

OPTIMAL DIVIDEND AND REINSURANCE STRATEGY OF A PROPERTY INSURANCE COMPANY UNDER CATASTROPHE RISK

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ABSTRACT. We consider an optimal control problem of a property insurance company with proportional reinsurance strategy. The insurance business brings in catastrophe risk, such as earthquake and flood. The catastrophe risk could be partly reduced by reinsurance. The management of the company controls the reinsurance rate and dividend payments process to maximize the expected present value of the dividends before bankruptcy. This is the first time to consider the catastrophe risk in property insurance model, which is more realistic. We establish the solution of the problem by the mixed singular-regular control of jump diffusions. We first derive the optimal retention ratio, the optimal dividend payments level, the optimal return function and the optimal control strategy of the property insurance company, then the impacts of the catastrophe risk and key model parameters on the optimal return function and the optimal control strategy of the company are discussed.

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1. Introduction

In this paper we consider a property insurance company in which the dividend payments process and risk exposure are controlled by the management. The property insurance business brings in catastrophe risk,

such as earthquake and flood. We assume that the company can only reduce its risk exposure by proportional insurance strategy for simplicity. The catastrophe risk could also be partly reduced by reinsurance. The regulation of the catastrophe risk determines to what extent the catastrophe risk could be eliminated, here we use reinsurance rate and adjusted risk rate in the regulation. We equate the value of the company to the expected present value of the dividend payments before bankruptcy.

This is a mixed singular-regular control on diffusion models with jumps. These optimization problems of diffusion models for property insurance companies that control their risk exposure by means of dividend payments have attracted significant interests recently. We refer readers to Radner and Shepp [22], Paulsen and Gjessing [21], Højgaard and Taksar[17, 19] and Asmussen[2]. Optimizing dividend payments is a classical problem starting from the early work of Borch[6, 7], Gerber[10]. For some applications of control theory in insurance mathematics, see Højgaard and Taksar[16, 18], Martin-löf[20], Asmussen and Taksar[4, 9] and He and Liang[13, 14, 15], Basse, Reddemann, Riegler and Schulenburg[8], Guo, Liu and Zhou [11] and other author's work. Recent surveys can be found in Taksar[23], Avanzi [3], Albrecher and Thonhauser[1].

Unfortunately, there is little work concerned with the catastrophe risk of the property insurance company in the problem of optimal risk control/dividend distribution via the reinsurance rate. In the real financial market, the property insurance business generally brings in catastrophe risk, such as earthquake and flood. The asset of the company evolves as a lévy process with jump diffusions. Harrison and Taksar [12] provides a good idea to solve this kind of problems. Bernt Øksendal and Agnès Sulem[5] study the stochastic control problem of jump diffusions. Enlightened by these innovative ideas, we can solve effectively the optimal control problem of the company under catastrophe risk. Firstly, we establish the control problem of the Lévy processes with jump diffusions which is a realistic model of the property insurance company facing catastrophe risk. Then we work out the solution of singular-regular control of the jump diffusions, that is, we establish the optimal return function, the optimal reinsurance rate and the optimal dividend strategy of the insurance company. Finally we study the impacts of some key model parameters on the optimal return function and the optimal dividend strategy.

The paper is organized as follows: In next section, we establish the mathematical control model of the insurance company facing catastrophe risk. In section 3, we work out a solution of HJB equations associated with the singular-regular control on Lévy processes with jump diffusions. In section 4, we establish the solution of the optimal control problem, i.e., we derive the optimal return function, the optimal reinsurance rate and the optimal dividend strategy of the property insurance company. In section 5, we use numerical calculations to discuss the influences of the key model parameters on the optimal retention ratio, the optimal dividend payments level, the optimal return function and optimal control strategy of the company. In section 6, we summarize main results of this paper.

2. Mathematical model with proportional reinsurance strategy under catastrophe risk

In this paper, we consider a property insurance company with proportional reinsurance strategy. The property insurance business brings in catastrophe risk, such as earthquake and flood. The catastrophe risk could only be partly reduced by reinsurance. The company's management can accommodate the profit and the risk by choosing dividend payments process and reinsurance rate.

The asset of the company evolves as the Lévy processes with jump diffusions. In this model, if there is no dividend payments and only the proportional reinsurance strategy is used to control the risk, then the asset of the property insurance company is approximated by the following processes (see Øksendal and Sulem[5]),

$$dR_t = \mu a(t)dt + \sigma a(t)dW_t + ka(t) \int_{\mathfrak{R}} z \tilde{N}(dt, dz),$$

where W_t is a standard Brownian motion, μ is the premium rate, and σ^2 is the volatility rate, it is a normal description of the property insurance company. $1 - a(t) \in [0, 1]$ is the proportional reinsurance rate. $\tilde{N}(dt, dz) = N(dt, dz) - I_{\{|z| < R\}} \nu(dz)dt$ is the compensated Poisson random measure of Lévy process $\{N_t\}$ with finite Lévy measure ν . The jump diffusions stand for the catastrophe risk produced by earthquake and flood in the property insurance business. The catastrophe risk could

be partly reduced by reinsurance strategy. Since the catastrophe risk is huge, the reinsurance strategy is not the same as the normal reinsurance. Denote k as the adjusted risk rate according to the reinsurance regulation of the catastrophe risk. k is a constant. Throughout this paper we assume that $k \in (0, \frac{\mu}{2 \int_{\mathfrak{R}} z \nu(dz)}]$, which ensures that the company does not go into bankruptcy as soon as the catastrophe risk appears.

To give a mathematical foundation of the optimization problem, we fixed a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\{W_t\}$ is a standard Brownian motion, $\tilde{N}(dt, dz) = N(dt, dz) - I_{\{|z| < R\}} \nu(dz)dt$ is also the compensated Poisson random measure of Lévy process $\{N_t\}$ with finite Lévy measure ν on this probability space. \mathcal{F}_t represents the information available at time t and any decision is made based on this information. In our model, we denote L_t as the cumulative amount of dividend payments from time 0 to time t . We assume that the dividend payments process L_t is an \mathcal{F}_t -adapted, non-decreasing and right-continuous with left limits.

A control strategy π is described by a pair of \mathcal{F}_t -adapted stochastic processes $\{a_\pi, L^\pi\}$. A strategy $\pi = \{a_\pi(t), L_t^\pi\}$ is called admissible if $0 \leq a_\pi(t) \leq 1$ and L_t^π is a nonnegative, non-decreasing and right-continuous function. We denote Π the set of all admissible policies. When a admissible strategy π is applied, we can rewrite the asset of the insurance company by the following processes,

$$dR_t^\pi = \mu a_\pi(t)dt + \sigma a_\pi(t)dW_t + k a_\pi(t) \int_{\mathfrak{R}} z \tilde{N}(dt, dz) - dL_t^\pi, \quad R_0^\pi = x.$$

In this case, we consider transaction cost in the dividend procedures. To simplify the problem, we consider the proportional transaction cost, that is, if the company pays l , as dividend payments, then the shareholders can get βl , $\beta < 1$. The company is considered bankruptcy as soon as its asset falls below 0. We define the bankrupt time as $\tau_\pi = \inf\{t \geq 0 : R_t^\pi \leq 0\}$. τ_π is clearly an \mathcal{F}_t -stopping time.

The performance function associated with each π is defined by

$$J(s, x, \pi) = \mathbf{E} \left[\int_0^{\tau_\pi} e^{-c(s+t)} \beta dL_t^\pi \right], \quad (2.1)$$

and the optimal return function is

$$V(s, x) = \sup_{\pi \in \Pi} \{J(s, x, \pi)\}, \quad (2.2)$$

where c denotes the discount rate. If a strategy π^* is such that $J(s, x, \pi^*) = V(s, x)$, then we call π^* , $a_{\pi^*}(t)$ and $L_t^{\pi^*}$ the optimal dividend strategy, the optimal retention ratio and the optimal dividend payments process, respectively. This paper aims at working out the optimal strategy as well as the optimal return function, and then discussing impacts of key model parameters (e.g. k , ν , μ and σ^2) on $V(s, x)$, $a_{\pi^*}(t)$ and $L_t^{\pi^*}$.

3. The solution of HJB equations for (2.1) and (2.2)

In order to solve the optimal stochastic control problem (2.1) and (2.2) of jump diffusions in next section, we establish a solution of HJB equation associated with the control problem in this section. The main result of this section is the following.

Theorem 3.1. *Assume that the Lévy measure ν and the adjusted risk rate k satisfy $0 < \nu(\mathfrak{R}) < +\infty$, $0 < \int_{\mathfrak{R}} z\nu(dz) < +\infty$ and $0 < k \leq \frac{\mu}{2 \int_{\mathfrak{R}} z\nu(dz)}$. Let $\phi(s, x)$ be the function defined by*

$$\phi(s, x) = e^{-cs}\psi(x) \text{ and}$$

$$\psi(x) = \begin{cases} \psi_1(x) = C_1x^\gamma, & 0 \leq x \leq x_0, \\ \psi_2(x) = C_3e^{d_-x} + C_4e^{d_+x}, & x_0 \leq x \leq x^*, \\ \psi_3(x) = \beta(x - x^*) + \psi_2(x^*), & x \geq x^*, \end{cases} \quad (3.1)$$

where $x_0 = \frac{(1-\gamma)\sigma^2}{\mu}$, γ , d_- and d_+ are solutions of (3.10) and (3.13) below with

$$\frac{x_0}{\gamma} + \frac{1}{|d_-|} - \frac{1}{d_+} < 0. \quad (3.2)$$

x^* , C_1 , C_2 and C_3 are determined by (3.21), (3.22), (3.16) and (3.17) below, respectively. Then $\phi(s, x) \in C^2$ and is a solution of the following HJB equation

$$\max \left\{ -\frac{\partial \phi}{\partial s}(s, x) + \beta e^{-cs}, \max_{a \in [0,1]} \{\mathcal{A}\phi\} \right\} = 0, \quad (3.3)$$

where

$$\begin{aligned} \mathcal{A}\phi &= \frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial x} a\mu + \frac{1}{2} a^2 \sigma^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathfrak{R}} \{ \phi(s, x + akz) - \phi(s, x) \\ &\quad - akz \frac{\partial \phi}{\partial x}(s, x) \} \nu(dz). \end{aligned}$$

Proof. Define D as

$$D = \{(s, x) : -\frac{\partial\phi}{\partial x}(s, x) + \beta e^{-cs} < 0\}.$$

We guess that

$$D = \{(s, x) : s \geq 0, 0 < x < x^*\}$$

for some unidentified x^* . Inside D , the ϕ satisfies

$$\max_{a \in [0,1]} \{\mathcal{A}\phi\} = 0, \quad (3.4)$$

i.e.,

$$\begin{aligned} \max_{a \in [0,1]} \left\{ \frac{\partial\phi}{\partial s} + \frac{\partial\phi}{\partial x} a \mu + \frac{1}{2} a^2 \sigma^2 \frac{\partial^2\phi}{\partial x^2} + \int_{\mathfrak{R}} \{ \phi(s, x + akz) - \phi(s, x) \right. \\ \left. - akz \frac{\partial\phi}{\partial x}(s, x) \} \nu(dz) \right\} = 0. \end{aligned} \quad (3.5)$$

Differentiating $\mathcal{A}\phi = 0$ w.r.t. a , we get

$$\frac{\partial\phi}{\partial x} \mu + a \sigma^2 \frac{\partial^2\phi}{\partial x^2} = 0. \quad (3.6)$$

The equation (3.6) implies that the maximizer of the right-hand side of the equation (3.5), $a(x)$, is the following

$$a(x) = -\frac{\mu \frac{\partial\phi}{\partial x}}{\sigma^2 \frac{\partial^2\phi}{\partial x^2}}. \quad (3.7)$$

Putting the expression (3.7) into the equation (3.5), we derive

$$\begin{aligned} \frac{\partial\phi}{\partial s} - \frac{1}{2} \frac{\mu^2 (\frac{\partial\phi}{\partial x})^2}{\sigma^2 \frac{\partial^2\phi}{\partial x^2}} + \int_{\mathfrak{R}} \left\{ \phi\left(s, x - \frac{\mu \frac{\partial\phi}{\partial x}}{\sigma^2 \frac{\partial^2\phi}{\partial x^2}} kz\right) - \phi(s, x) \right. \\ \left. + \frac{\mu \frac{\partial\phi}{\partial x}}{\sigma^2 \frac{\partial^2\phi}{\partial x^2}} kz \frac{\partial\phi}{\partial x}(s, x) \right\} \nu(dz) = 0. \end{aligned} \quad (3.8)$$

Define $\phi = e^{-cs}\psi(x)$, then it is easy to see from (3.8) that the function $\psi(x)$ satisfies

$$-c\psi - \frac{1}{2} \frac{\mu^2 (\psi')^2}{\sigma^2 \psi''} + \int_{\mathfrak{R}} \left\{ \psi\left(x - \frac{\mu \psi'}{\sigma^2 \psi''} kz\right) - \psi + \frac{\mu (\psi')^2}{\sigma^2 \psi''} kz \right\} \nu(dz) = 0. \quad (3.9)$$

Because $a(x) \in [0, 1)$, $0 \leq x \leq x_0$ and $a(x) = 1$, $x \geq x_0$ for some $x_0 \geq 0$, we guess that $\psi(x) = \psi_1(x) := C_1 x^\gamma + C_2$, $0 \leq x \leq x_0$. Using $\psi(0) = 0$,

we have $\psi(x) = C_1 x^\gamma$. Putting it into (3.9), we derive the following equation

$$-c - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{\gamma}{\gamma - 1} + \int_{\mathfrak{R}} \left\{ \left(1 - \frac{\mu}{\sigma^2} \frac{1}{\gamma - 1} kz\right)^\gamma - 1 + \frac{\mu}{\sigma^2} \frac{\gamma}{\gamma - 1} kz \right\} \nu(dz) = 0. \quad (3.10)$$

By the assumption of Lévy measure ν every term in the (3.10) is well-defined. Let $h(\gamma)$ denote the left hand side of the (3.10). Then by the assumption of k we have $h(1_-) := \lim_{\gamma < 1, \gamma \rightarrow 1} \{h(\gamma)\} = +\infty$ and $h(0) = -c < 0$. So there is at least a γ to solve the equation (3.10). Thus $\psi_1(x) = C_1 x^\gamma$ and $a(x) = \frac{\mu x}{\sigma^2(1-\gamma)}$ for $0 \leq x \leq x_0 = \frac{(1-\gamma)\sigma^2}{\mu}$ because of $a(x) \in [0, 1]$.

If $x_0 \leq x \leq x^*$, then $a(x) = 1$ and the (3.5) becomes

$$\frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial x} \mu + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathfrak{R}} \left\{ \phi(s, x + kz) - \phi(s, x) - kz \frac{\partial \phi}{\partial x}(s, x) \right\} \nu(dz) = 0. \quad (3.11)$$

Define $\phi(x) = \phi_2(x) := e^{-cs} \psi_2(x)$ for $x_0 \leq x \leq x^*$, then we derive from the (3.11) that

$$\frac{1}{2} \sigma^2 \psi_2''(x) + \mu \psi_2'(x) - c \psi_2(x) + \int_{\mathfrak{R}} \left\{ \psi_2(x + kz) - \psi_2(x) - kz \psi_2'(x) \right\} \nu(dz) = 0. \quad (3.12)$$

We guess that

$$\psi_2(x) = e^{dx} \quad \text{for some constant } d \in \mathfrak{R}$$

and further get the equation

$$l(d) := \frac{1}{2} \sigma^2 d^2 + \mu d - c + \int_{\mathfrak{R}} \left\{ e^{kdz} - 1 - kdz \right\} \nu(dz) = 0. \quad (3.13)$$

Since $l(0) < 0$ and $\lim_{d \rightarrow +\infty} l(d) = \lim_{d \rightarrow -\infty} l(d) = +\infty$, the equation (3.13) has two solutions d_- and d_+ with $d_- < 0 < d_+$, and so the $\psi_2(x)$ should have the following form

$$\psi_2(x) = C_3 e^{d_- x} + C_4 e^{d_+ x} \quad \text{for } x_0 \leq x \leq x^*$$

where C_3 and C_4 are constants.

For $x \geq x^*$, the solution $\phi = e^{-cs} \psi_3(x)$ and

$$\psi_3(x) = \beta(x - x^*) + \psi_2(x^*) \quad \text{for } x \geq x^*.$$

Since ψ' and ψ'' are continuous at x^* ,

$$\psi'_2(x^*) = \psi'_3(x^*), \quad (3.14)$$

$$\psi''_2(x^*) = \psi''_3(x^*). \quad (3.15)$$

So

$$C_3(x^*)d_-e^{d_-x^*} + C_4(x^*)d_+e^{d_+x^*} = \beta,$$

$$C_3(x^*)d_-^2e^{d_-x^*} + C_4(x^*)d_+^2e^{d_+x^*} = 0.$$

Solving the last two equations, we have

$$C_3(x^*) = \frac{\beta d_+}{e^{d_-x^*}d_-(d_+ - d_-)} < 0, \quad (3.16)$$

$$C_4(x^*) = \frac{\beta d_-}{e^{d_+x^*}d_+(d_- - d_+)} > 0. \quad (3.17)$$

Also, since ψ and ψ' are continuous at x_0 ,

$$\psi_1(x_0) = \psi_2(x_0),$$

$$\psi'_1(x_0) = \psi'_2(x_0),$$

that is,

$$C_1x_0^\gamma = C_3(x^*)e^{d_-x_0} + C_4(x^*)e^{d_+x_0}, \quad (3.18)$$

$$C_1\gamma x_0^{\gamma-1} = C_3(x^*)d_-e^{d_-x_0} + C_4(x^*)d_+e^{d_+x_0}. \quad (3.19)$$

We deduce from the equations (3.18) and (3.19) that

$$\begin{aligned} q(x^*) &:= \left(\frac{x_0}{\gamma} - \frac{1}{d_-}\right) \frac{\beta d_+}{(d_+ - d_-)} e^{d_-(x_0-x^*)} \\ &\quad - \left(\frac{x_0}{\gamma} - \frac{1}{d_+}\right) \frac{\beta d_-}{(d_+ - d_-)} e^{d_+(x_0-x^*)} = 0. \end{aligned}$$

We claim that the x^* satisfying the last equation does exist. In fact, differentiating $q(x)$, we have

$$\begin{aligned} q'(x) &= -\left(\frac{x_0}{\gamma}d_- - 1\right) \frac{\beta d_+}{(d_+ - d_-)} e^{d_-(x_0-x)} - \left(\frac{x_0}{\gamma}d_+ - 1\right) \frac{\beta d_-}{(d_- - d_+)} e^{d_+(x_0-x)} \\ &= -\beta \left\{ \frac{x_0 d_+ d_-}{\gamma(d_+ - d_-)} (e^{d_-(x_0-x)} - e^{d_+(x_0-x)}) + \frac{d_- e^{d_+(x_0-x)} - d_+ e^{d_-(x_0-x)}}{d_+ - d_-} \right\} > 0 \end{aligned}$$

for $x > x_0$. So $q(x)$ is an increasing function of x and reaches its minimum at x_0 . Furthermore, by (3.2) we have

$$q(x_0) = \left(\frac{x_0}{\gamma} - \frac{1}{d_-}\right) \frac{\beta d_+}{(d_+ - d_-)} - \left(\frac{x_0}{\gamma} - \frac{1}{d_+}\right) \frac{\beta d_-}{(d_+ - d_-)} < 0. \quad (3.20)$$

Also $\lim_{x \rightarrow +\infty} q(x) = +\infty$. Thus there exists an $x^* (> x_0)$ satisfying $q(x^*) = 0$. Solving the equation $q(x^*) = 0$, we get

$$x^* = x_0 - \frac{1}{d_+ - d_-} \ln \left\{ \frac{d_+^2 (d_- x_0 - \gamma)}{d_-^2 (d_+ x_0 - \gamma)} \right\}. \quad (3.21)$$

Clearly, (3.2) implies that $0 < \frac{d_+^2 (d_- x_0 - \gamma)}{d_-^2 (d_+ x_0 - \gamma)} < 1$, so $x^* > x_0$. Moreover,

$$C_1(x^*) = \frac{\beta d_+}{x_0^\gamma e^{d_- x^*} d_- (d_+ - d_-)} e^{d_- x_0} + \frac{\beta d_-}{x_0^\gamma e^{d_+ x^*} d_+ (d_- - d_+)} e^{d_+ x_0} > 0. \quad (3.22)$$

Therefore the function $\phi(s, x)$ defined by the (3.3) should be the following form

$$\phi(s, x) = e^{-cs} \psi(x) \quad \text{and}$$

$$\psi(x) = \begin{cases} \psi_1(x) = C_1(x^*) x^\gamma, & 0 \leq x \leq x_0, \\ \psi_2(x) = C_3(x^*) e^{d_- x} + C_4(x^*) e^{d_+ x}, & x_0 \leq x \leq x^*, \\ \psi_3(x) = \beta(x - x^*) + \psi_2(x^*), & x \geq x^*, \end{cases} \quad (3.23)$$

where $x_0 = \frac{(1-\gamma)\sigma^2}{\mu}$. x^* , γ , d_- and d_+ are solutions of (3.21), (3.10) and (3.13), and C_1 , C_2 and C_3 are determined by (3.22), (3.16) and (3.17), respectively.

The problem remained is to approve the following inequalities.

For $0 \leq x \leq x^*$,

$$- \frac{\partial \phi}{\partial x}(s, x) + \beta e^{-cs} < 0, \quad (3.24)$$

$$\begin{aligned} \max_{a \in [0,1]} \left\{ \frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial x} a \mu + \frac{1}{2} a^2 \sigma^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathfrak{R}} \{ \phi(s, x + akz) - \phi(s, x) \right. \\ \left. - akz \frac{\partial \phi}{\partial x}(s, x) \} \nu(dz) \right\} \leq 0. \end{aligned} \quad (3.25)$$

For $x \geq x^*$,

$$- \frac{\partial \phi}{\partial x}(s, x) + \beta e^{-cs} = 0, \quad (3.26)$$

$$\begin{aligned} \max_{a \in [0,1]} \left\{ \frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial x} a \mu + \frac{1}{2} a^2 \sigma^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathfrak{R}} \{ \phi(s, x + akz) - \phi(s, x) \right. \\ \left. - akz \frac{\partial \phi}{\partial x}(s, x) \} \nu(dz) \right\} \leq 0. \end{aligned} \quad (3.27)$$

Since

$$\begin{aligned}\phi_1''(x) &= e^{-cs}C_1\gamma(\gamma-1)x^{\gamma-2} < 0, \\ \phi_2''(x) &= e^{-cs}\frac{\beta d_+ d_-}{d_+ - d_-}(e^{d_-(x-x^*)} - e^{d_+(x-x^*)}) < 0\end{aligned}$$

for $\gamma < 1$ and $x \leq x^*$, the inequality (3.24) is trivial due to $\phi(x) \in C^2$ is a convex function, and the inequality (3.26) is a direct consequence of $\phi_3''(x) = \beta$ for $x \geq x^*$.

For $0 \leq x \leq x_0$, by the expression of ϕ , $\max_{a \in [0,1]} \{\mathcal{A}\phi\} = 0$ is obvious.

For $x_0 \leq x \leq x^*$, the inequality (3.25) is equal to

$$\begin{aligned}\max_{a \in [0,1]} \left\{ \frac{1}{2}a^2\sigma^2\psi_2''(x) + a\mu\psi_2'(x) - c\psi_2(x) + \int_{\mathfrak{R}} \{\psi_2(x+akz) - \psi_2(x) \right. \\ \left. - akz\psi_2'(x)\} \nu(dz) \right\} \leq 0.\end{aligned}\quad (3.28)$$

Denote the function in bracket $\{\cdot\}$ at the left side of the inequality (3.28) as $p(a)$, we will prove that $p(a)$ is an increasing function of a .

$$\begin{aligned}p'(a) &= a\sigma^2\psi_2''(x) + \mu\psi_2'(x) = a\sigma^2[C_3(d_-)^2e^{d_-x} + C_4(d_+)^2e^{d_+x}] \\ &+ \mu[C_3d_-e^{d_-x} + C_4d_+e^{d_+x}] = \frac{\beta d_+ d_-}{d_+ - d_-}(e^{d_-(x-x^*)} - e^{d_+(x-x^*)}) \\ &+ \mu[C_3d_-e^{d_-x} + C_4d_+e^{d_+x}] \geq 0.\end{aligned}$$

as $d_- < 0$, $d_+ > 0$, $x \leq x^*$, $C_3 < 0$, and $C_4 > 0$. Then $p(a) \leq p(1) = 0$ for $0 \leq a \leq 1$.

For $x \geq x^*$, the inequality(3.27) is equal to

$$\begin{aligned}\max_{a \in [0,1]} \frac{1}{2}a^2\sigma^2\psi_3''(x) + a\mu\psi_3'(x) - c\psi_3(x) + \int_{\mathfrak{R}} \{\psi_3(x+akz) - \psi_3(x) \\ - akz\psi_3'(x)\} \nu(dz) = a\mu\beta - c\beta(x-x^*) - c\psi_2(x^*) \\ \leq \mu\beta - c\psi_2(x^*) - c\beta(x-x^*) \leq 0\end{aligned}$$

due to $x \geq x^*$ and $\mu\beta - c\psi_2(x^*) = 0$. So we end the proof. \square

4. The solution of the optimal control problem with jump diffusions

We now give a verification theorem for singular -regular control problem(2.1) and (2.2). We first prove the following.

Theorem 4.1. *Let $W(s, x)$ satisfy the following HJB equation,*

$$\max\left\{-\frac{\partial W}{\partial x}(t, x) + \beta e^{-ct}, \max_{a \in [0,1]} \{\mathcal{A}W(t, x)\}\right\} = 0 \quad (4.1)$$

for $t \geq 0$ and $x \geq 0$,

$$W(t, 0) = 0 \quad \text{for any } t \geq 0. \quad (4.2)$$

Then $W(s, x) \geq J(s, x, \pi)$ for any admissible strategy π and $(s, x) \in \mathfrak{R}_+^2$.

Proof. For any fixed strategy π , let $\Lambda = \{s : L_{s-}^\pi \neq L_s^\pi\}$, $\hat{L} = \sum_{s \in \Lambda, s \leq t} (L_s^\pi - L_{s-}^\pi)$ be the discontinuous part of L_s^π and $\tilde{L}_t^\pi = L_t^\pi - \hat{L}_t^\pi$ be the continuous part of L_s^π . Let τ_π be the first time that the corresponding cash flow R_t^π defined by (2.2) hit $(-\infty, 0)$. Then, by applying the generalized Itô formula to the stochastic process $Y^\pi(t) := (s + t, R_t^\pi)^T$ and the function $W(s, x)$, we have

$$\begin{aligned} & \mathbf{E}[W(s + t \wedge \tau_\pi, R_{t \wedge \tau_\pi}^\pi)] \\ &= W(s, x) + \mathbf{E}\left[\int_0^{t \wedge \tau_\pi} \mathcal{A}W(s + u, R_u^\pi) du \right. \\ & \quad \left. - \int_0^{t \wedge \tau_\pi} \frac{\partial W(s + u, R_u^\pi)}{\partial x} dL_u^{(c)} + \sum_{0 < t_n \leq t \wedge \tau_\pi} \Delta_L W(s + t_n, R_{t_n}^\pi)\right], \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \mathcal{A}W(s, x) &= \frac{\partial W}{\partial s} + a\mu \frac{\partial W}{\partial x} + \frac{1}{2} a^2 \sigma^2 \frac{\partial^2 W}{\partial x^2} + \int_{\mathfrak{R}} \{W(s, x + akz) - W(s, x) \\ & \quad - akz \frac{\partial W}{\partial x}(s, x)\} \nu(dz), \end{aligned}$$

$$\Delta_L W(s + t_n, R_{t_n}^\pi) := W(Y^\pi(t_n)) - W(Y^\pi(t_n^-) + \Delta_N Y^\pi(t_n)),$$

$$\Delta_N Y^\pi(t_n) := (0, ka_\pi(t_n) \int_{\mathfrak{R}} z \tilde{N}(\{t_n\}, dz)),$$

$\{t_k\}$ is the set of jumping times of L .

Using $\mathcal{A}W \leq 0$ in the equation (4.3), we see that

$$\begin{aligned} & \mathbf{E}[W(s + t \wedge \tau_\pi, R_{t \wedge \tau_\pi}^\pi)] \leq W(s, x) \\ & - \mathbf{E}\left[\int_0^{t \wedge \tau_\pi} \frac{\partial W(s + u, R_u^\pi)}{\partial x} dL_u^{(c)} - \sum_{s < t_n \leq t \wedge \tau_\pi} \Delta_L W(Y_{t_n}^\pi)\right]. \end{aligned} \quad (4.4)$$

By the mean value theorem we have

$$\Delta_L W(Y_{t_n}^\pi) = -\frac{\partial W}{\partial x}(\hat{Y}_{t_n}^{(n)}) \Delta L(t_n),$$

where $\hat{Y}_{t_n}^{(n)}$ is some point on the straight line between $Y_{t_n}^\pi$ and $Y_{t_n}^\pi + \Delta_N(Y_{t_n}^\pi)$. Since $W'(Y_u^\pi) \geq \beta e^{-c(s+u)}$,

$$\Delta_L W(Y_{t_n}^\pi) \leq -\beta e^{-c(s+t_n)}(L_{t_n}^\pi - L_{t_n-}^\pi),$$

which, together with the inequality (4.4), implies

$$\mathbf{E}[W(s+t \wedge \tau_\pi, R_{t \wedge \tau_\pi}^\pi)] + \mathbf{E}\left\{\int_0^{t \wedge \tau_\pi} \beta e^{-c(s+u)} dL_u^\pi\right\} \leq W(s, x). \quad (4.5)$$

By the definition of τ_π , the boundary condition (4.2) and $W'(Y_u^\pi) \geq \beta e^{-c(s+u)}$, it is easy to prove that $\liminf_{t \rightarrow \infty} W(Y_t)I_{\{\tau_\pi = \infty\}} = 0$ and

$$\begin{aligned} \liminf_{t \rightarrow \infty} W(s+t \wedge \tau_\pi, R_{t \wedge \tau_\pi}^\pi) &= W(s+\tau_\pi, 0)I_{\{\tau_\pi < \infty\}} + \liminf_{t \rightarrow \infty} W(Y_t)I_{\{\tau_\pi = \infty\}} \\ &\geq W(s+\tau_\pi, 0)I_{\{\tau_\pi < \infty\}} = 0. \end{aligned} \quad (4.6)$$

So, we deduce from the inequalities (4.5) and (4.6) that

$$J(s, x, \pi) = \mathbf{E}\left\{\int_0^{\tau_\pi} e^{-c(s+t)} \beta dL_t^\pi\right\} \leq W(s, x),$$

thus we complete the proof. \square

Let

$$a(x) = \begin{cases} \frac{\mu x}{\sigma^2(1-\gamma)}, & x < x_0, \\ 1, & x \geq x_0 \end{cases}$$

where $x_0 = \frac{(1-\gamma)\sigma^2}{\mu}$. We call $a(x)$ the feedback control function of the control problem (2.1) and (2.2).

We can now state the main result of this paper.

Theorem 4.2. *Assume that (3.2) holds, the Lévy measure ν and the adjusted risk rate k satisfy $0 < \nu(\mathfrak{R}) < +\infty$, $0 < \int_{\mathfrak{R}} z\nu(dz) < +\infty$ and $0 < k \leq \frac{\mu}{2 \int_{\mathfrak{R}} z\nu(dz)}$. Then the optimal return function and the optimal dividend strategy of the control problem (2.1) and (2.2) are $V(s, x) = \phi(s, x) = e^{-cs}\psi(x)$ and $\pi^* = (a(R_t^{\pi^*}), L_t^{\pi^*})$, respectively, where $(R_t^{\pi^*}, L_t^{\pi^*})$ is uniquely determined by the following stochastic differential equations with reflection,*

$$\begin{cases} R_t^{\pi^*} = x + \int_0^t \mu a(R_s^{\pi^*}) ds + \int_0^t \sigma a(R_s^{\pi^*}) dW_s + k \int_0^t \int_{\mathfrak{R}} a(R_s^{\pi^*}) z \tilde{N}(ds, dz) \\ -L_t^{\pi^*}, \\ R_t^{\pi^*} \leq x^*, \\ \int_0^\infty I_{\{t: R_t^{\pi^*} < x^*\}}(t) dL_t^{\pi^*} = 0, \end{cases} \quad (4.7)$$

$\psi(x)$ is the function defined by (3.1) and the optimal dividend payments level x^* is given by (3.21).

Proof. Since the function $\phi(s, x)$ satisfies the HJB equations (3.3), it is not hard to see that $\phi(s, x)$ also satisfies conditions in Theorem 4.1. So $\phi(s, x) \geq J(s, x, \pi)$ for any π , i.e.,

$$\phi(s, x) \geq V(s, x). \quad (4.8)$$

Next, we will prove $V(s, x) = \phi(s, x) = J(s, x, \pi^*)$ corresponding to π^* . By applying the generalized Itô formula, noting that the construction of $\phi(s, x)$ and the last two equations in (4.7), we deduce from the inequality (3.24) and the equations (4.1) that $\mathcal{A}\phi(s + t, R_t^{\pi^*}) = 0$ for any $t \geq 0$, $\int_0^{t \wedge \tau^*} \frac{\partial \phi(Y_u^{\pi^*})}{\partial x} dL_u^{(c)} = \int_0^{t \wedge \tau^*} \beta e^{-c(s+u)} dL_u^{(c)}$ and $\sum_{s < t_n \leq \tau^*} \Delta_L \phi(Y_{t_n}^{\pi^*}) = - \sum_{s < t_n \leq \tau^*} \frac{\partial \phi}{\partial x}(s + t_n, x^*) \Delta L(t_n) = - \sum_{s < t_n \leq \tau^*} \beta e^{-c(s+t_n)} \Delta L(t_n)$, where $\tau^* = \inf\{t \geq 0 : R_t^{\pi^*} < 0\}$. So

$$\begin{aligned} \mathbf{E}[\phi(s + t \wedge \tau^*, R_{t \wedge \tau^*}^{\pi^*})] &= \phi(s, x) + \mathbf{E}\left[\int_0^{t \wedge \tau^*} \mathcal{A}\phi(Y_u^{\pi^*}) du \right. \\ &\quad \left. - \int_0^{t \wedge \tau^*} \frac{\partial \phi(Y_u^{\pi^*})}{\partial x} dL_u^{(c)} + \sum_{s < t_n \leq \tau^*} \Delta_L \phi(Y_{t_n}^{\pi^*})\right] \\ &= \phi(s, x) - \mathbf{E}\left[\int_0^{t \wedge \tau^*} \beta e^{-c(s+u)} dL_u^{(c)} \right. \\ &\quad \left. + \sum_{s < t_n \leq \tau^*} \beta e^{-c(s+t_n)} \Delta L(t_n)\right] \\ &= \phi(s, x) - \mathbf{E}\left[\int_0^{t \wedge \tau^*} \beta e^{-c(s+u)} dL_u^{\pi^*}\right]. \quad (4.9) \end{aligned}$$

Since $\lim_{t \rightarrow \infty} \phi(s + t \wedge \tau^*, R_{t \wedge \tau^*}^{\pi^*}) = \lim_{t \rightarrow \infty} e^{-c(s+t \wedge \tau^*)} \psi(R_{t \wedge \tau^*}^{\pi^*}) = e^{-c(s+\tau^*)} \psi(R_{\tau^*}^{\pi^*}) = e^{-c(s+\tau^*)} \psi(0) = 0$, we see from the inequality (4.8) and the equation (4.9) that

$$V(s, x) \leq \phi(s, x) = \lim_{t \rightarrow \infty} \mathbf{E}\left[\int_0^{t \wedge \tau^*} e^{-cs} \beta_1 dL_s^{\pi^*}\right] = J(s, x, \pi^*) \leq V(s, x).$$

So $V(s, x) = \phi(s, x) = J(s, x, \pi^*)$, that is, $\phi(s, x)$ is the optimal return function, π^* is the optimal dividend strategy and x^* is the optimal dividend payments level. Thus the proof has been done. \square

5. Numerical examples

In this section, based on Theorem 4.2, we present some numerical examples, together with the feedback control function $a(x)$ and the comparison theorem for SDE, to portray how the key model parameters (e.g. k , μ , σ^2 and ν) impact on $V(s, x)$ and the optimal control strategy π^* , that is, $a_{\pi^*}(t)$ and $L_t^{\pi^*}$, respectively.

Example 5.1. Let $\nu(dz) = e^{-z}I_{\{z \geq 0\}}(z)dz$. Figure 1 below explains that the adjusted risk rate will increase the optimal dividend payments level $x^*(k)$, so to avoid bankruptcy the company should decrease the times of dividend or increase k if possible, that is, the company needs to maintain the cash inside the company to cover the catastrophe risk, so it pays dividend at a higher level. On the other hand, $L_t^{\pi^*}$ decreases with k by (4.7), $R_t^{\pi^*}$ increases with k , so we see that $a_{\pi^*}(t)$ also increases with k . In fact, the catastrophe risk business brings in more risk as well as more income, and the higher asset level raises the risk sustainment of the company. It could reduce its reinsurance level (i.e., $1 - a_{\pi^*}(t)$) according with the optimal control strategy π^* .

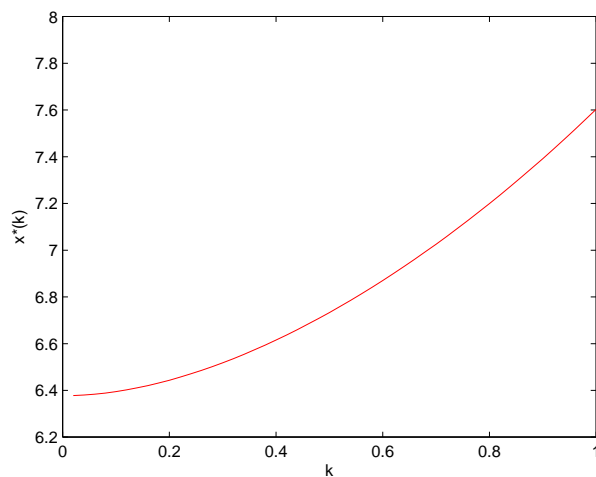


FIGURE 1. The optimal dividend payments level $x^*(k)$ as a function of k . The parameter values are $\sigma^2 = 5$, $c = 0.05$, $\mu = 2$, $s = 0$.

Example 5.2. Let $\nu(dz) = e^{-z}I_{\{z \geq 0\}}(z)dz$. Figure 2 below states that the property insurance company's profit increases with the initial capital

x and the adjusted risk rate k . So the property insurance company can get some return from its catastrophe risk insurance business, but the return's increment is small by adjusting k . However, the company can receive a good public reputation by constant k , and interest from the catastrophe insurance business.

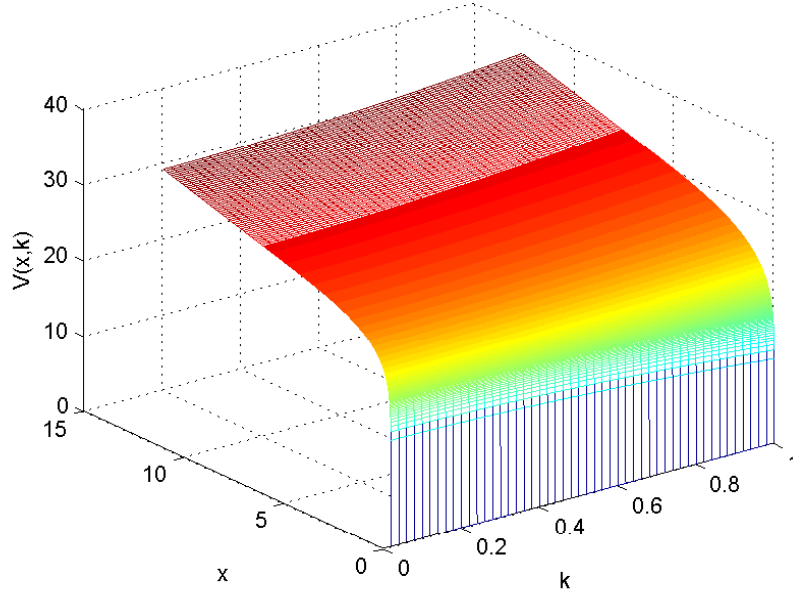


FIGURE 2. The optimal return function $V(x, k)$ as a function of x and k . The parameter values are $\sigma^2 = 5$, $c = 0.05$, $\beta = 0.8$, $s = 0$, $\mu = 2$.

Example 5.3. Let $\nu(dz) = e^{-z} I_{\{z \geq 0\}}(z)$. Figure 3 below portrays that the optimal dividend payments level $x^*(\mu)$ decreases with the premium rate μ , so $L_t^{\pi^*}$ increases with μ , but $a_{\pi^*}(t)$ decreases with the premium rate. These facts mean that the higher growth rate of the insurance company's asset raise the company's risk tolerance level and the company could pay dividend at a lower level. Meanwhile, the company should adopt a higher reinsurance rate to avoid bankruptcy due to the lower dividend payments level.

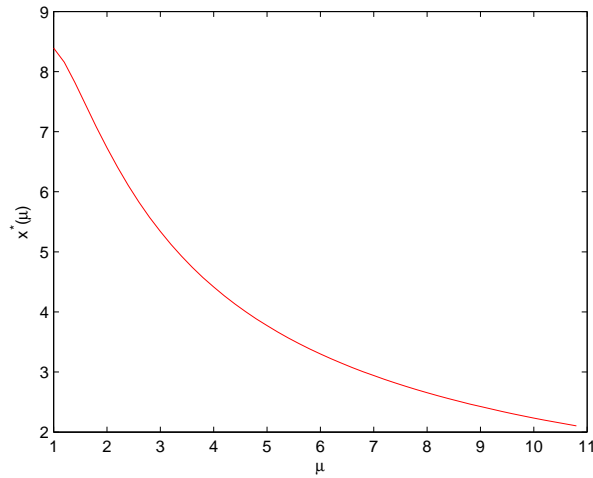


FIGURE 3. The optimal dividend payments level $x^*(\mu)$ as a function of μ . The parameter values are $\sigma^2 = 5$, $c = 0.05$, $s = 0$, $k = 0.5$.

Example 5.4. Let $v(dz) = e^{-z}I_{\{z \geq 0\}}(z)dz$. Figure 4 states that the optimal return function $V(x, \mu)$ is an increasing function of μ , and high premium rate can notably increase the company's return, that is, a higher growth rate of the insurance company's asset results in a higher return.

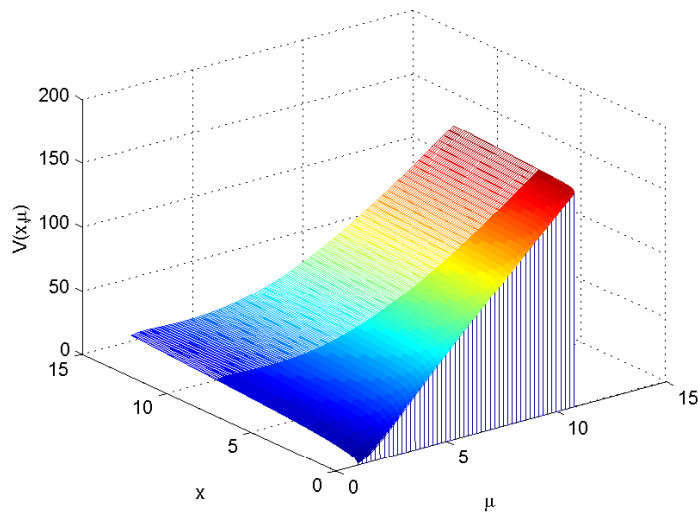


FIGURE 4. The optimal return function $V(x, \mu)$ as a function of x and μ . The parameter values are $\sigma^2 = 5$, $c = 0.05$, $\beta = 0.8$, $s = 0$, $k = 0.5$.

Example 5.5. Let $\nu(dz) = e^{-z}I_{\{z \geq 0\}}(z)dz$. Figure 5 below portrays that the optimal dividend payments level $x^*(\sigma^2)$ increases with the risk volatility rate σ^2 of normal insurance business, so L_t^* decreases with σ^2 , but $a_{\pi^*}(t)$ increases it. These mean that the higher volatility make the insurance company's asset reduce the company's risk tolerance level and the company prefer to maintain the cash inside the company to cover the risk. Meanwhile, the company should adopt a lower reinsurance rate to get lower optimal dividend payments level.

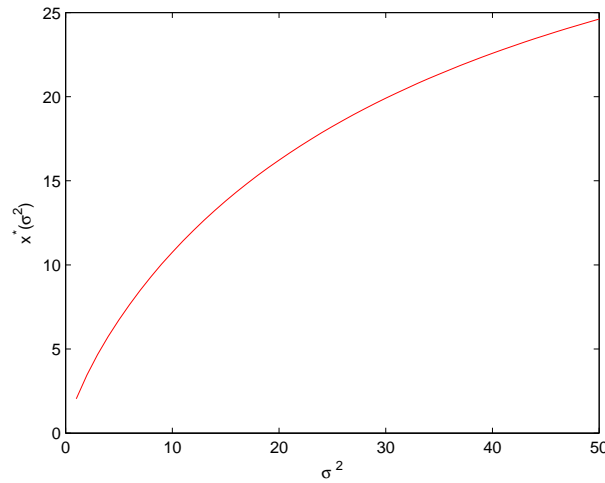


FIGURE 5. The optimal dividend payments level $x^*(\sigma^2)$ as a function of σ^2 . The parameter values are $c = 0.05$, $s = 0$, $k = 0.5$, $\mu = 2$.

Example 5.6. Let $\nu(dz) = e^{-z}I_{\{z \geq 0\}}(z)dz$. Figure 6 below states that the increment of the optimal return function $V(x, \sigma^2)$ due to σ^2 is very large, so higher risk can also notably increase the company's return.

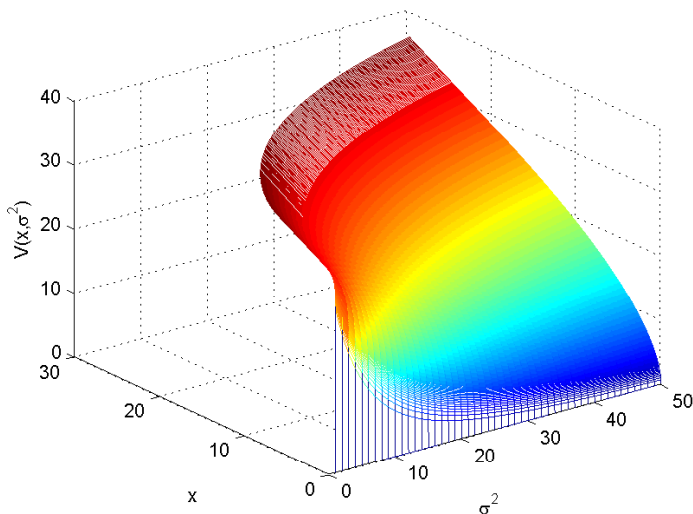


FIGURE 6. The optimal return function $V(x, \sigma^2)$ as a function of x and σ^2 . The parameter values are $c = 0.05$, $\beta = 0.8$, $s = 0$, $k = 0.5$, $\mu = 2$.

Example 5.7. Let $\nu_t(dz) = e^{-tz} I_{\{z \geq 0\}}(z) dz$ ($t \geq 1$). Figure 7 below portrays that the optimal dividend payments level $x^*(t)$ has obvious decrements on $[1, 4]$, but on $[4, +\infty)$ the optimal dividend payments level has no visibly changes, so $a_{\pi^*}(\cdot)$ and L_{π^*} change greatly for different Lévy measures $\nu_t(dz) = e^{-tz} I_{\{z \geq 0\}}(z) dz$, $t \in [1, 4]$. However, they are almost same for different Lévy measures $\nu_t(dz) = e^{-tz} I_{\{z \geq 0\}}(z) dz$, $t \geq 4$.

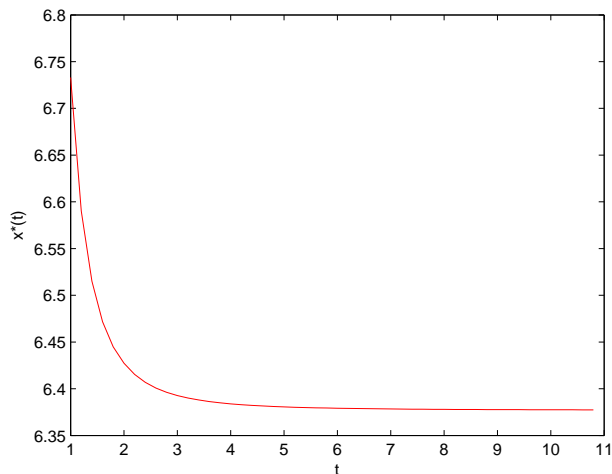


FIGURE 7. The optimal dividend payments level $x^*(t)$ as a function of t . The parameter values are $\sigma^2 = 5$, $c = 0.05$, $s = 0$, $k = 0.5$, $\mu = 2$.

Example 5.8. Let $\nu_t(dz) = e^{-tz} I_{\{z \geq 0\}}(z) dz$. Figure 8 below states that the change of the optimal return function $V(x, t)$ for different t is not distinct. So the optimal return function $V(x, t)$ is nearly stable for different Lévy measures $\nu_t(dz) = e^{-tz} I_{\{z > 0\}}(z) dz$ ($t > 0$).

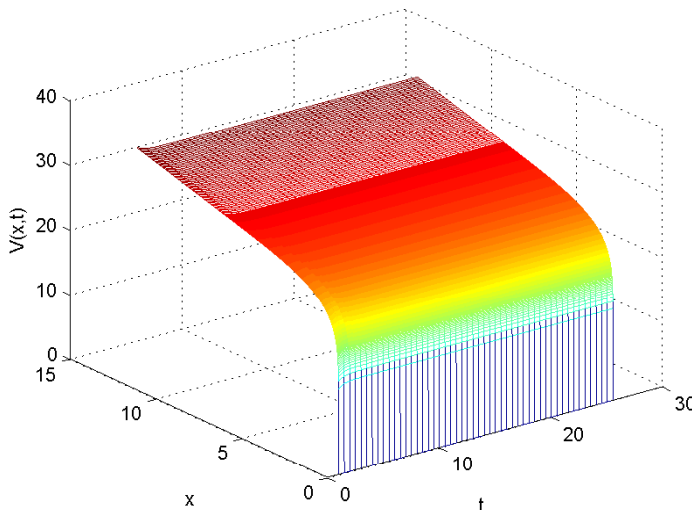


FIGURE 8. The optimal return function $V(x, t)$ as a function of x and t . The parameter values are $\sigma^2 = 5$, $c = 0.05$, $\beta = 0.8$, $s = 0$, $k = 0.5$, $\mu = 2$.)

6. Conclusion

We consider the optimal dividend and the reinsurance strategy of a property insurance company. The property insurance business brings in catastrophe risk, such as earthquake and flood. The catastrophe risk could be partly reduced by reinsurance. Due to the huge risk, the company needs to add an adjusted risk rate in the regulation. The management of the company controls the reinsurance rate and dividend payments to maximize the expected present value of the dividends before bankruptcy. This is the first time to consider the catastrophe risk in an insurance model, which is more realistic. The catastrophe risk is modeled as the jump process in the stochastic control problem. In order to find the solution of the problem, we explore the mixed singular-regular control methods of jump diffusions. We establish the optimal reinsurance rate, the optimal dividend strategies and explicit the optimal return function of the company. The influences of the catastrophe risk and the reinsurance regulation of the catastrophe risk on the optimal control strategy

of the insurance company are also discussed. Based on the main results we have just established, we present some numerical examples to analyze in detail how the key model parameters impact on the optimal retention ratio, the optimal dividend payments strategies and the optimal return of the company.

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