymmetry are less familiar than those of theories with extended symmetries. The latter theories seem easier to twist, using the possible mixings between the  $R$ -symmetry and the Lorentz symmetry.

For  $N=1$  theories, there is no  $R$  symmetry to be used. However, in Kähler manifolds, the Lorentz symmetry is reduced down to  $SU(D) \subset SO(2D)$ , so that the spinors can be decomposed into holomorphic and antiholomorphic forms and a twist can be often performed. We refer to [1] [2] [3] [4] [5] [6][7] [8] for works related to  $N=1$  theories in twisted form, and to [9] for a  $Spin(7) \subset SO(10)$  invariant off-shell description in  $d=10$ .

Here we directly build the  $N=1$ ,  $d=10$  superYang–Mills theory in a twisted form, using  $SU(5)$  invariant decomposition of Majorana–Weyl spinors in 10 dimensions. We build the action and its symmetries by using a unification between twisted components of the Majorana–Weyl gluino and the antiholomorphic part of the gauge field. In this construction, the Batalin–Vilkowiski formalism is of great help. A Chern–Simons action with off-shell closed twisted scalar supersymmetry operator  $Q$  emerges quite naturally, using a generalization of previous works [11]. A striking property is found : the  $N=1$ ,  $d=10$  action is the sum of a term in the cohomology of  $Q$  and of a  $Q$ -exact term. Both terms are related by a twisted vector supersymmetry. Off-shell closure is obtained modulo Yang–Mills gauge invariance, but can be enforced exactly by introducing shadows [8].

The twisted scalar and vector supersymmetries that we exhibit can be also obtained by a brute force twist of the known on-shell closed 10-dimensional superYang–Mills supersymmetry transformations, using  $SU(5) \subset SO(10)$  decompositions of fields and symmetry generators. In this way one misses however the geometrical aspects. The main point of this paper is in fact the central role played by the Chern–Simons term and the observation that the  $N=1$ ,  $d=10$  superYang–Mills action is determined by only 6 twisted generators, with a possible twisted superspace  $6=1+5$  fermionic directions. The later can be constructed as in [8][9][10].

## 2 The d=10 superYang–Mills action as a supersymmetric gauge-fixing of the fermionic Chern–Simons action $\chi_{0,2}D_{0,1}\chi_{0,2}$

The 16 components of 10-dimensional Majorana–Weyl spinor can be decomposed in a  $SU(5) \subset SO(10)$  invariant way as a set of holomorphic and antiholomorphic forms

$$\lambda \sim \chi_{0,0} \equiv \chi, \Psi_{1,0} \equiv dz^m \Psi_m, \chi_{0,2} \equiv \chi_{\bar{m}\bar{n}} dz^{\bar{m}} dz^{\bar{n}} \quad (1)$$

with  $16 = 1 \oplus 5 \oplus 10$ . The  $z^m$  are complex coordinates, with  $SU(5)$  indices  $m, n, p, ..$  running from 1 to 5.  $z^{\bar{m}}, z^{\bar{n}}, ..$  are their complex conjugates.  $\Omega_{5,0}$  is the complex structure of the manifold,  $d^{10}x = \Omega_{5,0} dz^1 dz^2 dz^3 dz^4 dz^5$ . The Kähler 2-form is  $J_{\bar{n}m} = -J_{m\bar{n}}$ ,  $J_{mn} = J_{\bar{m}\bar{n}} = 0$ , and  $J \equiv J_{\bar{m}\bar{n}} dz^{\bar{m}} dz^{\bar{n}}$ . It can be used as a metric, with the notation  $X_m Y_{\bar{m}} \equiv g^{m\bar{n}} X_m Y_{\bar{n}}$ .

The 10-dimensional Dirac Lagrangian can be written in a twisted form as

$$\text{Tr } \lambda \not{D} \lambda = \text{Tr} (\epsilon_{mnpqr} \chi_{\bar{m}\bar{n}} D_{\bar{p}} \chi_{\bar{q}\bar{r}} + \chi_{\bar{m}\bar{n}} D_m \Psi_n + \chi D_{\bar{m}} \Psi_m) \quad (2)$$

Moreover, the Yang–Mills Lagrangian  $\text{Tr } F_{\mu\nu} F^{\mu\nu}$  can be written as

$$\text{Tr} (F_{mn} F_{\bar{m}\bar{n}} + (F_{\bar{m}}^m)^2) \sim \text{Tr} (F_{mn} F_{\bar{m}\bar{n}} - \frac{h^2}{2} + h F_{\bar{m}}^m) \quad (3)$$

modulo a boundary term  $J \wedge J \wedge J \wedge \text{Tr} (F \wedge F)$ . Here  $h$  is an auxiliary field and  $A = A_{1,0} + A_{0,1}$ ,  $A_{1,0} = A_m dz^m$ ,  $A_{0,1} = A_{\bar{m}} dz^{\bar{m}}$ . So the d=10 supersymmetric Yang–Mills action can be written in a twisted form as the sum of both expressions (2) and (3). The aim is to directly build this sum and its symmetries in a TQFT formalism.

The action will be expressed as the sum of a term in the cohomology of  $Q$  and of a  $Q$ -exact term, where  $Q$  is a scalar twisted supersymmetry generator. Moreover, the former is the 10-dimensional projection of a fermionic Chern–Simons term. The occurrence of a term in the cohomology of  $Q$  is specific to 10-dimensions. Dimensionally reduced actions with maximal supersymmetry, such as the d=4, N=4 theory, are purely  $Q$ -exact terms in lower dimensions.

We will unify the fields  $A_{\bar{m}}$  and  $\chi_{\bar{m}\bar{n}}$  as elements of a generalised 1-form  $\mathcal{A}$  and find a closed nilpotent scalar supersymmetry that acts naturally on this 1-form. This field unification within Chern–Simons or BF theories has been noticed in other papers [11]. The method will determine at once the supersymmetry and the twisted action.

To proceed, we consider the following unified one-form that is made of purely anti-holomorphic forms, all valued in the Lie algebra of a given gauge group.

$$\mathcal{A} = {}^*c_{0,5}^{-4} + {}^*A_{0,4}^{-3} + {}^*\chi_{0,3}^{-2} + \chi_{0,2}^{-1} + A_{0,1}^0 + c^1 \quad (4)$$

The grading is sum of the above index that is the shadow number (it was called ghost number as in the old TQFT language) plus the ordinary form degree. The notation  ${}^*\varphi$  means that  ${}^*\varphi$  is a Batalin–Vilkowiski (BV) antifield <sup>4</sup>. If one uses indices, one has  $\mathcal{A} = {}^*c_{\bar{m}\bar{n}\bar{p}\bar{r}\bar{s}} dz^{\bar{m}} dz^{\bar{n}} dz^{\bar{p}} dz^{\bar{r}} dz^{\bar{s}} + {}^*A_{\bar{m}\bar{n}\bar{p}\bar{r}} dz^{\bar{m}} dz^{\bar{n}} dz^{\bar{p}} dz^{\bar{r}} + {}^*\chi_{\bar{m}\bar{n}\bar{p}} dz^{\bar{m}} dz^{\bar{n}} dz^{\bar{p}} + \chi_{\bar{m}\bar{n}}^{-1} dz^{\bar{m}} dz^{\bar{n}} + A_{\bar{m}} dz^{\bar{m}} + c^1$ . We then consider the Chern–Simons form

$$\Delta = \text{Tr} \left( \mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A} \mathcal{A} \mathcal{A} \right) \quad (5)$$

and its action projected on a d=10 manifold with holonomy  $SU(5) \subset SO(10)$ ,

$$I = \int \Omega_{5,0} \Delta = \int \Omega_{5,0} \text{Tr} \left( \chi_{0,2} \overline{D} \chi_{0,2} + {}^*\chi_{0,3} (F_{0,2} + [c, \chi_{0,2}]) - {}^*A_{0,4} \overline{D} c - {}^*c_{0,5} c c \right) \quad (6)$$

where  $\overline{D} \equiv dz^{\bar{m}} d_{A_{\bar{m}}} = dz^{\bar{m}} (\partial_{\bar{m}} + A_{\bar{m}})$ . Because of the Chern–Simons formula  $d\Delta = \text{Tr} F_{\mathcal{A}} F_{\mathcal{A}}$ ,  $I$  satisfies a master equation, and can be interpreted as a BV action. The nilpotent symmetry of the action, expressed by the fermionic generator  $\delta$ , is obtained in a standard way by the generalized equations of motion of  $I$ ,

$$\delta\varphi = \frac{\delta I}{\delta {}^*\varphi} \quad \delta {}^*\varphi = -\frac{\delta I}{\delta\varphi} \quad (7)$$

$I$  needs a BV gauge-fixing, that is, the introduction of a gauge-fixing function  $\mathcal{Z}[\varphi]$  for fixing the antifields  ${}^*\varphi$  in function of the  $\varphi$ 's and get a quantum field theory for  $\varphi$

$${}^*\varphi = \frac{\delta \mathcal{Z}[\varphi]}{\delta\varphi} \quad (8)$$

The choice of the local functional  $\mathcal{Z}[\varphi]$  will be justified by power counting and symmetry requirements, with the demand of a vector symmetry  $Q_{\bar{m}}$  of the gauge-fixed action that anticommutes with  $Q$ . This will warrantee a Poincaré supersymmetry interpretation.

The antifield independent part  $\Omega_{5,0} \text{Tr} \chi_{0,2} \overline{D} \chi_{0,2}$  of  $I$  can be called the “classical” Lagrangian. It is a fermionic generalisation of the Chern–Simons action. It is  $Q$ -supersymmetric, with  $Q\chi_{0,2} = F_{0,2}$ , because of  $\overline{D}F_{0,2} = 0$ . It can be completed by

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<sup>4</sup> For the 3-dimensional Chern–Simons action, the last reference of [11] defined  $\mathcal{A} = {}^*c_3^{-2} + {}^*A_2^{-1} + A_1^0 + c^1$ , where  ${}^*A_2^{-1}$  and  ${}^*c_3^{-2}$  are the antifield of the gauge field  $A$  of the Faddeev Popov ghost  $c$ . One can presumably generalize the construction with  $SU(N)$ ,  $N > 5$ .

addition of the topological term  $JJJ\text{Tr } FF$ , so that, we understand the relevance of the “topological” Lagrangian

$$\Omega_{5,0}\text{Tr } \chi_{0,2}\overline{D}\chi_{0,2} + JJJ\text{Tr } FF \quad (9)$$

Had we started from such a topological Lagrangian and not understood the Chern–Simons structure, more work would have been needed to understand conventionally its TQFT gauge-fixing into the N=1, d=10 theory.

In the BV formalism, the so-called antifields  ${}^*\chi_{0,3}$ ,  ${}^*A_{0,4}$  and  ${}^*c_{0,5}$  are, respectively, the sources of  $\delta$ -supersymmetry transformations of the fields  $\chi_{0,2}$ ,  $A_{0,1}$  and  $c$ .

Eqs (7) give the scalar supersymmetry of the action  $I$ , with

$$\begin{aligned} \delta A_{\bar{m}} &= D_{\bar{m}}c, & \delta c &= -cc \\ \delta \chi_{\bar{m}\bar{n}} &= F_{\bar{m}\bar{n}} - [c, \chi_{\bar{m}\bar{n}}] \end{aligned} \quad (10)$$

$$\begin{aligned} \delta^* \chi_{0,3} &= \overline{D}\chi_{0,2} - [c, {}^*\chi_{0,3}] \\ \delta^* A_{0,4} &= \overline{D}^* \chi_{0,3} - [\chi_{0,2}, \chi_{0,2}] - [c, {}^*A_{0,4}] \\ \delta^* c_{0,5} &= \overline{D}^* A_{0,4} - [\chi_{0,2}, {}^*\chi_{0,3}] - [c, {}^*c_{0,5}] \end{aligned} \quad (11)$$

The property  $\delta^2 = 0$  is ensured by construction. The equivariant operator  $Q$  is obtained by molding out the Yang–Mills symmetry, that is, by setting  $c = {}^*c_{0,5} = 0$ . With this simplification, the BV Chern–Simons action is

$$I = \int \Omega_{5,0} \text{Tr } (\chi_{0,2}\overline{D}\chi_{0,2} + {}^*\chi_{0,3}F_{0,2}) \quad (12)$$

Its nilpotent twisted scalar supersymmetry generator  $Q$  is

$$\begin{aligned} QA_{0,1} &= 0 & Q^*A_{0,4} &= \overline{D}^* \chi_{0,3} - [\chi_{0,2}, \chi_{0,2}] \\ Q\chi_{0,2} &= F_{0,2} & Q^*\chi_{0,3} &= \overline{D}\chi_{0,2} \end{aligned} \quad (13)$$

The antifield  ${}^*B_{0,3}$  has a 2-form gauge invariance, due to the Bianchi identity.

$$\begin{aligned} s\chi_{0,2} &= \overline{D}L_{0,1} \\ s^*\chi_{0,3} &= \overline{D}M_{0,2} + [L_{0,1}, \chi_{0,2}] \end{aligned} \quad (14)$$

with  $sI = 0$ . It has a ghost of ghost of ghost degeneracy  $M_{0,2} \sim M_{0,2} + \overline{D}M_{0,1}$ ,  $M_{0,1} \sim M_{0,0} + \overline{D}M_{0,0}$ . This explains that the antifield  ${}^*\chi_{0,3}$  truly counts for  $4 = 10 - 5 + 1$  degrees freedom. After BV gauge-fixing,  ${}^*\chi_{0,3}$  will be expressed in function of an a holomorphic 1-form  $A_m$ , which also counts for  $4 = 5 - 1$  degrees freedom, modulo the gauge-invariance

$A_m \sim A_m + D_m \epsilon$ . This gauge symmetry will be preserved by the BV gauge function  $\chi_{0,2} F_{2,0}$ .

To eliminate the antifields, and obtain a theory with well-defined propagators for the fields (modulo the ordinary Yang–Mills gauge invariance), the standard BV routine suggest one to introduce two trivial BRST doublets,  $A_m, \Psi_m$ , and  $\chi, h$ . Their antifields are  ${}^*A_{\bar{m}}, {}^*\Psi_{\bar{m}}$ , and  ${}^*\chi, {}^*h$ . One thus adds the following trivial action  $I'$  to  $I$

$$I' = \int \text{Tr} ({}^*A_{\bar{m}}(\Psi_m - D_m c) + {}^*\chi(h - [c, \chi]) - {}^*h[c, h]) \quad (15)$$

Eqs. (7) give then the  $\delta$ -transformations for these doublets (and simple modifications for  $\delta^* c_{0,5}$  because of the  $c$  dependance in  $I'$ )

$$\begin{aligned} \delta A_m &= \Psi_m - D_m c & \delta^* A_{\bar{m}} &= -[c, {}^*A_{\bar{m}}] \\ \delta \Psi_m &= -[c, \Psi_m] & \delta^* \Psi_{\bar{m}} &= {}^*A_{\bar{m}} - [c, {}^*\Psi_{\bar{m}}] \\ \delta \chi &= h - [c, \chi] & \delta^* \chi &= -[c, {}^*\chi] \\ \delta h &= -[c, h] & \delta^* h &= -[c, {}^*h] + {}^*\chi \end{aligned} \quad (16)$$

For  $c = 0$ , one gets the Q supersymmetry

$$\begin{aligned} QA_m &= \Psi_m & Q\Psi_m &= 0 \\ Q\Psi_m &= 0 & Q^*\Psi_{\bar{m}} &= {}^*A_{\bar{m}} \\ Q\chi &= h & Q^*\chi &= 0 \\ Qh &= 0 & Q^*h &= {}^*\chi \end{aligned} \quad (17)$$

Finally,  $Q$  can be derived from the equivariant action

$$I_T = \int \text{Tr} (\Omega_{5,0} \text{Tr} (\chi_{0,2} \bar{D} \chi_{0,2} + {}^*\chi_{0,3} F_{0,2}) + \text{Tr} ({}^*A_{\bar{m}} \Psi_m + {}^*\chi h) \quad (18)$$

We choose the following  $s$ -invariant BV gauge-fixing fermion

$$\mathcal{Z}_V = \text{Tr} (\chi_{\bar{m}\bar{n}} F_{mn} + \chi(\frac{h}{2} + F_{m\bar{m}})) \quad (19)$$

Then, Eq. (8) fixes the antifields in functions of the fields

$$\begin{aligned} {}^*B_{\bar{m}\bar{n}\bar{p}} &= \epsilon_{\bar{m}\bar{n}\bar{p}\bar{r}\bar{s}} F_{rs} \\ {}^*A_{\bar{m}} &= D_n \chi_{\bar{n}\bar{m}} + D_{\bar{m}} \chi \\ {}^*\chi &= \frac{h}{2} + F_{m\bar{m}} \\ {}^*h &= 0 \end{aligned} \quad (20)$$

We will see shortly that this gauge-fixing function implies an additional  $SU(5)$ -vector symmetry with 5 parameters for the complete gauge-fixed action, which gives a larger symmetry with  $6=1+5$  parameters.

By substitution of the antifield values (20) in  $I_T$ , one finds the twisted  $N=1$ ,  $d=10$  action, modulo the topological term  $\int JJJ\text{Tr}(FF)$

$$\begin{aligned} I_T &= \int d^5z d^5\bar{z} \text{Tr} \left( F_{mn} F_{\bar{m}\bar{n}} - \frac{h^2}{2} + h F_{\bar{m}}^m + \epsilon_{mnpqr} \chi_{\bar{m}\bar{n}} D_{\bar{p}} \chi_{\bar{q}\bar{r}} + \chi_{\bar{m}\bar{n}} D_m \Psi_n + \chi D_{\bar{m}} \Psi_m \right) \\ &= \int \text{Tr} (F_{\mu\nu} F^{\mu\nu} + \lambda \not{D} \lambda) - \int JJJ\text{Tr}(FF) \end{aligned} \quad (21)$$

Moreover, after this elimination of the antifields,  $I_T$  splits into two distinguished terms

$$I_T = \int \text{Tr} \left( \epsilon_{mnpqr} \chi_{\bar{m}\bar{n}} D_{\bar{p}} \chi_{\bar{q}\bar{r}} + Q \left( \chi_{\bar{m}\bar{n}} F_{mn} + \chi \left( \frac{h}{2} + F_{m\bar{m}} \right) \right) \right) \quad (22)$$

The first Chern–Simons-like term  $\text{Tr} \epsilon_{mnpqr} \chi_{\bar{m}\bar{n}} D_{\bar{p}} \chi_{\bar{q}\bar{r}}$  is in the cohomology of  $Q$  and the second term is  $Q$ -exact. This decomposition of the  $N=1, d=10$  Lagrangian according to the cohomology of  $Q$  is quite interesting, and, moreover, it has been derived from the simplest BV action  $\int \Omega_{5,0} \text{Tr} (\chi_{0,2} D \chi_{0,2} + * \chi_{0,3} F_{0,2})$ . Scalar supersymmetry is a genuine consequence of the Bianchi identity, in a typically TQFT way.

One has  $10=9$  (for  $A$ )  $+1$  (for  $h$ ) bosonic degrees of freedom, modulo the gauge invariance of  $A$ . This equates the number of fermionic degrees of freedom in  $\chi_{\bar{m}\bar{n}}, \Psi_m, \chi$ , which is also  $10=4$  (for  $\chi_{\bar{m}\bar{n}}$ )  $+5$  (for  $\Psi_m$ )  $+1$  (for  $\chi$ ), if one counts 4 degrees of freedom for  $\chi_{\bar{m}\bar{n}}$ , taking into account the gauge symmetry  $\chi_{\bar{m}\bar{n}} \sim \chi_{\bar{m}\bar{n}} + \epsilon_{\bar{m}\bar{n}\bar{p}\bar{q}\bar{r}} D_{\bar{p}} M_{\bar{q}\bar{r}}$ . In a sense, the  $Q$ -exact part of the action corresponds to a balanced TQFT, with a gauge symmetry for  $\chi_{\bar{m}\bar{n}}$ .

Because of the obvious  $U(1)$  shadow number symmetry (the shadow charges are 1 for  $\Psi$ , -1 for  $\chi_{\bar{m}\bar{n}}$  and  $\chi$ , and 0 for all other fields), one has a global  $U(5) = SU(5) \times U(1)$  symmetry that commutes with the  $Q$  symmetry. However, the classical term  $\chi \bar{D} \chi$  has shadow number 2, and violates this  $U(1)$  symmetry in the Lagrangian. This may support the attracting idea of understanding the term  $\exp \int \chi_{0,2} \bar{D} \chi_{0,2}$  as an observable that can be inserted in the path integral<sup>5</sup> and also serves as a gauge-fixing for the gauge symmetry of the  $Q$ -exact term. Since the shadow number is only a  $SU(5) \subset SO(10)$ -invariant concept, one can however adopt the pragmatic attitude that it has to be conserved only modulo 2.

<sup>5</sup>This methodology has been already applied in lower dimensions[13].

### 3 Vector supersymmetry

One has the following vector symmetry

$$\begin{aligned}
 \delta_{\bar{p}} A_m &= g_{\bar{p}m} \chi & \delta_{\bar{p}} \chi &= 0 \\
 \delta_{\bar{p}} A_{\bar{m}} &= \chi_{\bar{p}\bar{m}} & \delta_{\bar{p}} h &= D_{\bar{p}} \chi \\
 \delta_{\bar{p}} \Psi_m &= F_{\bar{p}m} - g_{\bar{p}m} h & \delta_{\bar{p}} \chi_{\bar{m}\bar{n}} &= \epsilon_{\bar{m}\bar{n}\bar{p}\bar{q}\bar{r}} F_{\bar{q}\bar{r}}
 \end{aligned}$$

It satisfies  $\{\delta_{\bar{q}}, \delta_{\bar{p}}\} = 0$  but  $\{\delta, \delta_{\bar{p}}\} = \partial_{\bar{p}}$  only modulo equations of motion.

The vector symmetry is a symmetry of  $I_V = I + I'$ , but not of  $I$  and  $I'$  separately. Notice that this symmetry conserves the  $U(1)$  shadow symmetry only modulo shadow number 2. Thus the vector symmetry connect the  $Q$ -exact term and the term in the cohomology of  $Q$ . In fact, its requirement forces the above choice of the gauge fermion. For TQFT observables, it can be relaxed, since the mean values of  $Q$ -invariant observables donnot depend on the chosen coefficients in the  $Q$ -exact terms.

This shows that the  $N=1, d=10$  action is determined by a supersymmetry with only 6  $= (1 \text{ (scalar)} + 5 \text{ (vector)})$  parameters<sup>6 7</sup>. The 10 other symmetries occur as “accidental” extra supersymmetries, which enables the untwisting toward the Poincaré supersymmetric theory. Most of the proofs concerning the theory should be doable by using the core-supersymmetry with 6 generators, provided that the 10 other ones have no anomaly, which can be shown to be the case. A superspace with 6 fermionic directions can be clearly constructed, in the line of [8][9][10].

### 4 Topological observables

One has a set of  $Q$ -invariant observables because the cohomology of  $Q$  is non empty.

They follows from the existence of descent equations. They are obtainable by a simple rewriting of the BV equations for  $\int \Omega_{5,0} \text{Tr} (\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}\mathcal{A}\mathcal{A})$ , as follows (where  $\bar{\partial} \equiv dz^{\bar{m}} \partial_{\bar{m}}$ )

$$(Q + \bar{\partial})\mathcal{A} + \mathcal{A}\mathcal{A} = 0 \quad \text{that is} \quad Q\mathcal{A} = -\bar{\partial}\mathcal{A} - \mathcal{A}\mathcal{A} \quad (23)$$

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<sup>6</sup> This is in agreement with the former result that the  $N=4, d=4$  theory, that is, the compactification of the  $N=1, d=10$  theory in  $d=4$ , is also determined by 6  $= 1 \text{ scalar} + 1 \text{ scalar} + 4 \text{ vector}$  parameters [14].

<sup>7</sup> In superstring theory [15], there is also a six dimensional subalgebra of maximal supersymmetry, with manifest  $U(5) \subset SO(10)$  invariance, which points out furthermore the relevance of pure spinors.



One has

$$\begin{aligned}
Q\text{Tr} \left( \mathcal{A}\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\mathcal{A}\mathcal{A} \right) &= -\text{Tr} \left( \bar{\partial}\mathcal{A} + \mathcal{A}\mathcal{A} \right)Q\mathcal{A} = \text{Tr} \left( \bar{\partial}\mathcal{A} + \mathcal{A}\mathcal{A} \right)(d\mathcal{A} + \mathcal{A}\mathcal{A}) \\
&= -\bar{\partial}\text{Tr} \left( \mathcal{A}\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\mathcal{A}\mathcal{A} \right)
\end{aligned} \tag{24}$$

One has thus

$$(Q + \bar{\partial})\text{Tr} \left( \mathcal{A}\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\mathcal{A}\mathcal{A} \right) = 0 \tag{25}$$

so that one gets descent equations (they are actually very easy to check directly, using the Bianchi identity),

$$\begin{aligned}
Q\text{Tr} \left( \chi_{0,2}\bar{D}\chi_{0,2} \right) &= \bar{\partial}\text{Tr} \left( \chi_{0,2}F_{0,2} \right) \\
Q\text{Tr} \left( \chi_{0,2}F_{0,2} \right) &= \bar{\partial}\text{Tr} \left( A_{0,2}\bar{\partial}A_{0,2} + \frac{2}{3}A_{0,2}A_{0,2}A_{0,2} \right) \\
Q\text{Tr} \left( A_{0,2}\bar{\partial}A_{0,2} + \frac{2}{3}A_{0,2}A_{0,2}A_{0,2} \right) &= 0
\end{aligned} \tag{26}$$

We have therefore the following observables, defined as elements of the cohomology of the scalar supersymmetry

$$\begin{aligned}
&\int_{\mathcal{M}_{0,5}} \text{Tr} \left( \chi_{0,2}\bar{D}\chi_{0,2} \right) \\
&\int_{\mathcal{M}_{0,4}} \text{Tr} \left( \chi_{0,2}F_{0,2} \right) \\
&\int_{\mathcal{M}_{0,3}} \text{Tr} \left( A_{0,2}\bar{\partial}A_{0,2} + \frac{2}{3}A_{0,2}A_{0,2}A_{0,2} \right)
\end{aligned} \tag{27}$$

All gauge invariant functionals of  $A_{0,1}$  are also in the cohomology of  $Q$  since  $QA_{0,1} = 0$ . One has for instance the Wilson loops of the following type

$$\exp \int dz^{\bar{m}} A_{\bar{m}} \tag{28}$$

Because  $\Psi_m$  and  $A_m$  build a trivial  $Q$ -doublet, observables cannot depend on them.

## 5 More on the gauge degeneracy of $\text{Tr} \chi_{02}\bar{D}\chi_{02}$

In the abelian case, the  $Q$ -exact term can be understood as a gauge-fixing of the degenerate gauge symmetry of the action  $\text{Tr} \epsilon_{mnpqr}\chi_{\bar{m}\bar{n}}d_{\bar{p}}\chi_{\bar{q}\bar{r}}$ ,  $\chi_{0,2} \sim \chi_{0,2} + \bar{\partial}\epsilon_{01}$ ,  $\epsilon_{01} \sim \epsilon_{01} + \bar{\partial}\epsilon$  of the term  $\text{Tr} \epsilon_{mnpqr}\chi_{\bar{m}\bar{n}}d_{\bar{p}}\chi_{\bar{q}\bar{r}}$ . This symmetry leaves invariant the BV fermion  $\mathcal{Z}_V$ . We can build a BRST symmetry with ghosts, antighosts and Lagrange multipliers for  $\chi_{02}$ ,

that can be called equivariant with respect to ordinary  $U(1)$  (Yang–Mills) gauge transformations. It needs a ghost associated to  $\epsilon_{01}$ , that we identify with  $A_{01}$  ( $A_{\bar{m}}$  has the opposite even statistics to  $\epsilon_{\bar{m}}$ ). We identify  $A_{01}$  and  $\Psi_{10}$  as its antighost and Lagrange multiplier, respectively.

We thus have the following equivariant BRST symmetry of  $\chi_{02}\bar{\mathcal{D}}\chi_{02}$ , defined modulo ordinary gauge transformations  $A_{01} \sim A_{01} + \bar{\partial}c$  and  $A_{10} \sim A_{10} + \partial c$ ,

$$\begin{aligned} s\chi_{02} &= \bar{\partial}A_{01} & s\chi &= h \\ sA_{01} &= 0 & sh &= 0 \\ sA_{10} &= \Psi_{10} & s\Psi_{10} &= 0 \end{aligned} \quad (29)$$

This is an abelian BRST symmetry for a (anticommuting) 2-form and one can thus build the following BRST-exact gauge-fixing for the 2-form gauge symmetry,

$$\begin{aligned} & s(A_{[m}\partial_n]\chi_{\bar{m}\bar{n}} + \chi(\frac{h}{2} + \partial_{[\bar{m}}A_{m]})) \\ &= F_{mn}F_{\bar{m}\bar{n}} + \frac{b^2}{2} + bF_{\bar{m}m} - \chi_{\bar{m}\bar{n}}\partial_{[m}\Psi_{n]} - \chi\partial_{[\bar{m}}\Psi_{m]} \end{aligned} \quad (30)$$

It is equivariant with respect to the abelian Yang–Mills symmetry. The complete action

$$\chi_{02}\bar{\mathcal{D}}\chi_{02} + s(A_{[m}\partial_n]\chi_{\bar{m}\bar{n}} + \chi(\frac{h}{2} + \partial_{[\bar{m}}A_{m]})) \quad (31)$$

reproduces therefore the abelian version of supersymmetric action (21) that we previously built from other considerations, using the BV Chern–Simons action.

The non-abelian case needs more refinement. We only indicate that it needs the introduction of a 3-form gauge field, since the variation of  $\int \text{Tr } \chi_{02}\bar{\mathcal{D}}\chi_{02}$  under  $s\chi_{02} = \bar{\mathcal{D}}\epsilon_{01}$  is  $\int \text{Tr } [F_{02}, \epsilon_{01}]$ , which implies the introduction of a compensating term, resulting into the following classical action

$$I_{classical}(\chi_{02}, A_{01}, B_{03}) = \int \text{Tr } (\chi_{02}\bar{\mathcal{D}}\chi_{02} + B_{03}F_{02}) \quad (32)$$

Its gauge symmetry involves two non abelian parameters  $\epsilon_{01}$  and  $\epsilon_{02}$ ,

$$\begin{aligned} \delta\chi_{02} &= \bar{\mathcal{D}}\epsilon_{01} \\ \delta B_{03} &= \bar{\mathcal{D}}\epsilon_{02} - [\epsilon_{01}, F_{02}] \end{aligned} \quad (33)$$

Here  $B_{03}$  must be interpreted as a field, not an antifield. This action can be a priori quantized in two different ways. One possibility is a BRST invariant gauge-fixing, which results in a

different action than the non-abelian action (21). The other one is by using the equation of motion  $F_{02} = 0$  of  $B_{03}$  and eliminating  $A_{01}$  by solving this equation. By substitution in  $\chi_{02}\overline{D}\chi_{02}$ , this gives a sort of non-linear sigma model coupled to  $\chi_{02}$ . We will not discuss it here.

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