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ASYMPTOTIC STABILITY OF SOLUTIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract. An evolution problem for abstract differential equations is studied. The typical problem is:

$$\dot{u} = A(t)u + F(t,u), \quad t \geq 0;\ u(0) = u_0; \quad \dot{u} = \frac{du}{dt} \qquad (*)$$

Here $A(t)$ is a linear bounded operator in a Hilbert space H, and F is a nonlinear operator, $\|F(t,u)\| \leq c_0\|u\|^p$, $p > 1$, $c_0, p = const > 0$. It is assumed that $\mathrm{Re}(A(t)u,u) \leq -\gamma(t)\|u\|^2\ \forall u \in H$, where $\gamma(t) > 0$, and the case when $\lim_{t\to\infty}\gamma(t) = 0$ is also considered. An estimate of the rate of decay of solutions to problem (*) is given. The derivation of this estimate uses a nonlinear differential inequality.

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1 Introduction

Let

$$\dot{u}(t) = A(t)u + F(t,u), \quad t \geq 0; \quad \dot{u}(t) := \dot{u} = \frac{du}{dt}, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where $u \in H$, H is a Hilbert space, $A(t)$ is a bounded linear operator in H, $F(t,u)$ is a nonlinear operator,

$$\|F(t,u)\| \leq c_0\|u\|^p, \quad p > 1, \tag{1.3}$$

c_0 and p are positive constants, and $u_0 \in H$.

One says that $A(t) \in B(\rho, N)$ if every solution to the equation

$$\dot{v}(t) = A(t)v \tag{1.4}$$

satisfies the estimate

$$\|v(t)\| \leq Ne^{\rho(t-s)}\|v(s)\|, \qquad t \geq s \geq 0, \tag{1.5}$$

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where $N > 0$ and ρ are real numbers. This definition is discussed in [1] and goes back to P. Bohl (see the historical remarks in [1]). If $U(t,s)$ is an operator that solves the problem

$$\dot{U}(t,s) = A(t)U(t,s), \qquad t \geq s; \quad U(s,s) = I, \tag{1.6}$$

where I is the identity operator, then (1.5) is equivalent (by the Banach-Steinhaus theorem) to the estimate

$$\|U(t,s)\| \leq Ne^{\rho(t-s)}, \qquad t \geq s \geq 0. \tag{1.7}$$

Let us define, following [1], the notion of upper general exponent κ for the solutions to (1.4):

$$\kappa = \overline{\lim}_{t,s\to\infty} \frac{\ln \|U(t+s,s)\|}{t}, \qquad t,s \geq 0. \tag{1.8}$$

If $\kappa < 0$, then $\|v(t)\| = O(e^{-|\kappa|t})$ as $t \to \infty$, s being fixed.

The following result is obtained in [1], Theorem 3.1, Chapter 7.

Proposition 1.1. *If $\kappa < 0$ and assumption* (1.3) *holds, then the zero solution to equation* (1.1) *is asymptotically stable.*

Recall that the zero solution to equation (1.1) is called Lyapunov stable if for every $\varepsilon > 0$, one can find a $\delta = \delta(\varepsilon) > 0$, such that if $\|u_0\| \leq \delta$, then the solution to problem (1.1)–(1.2) satisfies the estimate $\sup_{t\geq 0} \|u(t)\| \leq \varepsilon$. If, in addition, $\lim_{t\to\infty} \|u(t)\| = 0$, then the zero solution to equation (1.1) is called asymptotically stable in the Lyapunov sense.

As one can see from our proof of Theorem 1.2, the condition of smallness of the initial data $\|u_0\| \leq \delta$ can be replaced by a different condition: if $\|u_0\|$ is arbitrary fixed, then one still derives the relation $\lim_{t\to\infty} \|u(t)\| = 0$ from (2.8) (see below), provided that c_0 is sufficiently small.

In Proposition 1.1, the exponent $\kappa < 0$ is a constant. For example, if $A(t) = A^*(t)$ is a selfadjoint compact operator, and $\lambda_j(t)$ are its eigenvalues, $\lambda_j(t) \leq \lambda_m(t) < 0$ if $j > m$, $j = 1,2,3,...$, then $\lambda_1(t) \leq \kappa < 0$.

Our goal is to derive an analog of Proposition 1.1 such that $\lim_{t\to\infty} \lambda_1(t) = 0$ is allowed, that is, we do not assume that the spectrum $\sigma(A(t))$ of $A(t)$ lies in a half-plane Re $z \leq \kappa$, where $\kappa < 0$ is a *fixed* constant independent of t.

It is known (see, e.g., [1]) that if A is a bounded linear operator in H with the spectrum $\sigma(A)$, which lies in the half-plane Re $z \leq -|\kappa|$, $|\kappa| > 0$, then there is a positive-definite operator W such that Re$WA = -V$, where V is an arbitrary given positive-definite operator in H. In other words, if $m = const > 0$, $\sigma(A) \subset \{z : Rez \leq -|\kappa| < 0\}$ and $V = V^* \geq m > 0$, that is, $(Vu,u) \geq m(u,u)\ \forall u \in H$, then the operator equation $A^*W + WA = -2V$ is solvable for W. In fact, there is an explicit formula for W: $W = 2\int_0^\infty e^{A^*t}Ve^{At}dt$ (see [1]). By ReA one understands the operator defined by the formula Re$A := A_R := (A+A^*)/2$, and $A = A_R + iA_I$, where A_R and A_I are selfadjoint operators that are called the real and imaginary parts of A. If $A_R \leq -a$, $a = const > 0$, then $\sigma(A)$ lies in the half-plane Re $z \leq -a$. The notation $A \leq -a$ means $(Au,u) \leq -a(u,u)\ \forall u \in H$.

The converse is not true: it is not true that if the spectrum of a linear bounded operator A lies in the half-plane Re $z \leq -a$, then the inequality $A_R \leq -a$ holds. A simple counterexample is given by the following 2×2 matrix A in $\mathbb{R}^2$, $A = \begin{pmatrix} 0 & b \\ -a & -1 \end{pmatrix}$. The eigenvalues of this matrix are $-0.5 \pm i\sqrt{ab-0.25}$, and if $ab \geq 0.25$, then the spectrum $\sigma(A)$ of A lies in the half-plane Re$z \leq -0.5$. On the other hand, if, for example, $a = 1$ and $b = 5$, $u_1 = u_2 = 0.5$, then $(A_R u,u) > 0$.

Inequality Re$(Au,u) \leq 0$ means that the operator $A(t)$ is *dissipative*. Such operators often arise in applications (see, e.g., [9]). The dissipativity property, defined by the above inequality, usually means that the energy in the system is dissipating, that is, the system is passive. In [7] a wide class of passive nonlinear networks is studied, see also [8], Chapter 3.

Our basic results on the stability of the solutions to problem (1.1)-(1.2) with dissipative operator $A(t)$ are formulated in Theorems 1.2 and 1.4. Theorem 1.2 contains an auxiliary result used in the proofs of Theorems 1.2 and 1.4. This result is of interest by itself and useful in applications.

Theorem 1.2. *Assume that $Re(Au,u) \le -|\kappa|\|u\|^2$ for every $u \in H$ and inequality* (1.3) *holds. Then the solution to problem* (1.1)$-$(1.2) *satisfies an estimate $\|u(t)\| = O(e^{-(|\kappa|-\varepsilon)t})$ as $t \to \infty$. Here $0 < \varepsilon < |\kappa|$ can be chosen arbitrarily small if $\|u_0\|$ is sufficiently small.*

This theorem implies asymptotic stability in the sense of Lyapunov of the zero solution to equation (1.1). Our proof of Theorem 1.2 is new and very short.

We first prove Theorem 1.2 and Theorem 1.4 in Section 2, because the ideas of our proofs of these theorems are quite similar. Theorem 1.4 contains a new result, and it is not assumed in the formulation of this theorem that the spectrum of $A(t)$ lies in a half-plane $\mathrm{Re}z \le -|\kappa|$ with $|\kappa| > 0$ being a constant independent of t.

Then we prove Theorem 1.3. The result of this theorem is used in the proofs of Theorems 1.2 and 1.4, and is of general interest. It gives a bound on solutions to a nonlinear differential inequality. Results of this type, but considerably less general, were used extensively in [6], where the Dynamical Systems Method (DSM) for solving operator equations, especially nonlinear equations, was developed.

The ideas of our proofs are quite different from these in [1].

Theorem 1.3. *Let $g(t) \ge 0$ be defined on an interval $[0,T)$, $T > 0$, and have a bounded derivative from the right at every point of this interval, $\dot{g}(t) := \lim_{s\to+0} \frac{g(t+s)-g(t)}{s}$. Assume that $g(t)$ satisfies the following inequality*

$$\dot{g}(t) \le -\gamma(t)g(t) + \alpha(t,g(t)) + \beta(t), \quad t \in [0,T); \quad g(0) = g_0, \tag{1.9}$$

where $\beta(t) \ge 0$ and $\gamma(t) \ge 0$ are continuous functions, defined on $[0,\infty)$, and $\alpha(t,v) \ge 0$ is defined on $[0,\infty) \times [0,\infty)$, $\alpha(t,v)$ is non-decreasing as a function of v, locally Lipschitz with respect to v, and continuous with respect to t on $\mathbb{R}_+ := [0,\infty)$.

If there exists a function $\mu > 0$, continuously differentiable on $\mathbb{R}_+$, such that

$$\alpha(t, \frac{1}{\mu(t)}) + \beta(t) \le \frac{1}{\mu(t)} \left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)} \right), \qquad \forall t \ge 0, \tag{1.10}$$

and

$$g(0) < \frac{1}{\mu(0)}, \tag{1.11}$$

then $g(t)$ exists for all $t \ge 0$, that is, $g(t)$ can be extended from $[0,T)$ to $[0,\infty)$, and $g(t)$ satisfies the following inequality:

$$0 \le g(t) < \frac{1}{\mu(t)}, \qquad \forall t \ge 0. \tag{1.12}$$

If $g(0) \le \frac{1}{\mu(0)}$, then $0 \le g(t) \le \frac{1}{\mu(t)}, \qquad \forall t \ge 0.$

Inequality (1.12) was formulated in [5] under some different assumptions, but not proved there. We sketch its proof at the end of this paper.

In [4] inequality (1.9) is studied in the case that includes $\alpha(t,g) = c_0 g^p$, where $p > 1$ and $c_0 > 0$ are constants, as a particular case: the coefficient c_0 in [4] was a function of time.

Our second stability result is the following theorem.

Theorem 1.4. *Assume that inequality* (1.3) *holds,*

$$Re(A(t)u,u) \le -\gamma(t)\|u\|^2, \qquad \forall t \ge 0, \tag{1.13}$$

and

$$\gamma(t) = \frac{c_1}{(1+t)^q}, \quad q \le 1; \quad c_1, q = const > 0. \tag{1.14}$$

Suppose that $\varepsilon \in (0, c_1)$ *is an arbitrary fixed number,* $\lambda = \left(\frac{c_0}{\varepsilon}\right)^{1/(p-1)}$, $(p-1)(c_1-\varepsilon) \ge q$, *and* $\|u(0)\| \le \frac{1}{\lambda}$.

Then the unique solution to (1.1)$-$(1.2) *exists on all of* $\mathbb{R}_+$ *and*

$$0 \le \|u(t)\| \le \frac{1}{\lambda(1+t)^{c_1-\varepsilon}}. \tag{1.15}$$

Remark 1. *One may change the formulation of Theorem 1.4 as follows: if for some positive constants* λ *and* $\nu > 0$ *inequalities* (2.12) *and* (2.7) *(see below) hold, then inequality* $\|u(t)\| \le \frac{1}{\lambda(1+t)^\nu}$ *holds for all* $t \ge 0$ *for the solution to problem* (1.1)$-$(1.2), *as follows from the proof of Theorem 1.4, given in Section 3.*

The rate of decay of the solution $u(t)$ as $t \to \infty$, obtained in Theorem 1.4, is not necessarily the best possible. The result in Theorem 1.4 is novel and interesting because no assumption of the type $\gamma(t) \ge \gamma_0 > 0$, where γ_0 is a constant, is made. This allows one to study, for instance, evolution problems with elliptic operators $A(t)$ the ellipticity constant $\lambda(t)$ of which may tend to zero as $t \to \infty$. Here $\lambda(t)$ is the smallest eigenvalue of the matrix $a_{ij}(t)$ of the elliptic operator $A(t)$. An example is given in Remark 2, at the end of the paper.

We have assumed above that $A(t)$ is a bounded linear operator, since this assumption is basic in the book [1], and in the Introduction to our paper a comparison was made with the results in [1]. However, boundedness of $A(t)$ was not used in our arguments. If $A(t)$ is a bounded linear operator satisfying the assumptions of Theorems 1.2 or 1.4, then one can guarantee the global existence of the solution to evolution problem (1.1)$-$(1.2). If $A(t)$ is an unbounded linear operator for which the global existence of $u(t)$ holds, then our arguments, which lead to estimate (1.15), remain valid. In the example given in Remark 2, the operator $A(t) = \gamma(t)(\Delta - I)$, where Δ is a selfadjoint realization of the Laplacian in $H = L^2(R^3)$, and I is the identity operator in H. For this $A(t)$ one knows that the solution $u(t)$ to problem (1.1)$-$(1.2) exists globally, so Theorem 1.4 is applicable.

In Section 2 proofs are given.

2 Proofs

Proof. (*Proof of Theorem 1.2*).

Multiply (1.1) by u, denote $g = g(t) := \|u(t)\|$, take the real part, and use the assumption (1.13) with $\gamma(t) = |\kappa| = const > 0$, to get

$$g\dot{g} \le -|\kappa| g^2 + c_0 g^{p+1}, \qquad p > 1. \tag{2.1}$$

If $g(t) > 0$ then the derivative $\dot{g}$ does exist, as one can easily check. If $g(t) = 0$ on an open subset of $\mathbb{R}_+$, then the derivative $\dot{g}$ does exist on this subset and $\dot{g}(t) = 0$ on this subset. If $g(t) = 0$ but in any neighborhood $(t-\delta, t+\delta)$ there are points at which g does not vanish, then by $\dot{g}$ we understand the derivative from the right, that is,

$$\dot{g}(t) := \lim_{s \to +0} \frac{g(t+s) - g(t)}{s} = \lim_{s \to +0} \frac{g(t+s)}{s}.$$

This limit does exist and is equal to $\|\dot{u}(t)\|$. Indeed, the function $u(t)$ is continuously differentiable, so

$$\lim_{s\to+0}\frac{\|u(t+s)\|}{s}=\lim_{s\to+0}\|\dot{u}(t)+o(1)\|=\|\dot{u}(t)\|.$$

The assumption about the existence of the bounded derivative $\dot{g}(t)$ from the right in Theorem 1.3 was made because the function $\|u(t)\|$ does not have, in general, a derivative in the usual sense at the points τ at which $\|u(\tau)\|=0$, no matter how smooth the function $u(t)$ is at the point τ. However, as we have proved above, the derivative $\dot{g}(t)$ from the right does exist always, if $u(t)$ is continuously differentiable at the point t.

Since $g\geq 0$, the inequality (2.1) yields inequality (1.9) with $\gamma(t)=|\kappa|=const>0$, $\beta(t)=0$, and $\alpha(t,g)=c_0g^p$. Inequality (1.10) takes the form

$$\frac{c_0}{\mu^p(t)}\leq\frac{1}{\mu(t)}\left(|\kappa|-\frac{\dot{\mu}(t)}{\mu(t)}\right),\qquad\forall t\geq 0. \tag{2.2}$$

Let

$$\mu(t)=\lambda e^{bt},\qquad\lambda,b=const>0, \tag{2.3}$$

and choose the constants λ and b later. Then inequality (2.2) takes the form

$$\frac{c_0}{\lambda^{p-1}e^{(p-1)bt}}+b\leq|\kappa|,\qquad\forall t\geq 0. \tag{2.4}$$

This inequality holds if

$$\frac{c_0}{\lambda^{p-1}}+b\leq|\kappa|. \tag{2.5}$$

Let $\varepsilon>0$ be an arbitrary small fixed number. Choose $b=|\kappa|-\varepsilon>0$. Then (2.5) holds if

$$\lambda\geq\left(\frac{c_0}{\varepsilon}\right)^{\frac{1}{p-1}}. \tag{2.6}$$

Condition (1.11) holds if

$$\|u_0\|=g(0)\leq\frac{1}{\lambda}. \tag{2.7}$$

From (2.6), (2.7) and (1.12) one gets

$$0\leq g(t)=\|u(t)\|\leq\frac{e^{-(|\kappa|-\varepsilon)t}}{\lambda},\qquad\forall t\geq 0. \tag{2.8}$$

Theorem 1.2 is proved. □

Proof. (*Proof of Theorem 1.4.*)

We start with inequality (2.2), let

$$\mu(t)=\lambda(1+t)^{\nu},\qquad\lambda,\nu=const>0, \tag{2.9}$$

and choose the constants λ and ν later. Inequality (2.2) holds if

$$\frac{c_0}{\lambda^{p-1}(1+t)^{(p-1)\nu}}+\frac{\nu}{1+t}\leq\frac{c_1}{(1+t)^q},\qquad\forall t\geq 0. \tag{2.10}$$

If

$$q\leq 1,\quad (p-1)\nu\geq q, \tag{2.11}$$

then inequality (2.10) holds if

$$\frac{c_0}{\lambda^{p-1}} + \nu \le c_1. \tag{2.12}$$

Let $\varepsilon > 0$ be an arbitrary small number. Choose

$$\nu = c_1 - \varepsilon. \tag{2.13}$$

Then (2.12) holds if (2.6) holds. Inequality (1.11) holds if (2.7) holds. Combining (2.6), (2.7) and (1.12), one obtains

$$0 \le \|u(t)\| = g(t) \le \frac{1}{\lambda(1+t)^{c_1-\varepsilon}}, \qquad \forall t \ge 0. \tag{2.14}$$

Choose $\lambda = \left(\frac{c_0}{\varepsilon}\right)^{\frac{1}{p-1}}$. Then inequality (2.12) holds because of (2.13). Inequality (1.11) holds because we have assumed in Theorem 1.4 that $\|u(0)\| \le \frac{1}{\lambda}$. Thus, the desired inequality (1.15) holds by Theorem 1.3.

Theorem 1.4 is proved. □

Proof. (*Proof of Theorem 1.3.*)

Define

$$v(t) := g(t)a(t), \quad a(t) := e^{\int_{t_0}^{t} \gamma(s)ds}, \quad \eta(t) := \frac{a(t)}{\mu(t)}. \tag{2.15}$$

Then inequality (1.9) takes the form

$$\dot v(t) \le a(t)[\alpha\left(t, \frac{v(t)}{a(t)}\right) + \beta(t)], \qquad v(0) = g(0) := g_0, \tag{2.16}$$

and

$$\dot\eta(t) = \frac{a(t)}{\mu(t)}[\gamma(t) - \frac{\dot\mu(t)}{\mu(t)}]. \tag{2.17}$$

From inequalities (1.11) and (1.10) one gets

$$v(0) < \frac{1}{\mu(0)} = \eta(0), \quad \dot v(0) \le \dot\eta(0). \tag{2.18}$$

Thus, $v(t) < \eta(t)$ on some interval $[0,T]$. Inequalities (2.16), (2.17), and (1.10) imply

$$\dot v(t) \le \dot\eta(t), \quad t \in [0,T]. \tag{2.19}$$

It follows from inequalities (2.18) and (2.19) that

$$v(t) < \eta(t), \qquad \forall t \ge 0. \tag{2.20}$$

From inequalities (2.20) and (2.15) one obtains

$$a(t)g(t) = v(t) < \eta(t) = \frac{a(t)}{\mu(t)}, \qquad \forall t \ge 0. \tag{2.21}$$

Since $a(t) > 0$, inequality (2.21) is equivalent to inequality (1.12). This essentially completes the major part of the proof of inequality (1.12). The last conclusion of Theorem 1.3 can be obtained by a standard limiting procedure.

Let us explain in detail why inequality (2.21) holds for all $t \ge 0$. The right-hand side of inequality (2.21) is defined for all $t \ge 0$. The function $g(t)$, a solution to inequality (1.9), exists on every interval on which $v(t)$ exists, and $v(t)$, the solution to inequality (2.16), exists on every interval on which the solution $w(t)$ to the problem

$$\dot{w}(t) = a(t)[\alpha(t, \frac{w(t)}{a(t)} + \beta(t)], \qquad w(0) = v(0) \tag{2.22}$$

exists. It follows from inequality (2.16) and equation (2.22) that $v(t) \le w(t)$ on every interval $[0,T)$ on which w exists. We have already proved that the solution to problem (2.22) (which also is a solution to problem (2.16)) satisfies the estimate

$$0 \le w(t) \le \frac{a(t)}{\mu(t)} \tag{2.23}$$

on every interval on which w exists. We claim that estimate (2.23) implies that w exists for all $t \ge 0$, in other words, that $T = \infty$. Indeed, according to the known result (see, e.g., [2], Theorem 3.1 in Chapter 2), if the maximal interval $[0,T)$ of the existence of the solution to problem (2.22) is finite, that is $T < \infty$, then $\lim_{t\to T-0} w(t) = \infty$. This, however, cannot happen because of the inequality (2.23), since the function $\frac{a(t)}{\mu(t)}$ is bounded for every $t \ge 0$.

Theorem 1.3 is proved. $\square$

Remark 2. Let $H = L^2(R^3)$, $A(t) = \gamma(t)A$, where $A = A^*$ is a selfadjoint operator in H which is the closure of a symmetric operator $\Delta - I$ with the domain of definition $C_0^\infty(R^3)$. Here Δ is the Laplacian. Let $\gamma(t)$ be defined in (1.14) with $c_1 = 1$, $q = 0.5$. let $\varepsilon = 0.01$, $p = 3$, $c_0 = 1$, $\lambda = 10$, $\nu = 0.99$. Assume that $\|u_0\| \le (0.99)^{-1}$. Theorem 1.4 yields the following estimate $\|u(t)\| \le 0.1(1+t)^{-0.99}$ for the solution $u(t)$ to problem (1.1)$-$(1.2) with the defined above $A(t)$ and a nonlinearity satisfying condition (1.3).

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