Entanglement entropy for even spheres

J.S.Dowker[1]

Theory Group,
School of Physics and Astronomy,
The University of Manchester,
Manchester, England

The coefficient of the logarithmic term in the entropy on even spheres is re–computed by the local technique of integrating the finite temperature energy density up to the horizon on static d–dimensional de Sitter space and thence finding the entropy by thermodynamics. Numeric evaluation yields the known answer *i.e.* (minus) the conformal anomaly on the d–sphere. The de Sitter quantities are obtained by conformal transformation of the Rindler ones, themselves obtained, for convenience, from those around a cosmic string. The expressions are given in terms of generalised Bernoulli polynomials for which an identity is derived. The arising spherical conformal anomaly is discussed and a formula is given for it for Branson's higher GJMS Laplacian, P_{2k}, as an oscillating polynomial in the level, k.

[1]dowker@man.ac.uk

1. Introduction.

In a previous work, [1], I have computed the universal logarithmic term that occurs in the expansion of the entanglement entropy associated with even spheres in the special case when the conically deformed Euclidean space–time used in the entropy recipe is the cyclically factored sphere, $\mathrm{S}^d/\mathbb{Z}_q$. Then the spatial surface is a $(d-2)$–sphere (with vanishing extrinsic curvature).

In this note I return to the same problem but the method this time uses the thermodynamic way of introducing a conical singularity, [2], and harks back, in detail, to a computation of the geometric entropy on de Sitter space, [3], by straightforward integration of the energy density off shell and then picking out the log term in the entropy before going back on shell.

This off shell procedure is well known and is used mostly in connection with the black hole geometry. Fursaev and Miele, [4], discuss the specific application to de Sitter space, relevant to the present situation. See also De Nardo *et al*, [5].

A recent paper by Marolf *et al* also sets up thermodynamics on static de Sitter by integrating the energy density holographically generated.

Calculations of spherical entanglement entropy have been performed by Casini and Huerta, [6], Solodukhin [7,8] and Lohmeyer *et al.*, [9].

2. The log coefficient in four dimensions

The geometry I treat is de Sitter space and the method, and formulae, are to be found in [3]. The novelty here is that the expansion away from the horizon, which there stopped at the most divergent term, is here taken far enough to give the logarithm. This is somewhat elementary but the different approach makes contact with the more geometric and holographic methods. Similar, but not identical, calculations have been done over the past fifteen years.

I start by writing the de Sitter metric in the static form,

$$ds^2=\frac{4a^2}{(1+Z^2)^2}\left(Z^2d(t/a)^2-dZ^2\right)-a^2\left(\frac{1-Z^2}{1+Z^2}\right)\left(d\theta^2+\sin\theta^2\,d\phi^2\right), \qquad (1)$$

where Z is related to the usual static radius r by $Z^2=(a-r)/(a+r)$. For convenience I set a, the radius of the de Sitter sphere, to unity. There is a slight computational advantage in using Z rather than r, and the metric exhibits a Rindler–like part. The part of the manifold I am interested in corresponds to the range,

$0 \le Z \le 1$, the lower limit being the location of the horizon where the field theory averages diverge. In this section I use a lower cut–off, $Z \ge \epsilon/2$, to define the global quantities. The basic idea is to compute the energy density, $\langle T_0^0 \rangle_\beta$, at a finite temperature, $T = 1/\beta$. Then integrate this over the region indicated to give the internal energy, $E(\beta)$, and next find the entropy from,

$$S(\beta) = \beta E'(\beta) - \int^\beta E'(\beta)\, d\beta = \int^\beta \beta\, dE'(\beta)\,, \tag{2}$$

where $E'(\beta)$ is the finite temperature correction to the energy.

The last act is to go on shell by setting $\beta = 2\pi$, the Gibbons–Hawking temperature associated with the horizon. At this temperature, static de Sitter turns into zero temperature de Sitter, [10], the state being the de Sitter invariant 'Euclidean' vacuum because the corresponding Green functions are related by the thermalisation, [11] [2]. See also Laflamme, [12].

Expressions for $\langle T_0^0 \rangle_\beta$ in four dimensions were given in [3]. There are various means of finding them. They can be calculated directly in de Sitter space from the re–periodised Green function, [11]. They can be found by conformally transforming the expressions in Rindler space, which can themselves be obtained from those around a cosmic string by transposition of coordinates. This last is the path I will follow here because the cosmic string results have already been calculated in d dimensions, [13]. I give a few subsidiary details. It will be noticed that B in the present situation plays the role that q did in [1].

For convenience I repeat the formulae on Rindler space with metric,

$$ds^2 = Z^2 d\,t^2 - dZ^2 - dx^2 - dy^2\,, \tag{3}$$

$$\begin{aligned}\langle T_0^0 \rangle_\beta &= \frac{1}{480\pi^2 Z^4}\,(B^2-1)(B^2+1)\,, \quad j=0\\ &= \frac{1}{1920\pi^2 Z^4}\,(B^2-1)(7B^2+17)\,, \quad j=1/2\\ &= \frac{1}{240\pi^2 Z^4}\,(B^2-1)(B^2+11)\,, \quad j=1\,,\end{aligned} \tag{4}$$

where $B = 2\pi/\beta$.

These quantities vanish on shell, *i.e.* when $B = 1$, which is a consequence of the subtraction of the Minkowski values. In fact the last terms in (4), the zero temperature values, do not contribute to the entropy, (2), as is easily seen, and we could simply concentrate on the remaining finite temperature corrections.

[2] However see [4].

The transition to de Sitter space uses the Bunch–Davies–Brown–Cassidy (BD–BC) relation between the averaged energy momentum tensor densities on conformally flat manifolds developed in [10], where the original references are given. It is not necessary to give this relation in detail. It connects tensor densities under conformal transformations in the expected manner, but there is an additional piece resulting from the integration of the conformal anomaly. For spaces conformal to Rindler, as here, with static conformal factors, the relation also holds for finite temperature quantities. The addition, however, is independent of temperature and can be ignored when calculating the entropy. The finite temperature corrections are thus homogeneously related by [3],

$$\sqrt{g}\,\langle T^0_0\rangle'_\beta = \sqrt{g}_R\,\langle T^{R\,0}_0\rangle'_\beta\,, \tag{5}$$

where R refers to Rindler (the expressions (4)) and the dash signifies the finite temperature corrections.

The relation of the coordinates implied in (5) is explained in [10]. The upshot is that the finite temperature corrections in static de Sitter can be obtained from (4) by the replacement,

$$\frac{1}{Z^4} \to \frac{(1+Z^2)^4}{16Z^4}\,. \tag{6}$$

The total de Sitter thermal energy is the integral,

$$\begin{aligned}E'_j(\beta) &= \int_{\epsilon/2}^1 \sqrt{g}\,dZ\,d\theta\,d\phi\,\langle T^0_0\rangle'_\beta \\ &= \frac{\big(P_j(B)-P_j(\infty)\big)}{16\pi}\int_{\epsilon/2}^1 \frac{(1-Z^2)^2}{Z^3}\,dZ\,,\end{aligned}$$

where $P_j(B)$ is the polynomial in (4), j indicating the spin *i.e.*

$$\langle T^{R,0}_0\rangle = \frac{P_j(B)}{16\pi^2\,Z^4}\,.$$

The constant, zero de Sitter temperature, term, $P_j(\infty)$, will not contribute to the entropy when (2) is constructed and hence can be dropped.

It then is easy to show that,

$$S_j(B) = \int^B \frac{1}{B}\,dP_j(B)\int_{\epsilon/2}^1 \frac{(1-Z^2)^2}{8Z^3}\,dZ\,, \tag{7}$$

[3] This is already evident for global quantities from the considerations in [14]

which, on shell, gives the simple expression, (using $P_j(1)=0$),

$$
\begin{aligned}
S_j(B)\bigg|_{\beta=2\pi} &= \int^1 dB\,\frac{P_j(B)}{B^2}\int_{\epsilon/2}^1 \frac{(1-Z^2)^2}{8Z^3}\,dZ \\
&= \left(\frac{1}{\epsilon^2}+\log(\epsilon/2)+\frac{\epsilon^2}{16}\right)\frac{1}{4}\int^1 \frac{dB}{B^2}\,P_j(B)\,.
\end{aligned}
\tag{8}
$$

I concentrate on the $\log\epsilon$ term as the others are not universal. Its coefficient is the number,

$$
C_j = \frac{1}{4}\int^1 \frac{dB}{B^2}\,P_j(B)\,,
$$

which is easily computed. I find

$$
C_0=\frac{1}{90}\,,\quad C_{1/2}=\frac{11}{180}\,,\quad C_1=\frac{16}{45}\,,
$$

and I should comment on these values.

The first two agree with (minus) the accepted values of the conformal anomaly on the four sphere if I note that I am using a Weyl neutrino, as in [10]. However the spin one value differs from the standard Maxwell conformal anomaly, $-62/90$, *e.g.* [15]. In [16], a relevant heat–kernel coefficient (that for transverse vectors) is given as $-2/45$.

3. Extension to higher dimensions

I now extend the procedure to d dimensions, but only for conformal scalars. The two factors in (8) extend separately. This is similar to the analysis of Solodukhin, [8], and Casini and Huerta, [6], where the formulae, following the geometry, likewise split into two.

The second integral in (8) is geometric and the extension to d dimensions consists of replacing the two–sphere metric in (1) by that for a $(d-2)$–sphere. This trivially alters the $\sqrt{|g|}$ measure.

The first integral is field–theoretic. I will again obtain it from the expressions for $\langle T_0^0\rangle$ around a cosmic string in d dimensions, [17], via the Rindler values which are related by,

$$
\langle T_0^0\rangle_{\text{Rindler}} = -(d-1)\,\langle T_0^0\rangle_{\text{string}}\,,
$$

using tracelessness of the local $\langle T_\mu^{\ \nu}\rangle$.

The general structure is now,

$$\langle T_0^0\rangle_{\rm Rindler} = \frac{P(d,B)}{2^d\pi^{d/2}Z^d}\,.$$

The BD–BC relation, (5), is still valid for the finite temperature corrections, which is all I need. It is fortunate the conformal anomaly contribution is not required since its d–dimensional form is unknown.

After the Rindler to de Sitter replacement, *cf* (6),

$$\frac{1}{Z^d}\to\frac{(1+Z^2)^d}{2^dZ^d}\,, \tag{9}$$

the expression for the entropy on shell becomes, as before,

$$S_j(\beta)\bigg|_{\beta=2\pi} = \frac{|S^{d-2}|}{2^{2d-3}\pi^{d/2-1}}\int^1 dB\,\frac{P(d,B)}{B^2}\int_{\epsilon/2}^1\frac{(1-Z^2)^{d-2}}{Z^{d-1}}\,dZ\,. \tag{10}$$

Picking out the log term in (10) I find its coefficient to be,

$$\begin{aligned} C(d) &= -(-1)^{d/2}\frac{|S^{d-2}|}{2^{2d-3}\pi^{d/2-1}}\binom{d-2}{d/2-1}\int^1 dB\,\frac{P(d,B)}{B^2}\\ &= (-1)^{d/2}\frac{1}{2^{d-2}\Gamma(d/2)}\int^1 dB\,\frac{P(d,B)}{B^2}\,. \end{aligned} \tag{11}$$

Perhaps a more fundamental quantity is the free energy and I just write down its off shell finite temperature correction,

$$F_j'(B) = \frac{|S^{d-2}|\,B}{2^{2d-4}\pi^{d/2-2}}\int^B dB\,\frac{P(d,B)}{B^2}\int_{\epsilon/2}^1\frac{(1-Z^2)^{d-2}}{Z^{d-1}}\,dZ\,. \tag{12}$$

The polynomial, $P(d,B)$, is actually the combination of two polynomials (see below). However, for the moment, I just present some explicit examples which can be read off from the formulae in [13], especially equns. (10) and (11).[4]

I list a few of the $P(d,B)$,

$$\begin{aligned} P(6,B) &= \frac{1}{945}(B^2-1)(10B^4+31B^2+31)\\ P(8,B) &= \frac{1}{4725}(B^2-1)(7B^2+17)(3B^4+10B^2+17)\\ P(10,B) &= \frac{4}{51975}(B^2-1)(30B^8+261B^6+1031B^4+2219B^2+2219)\,. \end{aligned} \tag{13}$$

[4] The expressions in [13] are for complex scalars. Hence there is a factor of two difference.

The polynomials do not contain a B^2 term, which is generally true and a consequence of conformal invariance (for scalars).

Compact expressions can be given in terms of generalised Bernoulli polynomials, as briefly mentioned in [13]. Algebra produces the explicit formula in d dimensions as the combination of two polynomials,

$$P(d,B) = -\frac{2^{d-1}(-1)^{d/2}\Gamma(d/2)}{d\,(d-2)!}\left(B_d^{(d+1)}(d/2|B,\mathbf{1}) + \frac{d(d-2)}{4}B_{d-2}^{(d-1)}(d/2-1|B,\mathbf{1})\right), \tag{14}$$

and it is this combination that accounts for the cancellation of the B^2 terms. Each term in (14) vanishes at $B = 1$ by virtue of a standard property of Bernoulli polynomials, [18], [19].

Substitution into (11) then produces, on shell, the log coefficient,

$$C(d) = \frac{2}{d\,(d-2)!}\int^1 \frac{dB}{B^2}\left(B_d^{(d+1)}(d/2|B,\mathbf{1}) + \frac{d(d-2)}{4}B_{d-2}^{(d-1)}(d/2-1|B,\mathbf{1})\right), \tag{15}$$

whose numerical evaluation yields (minus) the scalar conformal anomalies on the d–sphere, as do the calculations mentioned earlier. The bare evaluation itself gives no clue as to why this should be so. In our previous approach, [1], the analytical reason was that the derivative of the conformal anomaly on the orbifolded d–sphere was zero for the non–singular sphere, *i.e.* on shell, and I make some relevant remarks in the next section.

The conformal anomaly values are available in the old literature and, for convenience, and interest, I give a few remarks and specifics in the next section.

4. Conformal anomaly on spheres

As has been used above, the conformal anomaly is the constant term in the heat–kernel expansion.[5] There are a number of ways of finding this, based on the standard spectral information for the (conformal) Laplacian on d–spheres (which can be traced back to George Green in the 1830s).

There is no need to elaborate, as the work of Copeland and Toms, [20], covers the necessary ground. One technique uses the values of the corresponding ζ–function, the required analytic continuation of which was done by expressing it as a series of Hurwitz ζ–functions, which have known properties. Birmingham, [21],

[5] The *trace* anomaly is a constant times the conformal anomaly.

employs a similar approach. A list of the numerical results can be found in [20] for scalars and spinors.

Among other works, I mention the interesting one by Cappelli and D'Appolonio, [22], which employs basically the same method, only slightly more streamlined. It also lists the values.

A different organisation of the spectral information was given in [23,24] and resulted in the compact expression for the heat–kernel coefficients (the conformal anomaly) on a d–sphere,

$$C_{d/2} = \frac{1}{d!}\bigg(B_d^{(d)}(d/2-1\big|\mathbf{1}) + B_d^{(d)}(d/2\big|\mathbf{1})\bigg)\,, \tag{16}$$

which rapidly evaluates to the known numbers. Spelling out some calculational details, this evaluation can be done using the integral relation, [18], p.171,

$$B_\nu^{(n-1)}(x\,|\mathbf{1}) = \int_0^1 dt\, B_\nu^{(n)}(x+t\,|\mathbf{1})\,,$$

which produces, from (16),

$$C_{d/2} = \frac{1}{d!}\int_0^1 dt\,\bigg(B_d^{(d+1)}(d/2+t\,|\mathbf{1}) + B_d^{(d+1)}(d/2-t\,|\mathbf{1})\bigg)\,, \tag{17}$$

with the product form, (*e.g.* [18] p.186, [25], §8),

$$B_d^{(d+1)}(x\big|\,\mathbf{1}) = (x-1)(x-2)\ldots(x-d)\,.$$

This yields an expression identical to that given by Diaz, [26],

$$C_{d/2} = 2\frac{(-1)^d}{d!}\int_0^1 dt\prod_{i=0}^{d/2-1}(i^2-t^2)\,, \tag{18}$$

which he obtains on the basis of a holographic technique as the simplest example of a more general construction of the conformal anomaly associated with Branson's GJMS higher Laplacian operators, P_{2k}, on the sphere, *e.g.* Branson, [27], Graham, [28], Gover [29], Juhl, [30].

It is possible to show [6], using the approach of [24], that the general result at level k is,

$$\begin{aligned} C_{d/2}(k) &= \frac{1}{k\,d!}\sum_{j=0}^{k-1}\bigg(B_d^{(d)}(d/2+j\,|\mathbf{1}) + B_d^{(d)}(d/2+j+1\,|\mathbf{1})\bigg) \\ &= \frac{2(-1)^{d/2}}{k\,d!}\int_0^k dt\prod_{i=1}^{d/2-1}\big(i^2-t^2\big)\,, \end{aligned} \tag{19}$$

[6] The details will be presented at another time.

which evaluates to a polynomial in k, for a given d; for example,

$$\begin{aligned}
C_2(k) &= \frac{2}{3.\,5!}\,k^2(3k^2-5)\\
C_3(k) &= \frac{2}{3.\,7!}\,k^2(3k^4-21k^2+28)\\
C_4(k) &= \frac{2}{5.\,9!}\,k^2(5k^6-90k^4+441k^2-540)\\
C_5(k) &= \frac{2}{3.\,11!}\,k^2(3k^8-110k^6+1287k^4-5412k^2+6336)\\
C_6(k) &= \frac{2}{105.\,13!}\,k^2(105k^{10}-6825k^8+155155k^6-1490775k^4\\
&\qquad\qquad\qquad\qquad +5753748k^2-6552000)\,.
\end{aligned}\tag{20}$$

Evaluation gives agreement with the numbers found by Diaz using holography, [26], eqn.(7.1) and Table 3.

The extrema of these polynomials lie in the region $|k|<d/2$ and are located, for positive k, closely below the integers which fact corresponds to the alternating signs at the 'physical' values, $k=1,\ldots,d/2$.

In this approach, the operator existence condition on the level, $k\le d/2$ (for d even), is equivalent to the absence of a zero mode in the spectrum of P_{2k}. (Actually for the Neumann modes on the hemisphere.) It may be possible to relax this condition.

It is worth noting that (17), (18) and (19) do not involve the Bernoulli numbers, unlike some previously mentioned computations.

Returning to the simplest case, $k=1$, equating (16) and (15) gives an identity between generalised Bernoulli polynomials. In fact there is the *off shell* identity

$$\begin{aligned}
&\frac{1}{B}\bigg(B^{(d)}_d(d/2-1\,|\,B,\mathbf{1})+B^{(d)}_d(d/2\,|\,B,\mathbf{1})\bigg)=\\
&\qquad 2(d-1)\int^B\frac{dB}{B^2}\bigg(B^{(d+1)}_d(d/2|B,\mathbf{1})+\frac{d(d-2)}{4}B^{(d-1)}_{d-2}(d/2-1|B,\mathbf{1})\bigg)\,.
\end{aligned}\tag{21}$$

The left–hand side is $d!$ times the conformal anomaly on the orbifolded d-sphere (or periodic lune), $\mathrm{S}^d/\mathbb{Z}_B$, discussed in [1] where it was proved that it had an extremum when $B=1$, *i.e.* on shell. This can be confirmed here immediately since (each term in) the integrand vanishes on shell. Reference to (12) shows that this corresponds to an extremum for $\beta\,F$, in thermodynamic language.

By applying Barnes' derivative formula,

$$
\begin{aligned}
\frac{\partial}{\partial B}\frac{1}{B}B_k^{(d)}(a|\,B,\mathbf{1}) &= -\frac{1}{B^2}\,B_k^{(d+1)}(a+B|\,B,B,\mathbf{1})\,,\\
&= -(-1)^k\,\frac{1}{B^2}\,B_k^{(d+1)}(d-1-a+B|\,B,B,\mathbf{1})\,,
\end{aligned}
\tag{22}
$$

to (21), I obtain the off shell identity (d is even),

$$
\begin{aligned}
B_d^{(d+1)}(d/2\,|\,B,\mathbf{1}) &+ \frac{d(d-2)}{4}\,B_{d-2}^{(d-1)}(d/2-1\,|\,B,\mathbf{1})\\
&= -\frac{1}{d(d-1)}\,B_d^{(d+1)}(d/2+B\big|\,B,B,\mathbf{1})\,,
\end{aligned}
\tag{23}
$$

which is not obvious without further manipulation. It furnishes a simpler formula for the scalar conformal polynomials, $P(d,B)$, (14),

$$
P(d,B) = -\frac{2^{d-1}(-1)^{d/2}\Gamma(d/2)}{d^2\,(d-1)!}\,B_d^{(d+1)}(d/2+B\big|\,B,B,\mathbf{1})\,. \tag{24}
$$

5. Comments

The calculation of the entropy in section 4 has similarities with others in this topic, *e.g.* [8,6]. However, in these, the entropy is usually computed from the global effective action while I have employed a local method which involves integrating the thermal average of the canonical energy obtained, *e.g.*, from the coincidence limit of a bilinear operator acting on the thermal Green function. This gives the total internal energy from which the free energy can be determined and thence the entropy.

The numerical evaluation is algorithmic and offers no obvious reason why the log coefficient is (minus) the conformal anomaly, although, being conformally invariant, there is not much else it could be. In (15) one has a rather complicated formula for calculating the conformal anomaly on spheres.

I have taken a simple minded approach to the conical singularity and have integrated the local energy density using a basic cut–off, resorting neither to smoothing the geometry nor to integrating against a test function. These procedures might appear more rigorous, but have their own ambiguities. I believe the present method is sufficient to achieve its aim.

I also note that the formulae encountered are closely related to expressions occurring in the holographic calculations of Ryu and Takayanagi of the entanglement entropy for a disk, [31] §7.1.2, and also to those of Graham, [32], on volume renormalisation.

The finite temperature averaged stress tensor on static de Sitter that I have used does not agree with that derived by Marolf *et al*, [33], on the basis of a dS/CFT duality.

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