On the classical limit of quantum mechanics and the threat of fundamental graininess and chaos to the correspondence principle.

Mario Castagnino
Institutos de Astronoía y Fisica del Espacio y de Física Rosario.
Casilla de Correos 67, Sucursal 28,
1428 Buenos Aires, Argentina.

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Abstract

The aim of this paper is to review the classical limit of Quantum Mechanics and to precise the well known threat of chaos (and fundamental graininess) to the correspondence principle. We will introduce a formalism for this classical limit that allows ud find the surfaces defined by the constants of the motion in phase space. Then in the integrable case we will find the classical trajectories, and in the non-integrable one the fact that regular initial cells become "amoeboid-like". This deformations and their consequences can be considered as a threat to the correspondence principle. Essentially we present an analysis of the problem similar to the one of Omnès, but with a simpler mathematical structure.

1 Introduction

It seems that Einstein was the first one to realize that chaos was a threat to quantum mechanics [1] in a paper that was ignored by forty years [2]. A panoramic view of the this incompatibility of the classical chaos and quantum concepts (up to 1994) can be found in [1] and a recent review in [3]. Our first contribution to the subject was the introduction of a theory of the classical limit for closed quantum systems with Hamiltonian with continuos spectrum based in destructive interference (that we have called the "Self Induced Decoherence" -SID- and where we have used the Riemann-Lebesgue theorem [4]) and later we found a class of quantum chaotic systems (that may not contain all cases but certainly it contains the relevant ones) with chaotic classical limit [5]. With this idea in mind we study quantum chaos in papers [6] and extended the notions of non-integrable, ergodic and mixing quantum systems in paper [7]. These works were inspired in the landmark paper of Bellot and Earman [8]. The aim of this remarkable paper is precisely to show " how chaos puts some pressure on the correspondence principle (CP)" and the author says that there is not a "quick and convincing argument for the conclusion that the CP fails". Another important source of inspiration for us was the two books of Roland Omnès [9] and [10], precisely the characterization of quantum chaos as the evolution of a square cell to a distorted "amoeboid" cell (see figure 6.B). In this paper we will essentially follow this idea, with simpler mathematical methods, and we will try to precise the origin of the elongated, distorted and final amoeboid cells which, in fact, we consider the main threat to the CP.

The paper is organized as follows:

Section 2: We introduce the mathematical structures we will use.

In the next sections we will see that the classical limit can be obtained using three weapons: decoherence, Wigner transformation and the limit $\frac{\hbar}{S} \to 0$. Precisely:

Section 3. We review the decoherence alla SID for non-integrable quantum systems.

Section 4 We obtain the classical statistical limit, using Wigner transformation and the limit $\frac{\hbar}{S} \to 0$, and the classical surfaces defined by the constant of the motion in phase space. We show that, up to this point chaos is not threat to the CP.

Section 5 deals the fundamental graininess of quantum mechanics

Section 6 We find the classical trajectories for the integrable system and estimate the threat to the CP, in the non-integrable case. The main conclusion will be that fundamental graininess and chaos constitutes a real menace for the CP.

Section 7 We present our conclusions.

2 Mathematical background

In this section we will review, following ref. [5], the main mathematical concepts we will use in the paper.

2.1 Weak limit

Our presentation is based on the algebraic formalism of quantum mechanics ([11], [12]). Let us consider an algebra $\mathcal{A}$ of operators, whose self-adjoint elements $O = O^\dagger$ are the observables belonging to the space $\mathcal{O}$. The states ρ are linear functionals belonging to the dual space $\mathcal{O}'$, but they must satisfy the usual conditions: self-adjointness, positivity and normalization and therefore the state ρ belongs to a convex $\mathcal{S}$. If $\mathcal{A}$ is a C*-algebra, it can be represented by a Hilbert space (GNS theorem see [12]). If $\mathcal{A}$ is a nuclear algebra, it can be represented by a rigged Hilbert space, as proved by a generalization of the GNS theorem ([13], [14]). In this case, the van Hove states with a singular diagonal can be properly defined (see [15]; for a rigorous presentation of the formalism, see also [16]).

If we write the action of the functional ρ on the space $\mathcal{O}$ as $(\rho|O)$, then we can say that:

- The evolution $U_t\rho = \rho(t)$ has a *Weak-limit* if, for any $O \in \mathcal{O}$ and any $\rho \in \mathcal{S}$, there is a unique $\rho_* \in \mathcal{S}$ such that

$$\lim_{t\to\infty}(\rho(t)|O) = (\rho_*|O), \qquad \forall\, O \in \mathcal{O} \tag{1}$$

  We will symbolize this limit as

$$W - \lim_{t\to\infty}\rho(t) = \rho_* \tag{2}$$

- A particular useful weak limit can be obtained using the Riemann-Lebesgue theorem. The idea of destructive interference is embodied in this theorem, according to which, if $f(\nu) \in \mathbb{L}_1$, then

$$\lim_{t\to\infty}\int_a^b f(\nu)\, e^{-i\nu t}\, d\nu = 0 \tag{3}$$

  If we can express the action of a functional $\rho(t) \in \mathcal{S}$ on the operator $O \in \mathcal{O}$ as

$$(\rho(t)|O) = \int_a^b [A\,\delta(\nu) + f(\nu)]\; e^{-i\nu t}\, d\nu \tag{4}$$

  with $f(\nu) \in \mathbb{L}_1$, then

$$\lim_{t\to\infty}(\rho(t)|O) = \lim_{t\to\infty}\int_a^b [A\,\delta(\nu) + f(\nu)]\; e^{-i\nu t}\, d\nu = A = (\rho_*|O), \quad \forall\, O \in \mathcal{O} \tag{5}$$

  We will call this result "*Weak Riemann-Lebesgue limit*".

2.2 Generalized Projections.

As it is well known, in order to describe an irreversible process in terms of an unitary evolution it is necessary to break the underlying unitary evolution. The usual tool to do this is to introduce a coarse graining, that restricts the information of the system. But generically any information restriction can be obtained using a projection, which retains the "relevant" information and discards the "irrelevant" one of the considered system.

In fact, in its traditional form, the action of a projection is to eliminate some components of the state vector corresponding to the finest description (see [35]) to obtain a coarse grained one. If this idea is generalized, any restriction of information can be conceived as the result of a convenient projection. In fact, we can define a projector Π belonging to the space $\mathcal{O} \otimes \mathcal{O}'$ such that

$$\Pi \doteq \sum_j |O_j)(\,\rho_j| \tag{6}$$

where $(\rho_j| \in \mathcal{O}'$ satisfies $(\rho_j|O_k) = \delta_{jk}$ where $|O_k)\epsilon\mathcal{O}$ [1]. Therefore, the action of Π on $\rho \in \mathcal{O}'$ involves a projection leading to a state ρ_P such that

$$\rho_P \circeq \rho\,\Pi = \sum_j (\rho|O_j)(\,\rho_j| \tag{7}$$

where in ρ_P only contains the information that we can obtain from the observables $|O_k)\epsilon\mathcal{O}$

2.3 Weyl-Wigner-Moyal mapping.

Let $\Gamma = \mathcal{M}_{2(N+1)} \equiv \mathbb{R}^{2(N+1)}$ be the phase space. The functions over Γ will be called $f(\phi)$, where ϕ symbolizes the coordinates of Γ, $\phi = (q^1, ..., q^{N+1}, p_q^1, ..., p_q^{N+1})$. If we consider the operators $\widehat{f}$,$\widehat{g}$, ... $\in \widehat{\mathcal{A}}$ and the candidates to be the corresponding distribution functions $f(\phi), g(\phi), ... \in \mathcal{A}$, where $\widehat{\mathcal{A}}$ is the quantum algebra of operators and $\mathcal{A}$ is the classical algebra of distribution functions, the *Wigner transformation* reads (see [36], [37], [38])

$$symb\widehat{f} \circeq f(\phi) = \int \langle q+\Delta|\,\widehat{f}\,|q-\Delta\rangle e^{2i\frac{p\Delta}{\hbar}} d^{N+1}\Delta \tag{8}$$

We can also introduce the *star product* (see [39]),,

$$symb(\widehat{f}\,\widehat{g}) = symb\,\widehat{f}*symb\,\widehat{g} = (f*g)(\phi) = f(\phi)\exp\left(-\frac{i\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right)g(\phi) \tag{9}$$

and the *Moyal bracket,* that is, the symbol corresponding to the quantum commutator

$$\{f,g\}_{mb} = \frac{1}{i\hbar}(f*g-g*f) = symb\left(\frac{1}{i\hbar}[f,g]\right) \tag{10}$$

It can be proved that (see [36])

$$(f*g)(\phi) = f(\phi)g(\phi) + 0(\hbar),\ \{f,g\}_{mb} = \{f,g\}_{pb} + 0(\hbar^2) \tag{11}$$

To define the inverse $symb^{-1}$, we will use the *symmetrical* or *Weyl* ordering prescription, namely,

$$symb^{-1}[q^i(\phi), p^j(\phi)] \circeq \frac{1}{2}\left(\widehat{q}^i\widehat{p}^j + \widehat{p}^j\widehat{q}^i\right) \tag{12}$$

Therefore, by means of the transformations $symb$ and $symb^{-1}$, we have defined an isomorphism between the quantum algebra $\widehat{\mathcal{A}}$ and the "classical-like" algebra $\mathcal{A}_q$,

$$symb^{-1}: \mathcal{A}_q \to \widehat{\mathcal{A}}, \quad symb: \widehat{\mathcal{A}} \to \mathcal{A}_q \tag{13}$$

The mapping so defined is the *Weyl-Wigner-Moyal symbol.*[2]

[1] In fact, Π is a projector since

$$\Pi^2 = \sum_{jk} |O_j)(\,\rho_j|O_k)(\,\rho_k| = \sum_{jk} |O_j)\delta_{jk}(\,\rho_k| = \sum_j |O_j)(\,\rho_j| = \Pi$$

[2] When $\hbar \to 0$, then $\mathcal{A}_q \to \mathcal{A}$, where $\mathcal{A}$ is the classical algebra of observables over phase space.

The Wigner transformation for states is

$$\rho(\phi) = symb\,\widehat{\rho} = (2\pi\hbar)^{-(N+1)}\, symb_{(\text{for operators})}\,\widehat{\rho} \tag{14}$$

As it is well known, an important property of the Wigner transformation is that:

$$\langle \widehat{O}\rangle_{\widehat{\rho}} = (\widehat{\rho}|\widehat{O}) = (symb\,\widehat{\rho}\,|\; symb\,\widehat{O}) = \int d\phi^{2(N+1)}\rho(\phi)\,O(\phi) \tag{15}$$

This means that the definition of $\widehat{\rho} \in \widehat{\mathcal{A}'}$ as a functional on $\widehat{\mathcal{A}}$ is equivalent to the definition of $symb\,\rho \in \mathcal{A}_q'$ as a functional on $\mathcal{A}_q$.

3 Decoherence in non-integrable systems

3.1 Local CSCO

This subsection is a short version of the corresponding subsection of paper [5].

a.- In [5] we have proved that, when the quantum system is endowed with a CSCO of $N+1$ observables containing $\widehat{H}$, that defines an eigenbasis in terms of which the state of the system can be expressed, the corresponding classical system is *integrable*. In fact, if the CSCO is $\{\widehat{H}, \widehat{G}_1, ..., \widehat{G}_N\}$, the Moyal brackets of its elements are

$$\{G_I(\phi), G_J(\phi)\}_{mb} = symb\left(\frac{1}{i\hbar}[\widehat{G}_I, \widehat{G}_J]\right) = 0 \tag{16}$$

where $I,\ J = 0, 1, ..., N$, $\widehat{G}_0 = \widehat{H}$, and $\phi \in \mathcal{M} \equiv \mathbb{R}^{2(N+1)}$. Then, when $\hbar \to 0$, from Eq. (11) we know that

$$\{G_I(\phi), G_J(\phi)\}_{pb} = 0 \tag{17}$$

Thus, since $H(\phi) = G_0(\phi)$, the set $\{G_I(\phi)\}$ is a complete set of $N+1$ constants of motion in involution, *globally* defined all over $\mathcal{M}$; as a consequence, the system is *integrable*.

b.- We have also proved (see [5]) that, when the CSCO has $A+1 < N+1$ observables, a local CSCO $\{\widehat{H}, \widehat{G}_1, ..., \widehat{G}_A, \widehat{O}_{i(A+1)}, ..., \widehat{O}_{iN}\}$ can be defined for a maximal domain $\mathcal{D}_{\phi_i}$ around any point $\phi_i \in \Gamma \equiv \mathbb{R}^{2(N+1)}$, where Γ is the phase space of the system. In this case the system is *non-integrable*.

In order to prove this assertion, we have to recall the *Carathèodory-Jacobi theorem* (see [40], theorem 16.29) according to which, when a system with $N+1$ degrees of freedom has $A+1$ global constants of motion in involution $\{G_0(\phi), G_1(\phi), ..., G_A(\phi)\}$, then $N-A$ local constants of motion in involution $\{A_{i(A+1)}(\phi), ..., A_{iN}(\phi)\}$ can be defined in a maximal domain $\mathcal{D}_{\phi_i}$ around ϕ_i, for any $\phi_i \in \Gamma \equiv \mathbb{R}^{2(N+1)}$ (see also section 3.2 below).

Let us consider the particular case of a classical system with $N+1$ degrees of freedom, and whose only global constant of motion (for simplicity) is the Hamiltonian $H(\phi)$. The Carathèodory-Jacobi theorem states that, in this case, the system has N local constants of motion $A_{iI}(\phi)$, with $I=0,...,N$, in the maximal domain $\mathcal{D}_{\phi_i}$ around ϕ_i, for any $\phi_i \in \Gamma$.

If we want to translate these phase space functions into the quantum language, we have to apply the transformation $symb^{-1}$; this can be done in the case of the Hamiltonian, $\widehat{H} = symb^{-1}H(\phi)$, but not in the case of the $A_{iI}(\phi)$ because they are defined in a maximal domain $\mathcal{D}_{\phi_i} \subset \Gamma$ and the Weyl-Wigner-Moyal mapping can only be applied on phase space functions defined on the whole phase space Γ. To solve this problem, we can introduce a positive partition of the identity (see [41]),

$$1 = I(\phi) = \sum_i I_i(\phi) \tag{18}$$

where each $I_i(\phi)$ is the *characteristic* or *index* function

$$I_i(\phi) = \begin{cases} 1 \text{ if } \phi \in D_{\phi_i} \\ 0 \text{ if } \phi \notin D_{\phi_i} \end{cases} \tag{19}$$

and $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$, $D_{\phi_i} \cap D_{\phi_j} = \emptyset$, $\bigcup_i D_{\phi_i} = \Gamma$. Then we can define the functions $O_{iI}(\phi)$ as

$$O_{iI}(\phi) = A_{iI}(\phi)\, I_i(\phi) \tag{20}$$

Now the $O_{iI}(\phi)$ are defined for all $\phi \in \Gamma$; so, we can obtain the corresponding quantum operators as

$$\widehat{O}_{iI} = symb^{-1} O_{iI}(\phi) \tag{21}$$

Since the original functions $A_{iI}(\phi)$ are local constants of motion in the maximal domain $\mathcal{D}_{\phi_i}$, they make zero the corresponding Poisson brackets, with H, in such a domain and, a fortiori, in the non-maximal domain $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$. This means that the $O_{iI}(\phi)$ makes zero the corresponding Poisson brackets in the whole space space Γ. In fact, for $\phi \in D_{\phi_i}$, because $O_{iI}(\phi) = A_{iI}(\phi)$, and trivially for $\phi \notin D_{\phi_i}$. We also know that, in the macroscopic limit $\hbar \to 0$, the Poisson brackets can be identified with the Moyal brackets, that is, the phase space counterpart of the quantum commutator (see eq. (11)) [3]. Therefore, we can guarantee that all the observables of the set $\left\{\widehat{H}, \widehat{O}_{iI}\right\}$ commute with each other:

$$\left[\widehat{H}, \widehat{O}_{iI}\right] = 0 \qquad \left[\widehat{O}_{iI}, \widehat{O}_{iJ}\right] = 0 \tag{22}$$

for $I, J = 1$ to N and in all the D_{ϕ_i}. As a consequence, we will say that the set $\left\{\widehat{H}, \widehat{O}_{i1}, ..., \widehat{O}_{iN}\right\}$ is the *local CSCO* of $N+1$ observables corresponding to the

[3] Even if these reasoning is only valid in the limit $\hbar \to 0$ it is enough for our purposes since essentially we are trying to find classical limit.

domain $D_{\phi_i} \subset \Gamma$. If $\widehat{H}$ has a continuous spectrum $0 \leq \omega < \infty$, and the $\widehat{O}_{iI}$ a discrete one (just for simplicity) a generic observable $\widehat{O}$ can be decomposed as

$$\widehat{O} = \sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' \; \widetilde{O}_{im_{iI}\, m'_{iI}}(\omega, \omega') \, |\omega, m_{iI}\rangle\langle\omega', m'_{iI}| \tag{23}$$

where the $|\omega, m_{iI}\rangle = |\omega, m_{i1}, ..., m_{iN}\rangle$ are the eigenvectors of the local CSCO $\left\{\widehat{H}, \widehat{O}_{iI}\right\}$ corresponding to D_{ϕ_i}. Since it can be proved that (see [5]), for $i \neq j$,

$$\langle\omega, m_{iI} \,|\omega, m_{jI}\rangle = 0 \tag{24}$$

the decomposition of eq. (23) is orthonormal, and it generalizes the usual eigen-decomposition of the integrable case to the non-integrable case. Therefore, any $\widehat{O}_{iI}$ corresponding to the domain D_{ϕ_i} commutes with any $\widehat{O}_{jI}$ corresponding to the domain D_{ϕ_j} with $i \neq j$,[4]

$$\left[\widehat{O}_{iI}, \widehat{O}_{jJ}\right] = \delta_{ij}\, \delta_{IJ} \tag{25}$$

3.2 Continuity and differentiability.

In paper [7], we have used a “bump” smooth function $B_i(\phi)$, in each domain D_{ϕ_i} surrounded by a frontier zone F_{ϕ_i},such that $D_i(\phi) \cup F_{\phi_i} \subset \mathcal{D}_i(\phi)$, and we have defined a new partition of the identity (compare with (18)),

$$1 = I(\phi) = \sum_i B_i(\phi) \tag{26}$$

where each $B_i(\phi) \geq 0$ satisfies (compare with (19))

$$B_i(\phi) = \begin{cases} 1 \text{ if } \phi \in D_{\phi_i} \\ \epsilon[0,1] \text{ if } \phi \notin F_{\phi_i} \\ 0 \text{ if } \phi \notin D_{\phi_i} \cup F_{\phi_i} \end{cases} \tag{27}$$

and $F_{\phi_i} \subset \mathcal{F} = \bigcup_i F_{\phi_i}$ is the union of all the joining zones (see figure 1.A)[5]. Then if we change the definition $O_{iI}(\phi) = A_{iI}(\phi)\, I_i(\phi\ )$ (compare (20)) by

$$O_{iI}(\phi) = A_{iI}(\phi)\, B_i(\phi)$$

we would have smooth connections between $\mathcal{F}$ through the functions $O_{iI}(\phi)$ [6]. Namely to work with continuos and differential functions force us to introduce

[4]In this paper we have slightly changed the notation of paper [5], because we consider that the present notation is more explicit than the one.of that paper.

[5]Moreover, as we will discuss in section 5, quantum phase space has a fundamental graininess. Then the width of $\mathcal{F}$ must be of the order that we will define in that section, i.e. it must contain a box of the size $\Delta x \Delta p = \frac{1}{2}\hbar$.

[6]In some cases it can be shown that the discontinuities in the boundary zones introduces a $0(\hbar^2)$, which vanishes when $\hbar \to 0$ and, therefore, in this cases, the Moyal brackets can be replaced with Poisson brackets in such a limit (see [7])

continuity zones $\mathcal{F}$ and functions $B_i(\phi)$ in the frontier of the domains D_{ϕ_i} (figure1. A). Then we can use $C^r - functions$ (and eventually $C^\infty - functions$) in the whole treatment (see.[7]) . For simplicity, up to now, we have not considered these $\mathcal{F}-$zones, nevertheless we will be forced to use them in section 6 (figure 6.A).

Another kind of joining zones are used in the decomposition, in small square boxes, of a "cell" [9], i. e. the small boxes distributed in the "boundary of C" in figure 6.1 of the quoted book (see also between eqs. (6.6) and (6.79) of this book). This figure corresponds to our figure 1.B. But, as the D_{ϕ_i} are neither boxes nor cells (that will be introduce in section 5), $\mathcal{F}$ and the "boundary of C" are completely different concepts.

Figure 1.A. The domains and the frontier. $\Delta x \Delta p = \frac{1}{2}\hbar$. Figure 1.B A cell decomposed in small square boxes.

3.3 Decoherence

Let us consider a quantum system with a globally defined Hamiltonian $\widehat{H}$. In order to complete the CSCO, we can add constants of the motion locally defined as in the previous subsection. Thus, we have the CSCO $\left\{\widehat{H}, \widehat{O}_{iI}\right\}$, with $I = 1$ to N and i corresponding to all the necessary domains D_{ϕ_i} obtained from the partition of the phase space Γ.

a.- In paper [5] we have considered the case with continuous and discrete spectrum for $\widehat{H}$ and for the $\widehat{O}_{iI}$. For the sake of simplicity in this paper we will only consider the continuous spectrum $0 \leq \omega < \infty$ for $\widehat{H}$ and discrete spectra $m_{iI} \in \mathbb{N}$ for the $\widehat{O}_{iI}$. Then in the eigenbasis of $\widehat{H}$, the elements of any local CSCO can be expressed as (see Eq. (23))

$$\widehat{H} = \sum_{im_{iI}} \int_0^\infty \omega \, |\omega, m_{iI}\rangle\langle\omega, m_{iI}| \, d\omega \tag{28}$$

$$\widehat{O}_{iJ} = \sum_{im_{iI}} \int_0^\infty m_{iI} \, |\omega, m_{iI}\rangle\langle\omega, m_{iI}| \, d\omega \tag{29}$$

where m_{iI} is a shorthand for $m_{i1}, ..., m_{iN}$, and $\sum_{im_{iI}}$ is a shorthand for $\sum_i \sum_{m_{i1}} ... \sum_{im_{iN}}$.

With this notation,

$$\widehat{H} \, |\omega, m_{iI}\rangle = \omega \, |\omega, m_{iI}\rangle, \qquad \widehat{O}_{iI} \, |\omega, m_{iI}\rangle = m_{iI} \, |\omega, m_{iI}\rangle \tag{30}$$

where the set of vectors $\{|\omega, m_{iI}\rangle\}$, with $I = 1$ to N and i corresponding to all the domain D_{ϕ_i}, is an orthonormal basis (see Eq. (24)), i. e.:

$$\langle\omega, m_{iI} | \, \omega', m'_{iI}\rangle = \delta(\omega - \omega') \, \delta_{m_{iI} \, m'_{iI}} \tag{31}$$

b.- Also in the orthonormal basis $\{|\omega, m_{iI}\rangle\}$, a generic observable reads (see Eq. (23))

$$\widehat{O} = \sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' \widetilde{O}_{im_{iI}\, m'_{iI}}(\omega, \omega')\, |\omega, m_{iI}\rangle\langle\omega', m'_{iI}| \tag{32}$$

where $\widetilde{O}_{im_{iI}\, m'_{iI}}(\omega, \omega')$ is a generic *kernel* or *distribution* in ω, ω'. As in paper [5], we will restrict the set of observables (i.e. we make a projection like those of section 2.2 namely a generalized coarse-graining) by only considering the van Hove observables (see [15]) such that [7]

$$\widetilde{O}_{im_{iI}\, m'_{iI}}(\omega, \omega') = O_{im_{iI}\, m'_{iI}}(\omega)\, \delta(\omega - \omega') + O_{im_{iI}\, m'_{iI}}(\omega, \omega') \tag{33}$$

The first term in the r.h.s. of Eq. (33) is the *singular term* and the second one is the *regular term* since the $O_{im_{iI}\, m'_{iI}}(\omega, \omega')$ are "regular" , i. e. $\mathbb{L}_2$, functions of the variable ω - ω'. Then we will call $\widehat{\mathcal{O}}$ the subspace of observable, of our algebra $\widehat{\mathcal{A}}$, with these characteristics. Moreover we can define a projector Π, as those of section 2.2, that projects on $\widehat{\mathcal{O}}$. This projection will be our generalized coarse graining.

Therefore, the observables will read

$$\begin{aligned} \widehat{O} \;=\; & \sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega O_{im_{iI}\, m'_{iI}}(\omega)\, |\omega, m_{iI}\rangle\langle\omega, m'_{iI}| + \\ & \sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' O_{im_{iI}\, m'_{iI}}(\omega, \omega')\, |\omega, m_{iI}\rangle\langle\omega', m'_{iI}| \end{aligned} \tag{34}$$

Since the observables are the self-adjoint operators of the algebra, $\widehat{O}^\dagger = \widehat{O}$, they belong to a space $\widehat{\mathcal{O}} \subset \widehat{\mathcal{A}}$ whose basis $\{|\omega, m_{iI}, m'_{iI}), |\omega, \omega', m_{iI}, m'_{iI})\}$ is defined as

$$|\omega, m_{iI}, m'_{iI}) \doteq |\omega, m_{iI}\rangle\langle\omega, m'_{iI}|, \qquad |\omega, \omega', m_{iI}, m'_{iI}) \doteq |\omega, m_{iI}\rangle\langle\omega', m'_{iI}| \tag{35}$$

c.- The states belong to a convex set included in the dual of the space $\widehat{\mathcal{O}}$, $\widehat{\rho} \in \widehat{\mathcal{S}} \subset \widehat{\mathcal{O}'}$. The basis of $\widehat{\mathcal{O}'}$ is $\{(\omega, m_{iI}, m'_{iI}|, (\omega, \omega', m_{iI}, m'_{iI}|\}$, whose elements are defined as functionals by the equations

$$\begin{aligned} (\omega, m_{iI}, m'_{iI}\,|\, \eta, n_{iI}, n'_{iI}) &\doteq \delta(\omega - \eta)\, \delta_{m_{iI}\, n_{iI}}\, \delta_{m'_{iI}\, n'_{iI}} \\ (\omega, \omega', m_{iI}, m'_{iI}\,|\, \eta, \eta', n_{iI}, n'_{iI}) &\doteq \delta(\omega - \eta)\, \delta(\omega' - \eta')\, \delta_{m_{iI}\, n_{iI}}\, \delta_{m'_{iI}\, n'_{iI}} \\ (\omega, m_{iI}, m'_{iI}\,|\, \eta, \eta', n_{iI}, n'_{iI}) &\doteq 0 \end{aligned} \tag{36}$$

and the remaining $(\bullet\, |\bullet)$ are zero. Then, a generic state reads

[7] In papers [6] we have shown that this choice does not diminish the physical generality of the model

$$
\begin{aligned}
\widehat{\rho} \quad = \quad & \sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega \overline{\rho_{im_{iI}\, m'_{iI}}(\omega)}\, (\omega, m_{iI}, m'_{iI}| + \\
& \sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{im_{iI}\, m'_{iI}}(\omega,\omega')}\, (\omega,\omega', m_{iI}, m'_{iI}| \qquad (37)
\end{aligned}
$$

where the functions $\overline{\rho_{im_{iI}\, m'_{iI}}(\omega,\omega')}$ are "regular", i.e. $\mathbb{L}_2$ functions of the variable ω - ω'. We also require that $\widehat{\rho}^\dagger = \widehat{\rho}$, i.e.,

$$
\overline{\rho_{im_{iI}\, m'_{iI}}(\omega,\omega')} = \rho_{im'_{iI}\, m_{iI}}(\omega',\omega) \qquad (38)
$$

and that the $\rho_{im_{iI}\, m_{iI}}(\omega,\omega) \doteq \rho_{im_{iI}}(\omega)$ would be real and non-negative, satisfying the total probability condition,

$$
\rho_{im_{iI}}(\omega) \geq 0, \qquad tr\widehat{\rho} = (\widehat{\rho}|\widehat{I}) = \sum_{im_{iI}} \int_0^\infty d\omega \rho_{im_{iI}}(\omega) = 1 \qquad (39)
$$

where $\widehat{I} = \sum_{im_{iI}} \int_0^\infty d\omega |\omega, m_{iI}\rangle\langle\omega, m_{iI}|$ is the identity operator in $\widehat{\mathcal{O}}$.

d.- On the basis of these characterizations, the expectation value of any observable $\widehat{O} \in \widehat{\mathcal{O}}$ in the state $\widehat{\rho}(t) \in \widehat{\mathcal{S}}$ can be computed as

$$
\begin{aligned}
\langle \widehat{O} \rangle_{\widehat{\rho}(t)} = (\widehat{\rho}(t)|\widehat{O}) = \sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega \overline{\rho_{im_{iI}\, m'_{iI}}(\omega)}\, O_{im_{iI}\, m'_{iI}}(\omega) + \\
\sum_{im_{iI}\, m'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{im_{iI}\, m'_{iI}}(\omega,\omega')}\; e^{i(\omega-\omega')t/\hbar}\, O_{im_{iI}\, m'_{iI}}(\omega,\omega') \qquad (40)
\end{aligned}
$$

The requirement of "regularity", in variables $\omega - \omega'$, for the involved functions, i. e. $O_{im_{iI}\, m'_{iI}}(\omega,\omega') \;\epsilon\; \mathbb{L}_2$ and $\overline{\rho_{im_{iI}\, m'_{iI}}(\omega,\omega')} \;\epsilon\; \mathbb{L}_2$, as a consequence of Schwarz inequality, it means that $\overline{\rho_{im_{iI}\, m'_{iI}}(\omega,\omega')}\, O_{im_{iI}\, m'_{iI}}(\omega,\omega') \in \mathbb{L}^1$ in the variable $\nu = \omega - \omega'$, a property that we will use below. .

Now, for reasons that will be clear further on, it is convenient to choose a new basis $\{|\omega, p_{iI})\}$ that diagonalize the m-variables of ρ (of eq. (38)), for the case $\omega = \omega'$, through a unitary matrix U, which performs the transformation

$$
\rho_{im_{iI}\, m'_{iI}}(\omega) \to \rho_{ip_{iI}\, p'_{iI}}(\omega)\, \delta_{p_{iI}\, p'_{iI}} \doteq \rho_{ip_{iI}}(\omega) \qquad (41)
$$

Such transformation defines the new orthonormal basis $\{|\omega, p_{iI}\rangle\}$, where p_{iI} is a shorthand for $p_{i1}, ..., p_{iN}$, and $p_{iI} \in \mathbb{N}$. This basis corresponds to a new local CSCO $\left\{\widehat{H}, \widehat{P}_{iI}\right\}$. Therefore, in each D_{ϕ_i} we can deduce, from the equations (40) and (41), that the basis $\{|\omega, p_{iI}\rangle\}$ corresponds to the basis of observables. i. e. $\{|\omega, p_{iI}), |\omega, \omega', p_{iI}, p'_{iI})\}$, defined as in Eq. (35) but with the indices p instead of m, and also to the corresponding basis for the states is $\{(\omega, p_{iI}|, (\omega, \omega', p_{iI}, p'_{iI}|\}$.

Then when the observables $\widehat{P}_{iI}$ have discrete spectra, in the new basis the van Hove observables of our algebra $\widehat{\mathcal{A}}$ will read

$$\widehat{O} = \sum_{ip_{iI}} \int_0^\infty d\omega O_{ip_{iI}}(\omega)\,|\omega, p_{iI}) +$$

$$\sum_{ip_{iI}\,p'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' O_{ip_{iI}\,p'_{iI}}(\omega,\omega')\,|\omega,\omega', p_{iI}, p'_{iI}) \tag{42}$$

where the first term of the r.h.s is the *singular part* and the second terms the *regular part* of $\widehat{O}$. The states, in turn, will have the following form

$$\begin{aligned} \widehat{\rho} &= \sum_{ip_{iI}} \int_0^\infty d\omega \overline{\rho_{ip_{iI}}(\omega)}\,(\omega, p_{iI}| + \\ &\quad \sum_{ip_{iI}\,p'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{ip_{iI}\,p'_{iI}}(\omega,\omega')}\,(\omega,\omega', p_{iI}, p'_{iI}| \end{aligned} \tag{43}$$

where, again, the first term of the r.h.s. is the *singular part* and the second one is the *regular part* of $\widehat{\rho}$.

From the last two equations we have

$$(\widehat{\rho(t)}|\widehat{O}) = \sum_{ip_{iI}} \int_0^\infty d\omega \overline{\rho_{ip_{iI}}(\omega)}\,O_{ip_{iI}}(\omega) +$$

$$\sum_{ip_{iI}\,p'_{iI}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{ip_{iI}\,p'_{iI}}(\omega,\omega')}\; e^{i(\omega-\omega')t/\hbar}\, O_{ip_{iI}\,p'_{iI}}(\omega,\omega')$$

Then we can make the Riemann-Lebesgue limit to $(\widehat{\rho}|\widehat{O})$ since from the Schwarz inequality $O_{ip_{iI}\,p'_{iI}}(\omega,\omega')\overline{\rho_{ip_{iI}\,p'_{iI}}(\omega,\omega')}\epsilon\mathbb{L}_1$ in $\nu = (\omega-\omega')$ the regular part vanishes and only the singular part remains:

$$W - \lim_{t\to\infty} \widehat{\rho}(t) = \sum_{ip_{iI}} \int_0^\infty d\omega \overline{\rho_{ip_{iI}}(\omega)}\,(\omega, p_{iI}| = \widehat{\rho}_* \tag{44}$$

and we have decoherence in all the variables (ω, p_{iI}).

Here we have considered the case of observables $\widehat{P}_{iI}$ with discrete spectra; the case of $\widehat{P}_{iI}$ with continuos spectra is very similar (see [5]).

3.4 Comment

A comment is in order: Usually decoherence is studied in the case of open system surrounded by an environment, up to the point that some people believe that decoherence takes place in open systems. But also several authors have introduced, for different reasons, decoherence formalisms for closed system ([17]

- [26]). Related with the method used in this paper two important examples are given:

1- In paper [27], where a system that decoheres at high energy at the Hamiltonian basis is studied, and

2.-In paper [28], where complexity produces decoherence in a closed triangular box (in what we could call a Sinai-Young model).

Also we have developed our own theory for decoherence of closed systems, SID (see [29] - [32]). In paper [33] we show how our formalism explains the decoherence of the Sinai-Young model above. Recently it has been shown that also the gravitational field produces decoherence in the Hamiltonian basis [34]

4 The classical statistical limit

In order to obtain the classical statistical limit, it is necessary to compute the Wigner transformation of observables and states. For simplicity and symmetry

we will consider all the variables (ω, p_{iI}) continuous in this section

If we do this substitution, Eq. (43), reads

$$\widehat{\rho}(t) = \sum_i \int_{p_{iI}} dp_{iI}^N \int_0^\infty d\omega \, \overline{\rho_i(\omega, p_{iI})} \, (\omega, p_{iI}| +$$

$$\sum_i \int_{p_{iI}} dp_{iI}^N \int_{p'_{iI}} dp'^N_{iI} \int_0^\infty d\omega \int_0^\infty d\omega' \, \overline{\rho_i(\omega, \omega', p_{iI}, p'_{iI})} \, e^{i(\omega-\omega')t/\hbar} \, (\omega, \omega', p_{iI}, p'_{iI}| \tag{45}$$

Therefore, Eq. (44) can be written as

$$W - \lim_{t\to\infty} \widehat{\rho}(t) = \widehat{\rho}_* = \sum_i \int_{p_{iI}} dp_{iI}^N \int_0^\infty d\omega \overline{\rho_i(\omega, p_{iI})} \, (\omega, p_{iI}| \tag{46}$$

where $\widehat{\rho}_*$ is simply the singular component of $\widehat{\rho}(t)$, where the regular part has vanished as a consequence of the Riemann-Lebesgue theorem.

Now, the task is to find the classical distribution $\rho_*(\phi)$ resulting from the Wigner transformation of $\widehat{\rho}_*$ in the limit $\hbar \to 0$,

$$\rho_*(\phi) = symb \, \widehat{\rho}_* \tag{47}$$

where

$$\rho_*(\phi) = symb \, \widehat{\rho}_* = \sum_i \int_{p_{iI}} dp_{iI}^N \int_0^\infty d\omega \, \overline{\rho_i(\omega, p_{iI})} \;\; symb \, (\omega, p_{iI} \, | \tag{48}$$

So, the problem is reduced to compute $symb \, (\omega, p_{iI} \, |$.

As it is well known, in its traditional form the Wigner transformation yields the correct expectation value of any observable in a given state when we are dealing with regular functions (see Eq. (15)). In previous papers ([5], [42]) we have extended the Wigner transformation to singular functions in order to use it in functions like $(\omega, p_{iI}|$. Here we will briefly resume the results of these papers in two steps: first, we will consider the transformation of observables and, second, we will study the transformation of states.

4.1 Transformation of observables

As we have seen (see Eq.(42)), our van Hove observables $\widehat{O} \in \widehat{\mathcal{O}}$ have a singular part, i. e. $\widehat{O}_S$, and a regular part, i.e. $\widehat{O.}_R$. We will direct our attention to the singular operators $\widehat{O}_S$, since the regular operators $\widehat{O}_R$ "disappear" from the expectation values after decoherence, as explained in Section 2.3. $\widehat{O}_S$ reads:

$$\widehat{O}_S = \sum_i \int_{p_{iI}} dp_{iI}^N \int_0^\infty d\omega O_i(\omega, p_{iI})\, |\omega, p_{iI}) \tag{49}$$

Then, the Wigner transformation of $\widehat{O}_S$ can be computed as

$$O_S(\phi) = symb\, \widehat{O}_S \tag{50}$$

where

$$O_S(\phi) = symb\, \widehat{O}_S = \sum_i \int_{p_{iI}} dp_{iI}^N \int_0^\infty d\omega\, O_i(\omega, p_{iI}) \;\; symb\, |\omega, p_{iI}) \tag{51}$$

Now if we consider that the functions $O_i(\omega, p_{iI})$ are polynomials of functions of a certain space where the polynomials are dense it can be probed that

$$\widehat{O}_S = \sum_i O_{\phi_i}(\widehat{H}, \widehat{P}_{iI}) \; = \sum_i \widehat{O_{S\phi_i}}$$

where $\widehat{O_{S\phi_i}} = O_{S\phi_i}(\widehat{H}, \widehat{P}_{iI})$, and where $symb\widehat{O_{S\phi_i}} = symb O_{S\phi_i}(\widehat{H}, \widehat{P}_{iI}) = O_{S\phi_i}(H(\phi), P_{iI}(\phi)) + O(\hbar^2)$. Then if $O_{S\phi_i}(H(\phi), P_{iI}(\phi)) = \delta(\omega - \omega')\, \delta^N(p_{iI} - p'_{iI})$ we have (see paper [7] for details) that the function $symb\, |\omega, p_{iI})$ in the limit $\frac{\hbar}{S} \to 0$, is

$$symb\, |\omega, p_{iI}) = \delta(H(\phi) - \omega)\, \delta^N(P_{iI}(\phi) - p_{iI}) \tag{52}$$

where $H(\phi) = symb\widehat{H}$ and $P_{iI}(\phi) = symb\widehat{P_{iI}}$

4.2 Transformation of states

As in papers [5] and [42], in order to compute the $symb$ $(\omega, p_{iI}|$, we will define the Wigner transformation of the singular operator $\widehat{\rho}_S = \widehat{\rho}_*$ on the base of the only reasonable requirement that such a transformation would lead to the correct expectation value of any observable. Then we must postulate that it is (see Eq. (15)),

$$(symb\, \widehat{\rho}_S \mid symb\, \widehat{O}_S) \doteq (\widehat{\rho}_S \,|\, \widehat{O}_S) \tag{53}$$

These equations must also hold in the particular case in which $\widehat{O}_S = |\omega', p'_{iI})$, $\widehat{\rho}_S = (\omega, p_{iI}|$, for some D_{ϕ_i} (see Eq.(24)) i. e.:

$$(symb(\omega, p_{iI}| \mid symb\,|\omega', p'_{iI})) = (\omega, p_{iI}\,|\,\omega', p'_{iI}) \tag{54}$$

and all the remaining cross terms are zero for any domain D_{ϕ_j}, with $j \neq i$. But from Eq. (52) we know how to compute $symb\,|\omega', p'_{iI})$. Moreover, from the definition of the cobasis (see Eq. (36)) we know that

$$(\omega, p_{iI}\,|\,\omega', p'_{iI}) = \delta(\omega - \omega')\,\delta^N(p_{iI} - p'_{iI}) \tag{55}$$

Therefore in the limit $\frac{\hbar}{S} \to 0$ we have,

$$(symb(\omega, p_{iI}| \mid \delta(H(\phi) - \omega')\,\delta^N(P_{iI}(\phi) - p'_{iI})) = \delta(\omega - \omega')\,\delta^N(p_{iI} - p'_{iI}) \tag{56}$$

Then in paper [5] we have proved that (always in the $\frac{\hbar}{S} \to 0$ limit)

$$symb(\omega, p_{iI}| = \frac{\delta(H(\phi) - \omega)\,\delta^N(P_{iI}(\phi) - p_{iI})}{C_i(H, P_{iI})} \tag{57}$$

where $C_i(H, P_{iI})$ is the configuration volume of the region $\Gamma_{H,P_{iI}} \cap D_{\phi_i}$, being $\Gamma_{H,P_{iI}} \subset \Gamma$ the hypersurface defined by $H = const.$ and $P_{iI} = const.$ In this way we have obtained the *symb* of $|\omega, p_{iI})$ and $(\omega, p_{iI}|$ so the classical statistical limit is completed.

4.3 Convergence in phase space

Finally, we can introduce the results of Eq. (57) into Eq. (48), in order to obtain the classical distribution $\rho(\phi)$:

$$\rho_*(\phi) = \rho_S(\phi) = \sum_i \int_{p_{iI}} dp^N_{iI} \int_0^\infty d\omega\, \frac{\overline{\rho_i(\omega, p_{iI})}}{C_i(H, P_{iI})}\ \delta(H(\phi) - \omega)\,\delta^N(P_{iI}(\phi) - p_{iI}) \tag{58}$$

As a consequence, the Wigner transformation of the limits of Eq. (44) can be written as

$$W - \lim_{t\to\infty} \rho(\phi, t) = \rho_S(\phi) = \rho_*(\phi) =$$

$$\sum_i \int_{p_{iI}} dp^N_{iI} \int_0^\infty d\omega\, \frac{\overline{\rho_i(\omega, p_{iI})}}{C_i(H, P_{iI})}\ \delta(H(\phi) - \omega)\,\delta^N(P_{iI}(\phi) - p_{iI}) \tag{59}$$

Remember that all this is only valid in a domain D_ϕ defined in eq. (19) and that it would completely change if we change to another domain through a continuity zone $\mathcal{F}$ of section 3.2

Then we have obtained a convincing classical limit of the states, that decomposed as in eq. 59, it turns out to be sums of states peaked in the classical hypersurfaces of constant energy, $H(\phi) = \omega$, and where also the other constants of motions are constant, $P_{iI}(\phi) = p_{iI}$. This is an important step forward, to have obtained these classical surfaces as a limit of the quantum mechanics formalism. Up to here chaos has not produced any problem to the CP, even if the system is not integrable. The real problems will begin in the next section.

5 Graininess

Up to here we have found the hypersurfaces where the classical trajectories lay. Now we want to find the classical motions in these trajectories. Thus we need to define the notion of "a point that moves". But in quantum mechanics there is not such a thing. In fact it is well known that the commutation relations and its consequence, the indetermination principle, establishes a fundamental graininess in "quantum phase space". Precisely if we call $\widehat{J}$ and $\widehat{\Theta}$ two generic conjugated operators (e. g. in our case $\widehat{J}$ will be the constants of the motion $\widehat{H}$, $\widehat{P}_{iI}$ and $\widehat{\Theta}$ the corresponding configuration operators) we have

$$[\widehat{\Theta}, \widehat{J}] = i\hbar\widehat{I} \tag{60}$$

and therefore

$$\Delta_\Theta \Delta_J \geq \frac{\hbar}{2} \tag{61}$$

where, from now on, Δ_Θ and Δ_J are defined as the variances of some *typical* state $\widehat{\rho}$ the one with the smallest dimensions we can "determinate" (in the sense of Ballentine chapter [45]) in our experiment. With different choices for this ρ we will obtain different ratios Δ_Θ/Δ_J but the qualitative results will be the same. Then we will consider that the rectangular box $\Delta_\Theta\Delta_J$ of volume $\hbar$ (or the polyhedral box of volume $\hbar^{(n+1)}$ in the many dimensions case) will be the smallest volume that we can determinate with our measurement apparatus, precisely:

$$vol \Delta_\Theta \Delta_J = \hbar \text{ (or eventually } N_0\hbar)$$

for a phase space of two dimensions or

$$vol \prod \Delta_\Theta \prod \Delta_J = \hbar^{(n+1)} \text{ (or eventually } N_0\hbar^{)n+1)})$$

for a phase space of $2(n+1)$ dimensions, where N_0 is not a very large natural number (cf. [9]). This is the new feature of the "quantum phase space": its graininess and this fact will be the origin of the threat to the CP [8].

In Omnès' book [9] the cell produced by the fundamental graininess are described in the (x, p) coordinates, using a mathematical theory, the microlocal analysis, based in the work [46]. In our formalism we will change these (x, p) for the (J, Θ) coordinates where J are the constants of the motion and Θ the corresponding configuration variables and where the commutation relations (60) and their consequence the indetermination principle (61) will play the main role.

To see how the fundamental graininess works let us consider a closed simply connected set of a two dimensional phase space that we will call a cell C^T, with its continuous boundary B, (figure 1.B, or fig 6.1 of [9]). The coordinates (J, Θ) and a lattice of rectangular boxes $\Delta_\Theta\Delta_J$ (eventually $2(n+1)$ polyhedral boxes) define the two domains related with C^T: Σ, set of boxes that intersect B, and C, the set of the interior rectangular boxes of the cell C^T. Volume is well

[8] Fundamental graininess appears in many other disguises (see [43], [44], etc.)

defined in phase space of any dimension while (hyper) surfaces are not defined, so in order to compare the the size of the frontier with the size of the interior we can define the coefficient

$$\Omega = \frac{vol\Sigma}{volC}$$

It is quite clear that $\Omega \ll 1$ corresponds to a bulky cell while $\Omega \gg 1$ corresponds to an elongated and maybe deformed cell. It is also almost evident that if we want that a cell would somehow represent a real point it is necessary that $\Omega < 1$, because if $\Omega > 1$ the volume of the interior C is smaller than the volume of the "frontier" Σ, where we do not know for certain if its points belong or not to C^T since $B \subset \Sigma$. Thus in the case $\Omega \gg 1$ we completely lose the notion of real point and the description of the classical trajectories, as the motion of C^T, becomes impossible.

Analogously Omnès defines semiclassical projectors for each cell and shows that if Ω is very large the definition of these projectors lose all its meaning and the classicality is lost, namely he obtain a similar conclusion.

In the next section we will consider the cells and their evolution in several cases and estimate the corresponding Ω.

6 The classical trajectories

Up to this point we have obtained the classical distribution $\rho_*(\phi) = \rho_S(\phi)$ to which the system converges in phase space. This distribution defines hypersurfaces $H(\phi) = \omega$, $P_{iI}(\phi) = p_{iI}$ corresponding to the constant of the motion i.e. our the "momentum" variables. But such a distribution does not define the trajectories of "points" on those hypersurfaces, i. e., it does not fix definite values for the "configuration" variables (the variables canonically conjugated to $H(\phi)$ and $P_{iI}(\phi)$). This is reasonable to the extent that definite trajectories would violate the uncertainty principle. In fact we know that, if $\widehat{H}$ and $\widehat{P}_{iI}$ have definite values, then the values of the observables that do non-commute with them will be completely undefined.

Nevertheless, trajectory-like motions can be recovered by means of the Ehrenfest theorem, applied to the constants of motion and their conjugated variables. But also a better classical limit (and its limitations) can be obtained searching the trajectories, of the rectangular boxes (and later of the cells) we will consider as "points", integrating the Heisenberg equation, and then studying the deformations of the cells under the motion (as in [9]). Let us follow this second path, as in section 5, let us call, $\widehat{J}$ the "momentum" variables $\widehat{H}$ and $\widehat{P}_{iI}$ (constants of the motion), and $\widehat{\Theta}$ the corresponding conjugated "configuration" variables, all of them defined in the domain D_{ϕ_i}. The equations of motion, in the Heisenberg picture, read

$$\frac{d\widehat{J}}{dt} = \frac{i}{\hbar}[\widehat{H}, \widehat{J}] \qquad \frac{d\widehat{\Theta}}{dt} = \frac{i}{\hbar}[\widehat{H}, \widehat{\Theta}] \tag{62}$$

where as $[\widehat{H}, \widehat{J}] = 0$

$$\frac{d\widehat{J}}{dt} = 0 \qquad \frac{d\widehat{\Theta}}{dt} = \frac{i}{\hbar}[\widehat{H}, \widehat{\Theta}] \tag{63}$$

Within the domain D_ϕ we know that if we can consider the $\widehat{H}$ as a function (or a convergent sum) of the $\widehat{J}$, i. e.:

$$\widehat{H} = F(\widehat{J}) = \sum_n a_n \widehat{J}^n \tag{64}$$

and since $[\widehat{\Theta}, \widehat{J}] = i\hbar$ we have $[\widehat{\Theta}, \widehat{J}^n] = in\hbar \widehat{J}^{(n-1)}$ so

$$[\widehat{H}, \widehat{\Theta}] = \frac{d\widehat{H}}{d\widehat{J}}$$

where $\widehat{H}$ and $\widehat{J}$ are constant in time, so calling $\widehat{V}(0) = \frac{d\widehat{H}}{d\widehat{J}}$, which is another constant in time, we have

$$\widehat{J}(t) = \widehat{J}(0), \quad \widehat{\Theta}(t) = \widehat{\Theta}(0) + \widehat{V}(0)t$$

Then we can make the Wigner transformation from these equations and, since this transformation is linear, we have

$$J(\phi, t) = J(\phi, 0), \quad \Theta(\phi, t) = \Theta(\phi, 0) + V(\phi, 0)t \tag{65}$$

We will use this equation to follow the motion of the boxes and the cells in the phase space:

Let us first consider a *rectangular* (eventually $2(n+1)$polyhedral) moving box of size Δ_Θ, Δ_J with $\Delta_\Theta \Delta_J \sim \hbar$ (eventually $\hbar^{(n+1)}$), that we will symbolize by a small square in figures 2, 3, 4, and 5 (and just by a point in the figures 6.A and 6.B) and let us also consider the *typical* point-test-distribution function $symb\widehat{\rho}{=}\rho(\phi) = \rho(j, \theta)$, (see under eq. (61), also from now on $\phi = (j, \theta)$) with support contained in $\Delta_\Theta \Delta_J$, then let us define the mean values

$$\overline{j(t)} = \int_{\Delta_\Theta \Delta_J} J(j, \theta, t)\rho(j, \theta) dj d\theta; \quad \overline{\theta(t)} = \int_{\Delta_\Theta \Delta_J} \Theta(j, \theta, t)\rho(j, \theta) dj d\theta,$$

$$\overline{v(t)} = \int_{\Delta_\Theta \Delta_J} V(j, \theta, t)\rho(j, \theta) dj d\theta; \tag{66}$$

where the $\rho(j, \theta)$ is not a function of the time since we are in the Heisenberg picture. Now using eq. (65) we have

$$\overline{j}(\phi, t) = \overline{j}(\phi, 0), \quad \overline{\theta}(\phi, t) = \overline{\theta}(\phi, 0) + \overline{v}(\phi, 0)t \tag{67}$$

so our minimal rectangular box moves along a classical trajectory of our system.

Now our rectangular boxes are so small that we can not even consider their possible deformation. Precisely the Indetermination Principle makes this deformation merely hypothetical. Thus, from now on, we will consider that the rectangular boxes *are not in motion (and therefore they can not be deformed by motion)* and that they are the most elementary theoretical fixed notion of a point at $(\overline{j},\overline{\theta})$ [9]. In this way we have obtained the classical trajectories of theoretical points (i. e. eq.(67)) and we would have completed our quantum to classical limit (apparently CP is safe up to now).

But remember that the real physical points are not these rectangular boxes but the cells with $\Omega < 1$ that we must also consider, because real measurement devises cannot see the elementary rectangular boxes but bigger cells of dimensions far bigger than the Planck ones. In the next examples we will see what happens with these cells that we will consider as real points: the cells can be deformed by the motion (while the rectangular boxes always remain rigid). We will show the interplay of these theoretical points (boxes) and physical real points (cells) in some examples bellow:

1.- Then, as a first example, let us consider a two dimensional space within a domain D_ϕ (much larger than the cell that we will define below) and let as also consider the system of coordinates (J,Θ) and the corresponding trajectories when the Hamiltonian is a linear function, $\widehat{H}=a_0\widehat{I}+a_1\widehat{J}$, Then $\widehat{V}=a_1\widehat{I}$ so

$$J(\phi,t)=J(\phi,0),\quad \Theta(\phi,t)=\Theta(\phi,0)+a_1It$$

and, with the same reasoning as above the trajectories of the boxes (theoretical points) are

$$\overline{j}(t)=\overline{j}(0),\quad \overline{\theta}(t)=\overline{\theta}(0)+a_1t \tag{68}$$

Namely we obtain figure 2 and we have a uniform translation motion with constant velocity $\overline{v}[\overline{j}(0)]$ along all the trajectories.. Let us then consider two parallel lines with constant velocities $\overline{v}[\overline{j}_1]=\overline{v}[\overline{j_2}]$, thus the difference of velocities is

$$\overline{v}(\overline{j}_1)-\overline{v}(\overline{j_2})=0 \tag{69}$$

. Then if we consider an initial rectangular cell the motion will not deform the cell. Since there is no deformation of the cell Ω is rigid, thus if $\Omega<1$ in the initial cell Ω will be <1 in any transferred cell. Therefore, in this trivial case the cell will represent a physical real point moving according to eq. (68). Thus in this case we have completed our classical limit and the CP is safe.

Evolution of a cell with constant velocity.

2.- As a further example let us consider the same two dimensional space within a D_ϕ and let as consider the system of coordinates (J,Θ) and the corresponding trajectories when $\widehat{H}=a_0\widehat{I}+a_1\widehat{J}+a_2\widehat{J}^2$. Then $\widehat{V}=a_1\widehat{I}+a_2\widehat{J}$ so

$$J(\phi,t)=J(\phi,0),\quad \Theta(\phi,t)=\Theta(\phi,0)+[a_1I+2a_2J(\phi,0]t$$

[9] The rectangular moving cell defined after the eq. (65) will be the only rectangular objects that moves in this paper.

and, with the same reasoning as above.

$$\overline{j}(t)=\overline{j}(0),\quad \overline{\theta}(t)=\overline{\theta}(0)+[a_1+2a_2\overline{j}(0)]t$$

Namely we obtain figure 3 and we have a uniform motion with constant velocity $\overline{v}[\overline{j}(0)]=a_1+2a_2\overline{j}(0)$ along straight lines parallel to the axis θ..

$$\overline{\theta}(t)=\overline{\theta}(0)+\overline{v}[\overline{j}(0)]t$$

Let us then consider two parallel lines with constant velocities $\overline{v}(\overline{j}_1)\neq\overline{v}(\overline{j_2})$, thus the difference of velocities is

$$\overline{v}(\overline{j}_1)-\overline{v}(\overline{j_2})=2a_2(\overline{j}_1-\overline{j_2})=v \tag{70}$$

Let J,Θ be the dimension of the initial cell and Δ_J,Δ_Θ the dimension of the fix rectangular boxes. Then the length of the basis is constant and so $VolC$ also is constant. Then if we consider an initial rectangular box the motion will deform this cell in a parallelogram, where the height continue to be J and the base will now be $\Theta+\Delta\theta$, i. e. there is "elongation" $\Delta\theta$ (see figure 3), precisely

$$\Delta\theta=vt$$

Let us compute the evolution of Ω in this case: the number of new boxes that appears at time t will be

$$n=2\frac{\Delta\theta}{\Delta_\Theta}=2\frac{vt}{\Delta_\Theta}$$

Now

$$\Omega=\frac{Vol\Sigma}{VolC}=\frac{Vol\Sigma+\Delta Vol\Sigma}{VolC}=\frac{Vol\Sigma+n\Delta_J\Delta_\Theta}{VolC}$$

so

$$\Delta\Omega=\frac{n\Delta_J\Delta_\Theta}{VolC}=\frac{n\hbar}{VolC}=2\frac{v}{\Delta_\Theta}\frac{\hbar}{VolC}t=2\frac{\Delta\theta}{\Delta_\Theta}\frac{\hbar}{VolC}>0$$

Then:

a.- The increment $\Delta\Omega$ is proportional to the time t

b.- It is also proportional to the product of the ratio of the elongation $\Delta\theta$ measured in units of Δ_Θ.

c.- Finally it is proportional to $\frac{\hbar}{VolC}$ so in the macroscopic limit $\frac{\hbar}{VolC}\to 0$ we have $\Delta\Omega\to 0$ and the threat to CP disappears.

But the most important conclusion is that, in a generic case, even if $\frac{\hbar}{VolC}$ would be small but if it is far from the limit $\frac{\hbar}{VolC}\to 0$, after enough time we will have $\Omega\gg 1$. Then the cell ceases to be a good model for a point and it surely is the beginning of threat to the CP. This happens even if the system is integrable, namely, $D_\phi=\Gamma$ the phase space, and the Hamiltonian $\widehat{H}=a_0\widehat{I}+a_1\widehat{J}+a_2\widehat{J}^2$, e. g., simply be $\widehat{H}=\frac{1}{2m}\widehat{P}^2$, namely the one of a free particle. So fundamental graininess alone (with no chaos) can be a threat to the CP, in the case $\frac{\hbar}{VolC}>0$

Evolution of the cell with linear velociy

3.- In the most general case the Hamiltonian is $\widehat{H} = a_0\widehat{I} + a_1\widehat{J} + a_2\widehat{J}^2 + a_3\widehat{J}^3 + ...$ and eq. (69) becomes

$$\overline{v}(\overline{j}_1) - \overline{v}(\overline{j_2}) = 2a_2(\overline{j}_1 - \overline{j_2}) + 3a_3(\overline{j}_1^2 - \overline{j_2}^2) + ...$$

as described in figure 4 where there are not vertical deformations but there are strong horizontal ones. Then for Hamiltonians with power bigger than 2 the threat of chaos begins.

Evolution of a cell with non linear velociy

In fact, let us consider the case

$$\widehat{H} = \sum_{n=0}^{\infty} A_n(\widehat{J}) \exp in \frac{\widehat{J}}{\Delta_J}$$

then

$$\overline{v(j)} = \sum_{n=0}^{\infty} [A'_n(\overline{j}) + \frac{in}{\Delta_J} A_n(\overline{j})] \exp in \frac{\overline{j}}{\Delta_J} = \sum_{n=0}^{\infty} B_n(\overline{j}) \exp in \frac{\overline{j}}{\Delta_J}$$

and

$$\overline{\theta}(j,t) = \overline{\theta}(j,0) + \overline{v}(\overline{j})t = \overline{\theta}(\overline{j},0) + t \sum_{n=0}^{\infty} B_n(\overline{j}) \exp in \frac{\overline{j}}{\Delta_J}$$

Then the elongation will be

$$\Delta\theta = t \sum_{n=0}^{\infty} B_n(\overline{j}) \exp in \frac{\overline{j}}{\Delta_J}$$

Let us consider the simple case $B_m(\overline{j}) = const \neq 0$ and all other $B_n(\overline{j}) = 0$ (figure 5), then

$$\Delta\theta = tB_m \exp im \frac{\overline{j}}{\Delta_J} \text{ so } \operatorname{Re} \Delta\theta = tB_m \cos m \frac{\overline{j}}{\Delta_J}$$

and the wave longitude of the oscillation of the vertical boundary curves is $\lambda = \frac{\Delta_J}{m}$ and we can have $\lambda \ll \Delta_J$ if $m \gg 1$. Then we have

$$\Delta\Omega = \frac{\Delta vol\Sigma}{VolC} = 2\frac{J\Delta\theta}{J\Theta} = 2\frac{B_m}{\Theta}t$$

So when $t \to \infty$ then $\Delta\Omega \to \infty$, and we have a real threat to the CP with no redemption in the classical limit. And this can happen even in a not chaotic case since we can have $D_\phi = \Gamma$ [10].

Evolition of a cell with periodical velocity.

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[10] In if the cases 1, 2, and 3 we would take the $\widehat{H}$ as the free variable we would have $\widehat{J} = F^{-1}(\widehat{H})$, and, in the corresponding figures, $H = const.$ would appear in the vertical axis, and t, in the horizontal one, with the same qualitative results.

4.- But things get really worst if, instead of one D_ϕ, we consider two D_{ϕ_1} and D_{ϕ^2} and their joining zone $\mathcal{F}$, as in figure 6.A. Precisely let us suppose that in $D_{\phi 1}$ we have two parallel motions and only a parallelogram deformation as in point 2, and we use the (θ, j) coordinate of D_{ϕ_1}. But neither in $\mathcal{F}$ nor in D_{ϕ_2} the just quoted coordinate j is a constant of the motion, so in D_{ϕ_2} the motion becomes completely deformed as shown in the figure 6.A. Then if the motion goes through several joining zones $\mathcal{F}$ it is clear that the initial regular cell will become the amoeboid object of figure 6.B, where of course $\Omega \gg 1$. Remember that, for the sake of simplicity, the points of all these figures 6.A and 6.B have a volume $\hbar$ (or really $\hbar^{(n+1)}$ in the general case). Then when, as a consequence of chaos, the volume of the complex details of the amoeboid figure becomes of the order of $\hbar$ (or $\hbar^{(n+1)}$ in the general case) the classical limit representing the notion the original cell becomes meaningless as a result of chaos. Moreover in this case we could speculate that the square box becomes strongly deformed. But this kind of reasonings is forbidden by the Indetermination Principle and because in our treatment square boxes are considered rigid.

Another way to see that there is a real problem is to consider that the classical motion of the center of the initial cell (where the probabilities to find the particle are different from zero) as the real classical motion of a classical particle. Then in the chaotic case it may happen that at time t, the cell would get the amoeboid shape of figure 6.B. Now the center of the original cell turns out to be outside of the amoeboid figure. Then this center is in a zone of zero probability and cannot represent the motion of a real point-like classical particle anymore.

So chaos and fundamental graininess are a real threat to the classical limit of quantum mechanics and so for its interpretation.

Figure 6.A. A square cell scatterd by a frontier. Figure 6.B. A square cell becomes an ameoboidal cell

Example: the Henon-Heiles system and the high energy problem.

In the case of Henon-Heiles classical system ([47] page 121) with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{1}{2}y^3$$

We can observe that:

a.-The Hamiltonian is non integrable so in the whole phase space we will find something like figure 6.A.

b.- For energies $E = \frac{1}{12}$ (figure 44a of [47]) the tori are practically unbroken, as in case 3 above. But in large D_ϕ and in a physical case most likely $volD_\phi \gg \hbar^2$ and CP could be far from having practical problems with chaos at least for short periods of time. These D_ϕ become smaller for $E = \frac{1}{8}$ (figure 44b of [47]) and probably very tiny for $E = \frac{1}{6}$ (figure 44c of [47]) so in such cases we may have serious problems with chaos (i.e. those of case 4) since for real high energy we could have $volD_\phi \approx \hbar^2$. We can obtain these conclusions because our method allows us to evaluate the $volD_\phi$ on the surface defined by the constant of motion (tori) from the Poincaré sections.

So we conclude that when the D_ϕ in phase space are of the order of $\hbar$ CP has real problems. But also we see that for high energy there is not a generic well defined "high energy limit". The threat of chaos to the CP is thus explained. Moreover this example introduces the threat of chaos to the high energy limit.

7 Conclusion.

In this paper we have:

1.- Presented a new formalism to study the classical limit of quantum mechanics.

2.- Showed that somehow fundamental graininess alone is a threat to the CP (section 6 case 3).

3.- Demonstrated how chaos increases this threat.

4.- Proved that these threats also compromise the high energy limit of quantum mechanics

We conclude that in fact there is a threat of chaos and fundamental graininess to the CP and this thread can be elucidated studying the domains of definition of the constants of the motion (in the considered non-integrable system), the corresponding broken tori at different energies and the behavior of the cells for different Hamiltonians (as in case 1,2, and 3 of section 6).

Based in these results we could go on with the following speculation: In the classical level, the KAM theorem was the solution of the problem of the scarcity of chaos in the solar system, since the tori were broken but not badly broken. In the same way we could consider that the study of the size of the D_{ϕ_i}, for different levels of energy, could also explain the behavior of chaotic quantum systems and may be the scarcity of chaos in these systems. I. e. it may be that, many cases, the D_{ϕ_i} would be large enough to endow these systems with a quasi-integral chaotic behavior Along these lines we will continue our research.

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