ON THE SYMMETRY OF PRIMES

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Abstract. We prove a kind of "almost all symmetry" result for the primes, i.e. we give non-trivial bounds for the "symmetry integral", say $I_{\Lambda}(N,h)$, of the von Mangoldt function $\Lambda(n)$ ($:= \log p$ for prime-powers $n = p^r$, 0 otherwise). Here we get $I_{\Lambda}(N,h) \ll NhL^5$, with $L := \log N$; then, as a Corollary, we bound non-trivially the Selberg integral of the primes, i.e. the mean-square of $\sum_{x<n\le x+h} \Lambda(n) - h$, over $x \in [N,2N]$, to get the "Prime Number Theorem in short intervals" of (log-powers!) length $h \ge L^{11/2+\varepsilon}$ ($\varepsilon > 0$, arbitrarily small). We trust the improvement $c < \frac{11}{2}$ in the exponent.

1. Introduction and statement of the results.

We give, here, a concrete example of "*essentially bounded*" (see [C1]), i.e. bounded by arbitrarily small powers, arithmetic function for the problem of "*almost all*" (abbreviated a.a. now on) symmetry in short intervals (see [C1]), namely the von-Mangoldt function

$$\Lambda(n) \stackrel{def}{=} \begin{cases} \log p & \text{if } n = p^r, \ p \text{ prime and } r \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

We mean, by almost all the short intervals $[x-h, x+h]$ (or even $[x, x+h]$, here), all of them, for $x \in [N, 2N]$, except possibly $o(N)$ of them (everywhere in this paper $N \to \infty$) and "short" since $h = h(N) \to \infty$ and $h = o(N)$.

Then, the Selberg integral of the primes, namely

$$J(N,h) \stackrel{def}{=} \int_N^{2N} \Big| \sum_{x<n\le x+h} \Lambda(n) - h \Big|^2 dx,$$

counts the deviations of the number of primes in a.a. short intervals $[x, x+h]$, giving the well-known Prime Number Theorem in a.a.s.i. (short intervals)

$$(PNT \ \ a.a.s.i.) \qquad \pi(x+h) - \pi(x) \sim \frac{h}{\log x} \ \ \forall x \in [N,2N] \text{ but } \ o(N)$$

where $\pi(x) := |\{p \le x : p \ \text{ prime}\}|$ is the number of primes up to x, since

$$J(N,h) = o(Nh^2) \iff \text{PNT a.a.s.i. } [x, x+h]$$

while the symmetry integral of the primes, say (as usual, $t \ne 0 \Rightarrow \text{sgn}(t) := |t|/t$, $\text{sgn}(0) := 0$)

$$\int_N^{2N} \Big| \sum_{|n-x|\le h} \Lambda(n)\text{sgn}(n-x) \Big|^2 dx,$$

checks the a.a. symmetry of primes in short intervals $[x-h, x+h]$, around the center-point x.

Actually, for arithmetic functions $f : \mathbb{N} \to \mathbb{C}$, we define [C1] the discrete variant, $x \sim N$ is $N < x \le 2N$,

$$I_f(N,h) \stackrel{def}{=} \sum_{x\sim N} \Big| \sideset{}{'}\sum_{|n-x|\le h} f(n)\text{sgn}(n-x) \Big|^2,$$

where the dash means: the terms $n = x \pm h$ are taken with weight $\frac{1}{2}$; for essentially bounded f, this discrete mean-square is close to the continuous one, see [C1]. For example, $f = \Lambda$ gives, writing hereafter $L := \log N$,

$$\int_N^{2N} \Big| \sum_{|n-x|\le h} \Lambda(n)\text{sgn}(n-x) \Big|^2 dx \ll \int_N^{2N} \Big| \sideset{}{'}\sum_{|n-[x]|\le h} \Lambda(n)\text{sgn}(n-[x]) \Big|^2 dx + NL^2,$$

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from the trivial $\Lambda(n) \ll \log N$; then, since the integral on the right is the sum over $N \le x < 2N$, we have to count the terms for $x = N$ and $x = 2N$ as $\ll | \sum'_{|n-2N|\le h} \Lambda(n)\mathrm{sgn}(n-2N)|^2 \ll h^2L^2$, in order to get

$$\int_N^{2N} \Big| \sum_{|n-x|\le h} \Lambda(n)\mathrm{sgn}(n-x)\Big|^2 dx \ll \sum_{x\sim N} \Big| \sum_{|n-x|\le h}{}' \Lambda(n)\mathrm{sgn}(n-x)\Big|^2 + NL^2 + h^2L^2 \ll I_\Lambda(N,h) + (N+h^2)L^2,$$

where the remainders are $\ll NhL^2$, negligible (see Theorem, following), from the hypothesis of short intervals $h \to \infty$ and $h = o(N)$, when $N \to \infty$. In the same way,

$$I_\Lambda(N,h) \ll \int_N^{2N} \Big| \sum_{|n-x|\le h} \Lambda(n)\mathrm{sgn}(n-x)\Big|^2 dx + (N+h^2)L^2,$$

remainders still negligible, for the same reasons. So, we'll work with I_Λ, see Theorem proof beginning.

We give our main result.

THEOREM. *Fix* $\varepsilon > 0$, *small. Let* $N, h \in \mathbb{N}$, *with* $h \le N^{2/3}L$ *and* $h \to \infty$ *when* $N \to \infty$. *Then*

$$\int_N^{2N} \Big| \sum_{|n-x|\le h} \Lambda(n)\mathrm{sgn}(n-x)\Big|^2 dx \ll NhL^5 + N^\varepsilon h^3.$$

Also, in the same hypotheses, assuming $h \le \sqrt{N}/N^\varepsilon$,

$$\int_N^{2N} \Big| \sum_{|n-x|\le h} \Lambda(n)\mathrm{sgn}(n-x)\Big|^2 dx \ll NhL^5.$$

The new form of the Riemann-von Mangoldt formula, [CLap, Th.m], in [Lang, Th.m 4] then proves the

COROLLARY. *Fix* $\varepsilon > 0$, *small. Let* $N, H \in \mathbb{N}$, *with* $H = H(N) \ge L^{11/2+\varepsilon}$ *and* $N \to \infty$. *Then* PNT *for* a.a. *short intervals of length* H, *i.e.*

$$J(N,H) = o(NH^2).$$

We trust the possibility to get $J(N,H)$ lower bounds from $I_\Lambda(N,H)$ lower bounds, [C], in a future paper.

We'll prove the Theorem in §3 and the Corollary in §4. First, some elementary Lemmas.

2. LEMMAS.

Here $*$ is the Dirichlet product, μ Möbius function, $\mathbf{1}(n) = 1$ in his inversion formula $f = g * \mathbf{1} \Leftrightarrow g = f * \mu$. For a generic $f : \mathbb{N} \to \mathbb{C}$, with $g := f * \mu$ of finite support, say supp (g), the *Ramanujan coefficients*

$$R_\ell(f) \stackrel{def}{=} \sum_{m\equiv 0(\mathrm{mod}\ \ell)} \frac{g(m)}{m} \quad \forall \ell \in \mathbb{N}$$

are well-defined. If supp $(g) \subset [1,Q]$, $\|g\|_\infty := \max_{q\in \mathrm{supp}\,(g)} |g(q)|$ and $\mathbf{1}_\mathcal{D}$ is $\mathcal{D}$ characteristic function,

$$(0) \qquad |g| \ll \|g\|_\infty \Rightarrow R_\ell(g*\mathbf{1}) = \frac{1}{\ell}\sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \ll \|g\|_\infty R_\ell(\mathbf{1}_{[1,Q]} * \mathbf{1}) \ll \|g\|_\infty R_\ell(d) \ll \frac{L}{\ell}\|g\|_\infty.$$

Here $d = d(n) = (\mathbf{1} * \mathbf{1})(n)$ is the divisor function, supported in $[1, 3N]$, say, so uniformly $\forall Q \le 3N$

$$R_\ell(\mathbf{1}_{[1,Q]} * \mathbf{1}) = \frac{1}{\ell}\sum_{q\le \frac{Q}{\ell}} \frac{1}{q} \ll \frac{L}{\ell}, \quad R_\ell(d) = R_\ell(\mathbf{1}*\mathbf{1}) = \frac{1}{\ell}\sum_{q\le \frac{3N}{\ell}} \frac{1}{q} \ll \frac{L}{\ell}.$$

Define the Fourier coefficients $F_h^{\pm}$ as follows in §3, in the Theorem proof. Until next section, $f:\mathbb{N}\to\mathbb{R}$. Set

$$\Sigma_f^{(1)}(A) := \sum_{2<\ell,t\le Q}\sum R_\ell(f)R_t(f) \sideset{}{^*}\sum_{\substack{j\le \frac{\ell}{2} \\ \delta:=\left\|\frac{j}{\ell}-\frac{r}{t}\right\|>\frac{1}{A}}} \sideset{}{^*}\sum_{r\le\frac{t}{2}} F_h^{\pm}\Big(\frac{j}{\ell}\Big)\, F_h^{\pm}\Big(\frac{r}{t}\Big) \sum_{Q<x\le 2N} \cos 2\pi\delta x,$$

for $Q\ll N$; here, as usual, $\|\alpha\| := \min_{n\in\mathbb{Z}}|\alpha-n|$ is the *distance to* the *integers*. Here and in the sequel, $\sum^*$ denotes restriction to reduced residue classes: $(j,\ell)=1=(r,t)$, whence j/ℓ and r/t are Farey fractions.

Here we want to bound this $\Sigma_f^{(1)}(A)$ applying a well-spaced argument, resembling the one used to prove the Large Sieve inequality. This is possible, since the Farey fractions appearing here are both in $]0,1/2[$ (say, both positive). We wish to treat also the following term in the same way.

Defining in fact

$$\Sigma_f^{(2)}(A) := \sum_{2<\ell,t\le Q}\sum R_\ell(f)R_t(f) \sideset{}{^*}\sum_{\substack{j\le \frac{\ell}{2} \\ \sigma:=\left\|\frac{j}{\ell}+\frac{r}{t}\right\|>\frac{1}{A}}} \sideset{}{^*}\sum_{r\le\frac{t}{2}} F_h^{\pm}\Big(\frac{j}{\ell}\Big)\, F_h^{\pm}\Big(\frac{r}{t}\Big) \sum_{Q<x\le 2N} \cos 2\pi\sigma x,$$

this can be expressed in terms of δ again, changing sign to r and using the fact: $F_h^{\pm}$ is odd, see below (§3),

$$\Sigma_f^{(2)}(A) = -\sum_{2<\ell,t\le Q}\sum R_\ell(f)R_t(f) \sideset{}{^*}\sum_{\substack{j\le \frac{\ell}{2} \\ \delta:=\left\|\frac{j}{\ell}-\frac{r}{t}\right\|>\frac{1}{A}}} \sideset{}{^*}\sum_{-\frac{t}{2}\le r\le -1} F_h^{\pm}\Big(\frac{j}{\ell}\Big)\, F_h^{\pm}\Big(\frac{r}{t}\Big) \sum_{Q<x\le 2N} \cos 2\pi\delta x;$$

here we have the problem of two different Farey fractions in two different intervals, now, and this prevents us from applying the same well-spaced argument possible for the previous term; but this trouble can be avoided, expressing this double sum over Farey fractions in distinct intervals through double sums over distinct fractions in the same interval. In fact, here in $\Sigma_f^{(2)}(A)$, one is positive and the other is negative, whence, looking at all the cases for the signs of $\frac{j}{\ell}$ and $\frac{r}{t}$ (first, exchange them), we may write

$$2\Sigma_f^{(2)}(A) = -\sum_{2<\ell,t\le Q}\sum R_\ell(f)R_t(f) \sideset{}{^*}\sum_{\substack{|j|\le \frac{\ell}{2} \\ \delta:=\left\|\frac{j}{\ell}-\frac{r}{t}\right\|>\frac{1}{A}}} \sideset{}{^*}\sum_{|r|\le\frac{t}{2}} F_h^{\pm}\Big(\frac{j}{\ell}\Big)\, F_h^{\pm}\Big(\frac{r}{t}\Big) \sum_{Q<x\le 2N} \cos 2\pi\delta x + 2\Sigma_f^{(1)}(A),$$

obtaining: $\Sigma_f^{(1)}(A)-\Sigma_f^{(2)}(A) = \frac{1}{2}\Sigma_f(A)$, with Farey fractions $\mathcal{F}=\mathcal{F}_Q\subset[0,1]$ of denominators in $]2,Q]$, and

$$\Sigma_f(A) \stackrel{def}{=} \sum_{2<\ell,t\le Q}\sum R_\ell(f)R_t(f) \sideset{}{^*}\sum_{\substack{\frac{j}{\ell}\in\mathcal{F} \\ \delta:=\left\|\frac{j}{\ell}-\frac{r}{t}\right\|>\frac{1}{A}}} \sideset{}{^*}\sum_{\frac{r}{t}\in\mathcal{F}} F_h^{\pm}\Big(\frac{j}{\ell}\Big)\, F_h^{\pm}\Big(\frac{r}{t}\Big) \sum_{Q<x\le 2N} \cos 2\pi\delta x.$$

In all, we can bound the difference $\Sigma_f^{(1)}(A)-\Sigma_f^{(2)}(A)$ through $\Sigma_f(A)$ bound, following.

We can state and show our

Lemma A. *Let* $A,N,h,Q\in\mathbb{N}$, *with* $Q\le 2N$ *and* $A\to\infty$, $h\to\infty$, $h=o(N)$, *when* $N\to\infty$. *Assume* $g:\mathbb{N}\to\mathbb{R}$ *is supported in* $[1,Q]$. *Then*

$$\Sigma_{g*\mathbf{1}}(A) \ll AL\sum_{2<\ell\le 2h}\Bigg|\sum_{d\le\frac{Q}{\ell}}\frac{g(\ell d)}{d}\Bigg|^2 + ALh\sum_{2h<\ell\le Q}\frac{1}{\ell}\Bigg|\sum_{d\le\frac{Q}{\ell}}\frac{g(\ell d)}{d}\Bigg|^2$$

and, even better, as a consequence of Montgomery & Vaughan generalization of Hilbert's inequality,

$$\Sigma_{g*\mathbf{1}}(A) \ll A\sum_{2<\ell\le 2h}\Bigg|\sum_{d\le\frac{Q}{\ell}}\frac{g(\ell d)}{d}\Bigg|^2 + Ah\sum_{2h<\ell\le Q}\frac{1}{\ell}\Bigg|\sum_{d\le\frac{Q}{\ell}}\frac{g(\ell d)}{d}\Bigg|^2.$$

Remark. Of course, in case $Q \le 2h$ we have the second sum over ℓ empty, i.e. not counted.

PROOF. The first's [C0] elementary Lemma (only Cauchy inequality), see [CS,Lemma 2]. Corollary 2 [M] is:

$$\|\alpha_m - \alpha_n\| \ge \Delta > 0 \Rightarrow \left| \sum_{m \ne n} u_m \overline{u_n} \, \frac{\sin t(\alpha_m - \alpha_n)}{\sin \pi(\alpha_m - \alpha_n)} \right| \le \frac{1}{\Delta} \sum_m |u_m|^2, \ \ \forall t \in \mathbb{R} \ \forall u_m \in \mathbb{C},$$

which follows [MV] Hilbert's inequality: gives the second, once applied to $\Delta := 1/A$ well-spaced Farey fractions $\alpha_m := \frac{j}{\ell}$, $\alpha_n := \frac{r}{t}$, numbering them with $1 \le m, n \ll Q^2$, $u_m := R_\ell(f) F_h^{\pm}(\frac{j}{\ell})$; in fact, from

$$\sum_{Q < x \le 2N} \cos 2\pi \delta x = \sum_{Q < x \le 2N} \cos 2\pi(\alpha_m - \alpha_n)x = \frac{1}{2} \left[\frac{\sin t(\alpha_m - \alpha_n)}{\sin \pi(\alpha_m - \alpha_n)} \right]_{t=2\pi Q + \pi/2}^{t=4\pi N + \pi/2}$$

and, see (2), Theorem proof in §3, using $F_h^{\pm}$ is odd,

$$\frac{1}{\ell^2} \sum_{|j| \le \frac{\ell}{2}}^{*} \left| F_h^{\pm}\!\left(\frac{j}{\ell}\right) \right|^2 \le \frac{2}{\ell^2} \sum_{j \le \frac{\ell}{2}} F_h^{\pm}\!\left(\frac{j}{\ell}\right)^2 \ll \min\left(1, \frac{h}{\ell}\right),$$

recalling the above definition of Ramanujan coefficients $R_\ell(f) = R_\ell(g * \mathbf{1})$, with $\overline{R_\ell(f)} = R_\ell(g * \mathbf{1})$,

$$\sum_m |u_m|^2 = \sum_{2 < \ell \le Q} \left| \sum_{d \le \frac{Q}{\ell}} \frac{g(\ell d)}{d} \right|^2 \frac{1}{\ell^2} \sum_{|j| \le \frac{\ell}{2}}^{*} \left| F_h^{\pm}\!\left(\frac{j}{\ell}\right) \right|^2 \ll \sum_{2 < \ell \le Q} \left| \sum_{d \le \frac{Q}{\ell}} \frac{g(\ell d)}{d} \right|^2 \min\left(1, \frac{h}{\ell}\right). \ \square$$

We need, now, an upper bound for the symmetry integral of the divisor function $d(n)$; actually, we have it from the asymptotic results of [CS] (see Theorem 1 and Corollary 1 there), but in the hypothesis $h < \frac{\sqrt{N}}{2}$; here, we can confine to bounds, but in a longer range for h and we'll accomplish this in a faster way (no asymptotic estimates are required !). However, the tiny details of calculation come from [CS], like the idea to apply the Large Sieve inequality (here, use Lemma A).

We give and prove the following (see [C2] bounds)

LEMMA B. *Let* $N, h \in \mathbb{N}$ *with* $h \to \infty$ *and* $h \ll N^{2/3}L$, *when* $N \to \infty$. *Then*

$$I_d(N,h), \ \int_N^{2N} \Big| \sum_{|n-x| \le h} d(n) \mathrm{sgn}(n-x) \Big|^2 \ll N h L^3.$$

PROOF. Since $I_d(N,h)$ differs from the integral for two kind of terms, see the above, we estimate them, i.e.: that for $x = 2N$ giving the negligible $\ll N^{\varepsilon} h^2 \ll N h L^3$, due to $d(n) \ll_{\varepsilon} n^{\varepsilon/2}$ (see [D]), while we keep that for $n = x$ and $n = x \pm h$, see above & remark in §3 on the $\chi_q(x)$ "edges", giving $d(x)$ & $d(x \pm h)$:

$$d(n) = 2 \sum_{d|n, d < \sqrt{n}} 1 + \mathbf{1}_{\mathbb{N}}(\sqrt{n}) \Rightarrow \left| S_d^{\pm}(x) \right| \ll \left| \sum_{d \le \sqrt{x}} \chi_d(x) \right| + d(x) + d(x \pm h) + \sum_{\sqrt{x-h} \le d \le \sqrt{x+h}} \left(\frac{h}{d} + 1 \right),$$

see [CS] for details, with $S_d^{\pm}(x) = S_d^{\pm}(x,h) := \sum'_{|n-x| \le h} \mathrm{sgn}(n-x) d(n)$ the symmetry sum of $d(n)$, so

$$I_d(N,h) \ll \sum_{x \sim N} \left| \sum_{d \le \sqrt{x}} \chi_d(x) \right|^2 + \sum_{x \sim N} d(x)^2 + \sum_{x \sim N} d(x \pm h)^2 + \sum_{x \sim N} \left| \sum_{\sqrt{x-h} \le d \le \sqrt{x+h}} \left(\frac{h}{d} + 1 \right) \right|^2,$$

where, by Cauchy inequality, this last remainder contributes to $I_d(N,h)$ as, see esp. [CS],

$$\ll \sum_{x\sim N}\left(\frac{h}{\sqrt{N}}+1\right)\sum_{\sqrt{x-h}\le d\le\sqrt{x+h}}\left(\frac{h^2}{d^2}+1\right)\ll \frac{h^4}{N}+N\ll NhL^3,$$

using our hypotheses on h; the other remainder terms can be estimated using the elementary

$$\sum_{n\le x} d(n)^2=\sum_{n\le x}\sum_{d_1|n}\sum_{d_2|n}1=\sum_{d\le x}\sum_{m_1\le\frac{x}{d}}\sum_{\substack{m_2\le x/d\\(m_1,m_2)=1}}\left[\frac{x}{dm_1m_2}\right]\ll\sum_{d\le x}\sum_{m_1\le x}\sum_{m_2\le x}\frac{x}{dm_1m_2}=x\left(\sum_{d\le x}\frac{1}{d}\right)^3$$

for $[t]$ the integer part of t, $\forall t\in\mathbb{R}$, trivially from the trivial

$$\sum_{d\le x}\frac{1}{d}\le 1+\int_1^x\frac{dt}{t}\ll\log x,$$

to get

$$\sum_{x\sim N}d(x-h)^2+\sum_{x\sim N}d(x)^2+\sum_{x\sim N}d(x+h)^2\ll\sum_{n\le 3N}d(n)^2\ll NL^3\ll NhL^3,$$

since $h\to\infty$. Then we are left with (here $c^{\pm}_{j,q}=-\frac{i}{2q}F^{\pm}_h(j/q)$, see §3, compare [CS] coefficients $c_{j,q}$)

$$\sum_{x\sim N}\left|\sum_{q\le\sqrt{x}}\chi_q(x)\right|^2\ll\sum_{x\sim N}\left|\sum_{q\le\sqrt{N}}\chi_q(x)\right|^2+\sum_{x\sim N}\left|\sum_{\sqrt{N}<q\le\sqrt{x}}\chi_q(x)\right|^2\ll$$

$$\ll\sum_{x\sim N}\left|\sum_{2<d\le\sqrt{N}}\left(\sum_{k\le\frac{\sqrt{N}}{d}}\frac{1}{k}\right){\sum_{j\le d}}^{*}c^{\pm}_{j,d}\sin\frac{2\pi xj}{d}\right|^2+\sum_{x\sim N}\left|\sum_{2<d\le\sqrt{x}}\left(\sum_{\frac{\sqrt{N}}{d}<k\le\frac{\sqrt{x}}{d}}\frac{1}{k}\right){\sum_{j\le d}}^{*}c^{\pm}_{j,d}\sin\frac{2\pi xj}{d}\right|^2$$
$$=\Sigma_1+\Sigma_2,$$

say, applying for both of them ${\sum_{j\le d}}^{*}|c^{\pm}_{j,d}|^2\le\sum_{0<j<d}|c^{\pm}_{j,d}|^2\ll\min(1,h/d)$, compare (2) in §3,

$$\Sigma_1:=\sum_{x\sim N}\left|\sum_{2<d\le\sqrt{N}}\left(\sum_{k\le\frac{\sqrt{N}}{d}}\frac{1}{k}\right){\sum_{j\le d}}^{*}c^{\pm}_{j,d}\sin\frac{2\pi xj}{d}\right|^2\ll NhL^3,$$

Lemma A, second, or [CS,Lemma 1]; whilst, esp., Lemma A first bound or [C0,Lemma],[CS,Lemma 3]

$$\Sigma_2:=\sum_{x\sim N}\left|\sum_{2<d\le\sqrt{x}}\left(\sum_{\frac{\sqrt{N}}{d}<k\le\frac{\sqrt{x}}{d}}\frac{1}{k}\right){\sum_{j\le d}}^{*}c^{\pm}_{j,d}\sin\frac{2\pi xj}{d}\right|^2\ll NhL^2,$$

because w.r.t. Σ_1 we lose one L (see the Lemma A 1st-2nd bounds difference), but now (see [D])

$$\sum_{\frac{\sqrt{N}}{d}<k\le\frac{\sqrt{x}}{d}}\frac{1}{k}\ll 1$$

(recall $x\ll N$) gains L^2, with respect to [T]

$$\sum_{k\le\frac{\sqrt{N}}{d}}\frac{1}{k}\ll L.$$

Thus

$$I_d(N,h) \ll \Sigma_1 + \Sigma_2 + NhL^3 \ll NhL^3. \ \square$$

We explicitly remark that elementary methods can't go beyond the remainder $\mathcal{O}(h^4/N)$: this fixes the range of uniformity for the symmetry integral bound of the divisor function, see the above.

Finally, we obtain here that the terms with Farey fractions $\frac{j}{\ell}, \frac{r}{t}$ such that $\|j/\ell + r/t\| \le 1/A < 1/6N$ can't have $\ell, t > 2$, so they give empty sums (choosing denominators > 2, now on, comes from $F_h^{\pm}(1/2) = 0$).

We state and prove the following

Lemma C. *Let* $A, N \in \mathbb{N}$ *with* $A > 6N$. *Let* $j/\ell, r/t \in]0, 1/2]$ *be Farey fractions and* $\ell, t \le Q \le 3N$. *Then*

$$\ell, t > 2 \ \Rightarrow \ \left\| \frac{j}{\ell} + \frac{r}{t} \right\| > \frac{1}{A}.$$

proof. Assuming $\sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \le 1/A$ we'll get the absurd $\frac{j}{\ell} = \frac{1}{2} = \frac{r}{t}$ (in Farey fractions $\Rightarrow \ell, t = 2$). So,

$$\sigma \le \frac{1}{A} \ \Rightarrow \ 0 < \frac{j}{\ell} + \frac{r}{t} \le \frac{1}{A} \ \text{or we have} \ 0 \le 1 - \frac{j}{\ell} - \frac{r}{t} \le \frac{1}{A};$$

first case gives in particular $0 < j \le \frac{\ell}{A} \le \frac{Q}{A} \le \frac{3N}{A} < 1$, $0 < r \le \frac{t}{A} \le \frac{Q}{A} \le \frac{3N}{A} < 1$, i.e., absurd at once. Hence

$$0 \le \left(\frac{1}{2} - \frac{j}{\ell}\right) + \left(\frac{1}{2} - \frac{r}{t}\right) \le \frac{1}{A} \ \Rightarrow \ 0 \le \frac{1}{2} - \frac{j}{\ell} \le \frac{1}{A}, 0 \le \frac{1}{2} - \frac{r}{t} \le \frac{1}{A},$$

whence (use $\ell/A, t/A < 1$, here)

$$\frac{\ell}{2} - \frac{\ell}{A} \le j \le \frac{\ell}{2}, \ \frac{t}{2} - \frac{t}{A} \le r \le \frac{t}{2}, \ \Rightarrow \ j = \left[\frac{\ell}{2}\right], r = \left[\frac{t}{2}\right]$$

that, see the above for $1 - j/\ell - r/t$, give, this time from $A > 6N$,

$$0 \le \frac{1}{\ell}\left\{\frac{\ell}{2}\right\} + \frac{1}{t}\left\{\frac{t}{2}\right\} \le \frac{1}{A} \ \Rightarrow \ \{\ell/2\} = 0 = \{t/2\} \ \Rightarrow \ 2|\ell, 2|t \ \Rightarrow \ \frac{j}{\ell} = \frac{1}{2} = \frac{r}{t}. \ \square$$

3. Proof of the Theorem.

proof. Write $f = g * \mathbf{1}$, i.e. open $f(n) = \sum_{q|n} g(q)$: $q|n$, $n \le x+h \Rightarrow q \le x+h$,

$$I_f(N,h) = \sum_{x \sim N} \Big| \sum_{q \le x+h} g(q)\chi_q(x) \Big|^2,$$

with the "character-like"(compare [CS], esp.) $\chi_q(x)$, defined below $\forall q \in \mathbb{N}$ (vanishes whenever $q > x+h$):

$$\chi_q(x) \stackrel{def}{=} - \sideset{}{'}\sum_{\substack{|n-x| \le h \\ n \equiv 0 (\bmod q)}} \mathrm{sgn}(n-x) = - \sideset{}{'}\sum_{\frac{x-h}{q} \le m \le \frac{x+h}{q}} \mathrm{sgn}\left(m - \frac{x}{q}\right) \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\};$$

we remark that, actually, $\chi_q(x) = \mp\frac{1}{2} \Leftrightarrow q|x \pm h$, the "edges" of $\chi_q(x)$; also, cases $\chi_q(x) \ne 0$ are "rare".

Here we have, first, to prepare the symmetry sums to further calculations.

In fact, the symmetry sum of our f is, in the hypothesis $x > N+2h$,

$$S_f^{\pm}(x,h) \overset{def}{=} \sideset{}{'}\sum_{|n-x|\le h} \mathrm{sgn}(n-x)f(n) = -\sum_{q\le N+h} g(q)\chi_q(x) - \sum_{N+h<q<x-h} g(q)\chi_q(x) - \sum_{x-h\le q\le x+h} g(q)\chi_q(x);$$

from $x \le 2N < 2N+h$ we get $N+h<q<x-h \Rightarrow 1 < \frac{x-h}{q} < \frac{x+h}{q} < 2 \Rightarrow \chi_q(x) = 0$, since $\nexists m \in]1,2[$; and $x-h \le q \le x+h \Rightarrow m=1$, i.e. $-\sum_{x-h\le q\le x+h} g(q)\chi_q(x) = S_g^{\pm}(x,h)$. For general $g : \mathbb{N} \to \mathbb{C}$, $f = g * \mathbf{1}$

$$(1)\ \ S_f^{\pm}(x,h) = S_g^{\pm}(x,h) - \sum_{q\le N+h} g(q)\chi_q(x) \Rightarrow |S_{f-g}^{\pm}(x,h)| = |S_{g-f}^{\pm}(x,h)| = \Bigg|\sum_{q\le N+h} g(q)\chi_q(x)\Bigg|\ \forall x > N+2h.$$

Now on we'll work in order to express the sum over q in terms of Farey fractions, i.e. reduced fractions j/ℓ (meaning the g.c.d. (j,ℓ) is 1). For the sake of clarity, we assume that g and its support don't depend on x.

From the orthogonality of additive characters:

$$\chi_q(x) = -\sideset{}{'}\sum_{\substack{|s|\le h \\ s+x\equiv 0 (\mathrm{mod}\ q)}} \mathrm{sgn}(s) = \frac{1}{q}\sum_{j(\mathrm{mod}\ q)} \Big(-2i \sideset{}{'}\sum_{s\le h} \sin\frac{2\pi js}{q}\Big) e_q(xj),$$

where the symmetric dashed sum means: $s = \pm h$ terms have weight $\frac{1}{2}$ and the last sum halves only $s = h$;

$$\chi_q(x) = \frac{1}{q}\sum_{j\le q/2} \Big(4 \sideset{}{'}\sum_{s\le h} \sin\frac{2\pi js}{q}\Big) \sin\frac{2\pi xj}{q},$$

since $j=0$ gives 0, also, $j = q/2$ gives $\sin\frac{2\pi js}{q} = \sin \pi s = 0$, $\forall s \in \mathbb{N}$. We define, say, the *Fourier coefficients*

$$F_h^{\pm}\left(\frac{j}{q}\right) \overset{def}{=} 4 \sideset{}{'}\sum_{s\le h} \sin\frac{2\pi js}{q},$$

in the finite Fourier expansion (we need it for $j \le q/2$ for the following reason on the non-negativity of $F_h^{\pm}$):

$$\chi_q(x) = \frac{1}{q}\sum_{j\le q/2} F_h^{\pm}\left(\frac{j}{q}\right) \sin\frac{2\pi xj}{q},$$

where we see immediately that the Fourier coefficients are positive (better, non-negative):

$$\sideset{}{'}\sum_{s\le h} \sin\frac{2\pi js}{q} = \sum_{s\le h} \sin\frac{2\pi js}{q} - \frac{1}{2}\sin\frac{2\pi jh}{q} = \cot\frac{\pi j}{q}\sin^2\frac{\pi jh}{q},$$

from the geometric sum of $e(\alpha s)$, $\forall \alpha \notin \mathbb{Z}$, taking $\alpha := j/q$. Hence, $F_h^{\pm}$ is *odd and non-negative in* $]0,1/2[$

$$F_h^{\pm}\left(\frac{j}{q}\right) = 4\cot\frac{\pi j}{q}\sin^2\frac{\pi jh}{q} \ge 0 \quad \forall j \le \frac{q}{2}$$

but, also, $\|\alpha\| = \min(\{\alpha\}, 1-\{\alpha\})$ gives

$$F_h^{\pm}\left(\frac{j}{q}\right) = 4 \sideset{}{'}\sum_{s\le h} \sin\frac{2\pi js}{q} = 4\sum_{s\le q\|h/q\|} \sin\frac{2\pi js}{q} + \mathcal{O}(1) = -2i\sum_{|s|<q\|h/q\|} \mathrm{sgn}(s)e_q(js) + \mathcal{O}(1)$$

and we need, say, Parseval identity for these coefficients:

$$\frac{1}{q^2}\sum_{j\le q}\Big|\sum_{|s|<q\|\frac{h}{q}\|} \mathrm{sgn}(s)e_q(js)\Big|^2 = \sum_{|s_1|<q\|\frac{h}{q}\|} \mathrm{sgn}(s_1) \sum_{|s_2|<q\|\frac{h}{q}\|} \mathrm{sgn}(s_2)\frac{1}{q^2}\sum_{j\le q} e_q(j(s_1-s_2)) = \frac{1}{q}\sum_{0<|s|<q\|\frac{h}{q}\|} 1,$$

whence

$$\frac{1}{q^2}\sum_{j\le \frac{q}{2}} F_h^{\pm}\left(\frac{j}{q}\right)^2 \ll \min\left(1,\frac{h}{q}\right) \quad \forall q>2 \tag{2}$$

In order to apply a kind of Large Sieve inequality (see Lemma A) we need to express $\chi_q(x)$ in terms of Farey fractions (i.e., we need a kind of Ramanujan expansion for it), so we collect in terms of g.c.d. (j,q)

$$\chi_q(x)=\frac{1}{q}\sum_{\substack{d|q\\ d<q}}\sum_{\substack{j\le q/2\\ (j,q)=d}} F_h^{\pm}\left(\frac{j}{q}\right)\sin\frac{2\pi xj}{q}=\frac{1}{q}\sum_{\substack{d|q\\ d<q}}\sum_{\substack{j'\le q/2d\\ (j',q/d)=1}} F_h^{\pm}\left(\frac{j'}{q/d}\right)\sin\frac{2\pi xj'}{q/d}$$

and setting $\ell := q/d$ we get

$$\chi_q(x)=\frac{1}{q}\sum_{\substack{\ell|q\\ \ell>1}}\sum_{j\le\frac{\ell}{2}}{}^{*} F_h^{\pm}\left(\frac{j}{\ell}\right)\sin\frac{2\pi xj}{\ell}\quad\forall q\in\mathbb{N}$$

but, actually, since $F_h^{\pm}\left(\frac{1}{2}\right)=0$, we can discard the only denominator giving $1/2$ in Farey fractions, i.e. $\ell=2$:

$$\chi_q(x)=\frac{1}{q}\sum_{\substack{\ell|q\\ \ell>2}}\sum_{j\le\frac{\ell}{2}}{}^{*} F_h^{\pm}\left(\frac{j}{\ell}\right)\sin\frac{2\pi xj}{\ell}\quad\forall q\in\mathbb{N}.$$

Coming back to (1), for generic $f:\mathbb{N}\to\mathbb{C}$, with (choose $g:=f*\mu$ here) $f=g*\mathbf{1}$, the bound is

$$I_{f-g}(N,h)=I_{g-f}(N,h)\ll\sum_{N+2h<x\le 2N}\left|\sum_{2<\ell\le N+h}\sum_{d\le\frac{N+h}{\ell}}\frac{g(\ell d)}{\ell d}\sum_{j\le\frac{\ell}{2}}{}^{*}F_h^{\pm}\left(\frac{j}{\ell}\right)\sin\frac{2\pi xj}{\ell}\right|^2+h^3\|f-g\|_\infty^2,$$

where $\|f\|_\infty := \max_{n\le 3N}|f(n)|$; from Ramanujan coefficients definition, adapted here to $Q=N+h$, i.e.

$$R_\ell(f)=\sum_{d\le\frac{N+h}{\ell}}\frac{g(\ell d)}{\ell d},$$

we get

$$I_{f-g}(N,h)=I_{g-f}(N,h)\ll\sum_{x\sim N}\left|\sum_{2<\ell\le N+h}R_\ell(f)\sum_{j\le\frac{\ell}{2}}{}^{*}F_h^{\pm}\left(\frac{j}{\ell}\right)\sin\frac{2\pi xj}{\ell}\right|^2+h^3\|f-g\|_\infty^2. \tag{3}$$

We may say these symmetry integrals have this Fourier-Ramanujan expansion, for any $f:\mathbb{N}\to\mathbb{C}$, $g:=f*\mu$.

Now the idea is very simple, once opened the square and taken sum over x inside: distinguish between terms on the diagonal and "near the diagonal" (in a suitable sense) on one side, giving a kind of majorant principle, opposed to all the others, far from the diagonal, for which we apply a kind of well-spaced argument.

Of course, this can be done for general f. Here, we confine to the case $g=\Lambda$, $f=\Lambda*\mathbf{1}=\log$, with the abbreviation $Q\overset{def}{=}N+h$:

$$I_\Lambda(N,h)\ll\sum_{x\sim N}\left|\sum_{|n-x|\le h}{}^{\prime}\operatorname{sgn}(n-x)\log n\right|^2+\sum_{x\sim N}\left|\sum_{2<\ell\le Q}\Big(\sum_{d\le\frac{Q}{\ell}}\frac{\Lambda(\ell d)}{d}\Big)\frac{1}{\ell}\sum_{j\le\frac{\ell}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)\sin\frac{2\pi xj}{\ell}\right|^2+h^3L^2;$$

use $\log n=\log x+\mathcal{O}(h/x)$ in the first term, while $\Lambda(n)\ll L$ above and for the $N<x\le Q$ terms ("tails"),

$$I_\Lambda(N,h)\ll\sum_{Q<x\le 2N}\left|\sum_{2<\ell\le Q}\Big(\sum_{d\le\frac{Q}{\ell}}\frac{\Lambda(\ell d)}{d}\Big)\frac{1}{\ell}\sum_{j\le\frac{\ell}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)\sin\frac{2\pi xj}{\ell}\right|^2+h^3L^2+\frac{h^4}{N}.$$

Last term's negligible; we omit also $\mathcal{O}(h^3L^2)$, in final bound. Open the square, take the $x-$sum inside:

$$I_\Lambda(N,h)=\sum_{2<\ell,t\le Q}\sum\Big(\sum_{d\le\frac{Q}{\ell}}\frac{\Lambda(\ell d)}{d}\Big)\Big(\sum_{q\le\frac{Q}{t}}\frac{\Lambda(tq)}{q}\Big)\frac{1}{\ell}\sum_{j\le\frac{\ell}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)\frac{1}{t}\sum_{r\le\frac{t}{2}}{}^{*}F_h^{\pm}\Big(\frac{r}{t}\Big)\sum_{Q<x\le 2N}\sin\frac{2\pi xj}{\ell}\sin\frac{2\pi xr}{t}$$

$$=D_{\log}^{\pm}(N,h)+\sum_{2<\ell,t\le Q}\sum\Big(\sum_{d\le\frac{Q}{\ell}}\frac{\Lambda(\ell d)}{d}\Big)\Big(\sum_{q\le\frac{Q}{t}}\frac{\Lambda(tq)}{q}\Big)\frac{1}{\ell t}\sum_{\substack{j\le\frac{\ell}{2}\\ \frac{j}{\ell}\neq\frac{r}{t}}}{}^{*}\sum_{r\le\frac{t}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)F_h^{\pm}\Big(\frac{r}{t}\Big)\sum_{x},$$

where in case $\frac{j}{\ell}\neq\frac{r}{t}$ we set $\sum_x:=\frac{1}{2}\sum_{Q<x\le 2N}\cos 2\pi\delta x-\frac{1}{2}\sum_{Q<x\le 2N}\cos 2\pi\sigma x$, abbreviating (compare the above) $\delta:=\big\|\frac{j}{\ell}-\frac{r}{t}\big\|$, $\quad\sigma:=\big\|\frac{j}{\ell}+\frac{r}{t}\big\|$, (here $\delta\in]0,1/2[$, $\sigma\in]0,1/2]$ from $\ell,t>2$) and we define the diagonal

$$D_{\log}^{\pm}(N,h)\overset{def}{=}\sum_{2<\ell\le Q}\Big(\sum_{d\le\frac{Q}{\ell}}\frac{\Lambda(\ell d)}{d}\Big)^2\frac{1}{\ell^2}\sum_{j\le\frac{\ell}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)^2\sum_{Q<x\le 2N}\sin^2\frac{2\pi xj}{\ell}\ge 0.$$

However, we may say that the diagonal amounts to $\delta=0$. Now,

$$I_\Lambda(N,h)=D_{\log}^{\pm}(N,h)+\sum_{2<\ell,t\le Q}\sum R_\ell(\log)R_t(\log)\sum_{\substack{j\le\frac{\ell}{2}\\ \delta>0}}{}^{*}\sum_{r\le\frac{t}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)F_h^{\pm}\Big(\frac{r}{t}\Big)\sum_x=$$

$$=D_{\log}^{\pm}(N,h)+\frac{1}{2}\sum_{2<\ell,t\le Q}\sum R_\ell(\log)R_t(\log)\sum_{\substack{j\le\frac{\ell}{2}\\ 0<\delta\le 1/A}}{}^{*}\sum_{r\le\frac{t}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)F_h^{\pm}\Big(\frac{r}{t}\Big)\sum_{Q<x\le 2N}\cos 2\pi\delta x+$$

$$+\frac{1}{2}\Sigma_{\log}^{(1)}(A)-\frac{1}{2}\sum_{2<\ell,t\le Q}\sum R_\ell(\log)R_t(\log)\sum_{\substack{j\le\frac{\ell}{2}\\ \delta>0,\sigma>1/A}}{}^{*}\sum_{r\le\frac{t}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)F_h^{\pm}\Big(\frac{r}{t}\Big)\sum_{Q<x\le 2N}\cos 2\pi\sigma x,$$

from §2 definitions, since $A>6N$ in Lemma C implies no sum over $\sigma\le\frac{1}{A}$. From $\frac{j}{\ell}\neq\frac{1}{2}\Rightarrow\|2j/\ell\|\neq 0$

$$\sum_{2<\ell,t\le Q}\sum R_\ell(\log)R_t(\log)\sum_{\substack{j\le\frac{\ell}{2}\\ \delta>0,\sigma>1/A}}{}^{*}\sum_{r\le\frac{t}{2}}{}^{*}F_h^{\pm}\Big(\frac{j}{\ell}\Big)F_h^{\pm}\Big(\frac{r}{t}\Big)\sum_{Q<x\le 2N}\cos 2\pi\sigma x=\Sigma_{\log}^{(2)}(A)+\mathcal{O}\left(\sum_{2<\ell\le Q}\frac{L^4}{\ell^2}\sum_{j\le\frac{\ell}{2}}{}^{*}\frac{F_h^{\pm}\big(\frac{j}{\ell}\big)^2}{\|2j/\ell\|}\right)$$

using the trivial $R_\ell(\log)\ll L^2/\ell$, see (0), and the elementary in Lemma A proof (compare [D,Chap.25] too)

$$\sum_{Q<x\le 2N}\cos 2\pi\sigma x\ll\frac{1}{|\sin\pi\sigma|}\ll\frac{1}{\|\sigma\|},$$

where from the trivial bound $F_h^{\pm}(j/\ell)\ll h$ we get

$$\sum_{j\le\ell/2}{}^{*}\frac{F_h^{\pm}(j/\ell)^2}{\|2j/\ell\|}\ll h^2\Big(\ell\sum_{j\le\ell/4}\frac{1}{j}+\sum_{\ell/4<j<\ell/2}\frac{\ell}{\ell-2j}\Big)\ll h^2\ell\Big(L+\sum_{n<\ell/2}\frac{1}{n}\Big)\ll\ell h^2L.$$

This gives the negligible

$$\mathcal{O}\left(L^4\sum_{2<\ell\le Q}\frac{1}{\ell^2}\sum_{j\le\frac{\ell}{2}}{}^{*}\frac{F_h^{\pm}(j/\ell)^2}{\|2j/\ell\|}\right)=\mathcal{O}\left(h^2L^6\right).$$

Hence, in case $A \ll N$, using §2 initial remarks, i.e. $\Sigma_f^{(1)} - \Sigma_f^{(2)} \ll |\Sigma_f|$, with Lemma A, (0) & (2)

$$I_\Lambda(N,h) = D_{\log}^{\pm}(N,h) + \frac{1}{2}\sum_{2<\ell,t\le Q}\sum R_\ell(\log)R_t(\log) \sideset{}{^*}\sum_{\substack{j\le \frac{\ell}{2} \\ 0<\delta\le 1/A}} \sideset{}{^*}\sum_{r\le \frac{t}{2}} F_h^{\pm}\left(\frac{j}{\ell}\right) F_h^{\pm}\left(\frac{r}{t}\right) \sum_{Q<x\le 2N} \cos 2\pi\delta x + \mathcal{O}\left(NhL^5\right).$$

Recall the inner sum over x in the diagonal $D_{\log}^{\pm}$ is positive, like the sum $\sum_x \cos 2\pi\delta x$ for $0 < \delta \le 1/A$ which is positive, assuming $A > 8N$ (better, it's $\gg N$ whenever $A \ge 9N$); we may apply a majorant principle, here, with $R_\ell(\log) \ll LR_\ell(d)$ from (0), in order to get the following:

$$I_\Lambda(N,h) \ll L^2\left(D_d^{\pm}(N,h) + \frac{1}{2}\sum_{2<\ell,t\le Q}\sum R_\ell(d)R_t(d) \sideset{}{^*}\sum_{\substack{j\le \frac{\ell}{2} \\ 0<\delta\le 1/A}} \sideset{}{^*}\sum_{r\le \frac{t}{2}} F_h^{\pm}\left(\frac{j}{\ell}\right) F_h^{\pm}\left(\frac{r}{t}\right) \sum_{Q<x\le 2N} \cos 2\pi\delta x\right) + NhL^5.$$

The expression in parentheses is, making the same considerations as above with $f(n) = d(n)$ instead of $f(n) = \log n$, applying again Lemma A, same hypotheses on A, simply $I_d(N,h) + \mathcal{O}\left(NhL^3\right)$, because $I_{d-\mathbf{1}}(N,h) = I_d(N,h)$, applying (3) to $g = \mathbf{1}$, $f = g * \mathbf{1} = \mathbf{1} * \mathbf{1} = d$; then, from Lemma B, with hypotheses that set the range of $h-$upper bound, after inserting omitted terms, from $I_{d-\mathbf{1}}(N,h)$ and $d(n) \ll N^{\varepsilon/4}$, too:

$$I_\Lambda(N,h) \ll L^2(I_d(N,h) + NhL^3 + N^{\varepsilon/2}h^3) + NhL^5 + h^3L^2 \ll NhL^5 + N^{\varepsilon}h^3. \ \square$$

4. Proof of the Corollary.

In order to prove the Corollary, we first give a consequence of the Theorem of [CLap], i.e., see the Proposition, following, giving an explicit formula for $\psi(x) \stackrel{def}{=} \sum_{n\le x}\Lambda(n)$ in which the error-term has a very good behavior, both in the discrete and the continuous mean-square over $[N,2N]$.

We need, for this reason, to apply and adapt the Theorem of [CLap] to the present situation. First of all, see that instead of the weight G_Y, see [CLap], we may use the following modified version,

$$\widetilde{G}_Y(x,T,t) := \frac{1}{\int_{\frac{T}{2}}^{T}\phi_Y(\tau)d\tau}\int_{\frac{T}{2}}^{T}\phi_Y(\tau)\int_{\frac{\tau|x-t|}{x}}^{\infty}\frac{\sin u}{u}du\, d\tau,$$

since (recall $|x-t|\ll H=o(x)$, here) the formula $|\log\frac{x}{t}| = \frac{|x-t|}{x}+\mathcal{O}((x-t)^2/x^2)$ gives errors

$$\left|G_Y(x,T,t)-\widetilde{G}_Y(x,T,t)\right| \ll_Y T\left(\frac{|x-t|}{x}\right)^2$$

which contribute, in the final symmetry integrals, as

$$\left|I_{fG_Y}(N,H)-I_{f\widetilde{G}_Y}(N,H)\right| \ll_Y \frac{H^6T^2}{N^3}\|f\|_\infty^2 \;\Rightarrow\; \left|I_{\Lambda G_Y}(N,H)-I_{\Lambda\widetilde{G}_Y}(N,H)\right| \ll_Y \frac{H^6T^2L^2}{N^3}.$$

(We used the trivial bound $\Lambda(n)\ll L$: Brun-Tichmarsh inequality's poor for H smaller than N powers.)

Recall we abbreviate, as soon before (3) above, $\|f\|_\infty = \max_{n\le 3N}|f(n)|$.

The weight $\widetilde{G}_Y$ doesn't influence the symmetry integral, i.e. with the above definitions, we have the following

Lemma D. *Let* $A,B,C\ge 0$. *Assume* $L^\varepsilon \ll H \ll N^{1/2}$ *as* $N\to\infty$. *Then* $\forall f:\mathbb{N}\to\mathbb{C}$

$$I_f(N,h)\ll NhN^AL^B\log^C L,\ \forall h\in[L^\varepsilon,H] \;\Rightarrow\; I_{f\widetilde{G}_Y}(N,H)\ll_Y NHN^AL^B\log^C L + NL^2\|f\|_\infty^2.$$

proof. First of all, since $\widetilde{G}_Y \ll_Y 1$, compare [CLap], let's use the symmetry of n in $\widetilde{G}_Y$ with respect to x:

$$\sideset{}{'}\sum_{|n-x|\le H} f(n)\widetilde{G}_Y(x,T,n)\mathrm{sgn}(n-x) = \sum_{m\le H}(f(x+m)-f(x-m))\widetilde{G}_Y(x,T,x+m)+\mathcal{O}_Y(\|f\|_\infty)$$

and apply partial summation [T] :

$$\sideset{}{'}\sum_{|n-x|\le H} f(n)\widetilde{G}_Y(x,T,n)\mathrm{sgn}(n-x) = \widetilde{G}_Y(x,T,x+H)\sideset{}{'}\sum_{|n-x|\le H} f(n)\mathrm{sgn}(n-x)+\mathcal{O}_Y(\|f\|_\infty)$$

$$-\int_1^H \sideset{}{'}\sum_{|n-x|\le[t]} f(n)\mathrm{sgn}(n-x)\frac{d}{dt}\widetilde{G}_Y(x,T,x+t)dt+\mathcal{O}_Y\left(\|f\|_\infty\int_1^H\left|\frac{d}{dt}\widetilde{G}_Y(x,T,x+t)\right|dt\right).$$

Hence, abbreviating (see above) the "symmetry sum" $S_f^{\pm}(x,[t]) = \sum'_{|n-x|\le[t]} f(n)\mathrm{sgn}(n-x)$,

$$I_{f\widetilde{G}_Y}(N,H)\ll_Y I_f(N,H)+NL^2\|f\|_\infty^2+$$

$$+\int_1^H\int_1^H\sum_{x\sim N}S_f^{\pm}(x,[t_1])S_f^{\pm}(x,[t_2])\frac{d}{dt_1}\widetilde{G}_Y(x,T,x+t_1)\frac{d}{dt_2}\widetilde{G}_Y(x,T,x+t_2)dt_1dt_2,$$

due to $\widetilde{G}_Y(x,T,m)\ll_Y 1$ and opening of the square, after

$$\frac{d}{dt}\widetilde{G}_Y(x,T,x+t) = -\frac{1}{t}\frac{1}{\int_{\frac{T}{2}}^{T}\phi_Y(\tau)d\tau}\int_{\frac{T}{2}}^{T}\phi_Y(\tau)\sin\frac{t\tau}{x}d\tau \ll_Y \frac{1}{t}\ \ \forall t\ge 1;$$

then

$$I_{f\widetilde{G}_Y}(N,H) \ll_Y I_f(N,H) + NL^2\|f\|_\infty^2 + \left(\int_1^H \frac{1}{t}\sqrt{I_f(N,[t])}dt\right)^2 ,$$

applying the Cauchy inequality and, splitting the integral at L^ε, we get

$$I_{f\widetilde{G}_Y}(N,H) \ll_Y I_f(N,H) + NL^2\|f\|_\infty^2 + \left(\int_{L^\varepsilon}^H \frac{1}{t}\sqrt{I_f(N,[t])}dt\right)^2 ,$$

where we used the trivial $I_f(N,[t]) \ll Nt^2\|f\|_\infty^2$; applying our hypothesis finally gives

$$I_{f\widetilde{G}_Y}(N,H) \ll_Y NL^2\|f\|_\infty^2 + NHN^A L^B \log^C L. \ \square$$

We need a suitable corollary to the Theorem of [CLap] since that Corollary [CLap] is given for T limited to some $N-$powers; we want it for T as general as possible, like (see [CLap] for ϕ_Y, G_Y and w_Y) in the following

PROPOSITION. *Fix $Y \in \mathbb{N}$. Let $16 \le N \le x \le 2N$, $4 \le T \le N/4$, $1 \le M \le \min(T^{\frac{1}{Y+1}}, (\frac{N^{\frac{1}{16}}}{L^3})^{1/Y}, (\frac{T^{\frac{1}{5}}}{L^8})^{1/Y})$. Then*

$$\psi(x) = x - \sum_{|\gamma|\le T} w_Y\left(\frac{|\gamma|}{T}\right)\frac{x^\rho}{\rho} + E_Y(x,T,H),$$

where we assume $\frac{N}{T} \ll h \ll \frac{N}{T}$ and set $H := [Mh]$, for the "symmetry sum"

$$S^{\pm}_{\Lambda G_Y}(x,H) \overset{def}{=} \sideset{}{'}\sum_{|n-x|\le H} \Lambda(n)G_Y(x,T,n)\mathrm{sgn}(n-x),$$

with, in the hypothesis $H = o(N)$, both

$$\sum_{x\sim N} |E_Y(x,T,H)|^2 \ll_Y \sum_{x\sim N} \left|S^{\pm}_{\Lambda G_Y}(x,H)\right|^2 + NL + Nh^2\left(\frac{L}{M^Y}\right)^2$$

and

$$\int_N^{2N} |E_Y(x,T,H)|^2 dx \ll_Y \sum_{N\le x\le 2N} \left|S^{\pm}_{\Lambda G_Y}(x,H)\right|^2 + NL + Nh^2\left(\frac{L}{M^Y}\right)^2 .$$

PROOF. The same procedure from Theorem [CLap] to Corollary [CLap] gives a slight change, due to T range,

$$\psi(x) = x - \sum_{|\gamma|\le T} w_Y\left(\frac{|\gamma|}{T}\right)\frac{x^\rho}{\rho} + \frac{1}{\pi}S^{\pm}_{\Lambda G_Y}(x,H) + \mathcal{O}\left(\Lambda([x]-H) + \Lambda([x]) + \Lambda([x]+H) + 1\right) +$$

$$+\mathcal{O}_Y\left(NL/TM^Y\right),$$

one L more because $\log N/T \gg 1$, now (hence, a different M); the remainder $\mathcal{O}(NL)$ in the mean-squares is due to the terms:

$$|\psi_0(x) - \psi(x)| \ll \Lambda(x), \quad -\frac{\zeta'(0)}{\zeta(0)} \ll 1,$$

passing from [CLap] formula to the present, with those (see that $H \in \mathbb{N}$, here) $\Lambda(x-H), \Lambda(x), \Lambda(x+H)$, see R_1 [CLap], from Chebyshev inequality for ψ with $x \in \mathbb{N}$ and $H = o(N)$, all giving to mean-squares:

$$\ll \sum_{N\le x\le 2N} \Lambda^2(x-H) + \sum_{N\le x\le 2N} \Lambda^2(x) + \sum_{N\le x\le 2N} \Lambda^2(x+H) + N \ll L\sum_{n\le 3N} \Lambda(n) + N \ll NL. \ \square$$

We are ready to prove our Corollary. Hereafter $\varepsilon > 0$ is a fixed, arbitrarily small absolute constant.

PROOF. Take $L^{11/2+\varepsilon} \le H \le N^{1/2-\varepsilon}$. We want to estimate the $j-$sum in Th.4 [Lang], so the mean-square

$$I(N, T_j) := \int_N^{2N} |E_Y(x, T_j, [MH_j])|^2 \, dx,$$

in it, don't confuse with symmetry integral; $E_Y(x, T_j, [MH_j])$ is in the Proposition,$\frac{N}{T_j} \ll H_j \ll \frac{N}{T_j}$,say. We may apply in Th.4 [Lang] our formula, instead of [KP] one: in place of w there, we'll use w_Y here. (Estimates over the zeros are unaffected by these weights, both w and w_Y, since we use $w, w_Y \ll_Y 1$.) Here Kaczorowski & Perelli formula corresponds to $Y = 1$ in the Proposition; while $Y = [2/\varepsilon]$ gives $\mathcal{O}_Y(NL/T_j M^Y)$ negligible: choose $M := L^{\varepsilon/2}$, it's $\mathcal{O}_Y(H_j/L^B)$, $B > 1/2$, good. Remains $\mathcal{O}_Y(NL)$ in

$$(*) \quad \sum_{j \le J} \frac{H^2}{H_j^2} I(N, T_j) = \sum_{j \le J} \frac{H^2}{H_j^2} \int_N^{2N} |E_Y(x, T_j, [MH_j])|^2 dx \ll_Y \sum_{j \le J} \frac{H^2}{H_j^2} (I_{\Lambda G_Y}(N, [MH_j]) + NL) + M^2 H^2 L^3$$

[cit.] $\frac{N}{T_j} \ll H_j \ll \frac{N}{T_j}$; but $M^2 H^2 L^3 = o(NH^2)$ and $k_1 := L^{\varepsilon/4} \Rightarrow H^2 \sum_{j \le J} H_j^{-2} \mathcal{O}_Y(NL) = o(NH^2)$. We are left with the estimate of:

$$\ll_Y H^2 \sum_{j \le J} \frac{1}{H_j^2} I_{\Lambda G_Y}(N, [MH_j]),$$

may say, bounded as (see H_j definition in [Th.m 4, Lang])

$$\ll_Y H^2 \sum_{j \le J} \frac{1}{H_j^2} (I_{\Lambda \widetilde{G}_Y}(N, [MH_j]) + H_j^4 L^{2+3\varepsilon}/N) \ll_Y H^2 \sum_{j \le J} \frac{1}{H_j^2} NMH_j L^5 + o(NH^2),$$

as a consequence of our Theorem, after changing G_Y into $\widetilde{G}_Y$ and Lemma D, with $A = 0 = C$, $B = 5$. This term gives, into $(*)$, $\ll_Y NH^2 L^{5+\varepsilon/2} \sum_{j \ll L} H_j^{-1} \ll_Y NHL^{11/2+3\varepsilon/4} = o(NH^2)$, again, $k_1 = L^{\varepsilon/4}$. We are done, since $k_2 := L^{\frac{3\varepsilon}{4}}$ in other terms, after $(*)$, gives to Th.m 4 [Lang] a contribute to $J(N, H)$:

$$\ll NH^2(1/k_1 + (L/Hk_1)^2 + 1/k_2) = o(NH^2). \ \square$$

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