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Momentum conservation in dissipationless reduced-fluid dynamics

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The momentum conservation law for general dissipationless reduced-fluid (e.g., gyrofluid) models is derived by Noether method from a variational principle. The reduced-fluid momentum density and the reduced-fluid canonical momentum-stress tensor both exhibit polarization and magnetization effects as well as an internal torque associated with dynamical reduction. As an application, we derive an explicit gyrofluid toroidal angular-momentum conservation law for axisymmetric toroidal magnetized plasmas.

Nonlinear reduced-fluid models play an important role in our understanding of the complex dynamical behavior of strongly magnetized plasmas. These nonlinear reduced-fluid models, in which fast time scales (such as the compressional Alfvén time scale) have been asymptotically removed, include the reduced magnetohydrodynamic equations [1–3], the nonlinear gyrofluid equations [4, 5], and several truncated reduced-fluid models (such as the Hasegawa-Mima equation [6, 7] and the Hasegawa-Wakatani equations [8]). Because the space-time-scale orderings for these reduced-fluid models are compatible with the nonlinear gyrokinetic space-time-scale orderings [5], they provide a very useful complementary set of equations that yield simpler interpretations of low-frequency turbulent plasma dynamics in realistic geometries.

The self-regulation of anomalous transport processes by plasma flows in turbulent axisymmetric magnetized plasmas has been intensively investigated in the past decade. Because a strong coupling has been observed [9] between toroidal-momentum transport and energy transport in such plasmas, it is natural to investigate the link between these two global conservation laws through an application of the Noether method on a suitable Lagrangian density [10]. The purpose of the present Letter is to focus its attention on a momentum conservation law derived from a general reduced-fluid model [11] and then explicitly investigate the reduced toroidal angular-momentum transport in axisymmetric magnetic geometry derived from it.

The general variational formulation of nonlinear dissipationless reduced-fluid models is expressed in terms of a Lagrangian density $\mathcal{L}(\psi^\alpha)$ as a function of the multi-component field

$$\psi^\alpha \;\equiv\; (\Phi, \mathbf{A}, \mathbf{E}, \mathbf{B}; n, \mathbf{u}, p_\parallel, p_\perp). \tag{1}$$

Here, the electromagnetic fields $(\mathbf{E}, \mathbf{B})$ are defined in terms of the electromagnetic potentials $(\Phi, \mathbf{A})$ as

$$\mathbf{E} \;\equiv\; -\nabla\Phi \;-\; c^{-1}\partial\mathbf{A}/\partial t \quad\text{and}\quad \mathbf{B} \;\equiv\; \nabla\times\mathbf{A} \tag{2}$$

and the reduced-fluid moments $(n, \mathbf{u}, p_\parallel, p_\perp)$ are used for each plasma-particle species (with mass m and charge q). We note that the Lagrangian formalism does not accommodate higher-order fluid moments (e.g., heat fluxes) and, therefore, the issue of fluid closure is completely ignored [12]. These higher-order moments, as well as dissipative effects, can be added after the dissipationless reduced-fluid equations are derived by variational method [13] (although a variational procedure [14, 15] can be used to include heat fluxes in the pressure evolution equations).

We begin with the general reduced Lagrangian density

$$\begin{aligned}\mathcal{L}(\psi^\alpha) \;\equiv\; & \mathcal{L}_{\rm M}(\mathbf{E}, \mathbf{B}) \;+\; \mathcal{L}_\Psi(\Phi, \mathbf{A}; n, \mathbf{u}) \\ & +\; \mathcal{L}_{\rm F}(n, \mathbf{u}, p_\parallel, p_\perp; \mathbf{E}, \mathbf{B}),\end{aligned} \tag{3}$$

where the electromagnetic Lagrangian density is $\mathcal{L}_{\rm M} \equiv (|\mathbf{E}|^2 - |\mathbf{B}|^2)/8\pi$, the gauge-dependent interaction Lagrangian density (summed over particle species) is $\mathcal{L}_\Psi \equiv -\sum q\,n\,(\Phi - \mathbf{A}\cdot\mathbf{u}/c)$, and the reduced-fluid Lagrangian density $\mathcal{L}_{\rm F}$ depends on $(\mathbf{E}, \mathbf{B})$ only through the process of dynamical reduction [16].

The reduced plasma-electrodynamic equations associated with the reduced Lagrangian density (3) are divided into either constraint equations or dynamical equations. The electromagnetic constraint equations are

$$\nabla\cdot\mathbf{B} \;=\; 0 \;=\; \nabla\times\mathbf{E} \;+\; c^{-1}\partial\mathbf{B}/\partial t, \tag{4}$$

which are satisfied by the representation (2). The reduced-fluid constraint equations, on the other hand, are the continuity equation

$$\frac{\partial n}{\partial t} \;=\; -\,\nabla\cdot\Big(n\,\mathbf{u}\Big), \tag{5}$$

and the Chew-Goldberger-Low (CGL) pressure equations [17, 18]

$$\frac{\partial p_\parallel}{\partial t} \;=\; -\,\nabla\cdot\Big(p_\parallel\,\mathbf{u}\Big) \;-\; 2\,p_\parallel\,\widehat{\mathsf{b}}_0\widehat{\mathsf{b}}_0 : \nabla\mathbf{u}, \tag{6}$$

$$\frac{\partial p_\perp}{\partial t} \;=\; -\,\nabla\cdot\Big(p_\perp\,\mathbf{u}\Big) \;-\; p_\perp\,\Big(\mathbf{I} - \widehat{\mathsf{b}}_0\widehat{\mathsf{b}}_0\Big) : \nabla\mathbf{u}, \tag{7}$$

associated with the CGL pressure tensor

$$\mathsf{P} \;\equiv\; p_\parallel\,\widehat{\mathsf{b}}_0\,\widehat{\mathsf{b}}_0 \;+\; p_\perp\,(\mathbf{I} - \widehat{\mathsf{b}}_0\,\widehat{\mathsf{b}}_0), \tag{8}$$

where $\mathbf{B}_0 \equiv B_0\;\widehat{\mathsf{b}}_0$ denotes the quasi-static background magnetic field. The reduced-fluid velocity $\mathbf{u}$ appearing in Eqs. (5)-(7) will be determined from the variational principle

$$\int \delta\mathcal{L}\; d^4x \;=\; 0. \tag{9}$$

The reduced Maxwell equations and the reduced-fluid momentum equation are the dynamical equations derived from the reduced variational principle (9). First, the reduced Maxwell equations

$$\nabla\cdot\mathbf{D} \;=\; 4\pi\,\varrho, \tag{10}$$

$$\nabla\times\mathbf{H} \;-\; \frac{1}{c}\,\frac{\partial\mathbf{D}}{\partial t} \;=\; \frac{4\pi}{c}\,\mathbf{J}, \tag{11}$$

are expressed in terms of the reduced charge density $\varrho \equiv -\,\partial\mathcal{L}_\Psi/\partial\Phi = \sum qn$ and the reduced current density $\mathbf{J} \equiv c\,\partial\mathcal{L}_\Psi/\partial\mathbf{A} = \sum qn\,\mathbf{u}$, while the reduced electromagnetic fields $\mathbf{D} \equiv 4\pi\,\partial\mathcal{L}/\partial\mathbf{E} = \mathbf{E} + 4\pi\,\mathbf{P}$ and $\mathbf{H} \equiv -\,4\pi\,\partial\mathcal{L}/\partial\mathbf{B} = \mathbf{B} - 4\pi\,\mathbf{M}$ are expressed in terms of the reduced polarization and magnetization

$$(\mathbf{P},\;\mathbf{M}) \;\equiv\; \left(\frac{\partial\mathcal{L}_{\rm F}}{\partial\mathbf{E}},\;\frac{\partial\mathcal{L}_{\rm F}}{\partial\mathbf{B}}\right). \tag{12}$$

Equations (10)-(11) can also be expressed as

$$\nabla\cdot\mathbf{E} \;=\; 4\pi\;\Big(\varrho \;-\; \nabla\cdot\mathbf{P}\Big), \tag{13}$$

$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \;=\; \frac{4\pi}{c}\left(\mathbf{J} + \frac{\partial\mathbf{P}}{\partial t} + c\,\nabla\times\mathbf{M}\right), \tag{14}$$

where $\varrho_{\rm pol} \equiv -\,\nabla\cdot\mathbf{P}$ denotes the polarization density, $\mathbf{J}_{\rm pol} \equiv \partial\mathbf{P}/\partial t$ denotes the polarization current, and $\mathbf{J}_{\rm mag} \equiv c\,\nabla\times\mathbf{M}$ denotes the magnetization current.

Next, the reduced-fluid momentum equation is

$$\begin{aligned} n\,\frac{d\mathbf{p}}{dt} \;=\;& qn\,\left(\mathbf{E} \;+\; \frac{\mathbf{u}}{c}\times\mathbf{B}\right) \;+\; n\,(\nabla K \;-\; \nabla\mathbf{u}\cdot\mathbf{p}) \\ &-\;\left(p_\perp\,\nabla\gamma_\perp + \frac{p_\parallel}{2}\,\nabla\gamma_\parallel \;+\; \nabla\cdot\mathsf{P}_*\right), \end{aligned} \tag{15}$$

where $d/dt \equiv \partial/\partial t + \mathbf{u}\cdot\nabla$, the reduced-fluid kinetic energy K and the reduced-fluid kinetic momentum $\mathbf{p}$ are

$$\left(\begin{array}{c} K \\ \mathbf{p} \end{array}\right) \;\equiv\; \left(\begin{array}{c} \partial\mathcal{L}_{\rm F}/\partial n \\ n^{-1}\,\partial\mathcal{L}_{\rm F}/\partial\mathbf{u} \end{array}\right), \tag{16}$$

and the symmetric reduced pressure tensor

$$\mathsf{P}_* \;\equiv\; p_\parallel\,\gamma_\parallel\,\widehat{\mathsf{b}}_0\widehat{\mathsf{b}}_0 \;+\; p_\perp\,\gamma_\perp\,(\mathbf{I} - \widehat{\mathsf{b}}_0\widehat{\mathsf{b}}_0) \tag{17}$$

is defined in terms of the coefficients $\gamma_\parallel \equiv -\,2\,\partial\mathcal{L}_{\rm F}/\partial p_\parallel$ and $\gamma_\perp \equiv -\,\partial\mathcal{L}_{\rm F}/\partial p_\perp$. We note that this pressure tensor generalizes the CGL pressure tensor (8) and includes standard finite-Larmor-radius (FLR) corrections through $\gamma_\perp \neq 1$ [13].

The reduced equations (4)-(7), (10)-(11), and (15) satisfy the reduced momentum conservation law [11]

$$\frac{\partial\mathbf{\Pi}}{\partial t} \;+\; \nabla\cdot\mathsf{T} \;=\; \nabla'\overline{\mathcal{L}}, \tag{18}$$

where the reduced momentum density is

$$\mathbf{\Pi} \;\equiv\; \sum n\,\mathbf{p} \;+\; \frac{\mathbf{D}\times\mathbf{B}}{4\pi\,c}, \tag{19}$$

the reduced canonical momentum-stress tensor is

$$\begin{aligned} \mathsf{T} \;\equiv\;& \sum \mathsf{P}_* \;+\; \left(\mathcal{L}_{\rm F} \;-\; \sum\,\eta^a\,\frac{\partial\mathcal{L}_{\rm F}}{\partial\eta^a}\right)\,\mathbf{I} \\ &+\;\left[\frac{1}{8\pi}\left(|\mathbf{E}|^2 \;+\; |\mathbf{B}|^2\right) \;-\; \mathbf{B}\cdot\mathbf{M}\right]\,\mathbf{I} \\ &-\;\frac{1}{4\pi}\,\Big(\mathbf{E}\,\mathbf{E} \;+\; \mathbf{B}\,\mathbf{B}\Big) \\ &+\;\left[\sum\,n\,\mathbf{u}\,\mathbf{p} \;-\; \Big(\mathbf{P}\,\mathbf{E} \;-\; \mathbf{B}\,\mathbf{M}\Big)\right], \end{aligned} \tag{20}$$

and $\nabla'\overline{\mathcal{L}}$ denotes the spatial gradient of the reduced Lagrangian density $\overline{\mathcal{L}} \equiv \mathcal{L} - \mathcal{L}_\Psi$ with the dynamical fields (1) held constant. We note that, while the first three terms in the canonical momentum-stress tensor (20) are symmetric while the remaining terms (on the last line) are not. The antisymmetric part $(\mathsf{T}_{\sf A})_{ij} \equiv \frac{1}{2}\,(T_{ij} - T_{ji}) \equiv \frac{1}{2}\,\varepsilon_{ijk}\,\tau^k$ of the canonical momentum-stress tensor (20) can be expressed in terms of the reduced *internal torque* density

$$\boldsymbol{\tau} \;\equiv\; \sum\,n\,\mathbf{u}\times\mathbf{p} \;+\; \Big(\mathbf{E}\times\mathbf{P} \;+\; \mathbf{B}\times\mathbf{M}\Big), \tag{21}$$

which exhibits classical *zitterbewegung* effects [19] associated with the decoupling of the reduced-fluid momentum $\mathbf{p} \neq m\mathbf{u}$ and the reduced-fluid velocity $\mathbf{u}$ and the reduced polarization and magnetization effects. Here, the *internal* degrees of freedom of a gyrofluid *particle* are associated with the fast gyromotion that has been eliminated by dynamical reduction.

The symmetry of the momentum-stress tensor is physically connected to the conservation of angular momentum [10, 20], i.e., conservation of the total angular momentum (including internal angular momentum) explicitly requires a symmetric momentum-stress tensor. Since the left side of Eq. (18) is invariant under the transformation [21] $\mathbf{\Pi}' \equiv \mathbf{\Pi} + \nabla\cdot\mathsf{S}$ and $\mathsf{T}' \equiv \mathsf{T} - \partial\mathsf{S}/\partial t$, the second-rank antisymmetric tensor $\mathsf{S} \equiv \frac{1}{2}\,\varepsilon_{ijk}\,\sigma^k$ can be chosen so that $\mathsf{T}' \equiv \mathsf{T}_{\sf S} \equiv \frac{1}{2}(T_{ij} + T_{ji})$ is symmetric, i.e., the antisymmetric part $\mathsf{T}_{\sf A} \equiv \partial\mathsf{S}/\partial t$ yields the reduced internal angular momentum equation

$$\frac{\partial\boldsymbol{\sigma}}{\partial t} \;\equiv\; \boldsymbol{\tau}, \tag{22}$$

where $\boldsymbol{\sigma}$ denotes the *internal* (spin) angular momentum density, and the reduced momentum conservation law (18) becomes

$$\frac{\partial}{\partial t}\left(\mathbf{\Pi} \;-\; \frac{1}{2}\,\nabla\times\boldsymbol{\sigma}\right) \;+\; \nabla\cdot\mathsf{T}_{\sf S} \;=\; \nabla'\,\overline{\mathcal{L}}, \tag{23}$$

where we used the identity $\nabla\cdot\mathsf{S} \equiv -\,\frac{1}{2}\,\nabla\times\boldsymbol{\sigma}$.

By applying the Noether Theorem in axisymmetric tokamak geometry, where

$$\mathbf{B}_0 \;\equiv\; \nabla\varphi\times\nabla\psi \;+\; B_{0\varphi}(\psi)\,\nabla\varphi \tag{24}$$

and the background scalar fields are independent of the toroidal angle φ (i.e., $\partial'\overline{\mathcal{L}}/\partial\varphi \equiv 0$), we obtain the reduced toroidal angular-momentum transport equation [11]

$$\frac{\partial}{\partial t}\Big(\Pi_\varphi \;-\; \sigma_z\Big) \;+\; \nabla\cdot\left(\mathsf{T}\cdot\frac{\partial\mathbf{x}}{\partial\varphi}\right) \;=\; 0. \tag{25}$$

Here, the toroidal momentum density is

$$\begin{aligned}\Pi_\varphi \;&=\; \mathbf{\Pi}\cdot\frac{\partial\mathbf{x}}{\partial\varphi} \;\equiv\; \widehat{\mathsf{z}}\cdot\mathbf{x}\times\mathbf{\Pi} \qquad (26)\\ &=\; \sum n\,p_\varphi \;+\; \frac{(1+b_\parallel)\,D^\psi}{4\pi\,c} \;+\; \frac{\mathbf{D}\times\mathbf{B}_\perp}{4\pi\,c}\cdot\frac{\partial\mathbf{x}}{\partial\varphi},\end{aligned}$$

where the perturbed magnetic field is $\mathbf{B}-\mathbf{B}_0 \equiv b_\parallel\,\mathbf{B}_0 + \mathbf{B}_\perp$ and we used the identity $\mathbf{B}_0\times\partial\mathbf{x}/\partial\varphi \equiv \nabla\psi$.

A more useful expression for Eq. (25), however, is obtained in terms of the magnetic-flux average $\langle\cdots\rangle \equiv \mathcal{V}^{-1}\oint\mathcal{J}\,(\cdots)\,d\theta\,d\varphi$ as

$$\frac{\partial\langle\Pi_\varphi\rangle}{\partial t} \;+\; \frac{1}{\mathcal{V}}\,\frac{\partial}{\partial\psi}\Big(\mathcal{V}\;\langle T^\psi_{\;\;\varphi}\rangle\Big) \;=\; \langle\tau_z\rangle, \tag{27}$$

where $\mathcal{J} \equiv (\nabla\psi\times\nabla\theta\cdot\nabla\varphi)^{-1} = (\mathbf{B}_0\cdot\nabla\theta)^{-1}$ is the Jacobian for the magnetic coordinates (ψ,θ,φ) and $\mathcal{V} \equiv \oint\mathcal{J}\;d\theta\,d\varphi$. In Eq. (27), $\langle\tau_z\rangle$ denotes the reduced internal torque density and the surface-averaged toroidal angular-momentum flux is

$$\begin{aligned}T^\psi_{\;\;\varphi} \;&\equiv\; \nabla\psi\cdot\mathsf{T}\cdot\frac{\partial\mathbf{x}}{\partial\varphi} \qquad (28)\\ &=\; \sum n u^\psi\,p_\varphi \;-\; \frac{1}{4\pi}\,(D^\psi\,E_\varphi + B^\psi_\perp H_\varphi),\end{aligned}$$

where $B^\psi_\perp \equiv \mathbf{B}_\perp\cdot\nabla\psi$ denotes the ψ-component of the perpendicular component of the perturbed magnetic field (since $\mathbf{B}_0\cdot\nabla\psi \equiv 0$).

We now investigate the surface-averaged toroidal angular-momentum conservation law (27) by considering the gyrofluid Lagrangian density [15, 22]

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{8\pi}\left(|\mathbf{E}_\perp|^2 - |\mathbf{B}|^2\right)\\ &+ \sum\left[\,\frac{1}{2}\,mn\,u_\parallel^2 \;-\; \Big(n\,\mathcal{K}_\rho \;+\; \mathcal{P}\Big)\,\right]\\ &+ \sum\;qn\left[\,\frac{\mathbf{u}}{c}\cdot\Big(\mathbf{A}_0 + A_{\parallel\rho}\,\widehat{\mathsf{b}}_0\Big) \;-\; \Phi_\rho\,\right], \qquad (29)\end{aligned}$$

where $u_\parallel \equiv \mathbf{u}\cdot\widehat{\mathsf{b}}_0$ denotes the gyrofluid velocity along the unperturbed (background) magnetic-field lines and the pressure tensor (17) is $\mathsf{P}_* \equiv \mathsf{P}$ (i.e., $\gamma_\parallel = 1 = \gamma_\perp$) with $\mathcal{P} \equiv \frac{1}{2}\,\mathrm{Tr}(\mathsf{P}) = p_\perp + p_\parallel/2$. In Eq. (29), the zero-Larmor-radius limit of the gyrocenter dynamical reduction [5] introduces the nonlinear FLR-corrected potentials

$$\left(\begin{array}{c}\Phi_\rho\\ \\ A_{\parallel\rho}\end{array}\right) \;\equiv\; \left(\begin{array}{c}\Phi \;-\; \boldsymbol{\rho}_\perp\cdot\mathbf{E}_\perp\\ \\ A_\parallel \;-\; \widehat{\mathsf{b}}_0\cdot\boldsymbol{\rho}_\perp\times\mathbf{B}_\perp\end{array}\right), \tag{30}$$

which generate the nonlinear FLR-corrected electromagnetic fields

$$\left(\begin{array}{c}\mathbf{E}_\rho\\ \mathbf{B}_{\perp\rho}\end{array}\right) \;\equiv\; \left(\begin{array}{c}-\nabla\Phi_\rho - c^{-1}\widehat{\mathsf{b}}_0\,\partial A_{\parallel\rho}/\partial t\\ \nabla\times(A_{\parallel\rho}\;\widehat{\mathsf{b}}_0)\end{array}\right), \tag{31}$$

and the low-frequency ponderomotive potential

$$\mathcal{K}_\rho \;\equiv\; \frac{1}{2}\,m\,\Omega_0^2\,|\boldsymbol{\rho}_\perp|^2 \;\equiv\; \frac{m}{2}\,|\mathbf{U}_\perp|^2, \tag{32}$$

which generates the generalized ponderomotive force density $\nabla\cdot\mathsf{P}_\rho \equiv \nabla\cdot\mathsf{P} + n\,\nabla\mathcal{K}_\rho$. These nonlinear FLR-corrected fields are expressed in terms of the gyrofluid displacement [22]

$$\boldsymbol{\rho}_\perp \;=\; \frac{c}{B_0\Omega_0}\Big(\mathbf{E}_\perp \;+\; \frac{u_\parallel}{c}\,\widehat{\mathsf{b}}_0\times\mathbf{B}_\perp\Big) \;\equiv\; \frac{\widehat{\mathsf{b}}_0}{\Omega_0}\times\mathbf{U}_\perp. \tag{33}$$

The partial derivatives (16) of the gyrofluid Lagrangian density (29) yield the gyrofluid kinetic energy $K = m\,u_\parallel^2/2 + \mathcal{K}_\rho$, and the gyrofluid momentum

$$\mathbf{p} \;=\; m\,\left(u_\parallel \;+\; \mathbf{U}_\perp\cdot\frac{\mathbf{B}_\perp}{B_0}\right)\widehat{\mathsf{b}}_0 \;\equiv\; m\,u_\parallel^*\,\widehat{\mathsf{b}}_0, \tag{34}$$

where $u_\parallel^*$ defines the gyrofluid velocity along the perturbed magnetic-field lines. Lastly, the partial derivatives (12) yield the gyrofluid polarization and magnetization

$$\mathbf{P} \;\equiv\; \sum\;qn\,\boldsymbol{\rho}_\perp \;=\; \sum\;mn\,\frac{c\widehat{\mathsf{b}}_0}{B_0}\times\mathbf{U}_\perp, \tag{35}$$

$$\mathbf{M} \;\equiv\; \sum\;qn\,\boldsymbol{\rho}_\perp\times\frac{u_\parallel}{c}\,\widehat{\mathsf{b}}_0 \;=\; \sum\;mn\,\frac{u_\parallel}{B_0}\,\mathbf{U}_\perp, \tag{36}$$

which appear in the Maxwell equations (10)-(11). Note that the reduced internal torque (21) for this gyrofluid model is expressed as

$$\begin{aligned}\boldsymbol{\tau} \;=\;& \sum\frac{mnc}{B_0}\,(u_\parallel - u_\parallel^*)\,\Big(\mathbf{E}_\perp \;+\; \frac{u_\parallel}{c}\,\widehat{\mathsf{b}}_0\times\mathbf{B}_\perp\Big)\\ &+ \sum\;mn\,u_\parallel^*\,\Big(\mathbf{u}_\perp \;-\; \mathbf{U}_\perp\Big)\times\widehat{\mathsf{b}}_0, \qquad (37)\end{aligned}$$

where $\mathbf{E}_\perp\times\mathbf{P} + \mathbf{B}_\perp\times\mathbf{M} \equiv 0$.

The gyrofluid momentum equation (15) derived from the gyrofluid Lagrangian density (29) is expressed as

$$mn\,\widehat{\mathsf{b}}_0\,\frac{du_\parallel}{dt} \;=\; qn\,\Big(\mathbf{E}_\rho \;+\; \frac{\mathbf{u}}{c}\times\mathbf{B}^*_\rho\Big) \;-\; \nabla\cdot\mathsf{P}_\rho, \tag{38}$$

where $\mathbf{B}^*_\rho \equiv \mathbf{B}_0 + u_\parallel\,(B_0/\Omega_0)\;\nabla\times\widehat{\mathsf{b}}_0 + \mathbf{B}_{\perp\rho}$. Equation (38) contains both the gyrofluid parallel-force equation

$$mn\,\frac{du_\parallel}{dt} \;=\; \mathsf{b}^*_\rho\cdot\Big(qn\,\mathbf{E}_\rho \;-\; \nabla\cdot\mathsf{P}_\rho\Big), \tag{39}$$

where $\mathsf{b}^*_\rho \equiv \mathbf{B}^*_\rho/B_0$, and the gyrofluid velocity

$$\mathbf{u} \;=\; u_\parallel\;\mathsf{b}^*_\rho \;+\; \Big(qn\,\mathbf{E}_{\perp\rho} \;-\; \nabla\cdot\mathsf{P}_\rho\Big)\times\frac{\widehat{\mathsf{b}}_0}{mn\,\Omega_0}, \tag{40}$$

which includes the $E \times B$ velocity and the diamagnetic velocity and their nonlinear FLR corrections.

Next, we consider the gyrofluid version of the surface-averaged toroidal angular-momentum conservation law (27), where the gyrofluid momentum (34) is substituted in Eqs. (19)-(20), with $\mathsf{P}_* = \mathsf{P}$ and $\eta^a\,\partial\mathcal{L}_{\rm F}/\partial\eta^a \equiv \mathcal{L}_{\rm F}$ in Eq. (20). The gyrofluid version of the surface-averaged equation (27) is expressed in terms of the gyrofluid toroidal angular-momentum density (26), with $p_\varphi = m u_\parallel^*\, b_{0\varphi}$ and

$$D^\psi \;\equiv\; \left(1 + 4\pi \sum \frac{mnc^2}{B_0^2}\right) E^\psi + \left(4\pi \sum \frac{mn\,u_\parallel c}{B_0^2}\right) \mathbf{B}_\perp\cdot(\nabla\psi\times\widehat{\mathsf{b}}_0), \quad (41)$$

while the expression for Eq. (28) is

$$T^\psi_{\;\varphi} \;=\; \sum n\left(u^\psi\, p_\varphi \;-\; q\,\rho_\perp^\psi\, E_\varphi \;+\; mu_\parallel\,\frac{B_\perp^\psi}{B_0}\,U_{\perp\varphi}\right) - \frac{1}{4\pi}\left(E^\psi\, E_\varphi \;+\; B_\perp^\psi\, B_\varphi\right), \quad (42)$$

where we have separated the reduced polarization $P^\psi = \sum qn\,\rho_\perp^\psi$ and magnetization $M_\varphi = \sum mnu_\parallel U_{\perp\varphi}/B_0$ from the Maxwell stress tensor.

As an application of the gyrofluid model (29), we consider its electrostatic version ($\mathbf{E} = -\nabla\Phi$, $u_\parallel^* = u_\parallel$, and $\mathbf{B} = \mathbf{B}_0$) and use the quasi-neutrality condition $\varrho \equiv \nabla\cdot\mathbf{P}$ [valid for $4\pi\,(\sum mnc^2/B_0^2) \gg 1$]. The surface-averaged gyrofluid toroidal angular-momentum equation (27) therefore becomes

$$\frac{\partial}{\partial t}\left(\left\langle\Pi_{\varphi\parallel}\right\rangle \;+\; \frac{1}{c}\,\langle P^\psi\rangle \;-\; \langle\sigma_z\rangle\right) \;=\; -\,\frac{1}{\mathcal{V}}\,\frac{\partial}{\partial\psi}\left[\mathcal{V}\left(\left\langle\Gamma^\psi_{\varphi\parallel}\right\rangle \;+\; \left\langle P^\psi\,\frac{\partial\Phi}{\partial\varphi}\right\rangle\right)\right], \quad (43)$$

where $\Pi_{\varphi\parallel} \equiv (\sum mn\,u_\parallel)\,b_{0\varphi}$, $\Gamma^\psi_{\varphi\parallel} \equiv (\sum mn\,u_\parallel u^\psi)\,b_{0\varphi}$, and $P^\psi \equiv (\sum mnc^2/B_0^2)\,E^\psi$. This equation was recently obtained [23] (without the internal angular momentum σ_z) by direct evaluation of the time evolution of the surface-averaged gyrocenter moment $\langle\Pi^{\rm can}_\varphi\rangle \equiv \langle\Pi_{\varphi\parallel}\rangle - (\psi/c)\,\langle\varrho\rangle$ of the toroidal canonical momentum $mv_\parallel\, b_{0\varphi} - q\,\psi/c$, where the surface-averaged gyrofluid charge density $\langle\varrho\rangle = \mathcal{V}^{-1}\;\partial(\mathcal{V}\,\langle P^\psi\rangle)/\partial\psi$ is expressed in terms of the surface-averaged polarization component $\langle P^\psi\rangle$.

In this Letter, we have derived an exact toroidal angular-momentum conservation law (25) that clearly exhibits the role played by the reduced internal torque $\tau_z \equiv \partial\sigma_z/\partial t$ in possibly driving spontaneous toroidal rotation in axisymmetric tokamak plasmas.

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