Efficient Sampling of Band-limited Signals from Sine Wave Crossings

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***Abstract*— This paper presents an efficient method for reconstructing a band-limited signal in the discrete domain from its crossings with a sine wave. The method makes it possible to design A/D converters that only deliver the crossing timings, which are then used to interpolate the input signal at arbitrary instants. Potentially, it may allow for reductions in power consumption and complexity, as well as for an increase in the achievable sampling bandwidth. The reconstruction in the discrete domain is based on a recently-proposed modification of the Lagrange interpolator, which is readily implementable with linear complexity and efficiently, given that it re-uses known schemes for variable fractional-delay (VFD) filters. As a spin-off, the method allows one to perform spectral analysis from sine wave crossings with the complexity of the FFT. Finally, the results in the paper are validated in a numerical example.**

I. Introduction

The analog-to-discrete (A/D) conversion is the first step for the discrete-time processing of continuous signals. This conversion is fundamentally based on the Sampling Theorem, which states that a band-limited signal can be recovered from its regularly-spaced samples taken at least at twice the Nyquist rate. However, some authors early noticed that this recovery is also possible from the signal's zeros, or from its crossings with another signal like a sine wave, [1]–[4]. This is due to the fact that a band-limited signal is an entire function of exponential type, for which there is a factorization in terms of its roots akin to that of conventional polynomials, (Hadamard's factorization theorem [5, chapter 2]). A consequence of this is that it would be possible, in principle, to design A/D converters in which the sample quantization is substituted by a zero crossing detector and an accurate timing, [6]. This new procedure would eliminate the need to quantize any signal sample, so decreasing the complexity and power consumption of A/D converters, and potentially increasing the achievable sampling bandwidth. Besides, it would mainly re-use existing technologies, given that zero crossing detection is implicit in many existing systems. This last point can be readily seen in the current trend in A/D converter design, in which the sample amplitudes are turned into zero crossings, which can then be accurately detected with low-power consumption and high speed, [7], [8].

The main obstacle for this alternative procedure is how the signal should be reconstructed or processed in the discrete domain, since Hadamard's factorization theorem does not directly lead to efficient implementations, due to its slow convergence rate. Here, the usual approach in the literature consists in approximating the signal in a finite interval using a trigonometric polynomial, [6], [9], [10]. But then the interpolation error decreases only as $\mathrm{O}(1/N)$, while the complexity per interpolated value is $\mathrm{O}(N)$, where N is the number of crossings inside the interval. So, in this approach it is necessary to employ a large N to ensure an acceptable accuracy, with the associated high complexity.

The purpose of this paper is to present a method for overcoming this obstacle, that makes it possible to reconstruct the bandlimited signal from its sine wave crossings efficiently. The method is based on viewing the reconstruction as a problem of interpolation from nonuniform samples, to which the efficient technique in [11] is applied. Relative to the state of the art, it has several advantages:

- The complexity is reduced significantly. If in the approach in [6], [9], [10], a complexity $\mathrm{O}(N)$ (per interpolated value) gives an interpolation error $\mathrm{O}(1/N)$, with the proposed method a complexity $\mathrm{O}(N)$ gives an error $\mathrm{O}(\mathrm{e}^{-\pi(1-BT)N})$, where B is the signal's two-sided bandwidth and T is the average crossing separation. In practice this means that "any" accuracy can be achieved with a small N.
- The method is based on the evaluation of a fixed smooth function and on the Lagrange interpolator. Besides, it can be evaluated with cost $\mathrm{O}(N)$ per interpolated value, and can be implemented by re-using efficient designs for variable fractional-delay (VFD) filters, [11, Sec. IV].
- As a spin-off, the method permits one to perform spectral analysis from sine wave crossings with complexity $\mathrm{O}(N \log N)$, while the usual method has complexity $\mathrm{O}(N^2)$, [6, Sec. IV].

The paper has been organized as follows. The next section reviews the state of the art and presents the problem formulation. In it, it is shown that the reconstruction from sine wave crossings can be turned into an interpolation problem from nonuniform samples. Then, this last problem is addressed in Sec. III, where the solution adopted is that in the recent reference [11], which is based on the Lagrange interpolator. Afterward, Sec. IV contains the main development in the paper, which is a simple formula for reconstructing the signal

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from its sine wave crossings with high accuracy. Then, Sec. V addresses the problem of analyzing the spectrum from sine wave crossings in the light of the formula in Sec. IV. It turns out that this formula makes it possible to reduce the complexity from the usual $\mathrm{O}(N^2)$ order to order $\mathrm{O}(N\log N)$. Finally, Sec. VI validates the results in the paper through a numerical example.

II. State of the art and problem formulation

The usual representation for a real finite-energy signal $\mathrm{s}(t)$ with spectrum inside $[-B/2, B/2]$ is given by the Sampling Theorem, which states that $\mathrm{s}(t)$ can be perfectly reconstructed from its samples $\mathrm{s}(n/B)$ using the series

$$\mathrm{s}(n/B+u) = \sum_{p=-\infty}^{\infty} \mathrm{s}((n-p)/B)\mathrm{sinc}(p+Bu), \quad (1)$$

where n is an integer and u is any time shift following $-1/(2B) \leq u < 1/(2B)$. This series is the basis of most processing algorithms for continuous signals in the discrete domain. Several authors [1], [2], [4] soon noticed that $\mathrm{s}(t)$ can alternatively be viewed as a *polynomial of infinite degree*, which can be described in terms of its roots. The reason why is that a band-limited signal can be regarded as an analytic function over the whole complex plane, if the t variable is allowed to take complex values. Besides, this kind of function is bounded on the real axis (real t), and its maximum growth rate is that of $\mathrm{e}^{\pi B|t|}$ along the imaginary axis. It can be shown that this kind of signal admits the representation

$$\mathrm{s}(t) = Kt^a \prod_{p=1}^{\infty}(1-t/\tau_p), \quad (2)$$

where a is the integer order of the root at the origin, $a \geq 0$, K is a real constant, and the τ_p are complex roots which appear in complex conjugate pairs, $\tau_p \neq 0$, and may not be distinct, [12, Theorem VI]. In Eq. (2), the roots are taken so that their module increases with p, i.e, $|\tau_p| \leq |\tau_{p+1}|$, since the infinite product only converges conditionally.

Eq. (2) is the explicit root factorization of $\mathrm{s}(t)$ as infinite-degree polynomial, and from it it is obvious that the roots τ_p determine the signal except for the scale factor K. The root density in any circle $|t| < \tau$ follows the same rule as the sample density in (1), i.e, the circle $|t| < \tau$ contains either $2\tau B$ samples in (1) or $2\tau B$ roots in (2) asymptotically.

The factorization in Eq. (2) led to consider the possible implementation of A/D converters based on the detection of the signal's zeros. However, it was soon realized that such implementation faced two main problems, [4], [6]. The first was how the roots should be located efficiently, since they may have a non-null imaginary part. And the second was how the infinite product in (2) should be approximated, since in practice only a finite sequence of roots τ_p is known, and (2) converges slowly.

A solution for the first problem was readily found [4, Sec V], and consisted in subtracting a sinusoidal $A\sin(\pi Bt)$ to $\mathrm{s}(t)$, with amplitude A not smaller than that of $\mathrm{s}(t)$, i.e, $A \geq A_s$ where

$$A_s = \sup_{t \text{ real}} |\mathrm{s}(t)|. \quad (3)$$

This simple procedure actually solved the problem of locating the roots, because the subtraction of this sine wave "moves" all roots to the real axis, due to a theorem of Duffin and Schaeffer, [13]. Specifically, $\mathrm{s}(t) - A\sin(\pi Bt)$ can only have zeros on the real axis, and each of them can be viewed as a zero of $A\sin(\pi Bt)$, which has been shifted by at most $1/(2B)$. So, in notation, all zeros of $\mathrm{s}(t) - A\sin(\pi Bt)$ have the form $n/B + \delta_n$, where n is an integer and $|\delta_n| \leq 1/(2B)$, and two consecutive zeros may overlap only at instants $n/B + 1/(2B)$. Using this description, the factorization in Eq. (2) for $\mathrm{s}(t) - A\sin(\pi Bt)$ is

$$\mathrm{s}(t) - A\sin(\pi Bt) = K'(t_0 - t)\prod_{k=1}^{\infty}\Big(1-\frac{t}{t_{-k}}\Big)\Big(1-\frac{t}{t_k}\Big), \quad (4)$$

where K' is a real constant and

$$t_n \equiv n/B + \delta_n. \quad (5)$$

From Eq. (4), the basic design for the desired A/D converter was clear. First an oscillator would be used to generate the wave $A\sin(\pi t/B)$, which would then be subtracted from $\mathrm{s}(t)$. Afterward, the zero crossings would be detected using a gate, and the converter output would be the sequence of shifts δ_n in Eq. (5).

As to the second problem, the usual solution to date consists in approximating the signal using a trigonometric polynomial, [6]. In short, if the finite sequence of roots $t_n, t_{n+1}, \ldots, t_{n+M-1}$ is known, then $\mathrm{s}(t) - A\sin(\pi Bt)$ is interpolated using a trigonometric polynomial of order M which is zero at t_{n+m}, $0 \leq m < M$. However, this solution is not satisfactory since its accuracy is poor even for a large number of roots. This complexity issue is the main obstacle for achieving an efficient implementation.

An efficient solution for this second problem is derived in the next two sections, which is based on recent results on the interpolation of band-limited signals from nonuniform samples. The key point is to realize that approximating $\mathrm{s}(t)$ from the roots t_{n+m} of $\mathrm{s}(t) - A\sin(\pi Bt)$ is the same as approximating $\mathrm{s}(t)$ from its value at these instants, since $\mathrm{s}(t_{n+m}) = A\sin(\pi Bt_{n+m})$. So, this is actually a problem of interpolation from nonuniform samples. An efficient and accurate method for solving this kind of problem has been recently presented in [11], and this method is shortly outlined in the next section. Then, the A/D converter from sine wave crossings is described in Sec. IV.

III. Efficient signal interpolation from nonuniform samples

The reconstruction of a band-limited signal from discrete samples is usually performed from a uniform sampling grid. Yet it is well known in Sampling Theory that this kind of reconstruction is also possible from nonuniform samples, [14]. From a theoretical point of view, the uniform and nonuniform cases can be handled jointly using an extension of the Lagrange interpolation procedure. In rough terms, if a band-limited signal $\mathrm{s}(t)$ is known at instants $t_1, t_2, \ldots$, then it can be reconstructed using the series

$$\mathrm{s}(t)=\sum_{p=1}^{\infty}\mathrm{s}(t_p)\frac{\phi(t)}{\phi'(t_p)(t-t_p)}, \tag{6}$$

where $\phi(t)$ has a zero at each t_p and is given by

$$\phi(t)\equiv A{\prod_{p=1}^{\infty}}'(1-t/t_p)\mathrm{e}^{t/t_p}. \tag{7}$$

Here, the prime (') means that the factor should be replaced with t if $t_p=0$, and A is any constant, $A\neq 0$. A particular case of Eq. (6) is the uniform sampling case in (1) for which $\phi(t)=\sin(\pi Bt)$.

The usual Lagrange interpolation procedure can be readily identified in (6). In this formula, the kernel $\phi(t)$ has a zero at $t=t_p$ which is removed by the denominator $t-t_p$. Then the division by $\phi'(t_p)$ normalizes the function value to one at $t=t_p$. So, the summand $\phi(t)/(\phi'(t_p)(t-t_p))$ is equal to one at $t=t_p$ and equal to zero at $t=t_q$, $q\neq p$. Finally, the sum for all p gives the infinite Lagrange formula for $\mathrm{s}(t)$.

For Eq. (6) to hold, $\mathrm{s}(t)$ must have finite energy, there must be a minimum separation among the sampling points, $\inf_{p,q}|t_p-t_q|>0$, and the average density of the instants t_p must be larger than or equal to twice the Nyquist frequency of $\mathrm{s}(t)$, [15, Lecture 22].

The series in (6) is a fundamental tool for characterizing band-limited signals, but is far from giving efficient interpolators, implementable in practice. The problem is that only a finite sequence of samples is known in most cases and, obviously, the interpolation will only be accurate in a finite interval.

In a more realistic situation, it would be necessary to interpolate $\mathrm{s}(t)$ in a sequence of instants as the nonuniform samples become available and, besides, this interpolation should be performed from the last, say, $2P+1$ samples. Also, though the samples are taken at irregular instants, they should have a maximum separation from a grid with given spacing T (jittered sampling). Finally, only interpolators that are linear in the signal samples should be considered, so as to facilitate the implementation. In notation, these conditions can be expressed through the following conventions,

- The signal is to be interpolated at instant $nT+u$ with u following $-T/2\leq u<T/2$ and integer n.
- For this interpolation, only the nonuniform samples at instants $(n+p)T+\delta_{n+p}$ with $|p|\leq P$ are available, where $|\delta_{n+p}|<T/2$.
- The interpolator must be linear in the nonuniform samples.

The combination of these conditions leads us to consider interpolators of the form

$$\mathrm{s}(nT+u)\approx\sum_{p=-P}^{P}\mathrm{s}((n+p)T+\delta_{n+p})\mathrm{g}_p(u;n), \tag{8}$$

where $\mathrm{g}_p(u;n)$ is a known set of functions.

The Lagrange series in Eq. (6) can be manipulated so as to produce an interpolator like (8). For this, replace t with $nT+u$ and t_p with $pT+\delta_p$, and re-order the summation. The result is

$$\mathrm{s}(nT+u)=\sum_{p=-\infty}^{\infty}\mathrm{s}((n+p)T+\delta_{n+p})\cdot\frac{\phi(nT+u)}{\phi'((n+p)T+\delta_{n+p})(u-pT-\delta_{n+p})}. \tag{9}$$

Finally, truncate this sum at $\pm P$ to obtain the formula of the form in Eq. (8)

$$\mathrm{s}(nT+u)\approx\sum_{p=-P}^{P}\mathrm{s}((n+p)T+\delta_{n+p})\cdot\frac{\phi(nT+u)}{\phi'((n+p)T+\delta_{n+p})(u-pT-\delta_{n+p})}. \tag{10}$$

However, this approach would not be viable because $\phi(t)$ depends on the infinite set of shifts δ_n and on the index n, and also because the accuracy would be poor. This latter inconvenient is already well known for the sinc series in Eq. (1), [16].

The problem of making the approach in Eq. (10) viable and efficient has been recently solved in [11] satisfactorily. Fundamentally, the solution in this reference consists in regularizing the series in (6), so that the truncation at indices $\pm P$ introduces a negligible error. The key lies in realizing that there is always some sampling inefficiency (slight over-sampling), and in exploiting this fact so as to improve the performance of Eq. (10). In short, the solution given in this reference is the following. If the average separation of the sampling points is T and there is non-null over-sampling, $BT<1$, then the series in Eq. (6) can be applied to the product of $\mathrm{s}(t)$ with another signal $\mathrm{w}(t)$ of bandwidth $1/T-B$. The result reads

$$\mathrm{s}(nT+u)\mathrm{w}(u)=\sum_{p=-\infty}^{\infty}\mathrm{s}((n+p)T+\delta_{n+p})\mathrm{w}(pT+\delta_{n+p})\cdot\frac{\phi(nT+u)}{\phi'((n+p)T+\delta_{n+p})(u-pT-\delta_{n+p})}. \tag{11}$$

However, $\mathrm{w}(u)$ can be selected so that it is close to one for $|u|\leq T/2$ and close to zero for $|u|\geq(P+1)T-\delta$. This allows one to truncate (11) at indices $\pm P$, and then solve for $\mathrm{s}(nT+u)$, so as to obtain the interpolator

$$\mathrm{s}(nT+u)\approx\frac{1}{\mathrm{w}(u)}\sum_{p=-P}^{P}\mathrm{s}((n+p)T+\delta_{n+p})\mathrm{w}(pT+\delta_{n+p})\cdot\frac{\phi(nT+u)}{\phi'((n+p)T+\delta_{n+p})(u-pT-\delta_{n+p})}. \tag{12}$$

Now, since $\mathrm{w}(u)\approx 0$ for $|u|\geq(P+1)T-\delta$ and $\mathrm{w}(u)\approx 1$ for $|u|\leq T/2$, the neglected summands in passing from (11) to (12) have small amplitude. *So, Eq. (12) must be an accurate interpolator.* Actually, as shown in [17], the error of Eq. (12) decreases exponentially with P as $\mathrm{e}^{-\pi(1-BT)P}$ for a proper choice of $\mathrm{w}(u)$. In practice this means that any accuracy can be achieved in (12) using a small P.

As a simple example, assume that $BT=0.7$ and $|\mathrm{s}(t)|\leq 1$. Then, following the analysis in [11, Sec. III], the interpolation error of (12) for the function $\mathrm{w}(t)$ specified in Ap. I is well fitted by

$$\epsilon\,(\mathrm{dB}) \approx 4.12106 + 66.6044\,\delta - 9.35838\,\delta^2 - 8.30873P + 3.13419\,\delta P - 0.125803\,\delta^2 P, \quad (13)$$

where δ is a bound on the deviations δ_{n+p}, $\delta < T/2$. So, if $\delta = T/4$ the value $P = 10$ gives an interpolation error below $\epsilon = -55$ dB, and $P = 16$ gives an error below $\epsilon = -100$ dB. Any practical accuracy can be obtained by slightly increasing P for fixed δ. (For a detailed analysis, see the previous reference.)

As to the dependence of $\phi(t)$ on the unknown shifts δ_n, it was also shown in [11], that δ_{n+p} for $|p| > P$ can be regarded as zero with negligible performance loss. This way, the kernel $\phi(t)$ does not depend anymore on the unknown shifts δ_{n+p}, $|p| > P$. After some algebraic manipulations on (12), it was demonstrated in the previous reference that there exists a fixed smooth function $\gamma(t)$ (independent of all the δ_n) such that Eq. (12) yields the same result as the following procedure,

1) Compute $\gamma(pT + \delta_{n+p})$, $|p| \leq P$.
2) Multiply the sample values by the values computed in step 1), i.e, compute the products $\mathrm{s}(n + pT + \delta_{n+p})\gamma(pT + \delta_{n+p})$, $|p| \leq P$.
3) Compute the value of the Lagrange interpolator at instant u for abscissas $pT+\delta_{n+p}$, $|p| \leq P$, and values computed in step 2).
4) Compute $\gamma(u)$ and divide the output of step 3) by this value. This is the desired approximation to $\mathrm{s}(nT + u)$.

For the specific function $\gamma(t)$ in this paper, see Ap. I. The previous steps are summarized by the formula

$$\mathrm{s}(nT+u) \approx \frac{1}{\gamma(u)} \sum_{p=-P}^{P} \mathrm{s}((n+p)T + \delta_{n+p})\gamma(pT + \delta_{n+p}) \cdot \frac{\mathrm{L}_n(u)}{\mathrm{L}'_n(pT + \delta_{n+p})(u - pT - \delta_{n+p})} \quad (14)$$

where $\mathrm{L}_n(u)$ is the conventional Lagrange Kernel for the abscissas $pT + \delta_{n+p}$, $|p| \leq P$,

$$\mathrm{L}_n(u) \equiv \prod_{p=-P}^{P} u - pT - \delta_{n+p}. \quad (15)$$

IV. Efficient A/D conversion from sine wave crossings

As commented at the end of Sec. II, the problem of interpolating a signal $\mathrm{s}(t)$ from its sine wave crossings can be cast as that of interpolating $\mathrm{s}(t)$ from the nonuniform samples $\mathrm{s}(t_{n+m}) = A\sin(\pi B t_{n+m})$. Then, an efficient interpolation method from nonuniform samples was presented in the previous section. Yet this method is still not applicable to the initial problem in Sec. II, since it requires some sampling inefficiency, i.e, the average sampling period must be some $T > 0$ with $BT < 1$. This condition can be easily fulfilled simply by increasing the frequency of the sinusoidal to $1/T$. If this is done, the factorization equivalent to (4) is

$$\mathrm{s}(t) - A\sin(\pi t/T) = K'\Big(1 - \frac{t}{t_0}\Big) \prod_{k=1}^{\infty} \Big(1 - \frac{t}{t_{-k}}\Big)\Big(1 - \frac{t}{t_k}\Big), \quad (16)$$

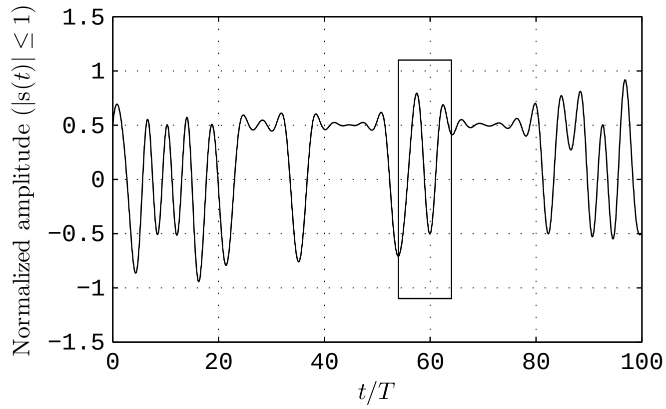


Fig. 1. Random BPSK signal with bandwidth $B = 0.7/T$ and peak amplitude 1. The symbols used to generate this signal were random ± 1 values, and the modulating pulse was a raised cosine with roll-off factor 0.2. The signal's peak amplitude was scaled to one.

where t_n must be re-defined as

$$t_n \equiv nT + \delta_n. \quad (17)$$

Besides, it is convenient to select an amplitude A, $A > A_s$, given that this way consecutive crossings cannot overlap. Indeed, it can be easily checked that if $A > A_s$ the shifts δ_n follow

$$|\delta_n| \leq \delta \ \text{ with } \ \delta \equiv (T/\pi)\arcsin(A_s/A), \quad (18)$$

and $\delta < T/2$.

With these conditions on T and A, the theorem of Duffin and Schaeffer in Sec. II implies that there is exactly one root in each of the intervals $[nT - \delta, nT + \delta]$, for any integer n. To verify this condition in a specific example, consider the BPSK signal in Fig. 1. This signal was generated by modulating a raised cosine pulse of roll-off 0.2 and bandwidth $B = 0.7/T$ with a sequence of random amplitudes ± 1. Then the signal's peak amplitude was scaled to 1. Fig. 2 shows the zone marked with a rectangle in Fig. 1, together with the sine wave $A\sin(\pi t/T)$. The condition in (18) means that the crossings with the sine wave can only take place inside the shaded rectangles, and there is exactly one in each of them, as can be seen in this example.

At this point, the derivation of an efficient interpolator comes down to applying the interpolator in the previous section to the instants $(n+p)T + \delta_{n+p}$, with their corresponding sample values

$$\begin{aligned}\mathrm{s}((n+p)T + \delta_{n+p}) &= A\sin(\pi(n + p + \delta_{n+p}/T)) \\ &= A(-1)^{n+p}\sin(\pi\delta_{n+p}/T).\end{aligned} \quad (19)$$

Substituting this expression into Eq. (14), the result is the desired interpolation formula

$$\mathrm{s}(nT+u) \approx \frac{(-1)^n A}{\gamma(u)} \sum_{p=-P}^{P} \sin(\pi\delta_{n+p}/T)\gamma(pT + \delta_{n+p}) \cdot \frac{\mathrm{L}_n(u)}{\mathrm{L}'_n(pT + \delta_{n+p})(u - pT - \delta_{n+p})}. \quad (20)$$

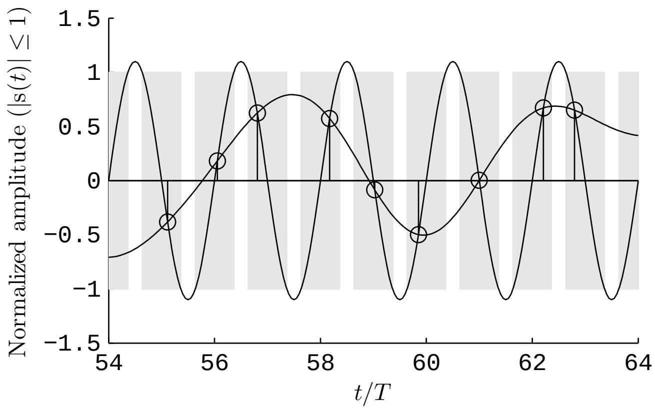
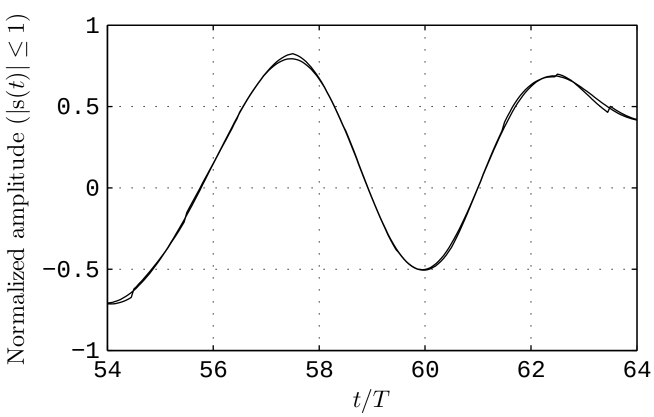
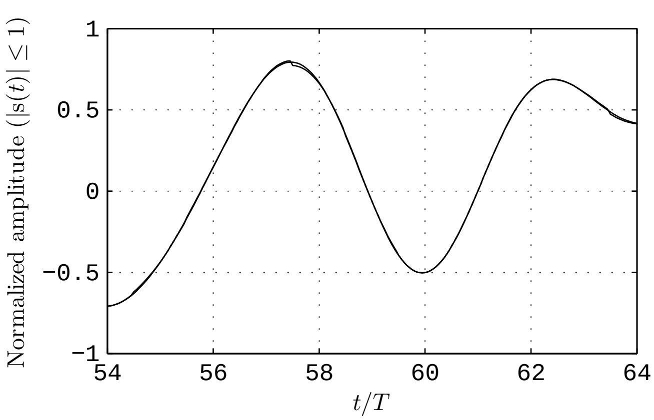


Fig. 2. Short piece of the BPSK signal in Fig. 1, overlapped with the sine wave $A\sin(\pi t/T)$, with $A = 1.1$. The stems indicate the positions of the crossings with the sine wave. These can only take place inside the rectangles, and there is exactly one in each of them.



(a) $P = 2$



(b) $P = 3$

Fig. 3. Sample BPSK signal in 2 and interpolated signal using (20) for $P = 2, 3$.

For the example in Fig. 2, Fig. 3(a) shows the signal and its interpolated version for $P = 2$. The discontinuities in the interpolated signal take place at $t = nT + T/2$, integer n, because the set of crossings used is different for each n. Notice however that just five crossings ($P = 2$) give a good accuracy. Fig. 3(b) shows the same comparison but for $P = 3$, (seven crossings). The difference between both signals is much smaller than in Fig. 3(a). For $P > 3$ the error becomes too small to be represented this way. For an error analysis see Sec. VI.

The formula in Eq. (20) yields discrete-time processing methods for delivering samples of $\mathrm{s}(t)$ with any spacing, simply by assigning proper values to n and u. The simplest case is for spacing T, simply by setting $u = 0$. For a generic grid of instants $n_1 T_1$ with $T_1 > 0$ and integer n_1, the grid sample $\mathrm{s}(n_1 T_1)$ is obtained from (20), simply by setting n and u equal to the modulo-T decomposition of $n_1 T_1$ in (20), i.e,

$$n = \lfloor n_1 T_1/T + 1/2 \rfloor \text{ and } u = n_1 T_1 - nT. \tag{21}$$

As to the efficient implementation of Eq. (20), it was shown in [11, Secs. IV] that this formula can be evaluated with cost just $\mathrm{O}(N)$. See also the numerical examples in these reference, (Sec. V).

V. Spectral estimation from sine wave crossings

The formula in (20) makes it possible to interpolate the input signal at any instant from its crossings with the sine wave, and the cost of this operation is just $\mathrm{O}(-\log\epsilon)$, where ϵ is a bound on the interpolation error. This is because the error of (20) decreases exponentially with trend $\mathrm{e}^{-\pi(1-BT)P}$. So, if ϵ is set below the working numerical error, Eq. (20) allows one to obtain one sample of $\mathrm{s}(t)$ with a small and constant computational cost. Therefore, the cost of computing N samples in a regular grid with arbitrary spacing T_1 is $\mathrm{O}(N)$. Once these samples are available, the situation is the usual one in which the signal's spectrum is estimated from regularly spaced samples, and any of the well-known techniques in spectral analysis becomes applicable, [18]. Since these techniques are based on the FFT whose complexity is $\mathrm{O}(N\log N)$, it is clear that the total complexity is also $\mathrm{O}(N\log N)$. A numerical example is presented in the next section.

VI. Numerical example

To validate the results in a specific example, a BPSK signal $\mathrm{z}(t)$ was generated with the following parameters,

- Modulating pulse: raised cosine with roll-off 0.2.
- Random amplitudes equal to ± 1.
- Total two-sided bandwidth $B = 0.7/T$.
- Time interval $I = [0, (N-1)T]$ with $N = 1024$.

Then several numerical experiments were conducted.

The first experiment consisted in interpolating $\mathrm{z}(t)$ in I from its sine wave crossings. The result is shown in Fig. 4 for $A = 1.1$ and $A = 16$, where the error norm is the maximum over I, that is, if $\hat{\mathrm{z}}(t)$ is the interpolated signal, then the ordinate in this figure is

$$\sup_{t\in I} |\mathrm{z}(t) - \hat{\mathrm{z}}(t)|. \tag{22}$$

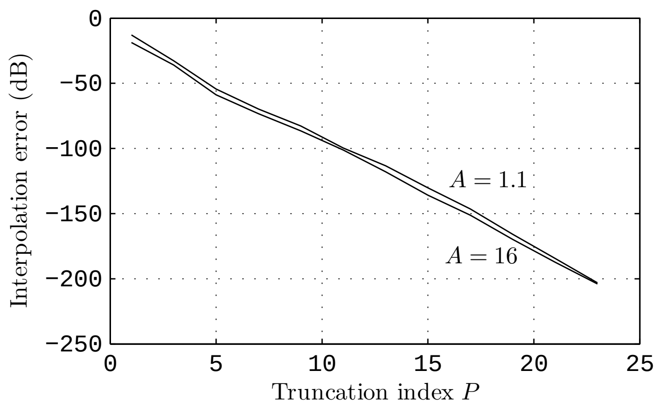


Fig. 4. Error in the interpolation of $\mathrm{z}(t)$ from its sine wave crossings versus index P. The error norm is the supremum $\sup_{t\in I} |\mathrm{z}(t) - \hat{\mathrm{z}}(t)|$ where $\hat{\mathrm{z}}(t)$ is the interpolated signal.

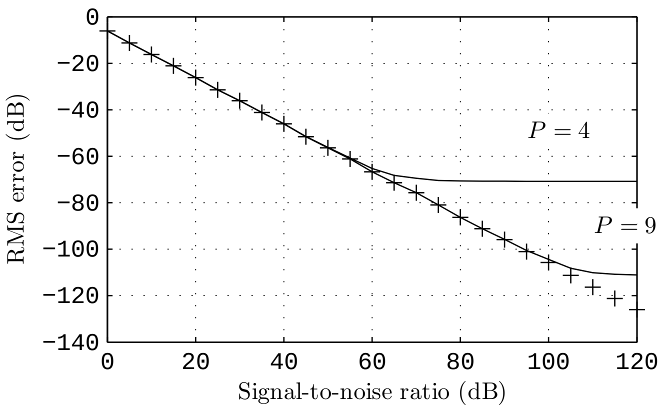


Fig. 5. RMS error versus the SNR in the interpolation of the samples $\mathrm{z}(nT)$ from the sine wave crossings of the noisy realization $\mathrm{z}(t) + \mathrm{w}(t)$.

Notice that this error decreases exponentially with P. Besides, the values of δ for $A = 1.1$ and $A = 16$ are $0.36T$ and $0.02T$, respectively, but the error is roughly the same in both cases. So, the fact that the sampling instants may differ from the grid nT (integer n) has a minimal effect on the performance.

In the second experiment, a white noise process $\mathrm{w}(t)$ of bandwidth B was added to $\mathrm{z}(t)$. Then, $\mathrm{z}(t)+\mathrm{w}(t)$ was sampled at instants nT (integer n) in I, and these samples were also interpolated from the sine wave crossings of $\mathrm{z}(t) + \mathrm{w}(t)$ for $A = 3$ and $P = 4$ and 9. Fig. 5 shows the resulting root-mean-square (RMS) error. The crosses ($+$) indicate the deviation of samples $\mathrm{z}(nT) + \mathrm{w}(nT)$. The other two curves are the RMS errors for $P = 4$ and $P = 9$, given by

$$\Big(\frac{1}{N}\sum_{nT\in I} |\mathrm{z}(nT) - \hat{\mathrm{z}}_1(nT)|^2\Big)^{1/2}, \tag{23}$$

where $\mathrm{z}_1(t)$ is the value interpolated from the sine wave crossings of the noisy signal $\mathrm{z}(t) + \mathrm{w}(t)$. The curve for either value of P overlaps the sample deviation up to an SNR threshold which is fixed by the specific value of P. So, below this threshold, the performance is the same if either the signal is directly sampled, or if it is interpolated from its sine wave crossings. The threshold can be fixed to an SNR as large as desired, simply by slightly increasing P, due to the exponential dependence of the interpolation error on P.

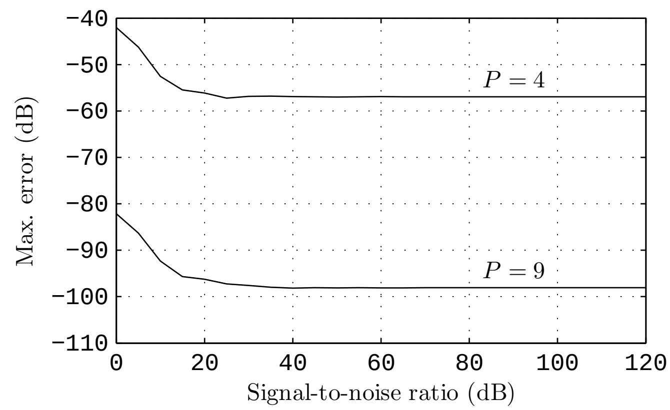


Fig. 6. Maximum difference between the samples $\mathrm{z}(nT) + \mathrm{w}(nT)$ and their interpolated values from sine wave crossings versus the SNR. The error norm if that defined in (24).

As to the spectral analysis, it is worth comparing the conventional procedure from uniform samples, with the one proposed in this paper from sine wave crossings. In the conventional procedure, the samples $\mathrm{z}(nT) + \mathrm{w}(nT)$ would be delivered by an A/D converter, and then any of the existing spectral analysis methods would be applied to these data, [18]. And in the proposed procedure, the A/D converter would deliver the sine wave crossing timings δ_n, then the uniform samples $\mathrm{z}_1(nT)$ would be computed using Eq. (20), and finally the spectral analysis would be the same as in the conventional procedure, i.e, it would be performed on the samples $\mathrm{z}_1(nT)$ instead of $\mathrm{z}(nT) + \mathrm{w}(nT)$. The fact is that the result of both procedures would be *the same* up to the numerical accuracy in use. This can be readily seen in Fig. 6, in which the error measure is the maximum difference between $\mathrm{z}(nT) + \mathrm{w}(nT)$ and $\mathrm{z}_1(nT)$,

$$\sup_{nT\in I} |\mathrm{z}(nT) + \mathrm{w}(nT) - \mathrm{z}_1(nT)|. \tag{24}$$

This coincidence is due to the fact that the interpolator in 20 is also reconstructing the noise realization $\mathrm{w}(t)$, since it is also a signal with bandwidth B. Fig. 7 shows the amplitude spectrum of the sequence $\mathrm{z}(nT)+\mathrm{w}(nT)$, where the maximum has been normalized to 0 dB. If this spectrum were computed from the sine wave crossings, the it would differ from that if Fig. 7 by the amplitude given in Fig. 8.

VII. Conclusions

A method has been presented that makes it possible to recover a band-limited signal from its crossings with a sine wave. It allows one to design A/D converters which only deliver the timing of the sine wave crossings, so allowing for a smaller complexity and power consumption in the converter, and potentially increasing the achievable sampling bandwidth. The method is based on viewing the problem as one of interpolation from nonuniform samples, to which a recent efficient technique is applied. This technique is based on the

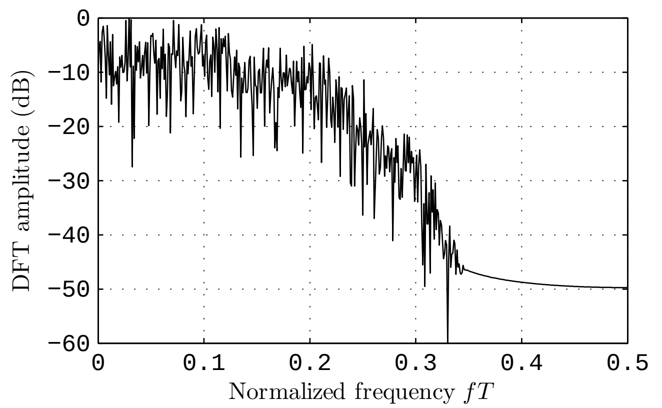


Fig. 7. Amplitude spectrum of sequence $\mathrm{z}(nT) + \mathrm{w}(nT)$ for nT in I. The spectrum's maximum value has been normalized to one.

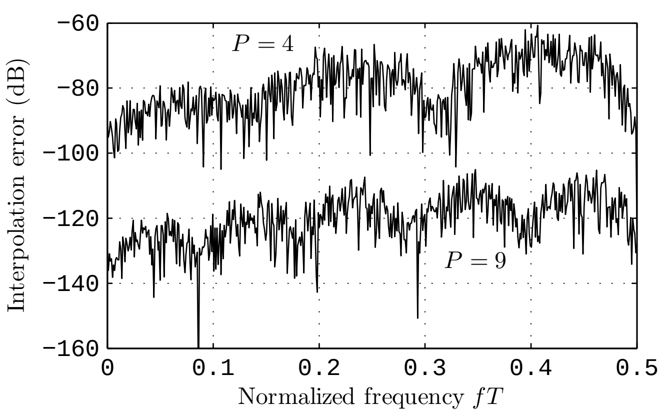


Fig. 8. Amplitude of the difference between the spectrum of $\mathrm{z}(nT)+\mathrm{w}(nT)$ and that of $\mathrm{z}_1(nT)$.

Lagrange interpolator and allows for efficient implementations based on current designs of VFD filters. As a spin-off, the method permits one to perform spectral analysis from the sine wave crossings with the complexity of the FFT. The method has been validated in a numerical example.

Appendix I
Weight function for the Lagrange interpolator

Following [11], the function $\mathrm{w}(t)$ in Sec. III is the inverse Fourier transform of the Kaiser-Bessel window,

$$\mathrm{w}(t) \equiv \frac{\mathrm{sinc}(B_w\sqrt{t^2 - T_w^2})}{\mathrm{sinc}(jB_wT_w)}, \tag{25}$$

where

$$B_w \equiv 1/T - B \text{ and } T_w = PT. \tag{26}$$

Note that in (25) the argument of the sinc functions may be pure imaginary. In this case, the sinc function can be evaluated from the hyperbolic sine since, for real a, it is

$$\mathrm{sinc}(ja) = \frac{\sin(j\pi a)}{j\pi a} = \frac{\mathrm{e}^{-\pi a} - \mathrm{e}^{\pi a}}{(2j)(j\pi a)} = \frac{\sinh(\pi a)}{\pi a}. \tag{27}$$

The weight function $\gamma(t)$ in Sec. III is given by

$$\gamma(t) \equiv \frac{(-1)^P}{(P!)^2}\frac{\mathrm{w}(t)\mathrm{L}_o(t)}{\sin(\pi t/T)}, \tag{28}$$

where $\mathrm{L}_o(t)$ is the Lagrange kernel for the instants pT, $|p| \leq P$,

$$\mathrm{L}_o(t) \equiv \prod_{p=-P}^{P} t - pT.$$

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