On the consistency of the quantum-like representation algorithm for hyperbolic interference

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Abstract

Recently quantum-like representation algorithm (QLRA) was introduced by A. Khrennikov [20]–[28] to solve the so-called "inverse Born's rule problem": to construct a representation of probabilistic data by a complex or more general (in particular, hyperbolic) probability amplitude which matches Born's rule or its generalizations. The outcome from QLRA is coupled to the formula of total probability with an additional term corresponding to trigonometric, hyperbolic or hyper-trigonometric interference. The consistency of QLRA for probabilistic data corresponding to trigonometric interference was recently proved [29]. We now complete the proof of the consistency of QLRA to cover hyperbolic interference as well. We will also discuss hyper trigonometric interference. The problem of consistency of QLRA arises, because formally the output of QLRA depends on the order of conditioning. For two observables (e.g., physical or biological) a and b, $b|a$- and $a|b$- conditional probabilities produce two representations, say in Hilbert spaces $H^{b|a}$ and $H^{a|b}$ (in this paper over the hyperbolic algebra). We prove that under "natural assumptions" these two representations are unitary equivalent (in the sense of hyperbolic Hilbert space).
Keywords Born's rule problem, hyperbolic interference, hyper trigonometric interference, inverse order of conditioning, quantum-like representation algorithm

1 Introduction

The interrelation between classical and quantum probabilities was early studied by von Neumann, see [1] and was followed by methods to generalize the probability theory to include quantum probabilities by Gudder, see [2]–[4]. For more recent and wide-ranging studies, see Svozil [5], [6], Fine [7], Garola et al.

[8]–[10], Dvurecenskij and Pulmanova [12], Ballentine [11], O. Nánásiová et al [13], [14], Allahverdyan et al [15], Khrennikov [30, 31]. The basic rule of QM is the Born's rule. Therefore the study of its origin is very important for quantum foundations. In a series of papers [20] –[28] Khrennikov studied so called "inverse Born's rule problem":

IBP (inverse Born problem): *To construct a representation of probabilistic data (of any origin) by a complex probability amplitude which matches Born's rule.*

The solution of IBP provides a possibility to represent probabilistic data by "wave functions" and operate with this data by using linear algebra (as we do in conventional QM). However, as it was found in [20] –[28], some data do not permit the complex wave representation. In this case probabilistic amplitudes valued in the hyperbolic algebra (a two dimensional Clifford algebra) should be used as well. A special algorithm, quantum-like representation algorithm (QLRA), was created to transfer probabilities into probabilistic amplitudes. Depending on the data these amplitudes are *complex, hyperbolic or hyper-complex.*

Formally, the output of QLRA depends on the order of conditioning of probabilities. For two observables a and b, $b|a$- and $a|b$- conditional probabilities produce two representations, say in Hilbert spaces (complex or hyperbolic) $H^{b|a}$ and $H^{a|b}$. In this paper we will be interested in hyperbolic amplitudes as outputs of QLRA and therefore consider the hyperbolic Hilbert space. The case of complex amplitudes has already been studied in [20]. It was shown that under "natural assumptions" these two conditional probabilistic representations are unitary equivalent. This result proved the consistency of QLRA for complex amplitudes, now we will study the case of hyperbolic amplitudes.

In a purely mathematical framework the problem of consistency of two representations is nothing else than construction of a special unitary operator in hyperbolic Hilbert space establishing the equivalence of two representations. This paper is also a contribution to mathematical physics over hyperbolic numbers, see, e.g., [32]–[44].

2 Inversion of Born's Rule

We consider the simplest situation. There are given two dichotomous observables of any context: $a = \alpha_1, \alpha_2$ and $b = \beta_1, \beta_2$. We set $X_a = \{\alpha_1, \alpha_2\}$ and $X_b = \{\beta_1, \beta_2\}$ – "spectra of observables".

We assume that there is given the matrix of transition probabilities $\mathbf{P}^{b|a} = (p^{b|a}_{\beta\alpha})$, where $p^{b|a}_{\beta\alpha} \equiv P(b = \beta|a = \alpha)$ is the probability to obtain the result $b = \beta$ under the condition that the result $a = \alpha$ has been obtained. There are also given probabilities $p^a_\alpha \equiv P(a = \alpha), \alpha \in X_a$, and $p^b_\beta \equiv P(b = \beta), \beta \in X_b$. Probabilistic data $C = \{p^a_\alpha, p^b_\beta\}$ are related to some experimental context (in the physics preparation procedure).

IBP is to represent this data by a probability amplitude ψ (in the simplest case it is complex-valued, but we are interested in more general amplitudes) such that Born's rule holds for both observables:

$$p_\beta^b = |\langle \psi, e_\beta^{b|a} \rangle|^2 \,, \qquad p_\alpha^a = |\langle \psi, e_\alpha^{b|a} \rangle|^2 \,, \tag{2.1}$$

where $\{e_\beta^{b|a}\}_{\beta \in X_b}$ and $\{e_\alpha^{b|a}\}_{\alpha \in X_a}$ are orthonormal bases for observables b and a, respectively (so the observables are diagonal in the respective bases).

In [20] –[28] the solution of IBP was given in the form of an algorithm which constructs a probability amplitude from the data. Formally, the output of this algorithm depends on the order of conditioning. By starting with the matrix of transition probabilities $\mathbf{P}^{a|b}$, instead of $\mathbf{P}^{b|a}$, we construct another probability amplitude $\psi^{a|b}$ (the amplitude in (2.1) should be denoted by $\psi^{b|a}$) and other bases, $\{e_\beta^{a|b}\}_{\beta \in X_b}$ and $\{e_\alpha^{a|b}\}_{\alpha \in X_a}$. We shall see that under natural assumptions these two representations are unitary equivalent.

3 QLRA

3.1 $H^{b|a}$-conditioning

Suppose that the matrix of transition probabilities $\mathbf{P}^{b|a}$ is given. In [20] –[28] the following formula for the interference of probabilities (generalizing the classical formula of total probability) was derived:

$$p_\beta^b \;=\; \sum_\alpha p_\alpha^a p_{\beta\alpha}^{b|a} + 2\lambda_\beta \sqrt{\prod_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}}, \tag{3.1}$$

where the "coefficient of interference" is given by

$$\lambda_\beta \;=\; \frac{p_\beta^b - \sum_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}}{2\sqrt{\prod_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}}} \,. \tag{3.2}$$

We will proceed under the conditions:

(1) $\mathbf{P}^{b|a}$ is doubly stochastic (for not doubly stochastic matrix, see section 5.3).

(2) Probabilistic data $C = \{p_\alpha^a, p_\beta^b\}$ consist of strictly positive probabilities.

(3) The absolute values of the coefficients of interference $\lambda_\beta, \beta \in X_b$, are larger than one: $|\lambda_\beta| > 1$.

Probabilistic data C such that $|\lambda_\beta| \leq 1$ are called *trigonometric*. In this case we have the conventional formula of trigonometric interference:

$$p_\beta^b \;=\; \sum_\alpha p_\alpha^a p_{\beta\alpha}^{b|a} + 2\lambda_\beta \sqrt{\prod_\alpha p_\alpha^a p_{\beta\alpha}^{b|a}} \,,$$

where

$$\lambda_\beta = \cos\theta_\beta \,. \tag{3.3}$$

The case of trigonometric interference (i.e. $|\lambda_\beta| \leq 1$) has been studied in [29]. Therefore, now we consider the case of hyperbolic interference: $|\lambda_\beta| > 1$. We represent this coefficient of interference by

$$\lambda_\beta = \epsilon_\beta \cosh \theta_\beta \tag{3.4}$$

where $\epsilon_\beta = \text{sign } \lambda_\beta$.

Furthermore, in the case of hyper-trigonometric interference (i.e. $|\lambda_{\beta_i}| > 1$ and $|\lambda_{\beta_j}| \leq 1$) we have $\lambda_{\beta_i} = \cos\theta_{\beta_i}$ and $\lambda_{\beta_j} = \epsilon_{\beta_j} \cosh\theta_{\beta_j}$, where $\epsilon_{\beta_j} = \text{sign } \lambda_{\beta_j}$, $i, j \in \{1, 2\}, i \neq j$.

Proposition 3.1. *Let $\mathbf{P}^{b|a}$ be doubly stochastic. Then the case of mixed hyper-trigonometric interference is excluded.*

This proposition follows straightforward from the equality

$$\lambda_{\beta_1} + \lambda_{\beta_2} = 0$$

and the condition that $\mathbf{P}^{b|a}$ is doubly stochastic, see (3.2). We have:

$$\begin{aligned} \lambda_{\beta_1} + \lambda_{\beta_2} &= \frac{p^b_{\beta_1} - \sum_\alpha p^a_\alpha p^{b|a}_{\beta_1\alpha}}{2\sqrt{\prod_\alpha p^a_\alpha p^{b|a}_{\beta_1\alpha}}} + \frac{p^b_{\beta_2} - \sum_\alpha p^a_\alpha p^{b|a}_{\beta_2\alpha}}{2\sqrt{\prod_\alpha p^a_\alpha p^{b|a}_{\beta_2\alpha}}} \\ &= \frac{p^b_{\beta_1} + p^b_{\beta_2} - p^a_{\alpha_1}\sum_\alpha p^{b|a}_{\beta_1\alpha} - p^a_{\alpha_2}\sum_\alpha p^{b|a}_{\beta_1\alpha}}{2\sqrt{\prod_\alpha p^a_\alpha p^{b|a}_{\beta_1\alpha}}} \\ &= 0\,. \end{aligned} \tag{3.5}$$

There is a contradiction between (3.5) and the definition of hyper-trigonometric interference:$|\lambda_{\beta_i}| > 1$ and $|\lambda_{\beta_j}| \leq 1$.

Therefore, we will focus on hyperbolic interference (since trigonometric interference has already been studied) and introduce the *hyperbolic algebra* $\mathbf{G}$; see appendix and [41]. Denote its generator (different from unit 1) by j :

$$j^2 = 1.$$

An element of $\mathbf{G}$ can be represented as $z = x + jy$, $x, y \in \mathbb{R}$. We introduce the hyperbolic exponential function

$$e^{j\theta} = \cosh\theta + j \sinh\theta, \quad \theta \in \mathbb{R}. \tag{3.6}$$

Define also $\bar{z} = x - jy$, it is apparent that $\bar{z} \in \mathbf{G}$. We also use the identities

$$\cosh\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sinh\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \tag{3.7}$$

Thus, by using the elementary formula:

$$D = A + B \pm 2\sqrt{AB}\cosh\theta = |\sqrt{A} \pm e^{j\theta}\sqrt{B}|^2, \quad A, B > 0\,, \theta \in \mathbb{R}\,, j^2 = 1 \tag{3.8}$$

for real numbers of A and B, we can represent the probability p^b_β as the square of the hyperbolic amplitude (Born's rule): $p^b_\beta = |\psi^{b|a}_\beta|^2$. Here

$$\psi^{b|a}_\beta \ = \ \sqrt{p^a_{\alpha_1} p^{b|a}_{\beta\alpha_1}} \pm e^{j\theta_\beta} \sqrt{p^a_{\alpha_2} p^{b|a}_{\beta\alpha_2}}\,, \quad \beta \in X_b\,. \tag{3.9}$$

The formula (3.9) gives the hyperbolic amplitude, the output of QLRA for any probabilistic data C if $|\lambda| > 1$. This is the normalized vector in the two dimensional hyperbolic Hilbert space[1], say $H^{b|a}$:

$$\psi^{b|a} = \psi^{b|a}_{\beta_1} e^{b|a}_{\beta_1} + \psi^{b|a}_{\beta_2} e^{b|a}_{\beta_2}, \tag{3.10}$$

where $e^{b|a}_{\beta_1} \ = \ (1\ 0)^T\,, \quad e^{b|a}_{\beta_2} \ = \ (0\ 1)^T\,.$

To solve **IBP** completely, we would like to have Born's rule not only for the b-variable, but also for the a-variable: $p^a_\alpha \ = \ |\langle \psi^{b|a}, e^{b|a}_\alpha\rangle|^2 \ , \alpha \in X_a$. Here the a-basis in the hyperbolic Hilbert space $H^{b|a}$ is given, see [20] –[28] for details, by

$$e^{b|a}_{\alpha_1} \ = \ \left(\begin{array}{c} \sqrt{p^{b|a}_{\beta_1\alpha_1}} \\ \sqrt{p^{b|a}_{\beta_2\alpha_1}} \end{array}\right), \quad e^{b|a}_{\alpha_2} \ = \ \left(\begin{array}{c} \sqrt{p^{b|a}_{\beta_1\alpha_2}} \\ -\sqrt{p^{b|a}_{\beta_2\alpha_2}} \end{array}\right). \tag{3.11}$$

This basis vectors are orthonormal, since $\mathbf{P}^{b|a}$ is assumed to be doubly stochastic. In this basis the hyperbolic amplitude $\psi^{b|a}$ is represented as

$$\psi^{b|a} = \sqrt{p^a_{\alpha_1}} e^{b|a}_{\alpha_1} \pm e^{j\theta_{\beta_1}} \sqrt{p^a_{\alpha_2}} e^{b|a}_{\alpha_2} \tag{3.12}$$

We recall that in QM two vectors (say ψ'_1, ψ'_2) define the same state ψ' if they differ by multipliers of the form $c = e^{i\varphi}$ (e.i. if $\psi'_1 = e^{i\varphi}\psi'_2$ for some φ). We will use a similar terminology for the case of the hyperbolic algebra: two vectors ψ_1, ψ_2 define the same state if $\psi_1 = \pm e^{j\gamma}\psi_2$. The consistency of this definition follows from the fact that

$$|\psi_2|^2 = |\pm e^{j\gamma}|^2 |\psi_2|^2 = |e^{j\gamma}\psi_2|^2 = |\psi_1|^2.$$

Thus measurements on these two states produce the same probability distribution.

Each hyperbolic amplitude $\psi^{b|a}$ produced by QLRA determines a *quantum-like state* (representing given probabilistic data) – the equivalence class $\Psi^{b|a}$ being determined by the representative $\psi^{b|a}$.

3.2 $H^{a|b}$-conditioning

For $a|b$-conditioning the state is represented by

$$\psi^{a|b}_\alpha \ = \ \sqrt{p^b_{\beta_1} p^{a|b}_{\alpha\beta_1}} \pm e^{j\theta_\alpha} \sqrt{p^b_{\beta_2} p^{a|b}_{\alpha\beta_2}}\,, \quad \alpha \in X_a\,. \tag{3.13}$$

[1]For the definition of hyperbolic Hilbert space, see appendix.

For any collection of probabilistic data C, QLRA produces the hyperbolic amplitude $\psi^{a|b}$ if $|\lambda| > 1$ (the normalized vector in the two dimensional hyperbolic Hilbert space, say $H^{a|b}$) :

$$\psi^{a|b} = \psi^{a|b}_{\alpha_1} e^{a|b}_{\alpha_1} + \psi^{a|b}_{\alpha_2} e^{a|b}_{\alpha_2}, \tag{3.14}$$

where $e^{b|a}_{\beta_1} = (1\ 0)^T$, $\quad e^{b|a}_{\beta_2} = (0\ 1)^T$. Here the b-basis in the hyperbolic Hilbert space $H^{a|b}$ is given by

$$e^{a|b}_{\beta_1} \ = \ \begin{pmatrix} \sqrt{p^{a|b}_{\alpha_1\beta_1}} \\ \sqrt{p^{a|b}_{\alpha_2\beta_1}} \end{pmatrix}, e^{a|b}_{\beta_2} \ = \ \begin{pmatrix} \sqrt{p^{a|b}_{\alpha_1\beta_2}} \\ -\sqrt{p^{b|a}_{\alpha_2\beta_2}} \end{pmatrix}. \tag{3.15}$$

In this basis the amplitude $\psi^{a|b}$ is represented as

$$\psi^{a|b} = \sqrt{p^b_{\beta_1}} e^{a|b}_{\beta_1} \pm e^{j\theta_{\alpha_1}} \sqrt{p^b_{\beta_2}} e^{b|a}_{\beta_2} \tag{3.16}$$

As in the case of $H^{b|a}$-representation, the *quantum-like state* (representing given probabilistic data) is defined as the equivalence class $\Psi^{a|b}$ with the representative $\psi^{a|b}$.

4 Unitary equivalence of $b|a$- and $a|b$-representations

Thus, as we have seen, by selecting two types of conditioning, we represented the probabilistic data $C = \{p^a_\alpha, p^b_\beta\}$ by two quantum-like states, $\Psi^{b|a}$ and $\Psi^{a|b}$. We are interested in the consistency of these representations.

We remark that any linear operator $W : H^{b|a} \to H^{a|b}$ induces the map of equivalence classes of the hyperbolic unit sphere[2] with respect to multipliers $c = \pm e^{j\gamma}$. We define the unitary operator $U^{a|b}_{b|a} : H^{b|a} \to H^{a|b}$ by $U(e^{b|a}_\alpha) = e^{a|b}_\alpha, \alpha \in X_a$. It induces the mentioned map of equivalent classes.

Theorem 4.1. *The operator $U^{a|b}_{b|a}$ maps $\Psi^{b|a}$ into $\Psi^{a|b}$ if and only if the following interrelation of symmetry takes place for the matrices of transition probabilities $\mathbf{P}^{b|a}$ and $\mathbf{P}^{a|b}$:*

$$p^{b|a}_{\beta\alpha} = p^{a|b}_{\alpha\beta}, \tag{4.1}$$

for all α and β from the spectra of observables a and b.

Proof. Take the representative of $\Psi^{b|a}$ given by (3.12). Then

$$U^{a|b}_{b|a} \psi^{b|a} = \sqrt{p^a_{\alpha_1}} e^{a|b}_{\alpha_1} \pm e^{j\theta_{\beta_1}} \sqrt{p^a_{\alpha_2}} e^{a|b}_{\alpha_2} \tag{4.2}$$

[2]The hyperbolic unit sphere is given by $|e^{j\theta}|^2 = \cosh^2\theta - \sinh^2\theta = 1$

Our aim is to show that this vector is equivalent to the vector $\psi^{a|b}$ given by (3.14). The coefficients of interference λ_α play in the $H^{a|b}$-representation the same role as the coefficients of interference λ_β played in $H^{b|a}$-representation:

$$p^a_{\alpha_1} = p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} + p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2} + \epsilon_{\lambda_{\alpha_1}} 2|\lambda_{\alpha_1}| \sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}} \tag{4.3}$$

$$\Leftrightarrow,$$

$$\lambda_{\alpha_1} = \frac{p^a_{\alpha_1} - p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} - p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}}{2\sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}}},$$

where $\epsilon_{\lambda_{\alpha_1}} = \text{sign } \lambda_{\alpha_1}$. We consider $|\lambda_{\alpha_1}| > 1$ and thus $|\lambda_{\alpha_1}| = \cosh\theta_{\alpha_1}$. We also calculate

$$\begin{aligned} \psi^{a|b}_{\alpha_2}\overline{\psi^{a|b}_{\alpha_1}} &= p^b_{\beta_1}\sqrt{p^{a|b}_{\alpha_1\beta_1} p^{a|b}_{\alpha_2\beta_1}} \\ &+ \epsilon_{\lambda_{\alpha_1}}\epsilon_{\lambda_{\alpha_2}} p^b_{\beta_2}\sqrt{p^{a|b}_{\alpha_2\beta_2} p^{a|b}_{\alpha_1\beta_2}} \\ &+ \epsilon_{\lambda_{\alpha_2}}(\cosh\theta_{\alpha_1} + j\sinh\theta_{\alpha_1})\sqrt{p^b_{\beta_2} p^{a|b}_{\alpha_2\beta_2} p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}} \\ &+ \epsilon_{\lambda_{\alpha_1}}(\cosh\theta_{\alpha_1} - j\sinh\theta_{\alpha_1})\sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_2\beta_1} p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}}, \end{aligned} \tag{4.4}$$

where $\psi^{a|b}_{\alpha_2} = \sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_2\beta_1}} \pm e^{j\theta_{\alpha_1}}\sqrt{p^b_{\beta_2} p^{a|b}_{\alpha_2\beta_2}}$ is given by (3.16). We also use

$$|\psi^{a|b}_{\alpha_i}|^2 = p^a_{\alpha_i} \Leftrightarrow \psi^{a|b}_{\alpha_i} = \pm\sqrt{p^a_{\alpha_i}}\left(\cosh\gamma_{\alpha_i} + j\sinh\gamma_{\alpha_i}\right),$$

where[3]

$$\gamma_{\alpha_i} = \arg\psi^a_{\alpha_i}, i \in \{1,2\}$$

and this implies

$$\psi^{a|b}_{\alpha_2}\overline{\psi^{a|b}_{\alpha_1}} = \pm\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\left(\cosh\left(\gamma_{\alpha_2} - \gamma_{\alpha_1}\right) + j\sinh\left(\gamma_{\alpha_2} - \gamma_{\alpha_1}\right)\right). \tag{4.5}$$

The real parts of the equations (4.4) and (4.5) give:

$$\begin{aligned} \pm\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cosh\left(\gamma_{\alpha_2} - \gamma_{\alpha_1}\right) &= p^b_{\beta_1}\sqrt{p^{a|b}_{\alpha_1\beta_1} p^{a|b}_{\alpha_2\beta_1}} - p^b_{\beta_2}\sqrt{p^{a|b}_{\alpha_2\beta_2} p^{a|b}_{\alpha_1\beta_2}} \\ &+ \epsilon_{\lambda_{\alpha_1}}\cosh\theta_{\alpha_1}\left(\sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_2\beta_1} p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}} - \sqrt{p^b_{\beta_2} p^{a|b}_{\alpha_2\beta_2} p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}}\right). \end{aligned} \tag{4.6}$$

Notice that $\lambda_{\alpha_2} = -\lambda_{\alpha_1}$ in (3.5) implies that

$$\epsilon_{\lambda_{\alpha_1}} = -\epsilon_{\lambda_{\alpha_2}}, \epsilon_{\lambda_{\alpha_1}}\epsilon_{\lambda_{\alpha_2}} = -1.$$

Moreover, since $p^b_{\beta_2} = 1 - p^b_{\beta_1}$ and $\mathbf{P}^{a|b}$ is doubly stochastic, i.e., $p^{a|b}_{\alpha_1\beta_2} = p^{a|b}_{\alpha_2\beta_1} = 1 - p^{a|b}_{\alpha_1\beta_1} = 1 - p^{a|b}_{\alpha_2\beta_2}$, we rewrite (4.6)

$$\begin{aligned} \pm\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\cosh\left(\gamma_{\alpha_2} - \gamma_{\alpha_1}\right) &= \left(2p^b_{\beta_1} - 1\right)\sqrt{p^{a|b}_{\alpha_1\beta_1}(1 - p^{a|b}_{\alpha_1\beta_1})} \\ &+ \epsilon_{\lambda_{\alpha_1}}\cosh\theta_{\alpha_1}\left(1 - 2p^{a|b}_{\alpha_1\beta_1}\right)\sqrt{(1 - p^b_{\beta_1})p^b_{\beta_1}}. \end{aligned} \tag{4.7}$$

[3]see appendix for definition of the argument (arg) in the hyperbolic algebra.

Then by (3.2) and (3.4) we obtain $\cosh\theta_{\beta_1}$:

$$\epsilon_{\lambda_{\beta_1}} \cosh\theta_{\beta_1} = \frac{p^b_{\beta_1} - p^a_{\alpha_1} p^{b|a}_{\beta_1\alpha_1} - p^a_{\alpha_2} p^{b|a}_{\beta_1\alpha_2}}{2\sqrt{p^a_{\alpha_1} p^{b|a}_{\beta_1\alpha_1} p^a_{\alpha_2} p^{b|a}_{\beta_1\alpha_2}}}. \tag{4.8}$$

Multiply (4.8) with $2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}$ and use again that $p^a_{\alpha_2} = 1 - p^a_{\alpha_1}$ and $\mathbf{P}^{a|b}$ is double stochastic and

$$\epsilon_{\lambda_{\beta_1}} 2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}} \cosh\theta_{\beta_1} = \frac{p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{b|a}_{\alpha_1\beta_1} - 2p^{b|a}_{\alpha_1\beta_1} p^a_{\alpha_1}}{\sqrt{p^{b|a}_{\alpha_1\beta_1} p^{b|a}_{\beta_1\alpha_2}}}. \tag{4.9}$$

We will show that $\pm\cosh(\gamma_{\alpha_2} - \gamma_{\alpha_1}) = \epsilon_{\lambda_{\beta_1}} \cosh\theta_{\beta_1}$ or equivalently, we show that

$$\epsilon_{\lambda_{\beta_1}} 2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}} \cosh(\gamma_{\alpha_2} - \gamma_{\alpha_1}) = 2\epsilon_{\lambda_{\beta_1}} \sqrt{p^a_{\alpha_1} p^a_{\alpha_2}} \cosh\theta_{\beta_1}. \tag{4.10}$$

We multiply $\pm\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}} \cosh(\gamma_{\alpha_2} - \gamma_{\alpha_1})$ by $2\sqrt{p^{a|b}_{\alpha_1\beta_1}(1 - p^{a|b}_{\alpha_1\beta_1})}$ in the left-hand side of (4.7). We get $LHS = \pm 2\sqrt{p^{a|b}_{\alpha_1\beta_1}(1 - p^{a|b}_{\alpha_1\beta_1})}\ \sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}\ \cosh(\gamma_{\alpha_2} - \gamma_{\alpha_1})$ and replace $\epsilon_{\lambda_{\alpha_1}} \cosh\theta_{\alpha_1}$ by $\frac{p^a_{\alpha_1} - p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} - (1-p^b_{\beta_1})(1-p^{a|b}_{\alpha_1\beta_1})}{2\sqrt{p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} p^b_{\beta_2} p^{a|b}_{\alpha_1\beta_2}}}$ in the right-hand side

$$\begin{aligned} LHS &= 2\left(2p^b_{\beta_1} - 1\right) p^{a|b}_{\alpha_1\beta_1}(1 - p^{a|b}_{\alpha_1\beta_1}) \\ &+ \left(p^a_{\alpha_1} - p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} - (1 - p^b_{\beta_1})(1 - p^{a|b}_{\alpha_1\beta_1})\right)\left(1 - 2p^{a|b}_{\alpha_1\beta_1}\right) \end{aligned} \tag{4.11}$$

We calculate the last term:

$$\begin{aligned} & \left(p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}\right)\left(1 - 2p^{a|b}_{\alpha_1\beta_1}\right) \\ = & \left(p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}\right) \\ - & 2p^{a|b}_{\alpha_1\beta_1}\left(p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}\right) \\ = & \left(p^a_{\alpha_1} - 1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1}\right) - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1} - 2p^{a|b}_{\alpha_1\beta_1} p^a_{\alpha_1} \\ - & 2p^{a|b}_{\alpha_1\beta_1}\left(-1 + p^b_{\beta_1} + p^{a|b}_{\alpha_1\beta_1} - 2p^b_{\beta_1} p^{a|b}_{\alpha_1\beta_1}\right). \end{aligned} \tag{4.12}$$

Moreover,

$$
\begin{aligned}
LHS &= 2\left(2p^b_{\beta_1}-1\right)p^{a|b}_{\alpha_1\beta_1}(1-p^{a|b}_{\alpha_1\beta_1}) \qquad (4.13)\\
&+ \left(p^a_{\alpha_1}-1+p^b_{\beta_1}+p^{a|b}_{\alpha_1\beta_1}-2p^{a|b}_{\alpha_1\beta_1}p^a_{\alpha_1}\right)\\
&- 2p^{a|b}_{\alpha_1\beta_1}\left(-1+p^b_{\beta_1}+p^{a|b}_{\alpha_1\beta_1}-2p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1}\right)-2p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1}\\
&= 2p^{a|b}_{\alpha_1\beta_1}\left(-1+2p^b_{\beta_1}+p^{a|b}_{\alpha_1\beta_1}-2p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1}\right)\\
&+ \left(p^a_{\alpha_1}-1+p^b_{\beta_1}+p^{a|b}_{\alpha_1\beta_1}-2p^{a|b}_{\alpha_1\beta_1}p^a_{\alpha_1}\right)\\
&- 2p^{a|b}_{\alpha_1\beta_1}\left(-1+2p^b_{\beta_1}+p^{a|b}_{\alpha_1\beta_1}-2p^b_{\beta_1}p^{a|b}_{\alpha_1\beta_1}\right)\\
&= p^a_{\alpha_1}-1+p^b_{\beta_1}+p^{a|b}_{\alpha_1\beta_1}-2p^{a|b}_{\alpha_1\beta_1}p^a_{\alpha_1}.
\end{aligned}
$$

Equations (4.9) and (4.13) imply that

$$
\pm\frac{p^a_{\alpha_1}-1+p^b_{\beta_1}+p^{b|a}_{\alpha_1\beta_1}-2p^{b|a}_{\alpha_1\beta_1}p^a_{\alpha_1}}{\sqrt{p^{b|a}_{\alpha_1\beta_1}p^{b|a}_{\beta_1\alpha_2}}}=\pm\frac{p^a_{\alpha_1}-1+p^b_{\beta_1}+p^{a|b}_{\alpha_1\beta_1}-2p^{a|b}_{\alpha_1\beta_1}p^a_{\alpha_1}}{\sqrt{p^{a|b}_{\alpha_1\beta_1}p^{a|b}_{\beta_1\alpha_2}}}
$$

$$
\Leftrightarrow \qquad (4.14)
$$

$$
p^{b|a}_{\alpha_1\beta_1}=p^{a|b}_{\alpha_1\beta_1}.
$$

Therefore we conclude that $\pm\cosh(\gamma_{\alpha_2}-\gamma_{\alpha_1})=\epsilon_{\lambda_{\beta_1}}\cosh\theta_{\beta_1}$ iff $\mathbf{P}^{b|a}=\mathbf{P}^{a|b}$. Let

$$
U^{a|b}_{b|a}=\begin{pmatrix}\sqrt{p^{b|a}_{\beta_1\alpha_1}} & \sqrt{p^{b|a}_{\beta_1\alpha_2}}\\ \sqrt{p^{b|a}_{\beta_2\alpha_1}} & -\sqrt{p^{b|a}_{\beta_2\alpha_2}}\end{pmatrix}. \qquad (4.15)
$$

We now show that this vector is equivalent to the vector $\psi^{a|b}$ given by (3.14).

$$
\begin{aligned}
U^{a|b}_{b|a}\psi^{b|a} &= \sqrt{p^a_{\alpha_1}}e^{a|b}_{\alpha_1}+\epsilon_{\lambda_{\beta_1}}e^{j\theta_{\beta_1}}\sqrt{p^a_{\alpha_2}}e^{a|b}_{\alpha_2} \qquad (4.16)\\
&= \sqrt{p^a_{\alpha_1}}e^{a|b}_{\alpha_1}+\epsilon_{\lambda_{\beta_1}}e^{j(\gamma_{\alpha_2}-\gamma_{\alpha_1})}\sqrt{p^a_{\alpha_2}}e^{a|b}_{\alpha_2}
\end{aligned}
$$

We use the fact that $\psi^{a|b}_{\alpha_i}=\pm\sqrt{p^a_{\alpha_i}}e^{j\gamma_{\alpha_i}}, i\in\{1,2\}$ into (3.14)

$$
\begin{aligned}
\psi^{a|b} &= \pm\sqrt{p^a_{\alpha_1}}e^{j\gamma_{\alpha_1}}e^{a|b}_{\alpha_1}\pm\sqrt{p^a_{\alpha_2}}e^{j\gamma_{\alpha_2}}e^{a|b}_{\alpha_2} \qquad (4.17)\\
&= \pm e^{i\gamma_{\alpha_1}}U^{a|b}_{b|a}\psi^{b|a}
\end{aligned}
$$

□

Thus the hyperbolic amplitudes $\psi^{a|b}$ and $U^{a|b}_{b|a}\psi^{b|a}$ differ only by the multiplicative factor $\pm e^{j\gamma_{\alpha_1}}$. Hence, they belong to the same equivalent class of vectors on the unit sphere. Thus they are two representatives of the same quantum state $\Psi^{b|a}$.

5 Appendix: Hyperbolic algebra and hyperbolic Hilbert space

5.1 Hyperbolic algebra

An element z belongs to the hyperbolic algebra $\mathbf{G}$ iff it has following form:

$$z = x + jy, \quad x, y \in \mathbf{R}$$

where $j^2 = 1$, $z_1 + z_2 = x_1 + x_2 + j(y_1 + y_2)$ and $z_1 z_2 = x_1 x_2 + y_1 + y_2 + j(y_1 x_2 + y_2 x_1)$. The hyperbolic conjugation is defined as $\bar{z} = x - jy$. We define the "square of the absolute value" as

$$|z|^2 = z\bar{z} = x^2 - y^2,$$

$|z|^2 \in \mathbf{G}$. In fact, $|z|^2 \in \mathbf{R}$. But $|z|$ is not well defined for z such that $|z|^2 < 0$. Therefore set

$$\mathbf{G}_+ = \{z \in : |z|^2 \geq 0\}.$$

and

$$\mathbf{G}_+^* = \{z \in : |z|^2 > 0\}.$$

We define the argument $\arg z$ of $z \in \mathbf{G}_+^*$ as

$$\arg z = \operatorname{arctanh} \frac{y}{x} = \frac{1}{2} \ln \frac{x + y}{x - y}.$$

Notice that $x \neq 0$, $x - y \neq 0$ and $\frac{x+y}{x-y} > 0$, since $z \in \mathbf{G}_+^*$.

5.2 Hyperbolic Hilbert space

A hyperbolic Hilbert space H is a $\mathbf{G}$-linear inner product space. Let $x, y, z \in H$ and $a, b \in \mathbf{G}$, then consider the inner product as a map from $H \times H \to \mathbf{G}$ having the following properties:

(1) Conjugate symmetry: $\langle x, y \rangle$ is the conjugate to $\langle y, x \rangle$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

(2) Linearity with respect to the first argument:

$$\langle ax + bz, y \rangle = a \langle x, y \rangle + b \langle z, y \rangle$$

(3) Nondegenerate:

$$\langle x, y \rangle = 0$$

for all $y \in H$ iff $x = 0$
In general, the norm $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$ is not well defined. But we will only need the square of the norm $\|\psi\|^2 = \langle \psi, \psi \rangle$.

5.3 Violation of Born's rule

Let us give a counterexample to illustrate the violation of Born's rule, if the transition probabilities matrix $\mathbf{P}^{b|a}$ is not doubly stochastic. We have that

$$\psi_\beta^{b|a} = \sqrt{p^a_{\alpha_1} p^{b|a}_{\beta\alpha_1}} \pm e^{j\theta_\beta}\sqrt{p^a_{\alpha_2} p^{b|a}_{\beta\alpha_2}}, \quad \beta \in X_b \tag{5.1}$$

and

$$\psi^{b|a} = \psi^{b|a}_{\beta_1} e^{b|a}_{\beta_1} + \psi^{b|a}_{\beta_2} e^{b|a}_{\beta_2}. \tag{5.2}$$

This will match Born's rule,

$$p^b_\beta = |\langle \psi^{b|a}, e^{b|a}_\beta \rangle|^2 \ , \beta \in X_b. \tag{5.3}$$

Moreover $p^b_{\beta_1} + p^b_{\beta_2} = 1$ and by (3.1)(3.1),

$$\begin{aligned} 1 = p^b_{\beta_1} + p^b_{\beta_2} = p^a_{\alpha_1}(p^{b|a}_{\beta_1\alpha_1} + p^{b|a}_{\beta_2\alpha_1}) + p^a_{\alpha_2}(p^{b|a}_{\beta_1\alpha_2} + p^{b|a}_{\beta_2\alpha_2}) \\ + 2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}(\lambda_1\sqrt{p^{b|a}_{\beta_1\alpha_1} p^{b|a}_{\beta_1\alpha_2}} + \lambda_2\sqrt{p^{b|a}_{\beta_2\alpha_1} p^{b|a}_{\beta_2\alpha_2}}). \end{aligned} \tag{5.4}$$

Let us select the transition probabilities matrix $\mathbf{P}^{b|a}$ not to be doubly stochastic, take $p^{b|a}_{\beta_1\alpha_1} = p^{b|a}_{\beta_1\alpha_2} = p$ and $p^{b|a}_{\beta_2\alpha_1} = p^{b|a}_{\beta_2\alpha_2} = q$ where $p+q=1,\ p \neq q,\ p,q > 0$. Then (5.4) becomes

$$1 = p^a_{\alpha_1} + p^a_{\alpha_2} + 2\sqrt{p^a_{\alpha_1} p^a_{\alpha_2}}(\lambda_1 p + \lambda_2 q) \Leftrightarrow \lambda_1 = -\frac{q}{p}\lambda_2 \tag{5.5}$$

Then (3.12) (3.11) will be

$$\psi^{b|a} = \sqrt{p^a_{\alpha_1}} e^{b|a}_{\alpha_1} \pm e^{j\theta_{\beta_1}}\sqrt{p^a_{\alpha_2}} e^{b|a}_{\alpha_2} \tag{5.6}$$

where

$$e^{b|a}_{\alpha_1} = \begin{pmatrix} \sqrt{p^{b|a}_{\beta_1\alpha_1}} \\ \sqrt{p^{b|a}_{\beta_2\alpha_1}} \end{pmatrix}, \quad e^{b|a}_{\alpha_2} = \begin{pmatrix} \sqrt{p^{b|a}_{\beta_1\alpha_2}} \\ -\frac{q}{p}\sqrt{p^{b|a}_{\beta_2\alpha_2}} \end{pmatrix}. \tag{5.7}$$

Thus $p^b_\beta = |\langle e^{b|a}_{\alpha_1}\rangle, e^{b|a}_{\alpha_2}\rangle|^2 = p - \frac{q^2}{p}$, where $e^{b|a}_{\alpha_1}$ and $e^{b|a}_{\alpha_2}$, are orthogonal, i.e. $|\langle e^{b|a}_{\alpha_1}\rangle, e^{b|a}_{\alpha_2}\rangle|^2 = 0$. This shows the violation of Born's rule by contradiction, since $p - \frac{q^2}{p} = 0 \Leftrightarrow p = \pm q$.

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