KÄHLER-EINSTEIN METRICS EMERGING FROM FREE FERMIONS AND STATISTICAL MECHANICS

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Abstract. We propose a statistical mechanical derivation of Kähler-Einstein metrics, i.e. solutions to Einstein's vacuum field equations in Euclidean signature (with a cosmological constant) on a compact Kähler manifold X. The microscopic theory is given by a canonical free fermion gas on X whose one-particle states are pluricanonical holomorphic sections on X (coinciding with higher spin states in the case of a Riemann surface). A heuristic, but hopefully physically illuminating, argument for the convergence in the thermodynamical (large N) limit is given, based on a recent mathematically rigorous result about exponentially small fluctuations of Slater determinants. Relations to effective bosonization and the Yau-Tian-Donaldson program in Kähler geometry are pointed out. The precise mathematical details will be investigated elsewhere.

1. Introduction

The basic laws of gravity have an intriguing similarity with the laws of thermodynamics and hydrodynamics - this has been pointed out at several occasions in the physics literature, in particular in connection to the study of black holes (see for example [7, 35, 54]). As a consequence one is lead to ask whether gravity can be seen as an emergent effect of an underlying microscopic theory in a thermodynamical limit [35]? The aim of this note is to propose a situation where this question can be answered in the affirmatively. We will consider Einstein's vacuum field equations in Euclidean signature on a compact manifold X, whose solutions are usually called *Einstein metrics* in the mathematics literature [4]. More precisely, these equations will be considered in the presence of a fixed background integrable complex structure J on X. It turns out that the underlying microscopic theory may then be realized as a certain free fermion gas on X and it will be shown how to recover an Einstein metric (with a non-zero cosmological constant) in the thermodynamical limit. The metric is singled out by the fact that it is Hermitian and Kähler with respect to J. In other words these are the *Kähler-Einstein metrics* which have been extensively studied during the last decade in the mathematics literature (for a recent survey see [44]).

Physically, metrics as above appear, for example, as gravitational instantons in Hawking's functional integral approach to quantum gravity [32, 51]. Although we will not restrict X to be a real four-manifold - the physically most relevant case - it is worth pointing out that in this latter case "most" Einstein metrics are Kähler-Einstein metrics. In particular, for a negative cosmological constant Λ it may actually be that *all* Einstein metrics are Kähler with respect to some complex structure J, up to diffeomorphism - as long as X admits some complex structure (the question was raised in [41]). This is for example the case for compact quotients of the unit ball, as shown in [40] using Seiberg-Witten gauge theory.

The main ingredients in the investigation of the thermodynamical limit below is the asymptotics of exponentially small fluctuations of Slater determinants for $N-$particle correlations of fermions on complex manifolds in [10] (building on [13, 14, 15]). On one hand, from a purely mathematical point of view these large N asymptotics concern large deviations for certain critical determinantal random point processes, which generalize Random Matrix ensembles previously extensively studied. On the other hand, from a physical point of view the result can be seen as an effective bosonization of a free fermion gas (see section 2.1.2), which in the case of a Riemann surface alternatively can be deduced from the exact bosonization results in [53, 18]. The large deviation result for the Slater determinant is then combined with a basic large deviation result for a non-interacting *classical* gas going back to Boltzmann's fundamental work on entropy (called Sanov's theorem in the mathematics literature).

It should however be pointed out that the argument in the present note which combines the two mathematically rigorous results refered to above is not completely rigorous. Basically, it involves an interchange of two limits which needs to be mathematically justified. The mathematical details, as well as various extensions, will be investigated elsewhere, but hopefully the heuristic derivation given here is illuminating from a physical point of view as it involves manipulations that are standard in the functional integral approach to quantum field theory.

Incidentally, in the case of a Riemann surface (i.e. the case when the *real* dimension D of X is two) the situation studied in the present note is closely related to the previous mathematical study of various $2D$ ensembles (point vortex systems, plasmas, self-gravitating systems, ...) from the point of view of mean field theory; see [17, 37] and references therein. In particular, the corresponding thermodynamical limit was studied in [17, 37] as a model of 2D turbulence. However, the higher dimensional situation studied in the present work is analytically considerably more complicated as the resulting limiting mean field equations are fully non-linear (see section 2.2). The reason is that the role of the Laplace operator on a Riemann surface is played by the non-linear *Monge-Ampère operator* for higher dimensional complex manifolds. A different "linear" higher-dimensional generalization of point vortex systems has previously been consider by Kiessling [38], where the role of the Laplace operator is played by the linear Paneitz operator. It involves *conformal geometry* of spheres rather than the *complex (holomorphic) geometry* considered here and the thermodynamical limit is a mean field limit of an explicit gas with logarithmic pair interactions.

It would be interesting to understand the relation between the present note and the ADS/CFT correspondence [1], which relates gravity in the bulk of a manifold to a conformal field theory on its boundary. This is a realization of t'Hooft's holographic principle. Such a principle has recently been put forward by Verlinde in [54] as the basis of an entropic explanation of gravity. As explained in the concluding section 2.4 the emergence of the Kähler-Einstein metric from the present microscopic model can be interpreted as coming from a *fermionic maximum entropy principle.*

From the mathematical point of view an important motivation for the present work comes from the Yau-Tian-Donaldson program which relates the analytic problem of existence of extremal metrics in a given Kähler class (i.e. Kähler-Einstein metrics in the case of the canonical class) to algebro-geometric stability conditions (notably $K-$stability; see [25, 26, 50, 44] and references therein). For example, the free energy functional derived below turns out to coincide, in the canonical case, with Mabuchi's $K-$energy, which is usually used to define various notions of $K-$stability. Moreover, the thermodynamical convergence towards a Kähler-Einstein volume form in section 2 is somewhat "dual" to the convergence of canonically balanced metrics conjectured by Donaldson in [27] and proved in [15] (see section 2.3).

As a conclusion one of the mathematical aims of the present paper is to introduce a "thermodynamical formalism" for Kähler-Einstein metrics and more generally for Monge-Ampère equations of mean field type that will be further investigated elsewhere.

Acknowledgment. Thanks to Kurt Johansson for stimulating my interest in general $\beta-$ensembles and to Bengt Nilsson for comments on a draft of the present paper.

1.1. **Geometric setup.** Let X be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. In other words, we are given a real manifold (X, J) of dimension $D = 2n$ equipped with an integrable complex structure J and admitting an Hermitian metric

$$\omega = \frac{i}{2} h_{ij} dz^i \wedge d\bar{z}^j$$

on the complex tangent bundle TX, which is closed: $d\omega = 0$. Identifying ω with a Riemannian metric g compatible with J, i.e. $g_{ij} = \mathrm{Re}\, h_{ij}$ the vacuum Einstein equations, in Euclidean signature, with a cosmological constant read:

$$\mathrm{Ric}\, \omega = \Lambda \omega \tag{1.1}$$

when $n > 1$ and for general n this is the equation for a Kähler-Einstein metric. After a scaling, we may assume that the cosmological constant Λ is $0, 1$ or -1. In the following we will be mainly concerned with the latter case, i.e. when the solution ω is a Kähler metric with constant *negative* Ricci curvature. As shown in the seminal works of Aubin [5] and Yau [56] such a metric ω exists precisely when the first Chern class $c_1(K_X)$ of the canonical line bundle $K_X := \Lambda^n(T^*X)$ is *positive,* which will henceforth be assumed. The Kähler-Einstein metric ω is then uniquely determined by the complex structure J and we will denote it by ω_{KE}. When $n = 1$, i.e. X is a Riemann surface, this amounts to the classical fact that X admits a metric of constant negative curvature precisely when X has genus at least two. This hyperbolic metric is unique in its conformal class (determined by the complex structure J)

The starting point of the existence proof of Aubin and Yau is the basic complex geometric fact that the metric ω_{KE} is uniquely determined by its volume form $\omega_{KE}^n/n!$, that we will normalize to become a probability measure:

$$\mu_{KE} := \frac{\omega_{KE}^n/n!}{Vn!}$$

In other words the tensor equation 1.1 reduces to a *scalar* equation (for the density of μ_{KE}) and the Kähler-Einstein metric ω_{KE} may then be recovered by

$$\omega_{KE} = \frac{i}{2\pi} \partial\bar{\partial} \log \mu_{KE}$$

i.e. as $\frac{i}{2\pi}$ times the curvature two form of the metric on the canonical line bundle K_X defined by μ_{KE}.

The question raised in the introduction may now be reformulated as *"Can the probability measure μ_{KE} be realized as the (macroscopic) expected distribution of particles in a thermodynamical limit of a (microscopic) statistical mechanical system canonically associated to* X? Moreover, the point is to be able to define the microscopic system without specifying any background metric structure so that the Einstein metric and hence (Euclidean) gravity would emerge macroscopically. It turns out that such a statistical mechanical system can indeed be realized by a certain free fermion gas on X, as explained below.

1.2. **General statistical mechanics formalism.** We start by recalling some basic statistical mechanical formalism. Mathematically, a (classical) gas of N identical particles (i.e. a *random point process with* N *particles*) is described by a *symmetric* probability measure $\mu^{(N)}$ on the $N-$fold product X^N *(the* $N-$*particle configuration space).* In local holomorphic coordinates $Z=(z_1,....,z_n)$ on the complex manifold X this means that

$$\mu^{(N)}=\rho^{(N)}(Z_1,...,Z_N)dV(Z_1)\wedge\cdots\wedge dV(Z_N)$$

where $dV(Z_1):=(\frac{i}{2})^n dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$ and where the local $N-$*point correlation function* $\rho^{(N)}$ is invariant under permutations of the Z_i :s. Pushing forward $\mu^{(N)}$ to X^j one then obtains the corresponding $j-$point correlation measures $\mu_j^{(N)}$ on X^j and their local densities $\rho_j^{(N)}$. We will be mainly concerned with the *one-point correlation measure* $\mu_1^{(N)}$ on X, i.e.

$$\mu_1^{(N)}:=\int_{X^{N-1}}\mu^{(N)}$$

In other words, its local density $\rho_1^{(N)}(Z)$ represents the probability of finding a particle in the infinitesimal box $dV(Z_1)$. Yet another (trivially) equivalent formulation representation of $\mu_1^{(N)}$ can be given:

$$\mu_1^{(N)}=\left\langle\frac{1}{N}\sum_i\delta_{x_i}\right\rangle,$$

where the brackets denote the ensemble mean (expectation) of the random variable

$$(x_1,...,x_N)\mapsto\frac{1}{N}\sum_i\delta_{x_i}\qquad(1.2)$$

with values in the space $\mathcal{M}_1(X)$ of probability measures on X. In other words, if ϕ denotes a fixed smooth function then

$$\int_X\phi\mu_1^{(N)}=\frac{1}{N}\sum_i\langle\phi(x_i)\rangle=\langle\phi(x_1)\rangle$$

We will next explain how to define $\mu^{(N)}$ so that the one-point correlation measurs convergen to the normalized volume form of the Kähler-Einstein metric:

$$\mu_1^{(N)}\to\mu_{KE}$$

in the large $N-$limit. More precisely, the convergence will hold in the weak topology on $\mathcal{M}_1(X)$, i.e.

$$\int_X \phi\mu_1^{(N)} \to \int \phi\mu_{KE}$$

for any fixed smooth function ϕ on X. This convergence can be interpreted as an answer to the question raised above. In fact, the argument will give a much stronger "exponential" convergence which in particular implies the asymptotic factorization of all $j-$point correlation functions (i.e. *propagation of chaos* holds).

1.3. **Line bundles and Slater determinants.** To define $\mu^{(N)}$ first recall that we have assumed that the canonical line bundle $K_X \to X$ is *positive* (i.e. ample in the sense of algebraic geometry). We next recall some basic facts about line bundles (see for example [18, 23] for introductions aimed at physicists). To any holomorphic line bundle $L \to X$ there is a naturally associated $N-$dimensional complex vector space $H^0(X, L)$ consisting of global holomorphic section of $L \to X$ and the limit we will be interested is when L is replaced by a large tensor power $L^{\otimes k}$. Since L is assumed ample it follows that the dimension $N = N_k$ (which will be the number of particles of our gas) grows with k in the following way:

$$N_k := \dim_{\mathbb{C}} H^0(X, L^{\otimes k}) = Vk^n + o(k^n)$$

where the *volume* $V > 0$. In particular,

$$N(= N_k) \to \infty \Leftrightarrow k \to \infty$$

We will often omit the subscript k in N_k.

In physics, $H^0(X, L)$ usually arises as the quantum ground state space of a single *chiral fermion* on X coupled to L [53, 18]. The corresponding $N-$particle space of fermions is then, according to Pauli's exclusion principle, represented by the top exterior power $\Lambda^N H^0(X, L)$. In other words this is the maximally filled many particle fermion state. As a consequence it is one-dimensional and may, up to scaling, be represented by the $N-$body state

$$\Psi(x_1, ...x_N) := \Psi_1(x_1) \wedge \cdots \wedge \Psi_N(x_N)$$

expressed in terms of a given base (Ψ_I) in $H^0(X, L)$, where $I = 1, ..., N$. Locally this means that Ψ may be written as a Slater determinant:

$$\Psi(Z_1, ..., Z_N) = \det(\Psi_I(Z_J)) \tag{1.3}$$

which hence transforms as a holomorphic section of the line bundle $L^{\boxtimes N}$ over X^N.

1.3.1. *Introducing metrics.* Usually, one equips L with an *Hermitian metric* h_0. Taking the point-wise norm $\|\Psi(Z)\|$ with respect to h_0 of a section Ψ of L hence gives a scalar function on X. Let us briefly recall the notion of curvature in this context. The (Chern) *curvature form* Θ of h_0 is the globally well-defined two-form on X locally defined as follows: if s is a local trivializing holomorphic section of L, then

$$\Theta := -\partial\bar{\partial}\log(\|s\|^2) \tag{1.4}$$

Physically, the curvature form Θ represents a background *magnetic* two-form of bidegree $(1, 1)$ to which the fermions are minimally coupled. More precisely, the holomorphic structure on L together with the Hermitian metric h_0 determines a unique unitary connection A on L, i.e. a $U(1)-$gauge potential such that its field strength $F_A = \Theta$ is of type $(1,1)$ [18, 23]. The metric h_0 is *positively curved* precisely when the *real* two-form

$$\omega := \frac{i}{2\pi}\Theta \tag{1.5}$$

is positive definite, i.e. when it defines a *Kähler metric* on X. The line bundle L is ample precisely when it admits some positively curved metric. The normalization above ensures that the cohomology class $[\omega]$, which represents the normalized *first Chern class* $c_1(L)$ is an integer class, i.e. it lies in the integer lattice $H^2(X,\mathbb{Z})$ of $H^2(X,\mathbb{R})$.

The Hermitian metic h_0 natually induces metrics on all tensor powers of L etc. Coming back to the Slater determinant above, the point-wise squared norm with respect to the metric h_0

$$\|\Psi(Z_1,...,Z_N)\|^2$$

is, from a physical point of view, proportional to the probability of finding (or creating) particles at the point $Z_1,...Z_N$ on X in the presense of the corresponding background magnetic field. To normalize it we also need to pick an integration measure μ_0 on X so that

$$\|\Psi(Z_1,...,Z_N)\|^2/\mathcal{Z}_N,\ \ \mathcal{Z}_N := \int_{X^N}\|\Psi\|^2\mu_0^{\otimes N}$$

is a probability density on X^N. Since, $\Lambda^N H^0(X,L)$ is one-dimensional the probability density above is in fact independent of the choice of base (Ψ_I) in $H^0(X,L)$, but, of course, it does depend on the metric h_0 on L (i.e. on the background magnetic field) and also on the integration measure μ_0 on X.

1.4. **The canonical background free ensemble .** The main point of the present note is the simple observation that in the particular case when L is the canonical line bundle K_X there is no need to specify any metric on K_X if one defines a probability measure on X^N by

$$\mu^{(N)} = (\Psi_1\wedge\bar{\Psi}_1\wedge\cdots\wedge\Psi_N\wedge\bar{\Psi}_N)^{1/k}/\mathcal{Z}_{\mathcal{N}}.$$

Indeed, it follows from the very definition of K_X that $(\Psi_1\wedge\bar{\Psi}_1\wedge\cdots\wedge\Psi_N\wedge\bar{\Psi}_N)^{1/k}$ transforms as a (degenerate) volume form on X^N and hence after dividing by

$$\mathcal{Z}_{\mathcal{N}} = \int_{X^N}(\Psi_1\wedge\bar{\Psi}_1\wedge\cdots\wedge\Psi_N\wedge\bar{\Psi}_N)^{1/k} = \int_{X^N}|\det(\Psi_I(Z_J))|^{2/k}dV(Z_1)\wedge\cdots\wedge dV(Z_N)$$

one obtains a probability measure $\mu^{(N)}$ on X^{N_k} which is canonically associated to $(X,K_X^{\otimes k})$, i.e. independent of the base (Ψ_I) in $H^0(X,K_X^{\otimes k})$. Note that when $n=1$, i.e. X is a Riemann surface of genus at least two the space $H^0(X,K_X^{\otimes k})$ arises as the space of spin $2k$ particles [53, 18] .

1.5. **General $\beta-$ensembles.** Before turning to the investigation of the thermodynamical convergence towards the Kähler-Einstein volume form μ_{KE} it should be pointed out that *integer* powers of Slater determinants have been used before to model the *fractional Quantum Hall effect* [39]. More generally we note that the previous construction may be generalized by introducing general $k-$dependent powers β_k in the Slater determinant. To see this we come back to the general setting of an ample line bundle $L \to X$ and now fix a background metric h_0 on L and a volume form μ_0 on X. To this geometric data we associate the probability measure

$$\mu^{(N_k)} = \|\Psi\|^{\beta_k} \mu_0^{\otimes N}/\mathcal{Z}_{\mathcal{N}}.$$

on X^N for a fixed choice of parameters β_k. The case of $L = K_X$ considered above is obtained by setting $\beta_k = 2/k$, fixing any metric h_0 on K_X and then letting $\mu_0 = 1/h_0$, which defines a volume form on X. Then it is easy to see that all factors of h_0 cancel out leading to the previous canonical construction above. Finally, note that if one defines the *Hamiltonian*

$$H^{(N)} := -\log \|\Psi\|$$

then $\mu^{(N_k)}$ may be represented as a Boltzmann-Gibbs ensemble

$$\mu^{(N_k)} = e^{-\beta_k H^{(N)}} \mu_0^{\otimes N}/\mathcal{Z}_{\mathcal{N}}, \tag{1.6}$$

of a *classical* system in thermal equilibrium with an external heat bath of temperature $T_k = 1/\beta_k$. From this point of view $\mathcal{Z}_{\mathcal{N}}$ is the *partition function* of the system. It depends of the choice of bases (Ψ_I) in $H^0(X, L^{\otimes k})$ (but $\mu^{(N_k)}$ does not, as explained above). For example, the case when $\beta_k = 1$, 2 or 4 appears in the study of the Random Matrix ensembles associated to the classical groups (see [36] and references therein).

It is worth emphasizing that the Hamiltonian $H^{(N)}$ above is not a sum of pair interactions (even to the leading order) when $n > 1$. This is closely related to the fact that the mean field equations obtained in section 2.2 are fully non-linear and it makes the analysis of the thermodynamical limit rather challenging.

2. Convergence in the thermodynamical limit

It will be illuminating to consider the general setting of the previous setting with

$$\beta_k = \beta/k$$

for a fixed parameter β (where $\beta = 2$ appears in the canonical background free case 1.4). As will be clear this is, in a certain sense, a mean field limit. As explain above we hence fix the geometric data (h_0, μ_0) consisting of Hermitian metric h_0 on $L \to X$ and a volume form μ_0 on X. Given this data we furthermore fix a base (Ψ_I) in $H^0(X, L^{\otimes k})$, for any k, which is orthonormal with respect to Hilbert space structure on $H^0(X, L^{\otimes k})$ induced by (h_0, μ_0) :

$$\langle f, g \rangle_X := \int_X \langle f, g \rangle \mu_0,$$

where the point-wise Hermitian product in the integrand is taken with respect h_0. In particular, the corresponding partition function is then (a power of) the induced $L^{\beta/k}$ norm of the corresponding Slater determinant Ψ (formula 1.3):

$$\mathcal{Z}_N := \int_{X^N} \|\Psi\|^{\beta/k} \mu_0^{\otimes N}$$

To prove the convergence we will use the techniques of the *theory of large deviations.* In a nutshell this is a formalism which allows one to give a meaning to the statement that a given sequence of probability measure $\mu^{(N)}$on X^N is "exponentially concentrated on a deterministic macroscopic measure μ_* with a rate functional $I(\mu)$" (see [52] for an introduction to the theory of large deviations, due to Cramér, Varadhan and others, emphasizing the links to statistical mechanics - relations to functional integrals are emphasized in [30]) . Heuristically, the idea is to think of the large $N-$limit of the $N-$particle space X^N of "microstates" as being approximated by a space of "macrostates", which is the space $\mathcal{M}_1(X)$ of all probability measures on X :

$$X^N \sim \mathcal{M}_1(X),$$

as $N \to \infty$. The exponential concentration referred to above may then be heuristically written as

$$\mu^{(N)} := \rho^{(N)}(Z_1, ..., Z_N)dV(Z_1) \wedge \cdots dV(Z_N) \sim e^{-NF(\mu)}\mathcal{D}\mu, \tag{2.1}$$

where $\mathcal{D}\mu$ denotes a (formal) probability measure on the infinite dimensional space $\mathcal{M}_1(X)$ (more generally, the exponent N could be replaced by a *rate* a_N which is usually a power of N). Exponential concentration around μ_* appears when $F(\mu) \geq 0$ with μ_* the unique minimizer of F. Mathematically, the "change of variables" from X^N to $\mathcal{M}_1(X)$ is made precise by using the embedding

$$j_N : X^N \to \mathcal{M}_1(X), \;\; j_N(x, ..., x_N) := \frac{1}{N}\sum_i \delta_{x_i}$$

and then pushing forward $\mu^{(N)}$ to $\mathcal{M}_1(X)$ with the map j_N, giving a probability measure $(j_N)_*\mu^{(N)}$ on $\mathcal{M}_1(X)$ (i.e. the law of the random variable 1.2). The precise meaning of 2.1, in the sense of large deviations, is then that

$$\lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N} \log \int_{\mathcal{B}_\delta(\mu)} (j_N)_*\mu^{(N)} = -I(\mu),$$

integrating over a small ball $\mathcal{B}_\delta(\mu)$ of radius δ centered at $\mu \in \mathcal{M}_1(X)$ (using any metric on $\mathcal{M}_1(X)$ which is compatible with the weak topology).

The idea is now to establish the asymptotics 2.1 for a certain *free energy functional* $F(\mu)$ which is minimized precisely on a measure μ_* which equals the Kähler-Einstein measure μ_{KE} in the canonical case introduced in section 1.4. In fact, in this latter case the functional $F(\mu)$ will turn out to be naturally identified with Mabuchi's K-energy, which plays an important role in Kähler geometry (as explained in section 2.3)

To this end we will combine two already established asymptotics, concerning the the case when $\beta_k = 0$ and $\beta_k = 2$ respectively. In the first case it is a classical result going back to the work of Boltzmann (called Sanov's theorem in the mathematics literature) that the asymptotics 2.1 hold with $-F(\mu)$ equal to the relative entropy functional $S(\mu)$:

$$\mu_0^{\otimes N} \sim e^{NS(\mu)}\mathcal{D}\mu, \tag{2.2}$$

where

$$S(\mu) := -\int_X \log(\frac{\mu}{\mu_0})\mu (\leq 0) \tag{2.3}$$

if μ has a density with respect to μ_0 and otherwise $S(\mu) = -\infty$. This result gives a precise meaning to Boltzmann's notion of entropy as proportional to the logarithmic number (or volume) of microstates corresponding to a given macrostate.

Next, in the case when $\beta_k = 2$ it was shown very recently in [10] that

$$e^{-2H^{(N)}(x_1,...,x_N)}\mu_0^{\otimes N} \sim e^{-kNE(\mu)}\mathcal{D}\mu \tag{2.4}$$

In the present work we are interested in the intermediate asymptotic regime where $\beta_k = \beta/k$. Decomposing the corresponding probability measure $\mu^{(N)}$ as

$$\mu^{(N)} := (e^{-2H^{(N)}(x_1,\ldots,x_N)})^{\beta/2k}\mu_0^{\otimes N}$$

or more precisely as

$$\mu^{(N)} := \left[(e^{-2H^{(N)}(x_1,\ldots,x_N)}\mu_0^{\otimes N})^{\beta/2k}\right]\left[\cdot(\mu_0^{\otimes N})^{1-\beta/2k}\right]$$

we can, at least heuristically, combine the asymptotics 2.2 and 2.4 to get

$$\mu^{(N)} \sim e^{-N\beta(E(\mu)-\frac{1}{\beta}S(\mu))}\mathcal{D}\mu$$

(in order to be mathematically rigorous this heuristic argument needs to be complemented with precise estimates justifying the "interchange" of the large N and small $\delta-$limits)

The convergence of of the one-point correlation measures $\mu_1^{(N)}$ towards the minimizer μ_* of the functional

$$F(\mu) := E(\mu) - \frac{1}{\beta}S(\mu))$$

can now be shown by standard arguments (given the existence and uniqueness of μ_* which we will deal with in section 2.2). First note that the partition function may be asymptotically calculated as

$$\mathcal{Z_N} \sim \int_{\mathcal{M}_1(X)} \mathcal{D}\mu e^{-\beta N(E(\mu)-\frac{1}{\beta}S(\mu))},$$

giving

$$-\frac{1}{\beta N}\log \mathcal{Z_N} \to \inf_{\mu\in\mathcal{M}_1(X)}(E(\mu) - \frac{1}{\beta}S(\mu)$$

Next, note that upon performing an overall scaling of the original base (Ψ_I) we may assume that the infimum above vanishes. Now fix a smooth function ϕ on X and consider the functional

$$\mathcal{F}_N(u) := -\log\left\langle e^{-(\phi(x_1)+\ldots+\phi(x_N))}\right\rangle := -\log\int_{X^N} e^{-(\phi(x_1)+\ldots+\phi(x_N))}\mu^{(N_k)}$$

The following basic general exact variational identity holds

$$\frac{1}{\beta N}\frac{d\mathcal{F}_N(t\phi)}{dt}_{t=0} = \int_X \mu_1^{(N_k)}\phi \tag{2.5}$$

Arguing precisely as above and using the trivial asymptotics

$$e^{-(\phi(x_1)+\ldots+\phi(x_N))} \sim e^{-N\int_X \phi\mu}, \tag{2.6}$$

hence gives

$$\frac{1}{\beta N}\mathcal{F}_N(t\phi) \to \inf_{\mu\in\mathcal{M}_1(X)}(E(\mu) - \frac{1}{\beta}S(\mu) + t\int_X \phi\mu)$$

Finally, differentiating with respect to t gives

$$\frac{1}{\beta N}\frac{d\mathcal{F}_N(t\phi)}{dt}_{t=0} \to 0 + \int_X \mu_*\phi$$

and hence, using 2.5, we finally get

$$\mu_1^{(N)} \to \mu_*$$

Next, we will show that the minimizer μ_* can be obtained by solving a mean field type equation which will reduce to the Kähler-Einstein equation in the canonical case. We will start by explaining the notion of pluricomplex energy $E(\mu)$ appearing in the asymptotics 2.4.

2.1. **The pluricomplex energy $E(\mu)$.** Assume now that the fixed metric h_0 on L has positive curvature, i.e. its normalized curvature form is a Kähler form that we denote by $\omega(=\omega_0)$. Since, h_0 is uniquely determined up to scaling by its curvature form ω and since the probability measure $\mu^{(N)}$ is insensitive to scaling of h_0 we may as well say that the geometric data defining the β_k- ensemble is (ω_0, μ_0).

Now any Kähler metric which is cohomologous to ω (i.e. in the class $[\omega] = c_1(L)$) may by the $\partial\bar{\partial}-$lemma be written as

$$\omega_\phi := \omega + \frac{i}{\pi}\partial\bar{\partial}\phi$$

for a smooth function u. In this way the space of all Kähler metrics in $c_1(L)$ be identified with the space of *Kähler potentials*

$$\mathcal{H}_\omega(X) := \{\phi \in \mathcal{C}^\infty(X) : \omega_\phi > 0\}$$

modulo constants (we will usually mod out by $\mathbb{R}$ sometimes without mentioning it explicitly). Geometrically, the space $\mathcal{H}_\omega(X)$ may be identified with the space of *all positively curved Hermitian metrics on* L and ω_u with the (normalized) curvature form of the metric

$$h_\phi := e^{-2\phi}h_0$$

on L corresponding to ϕ (as follows immedaitely from formula 1.4).

Thanks to Yau's solution of the Calabi conjecture one can also associate potentials to *volume forms* on X. Indeed, to any volume form μ on X (which we will always assume normalized so that $\int_X \mu = 1$) there is a unique *potential* $\phi(:= \phi_\mu)$ in $\mathcal{H}_\omega(X)/\mathbb{R}$ such that

$$\frac{\omega_\phi^n}{Vn!} = \mu,$$

where V is the *volume* of any Kähler metric in the class $c_1(L)$. The equation involves the n :th exterior power of ω_ϕ and is hence a non-linear generalization of the inhomogeneous Laplace equation, called the inhomogeneous (complex) *Monge-Ampère equation* (and the left hand side above is called the *Monge-Ampère measure* of ϕ).

The previous equation can also be given a variational formulation by noting that there is a functional $\mathcal{E}_\omega$ (we will often omit the subscript ω) on the space $\mathcal{H}_\omega(X)$ such that its first variation is given by

$$\delta\mathcal{E}(\phi) := d\mathcal{E}_\phi = \frac{\omega_\phi^n}{Vn!}, \tag{2.7}$$

where $d\mathcal{E}$ is the differential of $\mathcal{E}(\phi)$ seen as a one-form on $\mathcal{H}_\omega(X)$. The functional $\mathcal{E}$ is uniquely determined by the normalization $\mathcal{E}(0) = 0$ (singled out by the fixed reference Kähler metric ω). This is a well-known functional in Kähler geometry which seems to first have been introduced by Mabuchi ((it is denoted by $-F_\omega$ in the book [50]; similar functionals also appeared in the works of Aubin and Yau). We will call it the *Monge-Ampère action*, since it physically appears as an action generalizing the *Liouville action*, as explained in section 2.1.2. It is straight-forward to obtain an explicit formula for $\mathcal{E}(\phi)$ by integrating along the line segment $t\phi$ for $0 \leq t \leq 1$ and get

$$\mathcal{E}_\omega(\phi) = \frac{1}{(n+1)!V} \int_X \phi \sum_{i=1}^{n} (\omega^{n-j} \wedge \omega_\phi^j), \tag{2.8}$$

but we will only make use of the defining property 2.7 in the following.

The functional $\mathcal{E}$ is (strictly) concave on $\mathcal{H}_\omega(X)/\mathbb{R}$ (with respect to the flat metric) [50] and hence the potential ϕ_μ may be characterized as the unique (mod $\mathbb{R}$) maximizer of the the functional

$$\phi \mapsto \mathcal{E}(\phi) - \langle \phi, \mu \rangle \, ,$$

expressed in terms of the usual pairing

$$\langle \phi, \mu \rangle := \int_X \phi\mu \tag{2.9}$$

Finally, we can now, following [15], define the *pluricomplex energy* $E(\mu)$ of the measure μ as

$$E(\mu) := \sup_{\phi \in \mathcal{H}_\omega(X)} \mathcal{E}(\phi) - \langle \phi, \mu \rangle = \mathcal{E}(\phi_\mu) - \langle \phi_\mu, \mu \rangle$$

The first equality in fact makes sense for *any* (possibly singular) measure μ in $\mathcal{M}_1(X)$ and one says that μ has finite energy if $E(\mu) < \infty$.

When $n = 1$ one may actually take the sup defining E over *all* $\phi \in \mathcal{C}^\infty(X)$ (i.e. without imposing the constraint $\omega_\phi > 0$). Then the convex functional $E(\mu)$ is, by definition, the Legendre transform of the concave functional $\mathcal{E}(\phi)$ on $\mathcal{C}^\infty(X)$ (with a non-standard sign convention). It turns out that in the case $n > 1$ the functional $E(\mu)$ can also be realized as a Legendre transform by extending $\mathcal{E}(\phi)$ to another (concave and one time differentiable) functional $\mathcal{F}_\infty$ on $\mathcal{C}^\infty(X)$ [15, 10, 13] (which appears in the general asymptotics 2.11 below). This fact is an important ingredient in the variational approach to complex Monge-Ampère equations introduced in [15].

Note that the energy functional E certainly depends on the choice of fixed Kähler metric ω. In fact, it is not hard to see that $E(\mu) \geq 0$ with equality precisely if $\mu = \omega^n/Vn!$. Indeed, it follows from general principles (concerning Legendre transforms) that

$$\inf_{\mu \in \mathcal{M}_1(X)} E(\mu) = E((\delta\mathcal{E})(0)) = E(\frac{\omega^n}{Vn!}) = \mathcal{E}(0) - 0 = 0$$

It would hence be more appropriate to call $E(\mu)$ the *relative energy of* μ.

2.1.1. *The Riemann surface case.* It may be illuminating to consider the case when $n = 1$, i.e. when X is a Riemann surface. Then $\mathcal{E}(\phi)$ coincides with the functional sometimes referred to as the *Liouville action* in the physics literature [18, 43]:

$$\mathcal{E}(\phi) = \frac{1}{2} \int_X \phi(\omega_\phi + \omega)$$

and hence, taking the potential ϕ_μ to be normalized so that $\int \phi_\mu \omega = 0$ we get

$$E(\mu) = -\frac{i}{2\pi} \int_X \phi_\mu \partial\bar{\partial}\phi_\mu = \frac{i}{2\pi} \int \partial\phi_\mu \wedge \bar{\partial}\phi_\mu \tag{2.10}$$

which is essentially the usual electrostatic energy of the continuous charge distribution μ in the neutralizing background charge $-\omega$. Equivalently, if we define the

Green function $g(x,y)$ for the scalar Laplacian $\Delta = \omega^{-1}\frac{i}{\pi}\partial\bar{\partial}$ on X by the properties $g(x,y) = g(y,x)$ and

$$\frac{i}{\pi}\partial_x\bar{\partial}_x g(x,y) = \delta_x(y) - \omega(y), \quad \int_X g(x,y)\omega(y) = 0$$

then we have $\phi_\mu(x) = \int_X g(x,y)(\mu)(y)$ and hence

$$E(\mu) = -\frac{1}{2}\int_{X\times X} g(x,y)\mu(x)\otimes\mu(y).$$

2.1.2. *The asymptotics 2.4 for $\beta_k = 2$ and effective bosonization.* Let us briefly explain the idea behind the large deviation asymptotics 2.4 proved in [10]. The starting point is the basic observation that when $\beta_k = 2$ the one point correlation function $\rho_1^{(N)}$ can be represented as a *density of states function:*

$$\rho_1^{(N)}(Z) = \sum_{I=1}^{N} \|\Psi_I(Z)\|^2$$

(called the *Bergman kernel at the diagonal* in the mathematics literature). By a fundamental result of Bouche and Tian the leading asymptotics of the corresponding one point correlation measure are given by the Monge-Ampère measure:

$$\mu_1^{(N)} \to \omega^n/Vn!$$

(see [58] for a survey of Bergman kernel asymptotics and [23] for a physical point of view) Now using these asymptotics and perturbing by potentials ϕ in $\mathcal{H}_\omega(X)$ one can reverse the arguments used in the end of section 2 and get

$$\frac{1}{N}\mathcal{F}_N(\phi) \to \mathcal{E}(\phi), \quad \phi \in \mathcal{H}_\omega(X)$$

using the variational property of $\mathcal{E}$. Then an argument involving Legendre transforms gives the large deviation asymptotics 2.4, using that $E(\mu)$ can be realized as an (infinite dimensional) Legendre transform of $\mathcal{E}(\phi)$. More precisely, the argument uses the convergence of (perturbed) free energies

$$\frac{1}{N}\mathcal{F}_N(\phi) \to \mathcal{F}_\infty(\phi), \quad \phi \in \mathcal{C}^\infty(X) \tag{2.11}$$

for *any* smooth function ϕ (not necesserly with $\omega_\phi \geq 0$) for a certain functional $\mathcal{F}_\infty$ on $\mathcal{C}^\infty(X)$, whose Legendre transform is $E(\mu)$. The key point, as shown in [13], is that $\mathcal{F}_\infty$ is one time differentiable on $\mathcal{C}^\infty(X)$, which hence establishes the absence of a phase transition with respect to perturbations of ϕ for the $\beta_k = 2-$ensemble.

Incidentally, the large deviation asymptotics 2.4 can, from a physical point of view, be interpreted as an *effective bosonization* of a fermionic quantum field theory on X (but is should be pointed out that this is only an interpretation: no bosonization is actually used in the derivation 2.4 as explained above). In other words, the collective theory of N fermions is effectively described by a bosonic field theory, as $N \to \infty$. To see this first recall the representation of the Slater determinant 1.3 as a functional integral over Grassman fields

$$\|\Psi(x_1,...,x_N)\|^2 = C_N \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{-S_{ferm}(\Psi,\bar{\Psi})}\|\Psi(x_1)\|^2\cdots\|\Psi(x_1)\|^2, \tag{2.12}$$

integrating of over all complex spinors, i.e. smooth sections of the exterior algebra $\Lambda^{0,*}(T^*X)\otimes L^{\otimes k}$ and where $S_{ferm}(\Psi,\bar{\Psi})$ is the fermionic action

$$S_{ferm}(\Psi,\bar{\Psi})=\int_X \langle D_L\Psi,\Psi\rangle\,\mu_0,$$

expressed in terms of the Dirac operator D_L on $\Lambda^{0,*}(T^*X)\otimes L^{\otimes k}$ induced by the complex structure J, i.e. $D_L=\overline{\partial}+\overline{\partial}^*$ (see [18] for the Riemann surface case). The integer N is the dimension of the space of zero-modes of D_L on $\Lambda^{0,*}(T^*X)\otimes L^{\otimes k}$ which coincides with $H^0(X,L^{\otimes k})$ due to Kodaira vanishing in positive degrees (when $k>>1$). Moreover, the constant C_N in 2.12 can be expressed in terms of determinants of $(0,q)-$Laplacians coupled to $L^{\otimes k}$, i.e. as the Ray-Singer analytic torsion of the complex $\Lambda^{0,*}(T^*X)\otimes L^{\otimes k}$ [9].

Next, we express the right hand side in the asymptotics 2.4 as a *bosonic* path integral with action $S_{bos}(\phi)=-\mathcal{E}(\phi)$ and with insertions of $e^{-\phi(x)}$:

$$e^{kNE(\mu)}\sim\left(\int_{\mathcal{H}_\omega(X)}\mathcal{D}\phi e^{kN\mathcal{E}_\omega(\phi)}e^{-k\phi(x_1)}\cdots e^{-k\phi(x_N)}\right), \tag{2.13}$$

using 2.6 and that the integral may, to the leading order, by estimated by it largest value, which by definition is precisely the exponential of $kNE(\mu)$.

Combining 2.12 and 2.13 the asymptotics 2.4 can now be interpreted by the effective bosonization rule that insertion of $\|\Psi(x)\|^2$ in the fermionic path integral is effectively equivalent to insertion of $e^{-k\phi(x)}$ in the bosonic path integral, up to inversion of the bosonic functional integral. This asymptotic fermion-boson equivalence is strongly reminiscent of the exact equivalence established in [18, 53, 55] when $n=1$, i.e. when X is a Riemann surface. However, there are several differences that should be pointed out:

- *When the genus of X is at least one further "solotonic" terms have to be added to the action $-\mathcal{E}(k\phi)$.* These terms are lower-order in k and hence do not contribute to the asymptotic equivalence (this is closely related to the fact that ϕ is assumed to be circle valued in 2.4, i.e. its values are only defined mod 2π)
- *The insertion of $\|\Psi(x)\|^2$ is equivalent to insertion of $e^{ik\phi(x)}$ and there is no inversion.* In fact, in the case when $n=1$ we could as well have inserted $e^{i\phi(x)}$ in the functional integral 2.13 and applied a stationary phase approximation. This would have given the asymptotics $e^{-kNE(\mu)}$ directly and there would have been no need to invert the final expression. This is a consequence of analytic continuation. Indeed, $\mathcal{E}_{t\omega}(tu)$ is homogeneous of degree $n+1$ in t and hence applying the previous argument with $e^{-k\phi(x_1)}$ replaced by $e^{kt\phi(x_1)}$ and ω replaced by $t\omega$ gives the asymptotics

$$e^{t^2kNE(\mu)}\sim\int_{\mathcal{H}_\omega(X)}\mathcal{D}\phi e^{kN\mathcal{E}_{t\omega}(\phi)}e^{-ikt\phi(x_1)}\cdots e^{-ikt\phi(x_N)},$$

Finally, setting $t=-i$ (assuming that analytic continuation is valid) gives

$$e^{-kNE(\mu)}\sim\int_{\mathcal{H}_\omega(X)}\mathcal{D}\phi e^{kN\mathcal{E}_{-i\omega}(\phi)}e^{ik\phi(x_1)}\cdots e^{ik\phi(x_N)}, \tag{2.14}$$

where

$$\mathcal{E}_{-i\omega}(\phi)=\frac{1}{2}\int_X\phi(\frac{i}{\pi}\partial\bar{\partial}\phi)+\int_X\phi(-i)\omega),$$

Note that the first term above is real and equal to $-\int |\nabla\phi|^2\omega$, while the second term is *imaginary.* This decomposition hence corresponds (up to real-valued normalization factors) to the decomposition of the action as $S_{bos} = S_1 + S_2$ in [18] (formula 3.9).

- *One integrates over* all $\phi \in \mathcal{C}^\infty(X)$ *and not only the subset* $\mathcal{H}_\omega(X)$, *i.e. without the constraint* $\omega_\phi > 0$. In fact, when $n = 1$ we could as well have integrated over all $\phi \in \mathcal{C}^\infty(X)$ in 2.13. The reason is that, when $n = 1$, the sup defining $E(\mu)$ may be taken over all of $\mathcal{C}^\infty(X)$ without changing the maximal value. Basically, this follows from the fact that a stationary point anyway satisfies $\omega_u = \mu > 0$.

Note that combining the comments in the last two points above the bosonic functional integral with $t = -i$ becomes, in the limit when we replace the points $(x_1, ..x_N)$ with a measure μ,

$$\int \mathcal{D}\phi e^{-k^2\|\nabla\phi\|^2\omega} e^{ik\langle\mu-\omega,\phi\rangle}$$

which can be interpreted as an infinite dimensional *Fourier transform* of the *Gaussian free field.* Recalling that $\mu - \omega = \frac{i}{\pi}\partial\bar{\partial}\phi_\mu$ the asymptotics 2.14 then amounts to the basic fact that Fourier transforms preserve Gaussian functions (compare formula 2.10).

As for the case when $n > 1$ it seems rather intruging that insertion of $e^{ik\phi(x_1)}$, as discussed in the second point above, gives a boson-fermion equivalence (at least effectively) precisely when $n = 1$ mod 4 (so that $(-i)^{n+1} = -1$) and hence the next case after $n = 1$ appears when $n = 5$, i.e. when the real dimension of X is ten. In conclusion, it would be interesting to better understand the differences between the case when $n = 1$ and higher dimensions.

2.2. **The minimizer μ_* of the free energy functional $F(\mu)$ and mean field equations.** Recall that the free energy functional $F(\mu)$ (for a fixed parameter $\beta > 0$) on the space$\mathcal{M}_1(X)$ of probability measure son X is defined by

$$F(\mu) := E(\mu) - \frac{1}{\beta}S(\mu)$$

where E is the energy functional define in the previous section and $S(\mu)$ is the relative entropy 2.3. It follows from basic duality arguments that F is strictly convex on $\mathcal{M}_1(X)$ (or rather on the subset where F is finite) and hence admits at most one minimizer. Next we note that μ is a critical point for $F(\mu)$ on $\mathcal{M}_1(X)$ if and only if

$$-\phi_\mu + \frac{1}{\beta}\log(\mu/\mu_0) - Z_\mu = 0,$$

where ϕ_μ is the potential of μ and Z_μ is a normalizing constant. Indeed, using the defining properties of $\mathcal{E}$ and E respectively one obtains (by basic Legendre transform considerations) that

$$\delta E(\mu) = -\phi_\mu$$

as a one-form on the infinite dimensional submanifold $\mathcal{M}_1(X)$ of the vector space $\mathcal{M}(X)$ of all signed measures. Moreover, a simple calculation gives

$$\delta S(\mu) = -\log(\mu/\mu_0) + Z_\mu,$$

where Z_μ is a normalizing constant (coming from the constraint $\int_X \mu = 1$). Combining these two variational formulas gives

$$\delta F(\mu) = -\log(\mu/\mu_0) - \phi_\mu \tag{2.15}$$

up to a normalizing constant. In other words, μ is a critical point for for $F(\mu)$ on $\mathcal{M}_1(X)$ if and only if its potential ϕ solves the following non-linear partial differential equation:

$$\frac{\omega_\phi^n}{Vn!} = \frac{e^{\beta\phi}\mu_0}{Z_\phi} \tag{2.16}$$

As follows from a simple modification of the proof of the Aubin-Yau theorem there is a unique $\phi \in \mathcal{H}_\omega(X)/\mathbb{R}$ solving this equation (crucially using that $\beta > 0$) which by strict convexity is hence the unique maximizer of the free energy functional F. It is sometimes convenient to fix the normalization of the solution ϕ above by imposing that

$$\int_X e^{\beta\phi}\mu_0 = 1,$$

i.e. $\phi \in \mathcal{H}_\omega(X)$ is the *unique* solution to

$$\frac{\omega_\phi^n}{Vn!} = e^{\beta\phi}\mu_0 \tag{2.17}$$

It should be pointed out that when $n = 1$ the previous equation is often called the *mean field equation* [17, 37] and accordingly we will call it the *mean field Monge-Ampère equation* for a general dimension n.

Finally, coming back to the canonical case when $\beta = 2$ and $L = K_X$ we take, as explained in section 1.4 the geometric data (ω, μ_0) such that ω is the curvature form of the metric on K_X defined by the inverse $1/\mu_0$. This means that $\mu_0 = e^{2f_\omega}\omega^n/Vn!$, where f_ω is the *Ricci potential, i.e.*

$$\frac{i}{\pi}\partial\bar{\partial}f_\omega = \omega + \mathrm{Ric}\omega,\ \int_X e^{2f_\omega}\omega^n/Vn! = 1$$

Then the corresponding Monge-Ampère mean field equation reads

$$\omega_\phi^n = e^{2\phi}e^{2f_\omega}\omega^n$$

Hence, the solution ϕ is such that the Kähler metric ω_ϕ satisfies

$$\mathrm{Ric}\omega_\phi = -\omega_\phi,$$

i.e. ω_ϕ is a Kähler-Einstein metric with negative Ricci curvature. Coming back to the convergence of the one-correlation measures in the thermodynamical limit, considered in section 2, this means that the limiting measure μ_* indeed equals $\mu_{KE} := \omega_{KE}^n/Vn!$.

2.3. **Duality and relation to the Yau-Tian-Donaldson program and balanced metrics.** In this section we will briefly point out some relations to the influential Yau-Tian-Donaldson program in Kähler geometry [25, 26, 50, 44]. In a nutshell the idea of this program is to approximate Kähler-Einstein metrics (and more general extremal metrics), by a limit of finite dimensional objects of an algebro-geometrical nature. There are various versions of this program, but the one which is most relevant for the present paper is Donaldson's notion of *canonically balanced metrics* introduced in [27], which is particularly adapted to Kähler-Einstein metrics (as opposed to general extremal metrics).

To highlight the similarities let us first formulate a more general "$\beta-$analogue" of Donaldson's setting, starting with an ample line bundle $L \to X$. The main point is to replace the infinite dimensional space $\mathcal{H}_\omega(X)$ of Kähler potentials in $c_1(L)$ with its *quantization at level* k. This latter space, denoted by $\mathcal{H}_k$, is the space of all Hermitian metrics on the finite dimensional vector space $H^0(X, L^{\otimes k})$. Upon fixing a reference metric $\mathcal{H}_k$ is hence isomorphic to the symmetric space $GL(N_k)/U(N_k)$ of all Hermitian $N_k \times N_k$- matrices. There is a natural injection defined by the *Fubini-Study map* $FS_k(H)$ at level k :

$$FS_k : \mathcal{H}_k \to \mathcal{H}_\omega(X), \quad FS_k(H)(x) := \frac{1}{k} \log \sum_{I=1}^{N} \|\Psi_I(x)\|^2 , \tag{2.18}$$

expressed in terms of the point-wise norms with respect to the fixed metric $h_0^{\otimes k}$ on $L^{\otimes k}$ of a base (Ψ_I) in $H^0(X; L^{\otimes k})$ which is orthonormal with respect to H. Moreover, for any given β we may define a map in the reversed direction that we will call $Hilb_{k,\beta}$:

$$Hilb_{k,\beta} : \mathcal{H}_\omega(X) \to \mathcal{H}_k$$

defined as follows: $Hilb_{k,\beta}(\phi)$ is the Hermitian product (or equivalently, Hilbert norm) on $H^0(X, L^{\otimes k})$ defined by

$$\langle f, g \rangle_{Hilb_{k,\beta}(\phi)} := \int_X \langle f, g \rangle e^{-k\phi} e^{\beta\phi} \mu_0$$

(note that $\langle \cdot, \cdot \rangle e^{-k\phi}$ is the Hermitian metric on $L^{\otimes k}$ *naturally* associated to $\phi \in \mathcal{H}_\omega(X)$ and the remaining factor $e^{\beta\phi}\mu_0$ should be thought of as a specific choice of integration element depending on ϕ). An element H_k in $\mathcal{H}_k$ will be said to be *$\beta-$balanced at level* k with respect to (ω, μ_0) if is is a fixed point under the composed map

$$T_{k,\beta} := Hilb_{k,\beta}(\phi) \circ FS_k : \ \mathcal{H}_k \to \mathcal{H}_k. \tag{2.19}$$

on $\mathcal{H}_k$. Equivalently, this means that H_k is a critical point of the following functional $\mathcal{G}_k$ on $\mathcal{H}_k$:

$$\mathcal{G}_k(H) := -\frac{1}{kN} \log \det(H) - \frac{1}{\beta} \log \int_X e^{\beta FS_k(H)} \mu_0,$$

(after normalization). Repeating the arguments in the proof of Theorem 7.1 in [15] concerning the canonical case when $L = K_X$ ((the case referred to as S_+ in [15]) essentially word for word, one obtains the existence and uniqueness of a $H_k \in \mathcal{H}_k$ which is *$\beta-$balanced at level* k with respect to (ω, μ_0) and such that

$$FS_k(H) \to u_\beta,$$

in $\mathcal{H}_\omega(X)$ when $k \to \infty$ (or equivalently, $N \to \infty$) where u_β is the unique solution of the Monge-Ampère mean field equation 2.17, assuming $\beta > 0$. As explained in [15] the main point of the proof is to show that any limit point in $\mathcal{H}_\omega(X)/\mathbb{R}$ of the sequence $FS_k(H)$ is a maximizer of the following functional on $\mathcal{H}_\omega(X)$:

$$\mathcal{G}(\phi) := \mathcal{E}(\phi) - \frac{1}{\beta} \log \int_X e^{\beta\phi} \mu_0,$$

whose critical points are precisely the solutions of the Monge-Ampère mean field equation 2.16. Note that the functional $\mathcal{G}$ is invariant under the natural action by

$\mathbb{R}$, $\phi \to \phi + c$ and hence maximizing the functional

$$\mathcal{E}(\phi) - \frac{1}{\beta}\int_X e^{\beta\phi}\mu_0 \tag{2.20}$$

picks out the maximizers of $\mathcal{G}$ which satisfies the normalization

$$\int_X e^{\beta\phi}\mu_0 = 1$$

In the Riemann surface case the functional 2.20 with the exponential term is also sometimes referred to as the *Liouville action* (it appears for example in Polyakov's functional integral quantization of the bosonic string, further developed in [43])

To see the relation to the $\beta-$ensembles introduced in section 1.5 and their thermodynamical limit one should keep in mind the basic linear duality between *functions* ϕ and *measures* μ defined by the basic pairing 2.9 In turn, this pairing induces, using the Legendre transform a non-linear duality between *convex functionals* of ϕ on one hand and convex functionals of μ, on the other.

The roles of the spaces $\mathcal{H}_\omega(X)$ and $\mathcal{H}_k(X)$ are now played by the space $\mathcal{M}_1(X)$ and $\mathcal{M}(X^{N_k})$, respectively, where $\mathcal{M}_{N_k}(X)$ denotes the space of all symmetric probability measures on the product X^N (i.e. all N_k-particle random point processes on X). The analogue of the Fubini-Study map 2.18 is the map

$$\mathcal{M}(X^{N_k}) \to \mathcal{M}_1(X), \ \ \mu_N \mapsto (\mu_N)_1 := \left\langle \frac{1}{N}\sum_i \delta_{x_i} \right\rangle,$$

sending a random point process to its one-point correlation measure. Finally, the role of a $\beta-$balanced metric is now played by the measure $\mu^{(N_k)} \in \mathcal{M}(X^{N_k})$ defining the $\beta-$ensemble with N_k particles, which was expressed as a Boltzmann-Gibbs ensemble with Hamiltonian $H^{(N_k)}$ in formula 1.6. The point is that $\mu^{(N_k)}$ can also be defined by a variational principle. Indeed, by the $N-$particle Gibbs principle for canonical ensembles $\mu_{(N_k)}$ is the unique minimizer of the $N-$particle mean free energy functional on $\mathcal{M}(X^{N_k})$:

$$F^{(N)}(\mu_N) = \frac{1}{N}\int_{X^N} \mu_N H^{(N)} - \frac{1}{N}S(\mu_N, \mu_0^{\otimes N})$$

i.e. the difference between *mean energy* and *mean entropy.* There is also an analogue of the definition of a balanced metric as a fixed point of the map $T_{k,\beta}$ above. Indeed, it is well-known that any Gibbs-Boltzmann measure can be uniquely determined as a stationary state for a stochastic process μ_t on X^N defined by suitable Glauber (or Langevin) dynamics, but we will not develop this point of view here.

Interestingly, performing a Legendre transform of each of the two convex functionals on $\mathcal{M}_1(X)$ summing up to the free energy functional $F(\mu)$ (i.e. the pluricomplex energy $E(\mu)$ and minus entropy $-\frac{1}{\beta}S(\mu)$) yields a functional on $\mathcal{H}_\omega(X)$ which is nothing but the functional G above:

$$F = E + (-\frac{1}{\beta}S), \ \ G = E^* + (-\frac{1}{\beta}S)^*$$

It should be pointed out that in the canonical case (where the critical points of the functionals are Kähler-Einstein metrics) the two functionals F and G have already appeared in Kähler geometry from a different point of view. For example, the limiting free energy functional $F(\mu)$ on $\mathcal{M}_1(X)$ may be identified with Mabuchi's $K-$energy ν of a Kähler metric in $c_1(K_X)$:

$$F(\omega_\phi^n/Vn!) = \nu(\omega_\phi) \tag{2.21}$$

The functional ν was first introduced by Mabuchi as the functional on $\mathcal{H}_\omega(X)$ whose gradient with respect to the Mabuchi-Semmes-Donaldson Riemannian metric on $\mathcal{H}_\omega(X)$ is the scalar curvature minus its average [50, 44]. But this is easily seen to be equivalent to the variational property 2.15 of F and hence F and ν coincide under the identification above. The explicit formula for ν obtained from the identification 2.21 is in fact equivalent to an explicit formula for ν of Tian and Chen [50]. Moreover, the functional $-G$ coincides with the Ding-Tian functional [50].

Using Legendre transforms as above one arrives at new proofs and generalizations of various useful results in Kähler geometry. For example, it follows from general principles that

$$\inf_{\mu\in\mathcal{M}_1(X)} F(\mu) = \inf_{\phi\in\mathcal{H}_\omega(X)} (-G)(\phi)$$

which in the "canonical case" was first shown by Bando-Mabuchi [6]. These relations and extensions to negative β will be further investigated elsewhere. In the latter case the functional $F(\mu)$ may not be bounded from below when β is too large, which in the Kähler-Einstein case is closely related to lack of $K-$stability.

Finally, it seems worth pointing out that in the case when $\beta = 0$ the notion of balanced metrics still makes sense and was studied by Donaldson in [27] with a particular emphasize on the case when X is a Calabi-Yau form. Then μ may be canonically chosen as $i^{n^2}\Omega\wedge\bar{\Omega}/\int i^{n^2}\Omega\wedge\bar{\Omega}$ where Ω is non-vanishing holomorphic $n-$form on X and the curvature forms of the balanced metrics at level k then converge to the unique Ricci flat metric in $[\omega]$, whose existence was established in Yau's proof of the Calabi conjecture. Relation between these balanced metrics on Calabi-Yau manifolds and black holes were considered in [24]. However, in the case when $\beta = 0$ the $\beta-$ensembles introduced in the present work appear to be less interesting: they are pure Poisson processes without any connections to fermions. The case when β is *negative* is briefly discussed below in connection to Kähler-Einstein metrics with positive Ricci curvature.

2.4. **Conclusion and discussion.** To a given a compact manifold with a fixed integrable complex structure J we have associated a canonical $N-$particle free fermion gas whose one-particle correlation measures converge in the thermodynamical (large N) limit to the volume form of the Kähler-Einstein metric ω_{KE} associated to (X, J). More precisely, it was assumed that the canonical line bundle K_X be positive (i.e. ample), which corresponds to ω_{KE} having *negative* Ricci curvature and the one-particle quantum state space of the fermion gas was taken as the N_k-dimensional space $H^0(X, K_X^{\otimes k})$ of global holomorphic sections on X with values in $K_X^{\otimes k}$. The argument in fact gave precise exponentially small fluctuations around the Kähler-Einstein volume forms with a rate function F naturally identified with Mabuchi's $K-$energy (which plays an important rule in Kähler geometry). The convergence in the thermodynamical limit was obtained by introducing an *auxiliary* background Kähler form ω in the first Chern class $c_1(K_X)$ (or equivalently a metric on K_X) also determining a volume form on X. This lead to a decomposition of the rate functional F as

$$F(\mu)=E(\mu)-S(\mu),$$

(with both terms depending on the choice of ω). In terms of the statistical mechanics of a classical canonical Boltzmann-Gibbs ensemble E and S appeared as the limiting *mean energy* and *mean entropy,* respectively. Minimizing F then gave an equation of mean field type whose unique solution is given by the Kähler-Einstein volume form. An interpretation of the energy E in terms of *effective bosonization* was also given.

Heuristically, it seems that one could interpret the thermodynamical limit above as saying that the Kähler-Einstein metric emerges from a *fermionic maximum entropy principle*: the particles try to maximize their entropy (i.e. volume in configuration space) under the constraint that they behave as fermions and hence, according to Pauli's exclusion principle, cannot occupy the same space leading to an effective repulsion.

It would be interesting to understand the physical relevance of these results. For example, it would be useful to understand the role of the background complex structure J. One could try to work immediately on the universal space $\mathcal{X}$ over the moduli space $\mathcal{J}$ of all complex structures J on X [29], i.e. $\mathcal{X}$ is a holomorphic fibration over $\mathcal{J}$ such that the fiber over $[J]$ is simply (X,J). Then specifying a particular complex structure J is somewhat similar to the choice of a conformal gauge in Hawking's functional integral for Euclidean quantum gravity (see section 5 in [32] and also the very recent paper [33] of t' Hooft). One could ask (rather speculatively) whether the integral over the configuration spaces X^N of increasing dimension can be seen as a canonical regularization of a functional integral describing quantum fluctuations around Kähler-Einstein metrics. This could be compared with approach of CDT (causal dynamical triangulations); see [3] and references therein, but in the present case one only consider fluctuating Riemannian metrics which are Kähler with respect to some complex structure.

The precise mathematical details of the thermodynamical convergence will be investigated elsewhere, as well as the case when K_X is negative (so that any Kähler-Einstein metric must have positive Ricci curvature). In the latter case there are well-known obstructions to the existence of Kähler-Einstein metrics and the Yau-Tian-Donaldson program aims at showing that all obstruction may be formulated in terms of algebro-geometric stability (such as $K-$stability). From the point of view of the present paper this is related to the fact that the natural candidate for the $N-$particle ensemble in the case when K_X is negative may be formally written as

$$\mu^{(N)}=(\Psi_1(x_1)\wedge\bar{\Psi}_1(x_1)\wedge\cdots\wedge\Psi_N(x_N)\wedge\bar{\Psi}_N(x_N))^{-1/k}/\mathcal{Z_N}$$

where now (Ψ_I) is a base in $H^0(X,(K_X^{-1})^{\otimes k})$. However, because of the negative exponent above this singular volume form on X^N is usually non-integrable, due to singularities appearing when to points merge. But fixing an auxiliary metric on K_X^{-1} one can look at the $\beta-$ensemble

$$\mu_\beta^{(N)}=\left\|\Psi_1\wedge\bar{\Psi}_1\wedge\cdots\wedge\Psi_N\wedge\bar{\Psi}_N\right\|^{-k\gamma/2}/\mathcal{Z_N}$$

(with $\beta=-\gamma$ *negative*) which is integrable for γ sufficiently small. In fact, it can be shown that the measure is integrable as long as $\gamma<\alpha_X$, where α_X is Tian's $\alpha-$*invariant* [50] (also called the log canonical threshold in algebraic geometry).

As shown by Tian a sufficient (but not necessary) criterion for the existence of a Kähler-Einstein metric with positive Ricci curvature is that

$$\alpha_X > n/(n+1)$$

(and similarly for the more useful equivariant versions of α_X). In the case of complex surface (i.e. $n = 2$) this leads to a complete classification of Fano manifolds admitting a Kähler-Einstein metric [48, 49].

If Tian's condition above holds it seems natural to expect the one-point correlation measure of $\mu_\beta^{(N)}$ to converge to the corresponding Kähler-Einstein volume form when first $N \to \infty$ and then $\beta \to -2$ (at least when there are no holomorphic vector fields on X, so that the Kähler-Einstein metric is unique [6]). The simplest case appears when X is the Riemann sphere. Then the corresponding ensemble is explicitly given by a one component plasma (or equivalently a point vortex system) studied by Kiessling in [38] where a first order phase transition appears at $\beta = -2$. Kiessling also considered generalizations in other directions than the one explored here, namely to the conformal geometry of higher dimensional spheres where the Hamiltonian, where the Hamiltonian $H^{(N)}$ in is a sum of logarithmic pair interaction. As a consequence the corresponding mean field equations are quasi-linear (with the non-linearity coming from the exponential term), as opposed to the present setting where the fully non-linear Monge-Ampère operator appears.

Finally, it would also be interesting to detail the Glauber (Langevin) stochastic dynamics alluded to in the previous section and investigate a suitable "hydrodynamical" scaling limit (see for example the recent paper [22] for relations between Langevin dynamics and field theories with holomorphic factorization and supersymmetry). This should lead to a deterministic heat-equation type flow on the space of all (smooth) probability measures $\mathcal{M}_1(X)$, converging towards the Kähler-Einstein volume form. In fact, a "dual" (in the sense of the previous section) scaling limit of Donaldson's iteration of the map $T_{k,\beta}$ 2.19 was shown to converge to the Kähler-Ricci flow in [11].

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