A Vector Field Method Approach to Improved Decay for Solutions to the Wave Equation on a Slowly Rotating Kerr Black Hole

Jonathan Luk

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Abstract

We prove that sufficiently regular solutions to the wave equation $\Box_{g_K}\Phi = 0$ on the exterior of a sufficiently slowly rotating Kerr black hole obey the estimates $|\Phi| \leq C(t^*)^{-\frac{3}{2}+\eta}$ on a compact region of r. This is proved with the help of a new vector field commutator that is analogous to the scaling vector field on Minkowski and Schwarzschild spacetime. This result improves the known robust decay rates that are proved using the vector field method in the region of finite r and along the event horizon.

Contents

1 Introduction

A major open problem in general relativity is that of the nonlinear stability of Kerr spacetimes. These spacetimes are stationary axisymmetric asymptotically flat black hole solutions to the vacuum Einstein equations

$$R_{\mu\nu} = 0$$

in $3+1$ dimensions. They are parametrized by two parameters (M, a), representing respectively the mass and the specific angular momentum of a black hole. (See Section 2). It is conjectured that Kerr spacetimes are asymptotically stable. In the framework of the initial value problem, the stability of Kerr would mean that for any solution to the vacuum Einstein equations with initial data close to the initial data of a Kerr spacetime, its maximal Cauchy development has an exterior region that approaches a nearby, but possibly different, Kerr spacetime.

In order to study the stability of Kerr spacetimes, it is important to first understand the corresponding linear problem. One way to approach this is to study the linear scalar wave equation $\Box_{g_K}\Phi = 0$, where g_K is the metric on a fixed Kerr background and $\Box_{g_K}$ is the Laplace-Beltrami operator. This can be compared with the proofs of the nonlinear stability of the Minkowski spacetime in which a robust understanding of the quantitative decay properties of solutions to the linear wave equation plays a fundamental role [6], [18].

The Kerr family of spacetimes contains a one-parameter subfamily known as the Schwarzschild spacetimes for which $a = 0$. It is natural when studying the wave equation on Kerr spacetimes to begin by focusing on the wave equation on Schwarzschild spacetimes. Pointwise boundedness and decay of the solutions to the wave equation on Schwarzschild spacetimes has been proved in [27], [15], [20], [4], [10], [17], [3] [11], [24]. In particular Dafermos and Rodnianski used the vector field method to show that on the exterior region of the Schwarzschild spacetimes, including along the event horizon, solutions to the linear wave equation satisfy $|\Phi| \leq C(t^*)^{-1}$, where t^* is a regular coordinate (up to the event horizon) that approaches infinity towards null infinity. In an earlier work [19], we improved this decay rate. More precisely, we showed that sufficiently regular solutions to the wave equation $\Box_g\Phi = 0$ on the Schwarzschild black hole obey the estimates $|\Phi| \leq C_\eta(t^*)^{-\frac{3}{2}+\eta}$ for any $\eta > 0$ on a compact region of r, including along the event horizon and inside the black hole.

This paper generalizes the above result to Kerr spacetimes where $a \ll M$. For Kerr spacetimes satisfying this condition, Dafermos and Rodnianski [7], and subsequently Andersson and Blue [2], have proved a decay rate in the exterior region of the Kerr spacetime, including along the event horizon, of $|\Phi| \leq C(t^*)^{-1+\eta}$, where t^* is a regular coordinate to be defined later, and with t^* we will define a foliation of the exterior region of Kerr spacetime by the spacelike hypersurfaces Σ_{t^*}. Extending the methods in [19], we are able to improve this decay rate using the vector field method. In particular, we have

Theorem 1. *Suppose $\Box_{g_K}\Phi = 0$. Then for all $\eta > 0$ and all $M > 0$ there exists a_0 such that the following estimates hold on Kerr spacetimes with (M, a) for which $a \leq a_0$:*

1. *Improved Decay of Non-degenerate Energy*

$$\sum_{j=0}^{N} \int_{\Sigma_{t^*} \cap \{r \leq R\}} \left(D^j \Phi\right)^2 \leq C_R E_N(t^*)^{-3+\eta}.$$

2. *Improved Pointwise Decay*

$$\sum_{j=0}^{N} |D^j \Phi| \leq C_R E'_N(t^*)^{-\frac{3}{2}+\eta} \quad for \ r \leq R.$$

Here, D denotes derivatives in a regular coordinate system (See Section 2). E_N and E'_N depend only on some weighted Sobolev norm of the initial data.

A more precise version of this theorem will be given in Section 6. Our proof relies on an analogue of the scaling vector field on Minkowski spacetime. Recall that in Minkowski spacetime the vector field $S = t\partial_t + r\partial_r$ is conformally Killing and satisfies $[\Box_m, S] = 2\Box_m$. Hence any estimates that hold for Φ a solution to $\Box_m \Phi = 0$ would also hold for $S\Phi$. However, S has a weight that is increasing with t. Hence one can hope to prove a better estimate for Φ using the estimates for $S\Phi$. (See, for example, [16]).

In [19], we introduced an analogue of the scaling vector field on Schwarzschild spacetimes. We defined, in the Regge-Wheeler tortoise coordinate (see Section 2), the vector field $S = t\partial_t + r^*\partial_{r^*}$. In [19], we studied the commutator $[\Box_{g_S}, S]$ and showed that all the error terms can be controlled. Thus, up to a loss of t^η (for η arbitrarily small), $S\Phi$ obeys all the estimates of Φ that were proved in [10]. In particular, we showed that $S\Phi$, like Φ itself, obeys a local integrated decay estimate

$$\int_{t'}^{t} \int_{r_1}^{r_2} (D^k \Phi)^2 dr dt \leq C E_k(t')^{-2} \quad for \ t' \leq t \leq (1.1)t'.,$$

$$\int_{t'}^{t} \int_{r_1}^{r_2} \left(SD^k \Phi\right)^2 dr dt \leq C E_k(t')^{-2+\eta} \quad for \ t' \leq t \leq (1.1)t'.$$

From this we proved an improved decay of the L^2 norm of $D^k \Phi$. We will explain the main idea in the case $k = 0$. Firstly, the local integrated decay for Φ would already imply that on a sequence of t_i slices, with $t_i \leq t_{i+1} \leq (1.1)^2 t_i$, Φ obeys a better decay rate, namely $\Phi(t_i) \leq C t_i^{-\frac{3}{2}}$. We then introduced a new method to use the estimates for $S\Phi$, which can be explained heuristically as follows. Given any time t, we find the largest $t_i \leq t$ that has a better decay rate. Then we integrated from t_i to t using the vector field S. Notice at this point that S has a weight that grows like t. Hence we have, at least schematically,

$$\int_{r_1}^{r_2} \Phi(t)^2 dr \leq C \left(\int_{r_1}^{r_2} \Phi(t_i)^2 dr + t^{-1} | \int_{t_i}^{t} \int_{r_1}^{r_2} S\left(\Phi^2\right) dr dt | \right).$$

We then notice that the last term can be estimated by the local integrated decay estimates

$$| \int_{t_i}^{t} \int_{r_1}^{r_2} S\left(\Phi^2\right) dr dt | \leq C \left(\int_{t_i}^{t} \int_{r_1}^{r_2} \Phi^2 dr dt + \int_{t_i}^{t} \int_{r_1}^{r_2} \left(S\Phi\right)^2 dr dt \right) \leq C t^{-2+\eta}.$$

Putting these together, we would get

$$\int_{r_1}^{r_2} \Phi(t)^2 dr \leq C t^{-3+\eta}.$$

Using this method, we also showed the improved decay for the L^2 norm of higher derivatives. Pointwise decay estimate thus followed from standard Sobolev embedding.

In this paper we would like to carry out a similar argument. We introduce a scaling vector field (which we again call S) which is the same as in [19] at the asymptotically flat end, but is smooth up to and across the event horizon. We will prove a local integrated decay estimate for $S\Phi$ and use the argument in [19] as outlined above to prove an improved decay rate. The most difficult part of the argument is to control the

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error terms coming from the commutation of $\Box_{g_K}$ and (the modified) S, i.e., the term $[\Box_{g_K}, S]\Phi$. To control this, we need to use estimates for derivatives of Φ, which in turn is provided by the energy estimates for the homogeneous equation $\Box_{g_K}\Phi = 0$ proved in [7] and [8]. This term schematically looks like

$$[\Box_{g_K}, S]\Phi = O(1)\Box_{g_K}\Phi + O(r^{-2+\delta})(D^2\Phi + D\Phi + rD\slashed{\nabla}\Phi). \tag{1}$$

The term $O(1)\Box_{g_K}\Phi$ vanishes since we are considering $\Box_{g_K}\Phi = 0$. We note that the other terms have the following two desirable features. Firstly, although S has a weight in t^*, the commutator is independent of t^*, which is a result of ∂_{t^*} being a Killing vector field. Secondly, these terms decay as $r \to \infty$, which is a result of the asymptotic flatness of Kerr spacetimes. The last term would appear to have less decay in r, which is also the case in Schwarzschild spacetimes. In that case, we controlled this term in [19] by commuting the equation with Ω, the generators of the spherical symmetry of Schwarzschild spacetimes. The quantity $\Omega\Phi$ would then give us control over an extra power of r. One difficulty that arises in the case of Kerr spacetimes is that they are not spherically symmetric. Nevertheless, following [8], we can construct an analog of Ω, call it $\tilde{\Omega}$, which is an asymptotic symmetry, i.e., the commutator $[\Box_{g_K}, \tilde{\Omega}]$ would decay in r. The non-degenerate energy of $\tilde{\Omega}\Phi$ would then control the last term in the above expression. Moreover, it is sufficient to define $\tilde{\Omega}$ only when r is very large since otherwise the factor in r can be absorbed by constants. Notice however that in a finite region of r, the commutator $[\Box_{g_K}, S]$ would in general be large.

In order to understand what quantities of $S\Phi$ have to be controlled, we re-derive the energy estimates in [8] in the slightly more general case of the inhomogeneous equation $\Box_{g_K}\Phi = G$. This would also immediately imply that for the linear inhomogeneous equation $\Box_{g_K}\Phi = G$ with sufficiently regular and sufficiently decaying (both in space and time) G, the solution Φ, assuming that the initial data is sufficiently regular, would decay with a rate of $(t^*)^{-1+\eta}$, precisely as that in [8]. We will then apply this to the equations for $\tilde{\Omega}\Phi$ and $S\Phi$. In order to derive these energy estimates, we will use the (non-Killing) vector field multipliers N and Z. N is a modification of ∂_{t^*} so that it is timelike everywhere, including near the event horizon. The use of N tackles the issue of superradiance, a difficulty that arises from the spacelike nature of ∂_{t^*} near the event horizon. Z is an analogue of the conformal vector field $u^2\partial_u + v^2\partial_v$ in Minkowski spacetime and is used to prove decay.

Since we will use vector field multipliers that has weights in t^* and r, in order to prove the energy estimates at $t^* = \tau$ for the inhomogeneous equation would have to control the term (as well as other similar or more easily controlled terms):

$$\iint_{\mathcal{R}(\tau_0, \tau)} (t^*)^2 r^{1+\delta} G^2,$$

where the integration over space and the t^* interval $[\tau_0, \tau]$. In order to prove the energy estimates for $S\Phi$, we need to show that for G as in (1), this is bounded by $C(\tau)^\eta$. We split this into two parts: $r \leq \frac{t^*}{2}$ and $r \geq \frac{t^*}{2}$. For $r \leq \frac{t^*}{2}$, first we notice that since G decays in r, we can replace $r^{1+\delta}$ by $r^{-3+2\delta}$. Then, we use the fact that

$$\sum_{k=1}^{N} \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r \leq \frac{t^*}{2}\}} r^{-1+\delta}(D^k\Phi)^2 \leq C\tau^{-2+\eta}.$$

Hence if we sum up the whole integral by integrating in $[\tau_0, (1.1)\tau_0]$, $[(1.1)\tau_0, (1.1)^2\tau_0]$ etc., we will get a bound of

$$\sum_{i=0}^{\lfloor \log \tau \rfloor + 1} (1.1)^i \tau_0 \sim_\eta \tau^\eta.$$

For $r \geq \frac{t^*}{2}$, we do not have a decay estimate for the integrated in time estimate. However, we would still have an almost boundedness estimate:

$$\sum_{k=1}^{N} \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r \geq \frac{t^*}{2}\}} r^{-1+\delta}(D^k\Phi)^2 \leq C\tau^\eta.$$

Notice, moreover, that $G^2 \sim r^{-3+\delta}(D^k\Phi)^2$ and this region we have $r^{-3+\delta} \leq (t^*)^{-2}r^{-1+\delta}$. Hence we again have

$$\sum_{k=1}^{N} \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r \geq \frac{t^*}{2}\}} G^2 \leq C\tau^{-2+\eta},$$

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and the required estimate followed in the same manner as in the case $r \leq \frac{t^*}{2}$.

With the modified S, which is smooth up to the event horizon (contrary to [19]), we can prove the improved decay estimates for the L^2 norm of Φ and $D\Phi$ once these error terms are controlled. Using the commutation with the Killing vector ∂_{t^*}, we would also have control for L^2 norm of $D\partial_{t^*}^k \Phi$. Away from the event horizon, this is sufficient to control all other derivatives by elliptic estimates. However, since near the event horizon, ∂_{t^*} is not Killing, we would not have control over other derivatives. Here, we follow [7] and [8] and commute the equation with a version of the red-shift vector field - $\hat{Y}$. Once we control $D\hat{Y}^k\partial_{t^*}^j \Phi$ we can use the wave equation to control (any derivatives of) $\slashed{\Delta}\Phi$, where $\slashed{\Delta}$ is elliptic on the sphere. We can thus control derivatives of Φ in any directions. We will show, moreover, that the commutator $[\Box_{g_K}, \hat{Y}]$ has the property that the inhomogeneous terms can be controlled once we have controlled the L^2 norm of $D\partial_{t^*}^k \Phi$. This implies that $\hat{Y}\Phi$ would decay in the same rate as $\partial_{t^*}\Phi$ for which we have already derived an improved decay rate.

We now turn to some history of this problem. We mention some results on Kerr spacetimes with $a > 0$ here and refer the readers to [8], [19] for references on the corresponding problem on Schwarzschild spacetimes. There has been a large literature on the mode stability and nonquantitative decay of azimuthal modes (See for example [22], [14], [28], [12], [13] and references in [8]). The first global result for the Cauchy problem was obtained by Dafermos-Rodnianski in [7], in which they proved that for a class of small, axisymmetric, stationary perturbations of Schwarzschild spacetime, which include Kerr spacetimes that rotate sufficiently slowly, solutions to the wave equation are uniformly bounded. Similar results were obtained later using an integrated decay estimate on slowly rotating Kerr spacetimes by Tataru-Tohaneanu [25]. Using the integrated decay estimate, Tohaneanu also proved Strichartz estimates [26].

Decay for general solutions to the wave equation on sufficiently slowly rotating Kerr spacetimes was first proved by Dafermos-Rodnianski [8] with a quantitative rate of $|\Phi| \leq C(t^*)^{-1+Ca}$. A similar result was later obtained by [2] using a physical space construction to obtain an integrated decay estimate. In all of [25], [8] and [2], the integrated decay estimate is proved and plays an important role. All proofs of such estimates rely heavily on the separability of the wave equation, or equivalently, the existence of a Killing tensor on Kerr spacetime. In a recent work [9], Dafermos-Rodnianski shows that assuming the integrated decay estimate (non-degenerate up to the event horizon if it exists) and boundedness for the wave equation on an asymptotically flat spacetime, the decay rate $|\Phi| \leq C(t^*)^{-1}$ holds. This in particular improves the rates in [8] and [2]. In a similar framework, but assuming in addition exact stationarity, Tataru [24] proved a local decay rate of $(t^*)^{-3}$ using Fourier-analytic methods. This applies in particular to sufficiently slowly rotating Kerr spacetimes. Dafermos and Rodnianski have recently announced a proof for the decay of solutions to the wave equation on the full range of sub-extremal Kerr spacetimes $a < M$.

In view of the nonlinear problem, it is important to understand decay in a robust manner. In particular, past experience shows that refined decay estimates might not be stable in nonlinear problems. The vector field method is known to be robust and culminated in the proof of the stability of the Minkowski spacetime [6], [18]. We prove our decay result using the vector field method with the expectation that the method will be useful for studying nonlinear problems. As a model problem, we will study the semilinear equation with a null condition on a fixed slowly rotating Kerr background. In a forthcoming paper that we will show the global existence of solutions with small initial data for this class of equations. We will also study the asymptotic behavior of these solutions. The null condition, which is a special structure of the nonlinearity, has served as an important model for the proofs of the nonlinear stability of Minkowski spacetime and we hope that it will find relevance to the problem of the nonlinear stability of Kerr spacetime.

We end the introduction with an overview of the paper. In Section 2, we will introduce the Kerr geometry, including a few different coordinate systems that we will find useful in the rest of the paper. In Section 4, we introduce the (non-Killing) vector field commutators that will be used in this paper. These include the scaling vector field S, which is the main tool for obtaining improved decay rates in this paper. In Section 5, we introduce the formalism for vector field multipliers. We then have all the notations necessary to state the precise form of our main theorem in Section 6. In Section 7, 8 and 9, we prove the main energy estimates using the vector field multipliers N, X and Z. We write down the energy estimates in the most general form, allowing for the possibility to control the inhomogeneous terms in different energy norms. Such generality is unnecessary for the result in this paper, but will be useful in studying the null condition. Starting from Section 10, we return to the homogeneous equation. In Section 10, we write down the energy estimates proved in [8]. We then derive the energy estimates after commuting with $\hat{Y}$, $\tilde{\Omega}$ and S in Sections 11, 12 and

13 respectively. Finally, using the estimates for $S\Phi$, we prove the main theorem in Section 14.

2   Geometry of Kerr Spacetime

2.1   Kerr Coordinates

The Kerr metric in the Boyer-Lindquist coordinates takes the following form:

$$
\begin{aligned}
g_K = & -\left(1 - \frac{2M}{r\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)}\right)dt^2 + \frac{1 + \frac{a^2\cos^2\theta}{r^2}}{1 - \frac{2M}{r} + \frac{a^2}{r^2}}dr^2 + r^2\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)d\theta^2 \\
& + r^2\left(1 + \frac{a^2}{r^2} + \left(\frac{2M}{r}\right)\frac{a^2\sin^2\theta}{r^2\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)}\right)\sin^2\theta d\phi^2 - 4M\frac{a\sin^2\theta}{r\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)}dtd\phi.
\end{aligned}
\tag{2}
$$

In this paper, we will consider Kerr spacetimes with a small. It can then be thought of as a small perturbation of Schwarzschild spacetimes because by setting $a = 0$, we retrieve the metric for Schwarzschild spacetimes:

$$
g_S = -\left(1 - \frac{2M}{r_S}\right)dt_S^2 + \left(1 - \frac{2M}{r_S}\right)^{-1}dr_S^2 + r_S^2 d\theta^2 + r_S^2\sin^2\theta d\phi^2.
$$

The Cauchy development of Kerr spacetimes can be described schematically by taking a two dimensional slice as in the following diagram:

Notice that (2) represents the metric on the exterior region ($\mathcal{R}$ in the diagram). These coordinates are not regular at the event horizon. It will be helpful in the sequel to use different coordinate systems on Kerr spacetimes. From now on we will call the coordinate system on which the metric (2) is defined the Kerr (t, r, θ, ϕ) coordinates. We define a new coordinate system - the Kerr (t, r^*, θ, ϕ) coordinates by

$$
\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta},
$$

where $\Delta = r^2 - 2Mr + a^2$ is zero at the event horizon. In this coordinate system, the metric looks like

$$
\begin{aligned}
g_K = & -\left(1 - \frac{2M}{r(1 + \frac{a^2\cos^2\theta}{r^2})}\right)dt^2 + \Delta\left(r^2 + a^2\right)^2\left(r^2 + a^2\cos^2\theta\right)dr^{*2} + r^2\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)d\theta^2 \\
& + r^2\left(1 + \frac{a^2}{r^2} + (\frac{2M}{r})\frac{a^2\sin^2\theta}{r^2(1 + \frac{a^2\cos^2\theta}{r^2})}\right)\sin^2\theta d\phi^2 - 4M\frac{a\sin^2\theta}{r(1 + \frac{a^2\cos^2\theta}{r^2})}dtd\phi.
\end{aligned}
$$

Notice that since the definition of r^* depends only on r, it would be unambiguous to talk about the vector ∂_t.

2.2   Schwarzschild Coordinates

In order to compare calculations on Kerr spacetimes to calculations on Schwarzschild spacetimes, it is helpful to exhibit a diffeomorphism between the two. We do so by defining an explicit map between the coordinate functions (t, r, θ, ϕ) on a Kerr spacetime and the coordinate functions $(t_S, r_S, \theta_S, \phi_S)$ on a Schwarzschild spacetime with the same mass. These will be defined differently near and away from the event horizon. Take

$$
\chi(r) = \begin{cases} 1 & r \leq r_Y^- - \frac{r_Y^- - r_+}{2} \\ 0 & r \geq r_Y^- - \frac{r_Y^- - r_+}{4} \end{cases},
$$

where r_+, as above, is the larger root of $\Delta = r^2 - 2Mr + a^2$ and $r_Y^- > r_+$ is a constant to be determined later. With this $\chi(r)$, we can then define

$$
r_S^2 - 2Mr_S = r^2 - 2Mr + a^2,
$$

$$t_S + \chi(r_S)2M \log(r_S - 2M) = t + \chi(r)h(r), \quad \text{where } \frac{dh(r)}{dr} = \frac{2Mr}{r^2 - 2Mr + a^2},$$

$$\theta_S = \theta,$$

$$\phi_S = \phi + \chi(r)P(r), \quad \text{where } \frac{dP(r)}{dr} = \frac{a}{r^2 - 2Mr + a^2}.$$

Then, by identifying $(t_S, r_S, \theta_S, \phi_S)$ with the corresponding coordinate functions on Schwarzschild spacetimes, we have a diffeomorphism between Kerr spacetimes and Schwarzschild spacetimes. This coordinate system will be used and will be called the Schwarzschild $(t_S, r_S, \theta_S, \phi_S)$ coordinates on Kerr spacetimes. Once we have this diffeomorphism, we can put any system of Schwarzschild coordinates on Kerr spacetimes. These include the Schwarzschild $(t_S^*, r_S, \theta_S, \phi_S)$ coordinates, where $t_S^* = t_S + \chi(r_S)2M \log(r_S - 2M)$ and r_S, θ_S, ϕ_S are defined as above. We also define $t^* = t_S^* = t_S + \chi(r_S)2M \log(r_S - 2M)$ and use the Kerr (t^*, r, θ, ϕ^*) coordinates. Notice that $\partial_{t^*} = \partial_{t_S^*}$.

It is common to denote on Schwarzschild spacetimes $\mu = \frac{2M}{r_S}$. We will take the same notation on Kerr spacetimes, with the understand that it is always defined with respect to the Schwarzschild r_S coordinates. In particular $(1 - \mu)$ approaches 0 as $r \to r_+$ (the event horizon).

Another system of Schwarzschild coordinates can be defined by introducing a system of coordinates (x_S^A, x_S^B) on the standard unit 2-sphere such that $\partial_{x_S^A}, \partial_{x_S^B}$ are orthonormal with respect to the standard metric. We then define the Schwarzschild $(t_S^*, r_S, x_S^A, x_S^B)$ coordinates in the obvious manner. Using this coordinate system and the diffeomorphism as above, we have, for small a:

$$|(g_K)_{\alpha\beta} - (g_S)_{\alpha\beta}| \le \epsilon r^{-2}. \tag{3}$$

This smallness assumption will be used throughout this paper.

2.3 Null Frame near Event Horizon

Some extra cancellations for the estimates near the event horizon are best captured using a null frame. Hence we define a null frame $\{\hat{V}, \hat{Y}, E_1, E_2\}$ in the region $r \le r_Y^-$, where r_Y^- is to be determined later. On the event horizon,

$$V = \partial_{t^*} + \frac{a}{2Mr_+}\partial_{\phi^*}$$

is the Killing null generator. A direct computation shows that it satisfies

$$\nabla_V V = \kappa V,$$

where $\kappa : \mathcal{H} \to \mathbb{R}_+$ is strictly positive on the event horizon. We want to extend V to a null frame. On the event horizon, define $\hat{Y}$ first on a 2-sphere given by a fixed t^* to be null, orthogonal to the 2-sphere and require that $g_K(V, \hat{Y}) = -2$. Define also an orthonormal frame $\{E_1, E_2\}$ tangent to the fixed 2-sphere. Then push forward $\hat{Y}, E_1, E_2$ using the 1-parameter family of isomorphisms generated by the Killing vector field V. We thus get a null frame $\{V, \hat{Y}, E_1, E_2\}$ on the event horizon. We then extend this null frame off a fixed 2-sphere on the event horizon (with $\hat{V}|_{\mathcal{H}} = V$) by requiring

$$\nabla_{\hat{Y}} \hat{Y} = \nabla_{\hat{Y}} \hat{V} = \nabla_{\hat{Y}} E_A = 0, \tag{4}$$

where $A \in \{1, 2\}$. Then extend using the isomorphisms generated by V. We notice that the above equations hold everywhere. If we choose r_Y^- close enough to r_+, we still have, by Taylor's Theorem,

$$\nabla_{\hat{V}} \hat{V} = \kappa \hat{V} + b^Y \hat{Y} + b^1 E_1 + b^2 E_2, \tag{5}$$

where κ is a strictly positive number and $|b^\alpha| \le C(1 - \mu)$. Using the fact that $\hat{V}$ is invariant under the isomorphisms generated by V, one can choose the lower bound of κ to be independent of t^*.

The same construction in Schwarzschild spacetime would yield, in the Schwarzschild (t^*, r, θ, ϕ^*) coordinates

$$\hat{V} = (1 + \mu)\partial_{t_S^*} + (1 - \mu)\partial_{r_S}, \hat{Y} = \partial_{t_S^*} - \partial_{r_S}, E_1 = \frac{1}{r_S}\partial_{\theta_S}, E_2 = \frac{1}{r_S \sin\theta_S}\partial_{\phi_S}$$

Since we consider Kerr spacetimes on which the metric is close to that on a Schwarzschild spacetime, we have that, in (t^*, r, θ, ϕ^*) coordinates, the null frame can be expressed as

$$\hat{V} = (1 + \mu)\partial_{t^*} + (1 - \mu)\partial_r + O_1(\epsilon)\partial,$$

$$\hat{Y} = \partial_{t^*} - \partial_r + O_1(\epsilon)\partial,$$

$$E_1 = \frac{1}{r}\partial_\theta + O_1(\epsilon)\partial,$$

$$E_2 = \frac{1}{r\sin\theta}\partial_{\phi^*} + O_1(\epsilon)\partial.$$

Alternatively, if we write E_α, where $\alpha = 1, 2, 3, 4$, for the null frame, we have

$$(1 + \mu)\partial_{t^*} + (1 - \mu)\partial_r = \hat{V} + O_1(\epsilon)E_\alpha,$$

$$\partial_{t^*} - \partial_r = \hat{Y} + O_1(\epsilon)E_\alpha,$$

$$\partial_\theta = rE_1 + O_1(\epsilon)E_\alpha,$$

$$\partial_{\phi^*} = r\sin\theta E_2 + O_1(\epsilon)E_\alpha.$$

We also define the vector fields $\hat{V}, \hat{Y}, E_1, E_2$ outside $\{r \leq r_Y^-\}$ by requiring them to be compactly supported in $\{r \leq r_Y^+\}$ (for some r_Y^+ to be determined) and invariant under the one parameter families of isometries generated by ∂_{t^*} and ∂_{ϕ^*}. Notice that in particular there is no requirement on the metric components with respect to the vector fields in the region $\{r_Y^- \leq r \leq r_Y^+\}$.

3   Notations

3.1   Constants

Throughout this paper, we will use C to denote a large constant and c to denote a small constant. They can be different from line to line. We will also use A to denote bootstrap constants and we think of A to be large, i.e., $A \gg C$. We also use the notation $O_i(1)$ and $O_i(\epsilon)$ to denote terms that are bounded up to a constant by 1 and ϵ, with bounds that improve by r^{-1} for each derivative up to the i-th derivative. We will also use the notation $f \sim g$ to denote $cf \leq g \leq Cf$.

There are some constants that we will choose in the proof. The following are values of r in the Kerr coordinates:
$$r_+ < r_Y^- < r_Y^+ < \frac{11M}{4} < R_\Omega.$$
r_Y^- and r_Y^+ will be fixed in Remarks 4 and 3 respectively.

There are also smallness parameters which can be thought of as obeying

$$0 < \delta < \epsilon \ll \eta \ll e.$$

ϵ will always be used to denote the smallness of the specific angular momentum a of the spacetimes that we are working on. $\eta \sim C\epsilon$ will denote the loss in the decay rate of the solutions to the wave equation as compared to that on Schwarzschild spacetimes. e will be used to construct the non-degenerate energy. δ will simply be used as a small parameter whenever it is needed in the analysis. δ needs not be fixed from line to line.

3.2   Values of t^*

We will adopt the following as much as possible: t^* will denote a general value of t^*. In particular, it will be used as integration variables. τ will denote the t^* value for which we want an estimate. τ_0 will denote the t^* value where the initial data is posed. We will always assume $\tau_0 \geq 1$ and the reader can think of $\tau_0 = 1$. When integrating, we will often denote the endpoints by τ' and τ. Finally, at a few places we will need to choose a particular value of t^* in an interval. This is usually done to achieve the max or min of the energy quantities. Such choices will often be denoted as $\tilde{\tau}$.

3.3 Integration

Definition 1. *Define the following sets:*

1. $\Sigma_\tau = \{t^* = \tau\}$

2. $\mathcal{R}(\tau', \tau) = \{\tau' \leq t^* \leq \tau\}$

3. $\mathcal{H}(\tau', \tau) = \{r = r_+, \tau' \leq t^* \leq \tau\}$

When integrating on these sets, we will always integrate with respect to the volume form which we suppress. On Σ_τ the volume form is $\sqrt{det g_K}|_{\Sigma_\tau}$. On $\mathcal{R}(\tau', \tau)$, the volume form is $\sqrt{-det g_K}$. However, on the event horizon $\mathcal{H}(\tau', \tau)$, the surface is null and the metric is degenerate. Nevertheless, on $\mathcal{H}(\tau', \tau)$, the integrand will always be of the form $J_\mu n^\mu_{\Sigma_{\mathcal{H}^+}}$, where $n^\mu_{\Sigma_{\mathcal{H}^+}}$ is the normal to $\mathcal{H}(\tau', \tau)$. We will hence take the volume form corresponding to the (arbitrarily) chosen normal. Occasionally, we will also integrate over the topological 2-spheres given by fixing t and r. We will denote the volume form by $dA = \sqrt{det g_K}|_{\mathbb{S}^2}$.

When we write the integrals, we will often use $\iint$ to denote an integral over a spacetime region and use $\int$ denote an integral over a spacelike or null hypersurface.

Notice that the volume form on Σ_{t^*} can be compared with that on $\mathcal{R}(\tau', \tau)$. In particular, we have

$$\iint_{\mathcal{R}(\tau', \tau)} f \sim \int_{\tau'}^{\tau} \left(\int_{\Sigma_{t^*}} f \right) dt^*.$$

4 Vector Field Commutators

4.1 Vector Field Commutators under Metric Perturbations

Vector field commutators are stable when the metric is perturbed:

Proposition 1. *Consider either the Schwarzschild $(t_S^*, r_S, x_S^A, x_S^B)$ coordinates or $(t_S, r_S \geq r_Y^-, x_S^A, x_S^B)$ coordinates. Suppose V is a vector field defined on either of these coordinates. Then*

$$|[\Box_{g_K} - \Box_{g_S}, V]\Phi| \leq C\epsilon r^{-2}(\sum_{m=1}^{2} \sum_{k=1}^{2} \max_\alpha |\partial^m V^\alpha||\partial^k \Phi|),$$

where ∂ is the coordinate derivative for the coordinate system on which V is defined.

Proof. We rewrite
$$\Box_{g_S} = g_S^{\alpha\beta} \partial_\alpha \partial_\beta + \eta_S^\alpha \partial_\alpha,$$
$$\Box_{g_K} = g_K^{\alpha\beta} \partial_\alpha \partial_\beta + \eta_K^\alpha \partial_\alpha.$$
Using $|(g_K)_{\alpha\beta} - (g_S)_{\alpha\beta}| \leq \epsilon r^{-2}$ and $|\partial_\gamma((g_K)_{\alpha\beta} - (g_S)_{\alpha\beta})| \leq \epsilon r^{-2}$, we have $|\sqrt{-\det g_K} - \sqrt{-\det g_S}| \leq \epsilon r^{-2}$ and $|\partial_\alpha(\sqrt{-\det g_K} - \sqrt{-\det g_S})| \leq \epsilon r^{-2}$. Therefore,

$$\sup_{\alpha,\beta} |g_S^{\alpha\beta} - g_K^{\alpha\beta}| + \sup_\alpha |\eta_S^\alpha - \eta_K^\alpha| \leq C\epsilon r^{-2}.$$

Therefore,

$$|[\Box_{g_K} - \Box_{g_S}, V]\Phi| \leq |(g_K^{\alpha\beta} - g_S^{\alpha\beta})(\partial_\alpha \partial_\beta V^\gamma)\partial_\gamma \Phi| + 2|(g_K^{\alpha\beta} - g_S^{\alpha\beta})\partial_\alpha V^\gamma \partial_\beta \partial_\gamma \Phi| + |(\eta_S^\alpha - \eta_K^\alpha)(\partial_\alpha V^\gamma)\partial_\gamma \Phi|$$

$$\leq C\epsilon r^{-2}(\sum_{m=1}^{2} \sum_{k=1}^{2} \sup_\alpha |\partial^m V^\alpha||\partial^k \Phi|).$$

$\square$

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4.2 Commutator S

We construct a commuting vector field S on Schwarzschild that is different from [19] and is stable under perturbation.

Define $S = t_S^* \partial_{t_S^*} + h(r_S)\partial_{r_S}$, where $h(r) = \begin{cases} r - 2M & r \sim 2M \\ (r + 2M\log(r - 2M) - 3M - 2M\log M)(1 - \mu) & r \geq R \end{cases}$,

for some large R and is interpolated so that it is smooth and non-negative. Notice that for $r \geq R$, since $t^* = t$, this agrees with the definition in [19]. Therefore we have

$$[\Box_{g_S}, S] = \left(2 + \frac{r^*\mu}{r}\right)\Box_g + \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*} + 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\slashed{\Delta}. \tag{6}$$

In the coordinates (t^*, r, θ, ϕ),

$$\Box_{g_S} = -\alpha_1(r)\partial_t^2 + \alpha_2(r)\partial_r^2 + \alpha_3(r)\partial_r\partial_t + \alpha_4(r)\partial_t + \alpha_5(r)\partial_r + \slashed{\Delta}.$$

The crucial observation is that all α_i are smooth and bounded and depend only on r. Noting that α_i does not depend on t, we have

$$[\Box_{g_S}, S] = \beta_1(r)\partial_t^2 + \beta_2(r)\partial_r^2 + \beta_3(r)\partial_r\partial_t + \beta_4(r)\partial_t + \beta_5(r)\partial_r + \beta_6(r)\slashed{\Delta}.$$

Again, it is important to note that all β_i are smooth, bounded and depend only on r. The form of β_i for $r \geq R$ is given by (6).

We consider the same vector field S on Kerr. Using Proposition 1, and noting that $\partial^m S^\alpha$ is bounded for $m \geq 1$, we have

$$\left|[\Box_{g_K}, S]\Phi - \left(2 + \frac{r^*\mu}{r}\right)\Box_g\Phi - \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*}\Phi - 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\slashed{\Delta}\Phi\right| \leq C\epsilon r^{-2}(\sum_{k=1}^{2}|\partial^k\Phi|).$$

4.3 Commutator $\tilde{\Omega}_i$

Let Ω_i be a basis of vector fields of rotations in Schwarzschild spacetimes. An explicit realization can be $\Omega = \partial_\phi, \sin\phi\partial_\theta \pm \frac{\cos\phi\cos\theta}{\sin\theta}\partial_\phi$. Define $\tilde{\Omega}_i = \chi(r)\Omega_i$ to be cutoff so that it is supported in $\{r > R_\Omega\}$ and equals Ω_i for $r > R_\Omega + 1$ for some large R. On Schwarzschild spacetimes, Ω_i is Killing and therefore $\tilde{\Omega}_i$ is Killing for $r > R + 1$. Therefore,

$$[\Box_{g_S}, \tilde{\Omega}_i] = \tilde{\chi}(r)\left(\partial^2 + \partial\right),$$

where $\tilde{\chi}$ is some function depending only on r and is supported in $\{R_\Omega < r < R_\Omega + 1\}$.

Using Proposition 1, we have

$$|[\Box_{g_K}, \tilde{\Omega}_i]\Phi| \leq Cr^{-2}\left(|\partial^2\Phi| + |\partial\Phi|\right).$$

Moreover, since $\tilde{\Omega}_i$ vanishes for $r < R_\Omega$, we have trivially,

$$[\Box_{g_K}, \tilde{\Omega}_i]\Phi = 0, \quad \text{for } r < R_\Omega.$$

From now on, we will write $\tilde{\Omega}$ to denote $\sum_i \tilde{\Omega}_i$. We would like to point out that this commutator is useful to gain powers of r near spatial infinity. In particular we have

$$|\slashed{\nabla}\Phi| \leq \frac{C}{r}|\tilde{\Omega}\Phi|.$$

This extra power of r is essential to control the error terms arising from the commutation of $\Box_{g_K}$ with S.

4.4 Commutator $\hat{Y}$

Let $\hat{Y}$, as in Section 2.3, be a vector field that is null near the event horizon, normalized with respect to another null vector $\hat{V}$ and is cut off to be compactly supported in $\{r \le r_Y^+\}$.

Proposition 2. *On Kerr spacetimes such that ϵ is small, we have for $r \le r_Y^-$,*

$$|[\Box_{g_K}, \hat{Y}]\Phi - 4\kappa \hat{Y}^2 \Phi| \le C \left(|D\partial_{t^*}\Phi| + \epsilon |D^2\Phi| + |D\Phi| \right),$$

where $\kappa > c > 0$ is as in (5).

Proof. The principal term for the commutator $[\Box_{g_K}, \hat{Y}]\Phi$ is $2(\pi^{\hat{Y}})^{\mu\nu} D_\mu D_\nu \Phi$. We look at two terms that are useful in deriving the estimates.

$$\pi_{\hat{V}\hat{V}}^{\hat{Y}} = g(\nabla_{\hat{V}}\hat{Y}, \hat{V}) = -g(\hat{Y}, \nabla_{\hat{V}}\hat{V}) = 2\kappa$$

$$|\pi_{\hat{V}E_A}^{\hat{Y}}| = |\frac{1}{2}\left(g(\nabla_{\hat{V}}\hat{Y}, E_A) + g(\nabla_{E_A}\hat{Y}, \hat{V}) \right)| \le \epsilon$$

$\square$

5 The Basic Identities for Currents

5.1 Vector Field Multipliers

We consider the conservation laws for Φ satisfying $\Box_g \Phi = 0$. Define the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \Phi \partial_\alpha \Phi.$$

We note that $T_{\mu\nu}$ is symmetric and the wave equation implies that

$$\nabla^\mu T_{\mu\nu} = 0.$$

Given a vector field V^μ, we define the associated currents

$$J_\mu^V(\Phi) = V^\nu T_{\mu\nu}(\Phi),$$

$$K^V(\Phi) = \pi_{\mu\nu}^V T^{\mu\nu}(\Phi),$$

where $\pi_{\mu\nu}^V$ is the deformation tensor defined by

$$\pi_{\mu\nu}^V = \frac{1}{2} \left(\nabla_\mu V_\nu + \nabla_\nu V_\mu \right).$$

In particular, $K^V(\Phi) = \pi_{\mu\nu}^V = 0$ if V is Killing. Since the energy-momentum tensor is divergence-free,

$$\nabla^\mu J_\mu^V(\Phi) = K^V(\Phi).$$

We also define the modified currents

$$J_\mu^{V,w}(\Phi) = J_\mu^V(\Phi) + \frac{1}{8} \left(w \partial_\mu \Phi^2 - \partial_\mu w \Phi^2 \right),$$

$$K^{V,w}(\Phi) = K^V(\Phi) + \frac{1}{4} w \partial^\nu \Phi \partial_\nu \Phi - \frac{1}{8} \Box_g w \Phi^2.$$

Then

$$\nabla^\mu J_\mu^{V,w}(\Phi) = K^{V,w}(\Phi).$$

We integrate by parts with this in the region bounded by Σ_τ, $\Sigma_{\tau'}$ and $\mathcal{H}^+(\tau', \tau)$. We denote this region as $\mathcal{R}(\tau', \tau)$. Denoting the future-directed normal to Σ_τ by $n_{\Sigma_\tau}^\mu$, we have

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Proposition 3.

$$\int_{\Sigma_\tau} J_\mu^V(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^V(\Phi)\, n_{\mathcal{H}^+(\tau',\tau)}^\mu + \iint_{\mathcal{R}(\tau',\tau)} K^V(\Phi) = \int_{\Sigma_{\tau'}} J_\mu^V(\Phi)\, n_{\Sigma_{\tau'}}^\mu.$$

$$\int_{\Sigma_\tau} J_\mu^{V,w^V}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}^+(\tau',\tau)} J_\mu^{V,w^V}(\Phi)\, n_{\mathcal{H}^+(\tau',\tau)}^\mu + \iint_{\mathcal{R}(\tau',\tau)} K^{V,w^V}(\Phi) = \int_{\Sigma_{\tau'}} J_\mu^{V,w^V}(\Phi)\, n_{\Sigma_{\tau'}}^\mu.$$

One can similarly define the above quantities for the inhomogeneous wave equation $\Box_g \Phi = F$. In this case, the energy-momentum is no longer divergence free. Instead, we have

$$\nabla^\mu T_{\mu\nu} = F\partial_\nu \Phi.$$

In this case,

$$\nabla^\mu J_\mu^V(\Phi) = K^V(\Phi) + FV^\nu \partial_\nu \Phi.$$

For the modified current,

$$\nabla^\mu J_\mu^{V,w}(\Phi) = K^{V,w}(\Phi) - \frac{1}{4}Fw\Phi + FV^\nu \partial_\nu \Phi.$$

Proposition 4.

$$\int_{\Sigma_\tau} J_\mu^V(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^V(\Phi)\, n_{\mathcal{H}^+(\tau',\tau)}^\mu + \iint_{\mathcal{R}(\tau',\tau)} K^V(\Phi)$$

$$= \int_{\Sigma_{\tau'}} J_\mu^V(\Phi)\, n_{\Sigma_{\tau'}}^\mu - \int_{\mathcal{R}(\tau',\tau)} FV^\nu \partial_\nu \Phi.$$

$$\int_{\Sigma_\tau} J_\mu^{V,w^V}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{V,w^V}(\Phi)\, n_{\mathcal{H}^+(\tau',\tau)}^\mu + \iint_{\mathcal{R}(\tau',\tau)} K^{V,w^V}(\Phi)$$

$$= \int_{\Sigma_{\tau'}} J_\mu^{V,w^V}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \int_{\mathcal{R}(\tau',\tau)} \left(\frac{1}{4}Fw\Phi - FV^\nu \partial_\nu \Phi\right).$$

5.2 Vector Field Multipliers under Metric Perturbations

If we consider Kerr spacetimes such that ϵ is small, vector fields multipliers that are defined in the Schwarzschild (t^*, r, x^A, x^B) coordinates or Schwarzschild $(t, r \geq r_Y^-, x^A, x^B)$ coordinates are stable. We can consider a fixed vector field defined on the differentiable structure of a Schwarzschild exterior and compare the currents obtained using the Schwarzschild metric and the Kerr metric.

Proposition 5. *Consider either the Schwarzschild $(t_S^*, r_S, x_S^A, x_S^B)$ coordinates or $(t_S, r_S \geq r_Y^-, x_S^A, x_S^B)$ coordinates. Suppose V is a vector field defined on either of these coordinates.*

$$|(J_S^{V,w^V})_\mu(\Phi)\, n_{\Sigma_\tau}^\mu - (J_K^{V,w^V})_\mu(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C\epsilon r^{-2} \max_\alpha |V^\alpha|(\partial\Phi)^2,$$

$$|K_S^{V,w^V}(\Phi) - K_K^{V,w^V}(\Phi)| \leq C\epsilon r^{-2}\left((\sum_{k=0}^1 \max_\alpha |\partial^k V^\alpha| + |w|)(\partial\Phi)^2 + \sum_{m=1}^2 |\partial^m w||\Phi^2\right)$$

6 Statement of Main Theorem

With the currents defined, we can state our main theorem.

Main Theorem 1. *Suppose $\Box_{g_K}\Phi = 0$. Then for all $\eta > 0$, $R > r_+$ and all $M > 0$ there exists a_0 such that the following estimates hold in the region $\{r_+ \leq r \leq R\}$ on Kerr spacetimes with (M, a) for which $a \leq a_0$:*

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1. Improved Decay of Non-degenerate Energy

$$\sum_{j=0}^{\ell} \int_{\Sigma_\tau \cap \{r \le R\}} \left(D^j \Phi\right)^2$$

$$\le C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\le\ell+5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).$$

2. Improved Pointwise Decay

$$\sum_{j=0}^{\ell} |D^j \Phi|$$

$$\le C_R \tau^{-\frac{3}{2}+\eta} \left(\sum_{m=0}^{\ell+4} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\le\ell+7} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right)^{\frac{1}{2}}.$$

Here the vector field N will be defined in Section 7; the vector field Z with the modifying function w^Z will be defined in Section 9.

Remark 1. *We will show that although $J_\mu^{Z,w^Z}(\Phi) n_{\Sigma_{t^*}}^\mu$ is not always positive, $J_\mu^{Z+CN,w^Z}(\Phi) n_{\Sigma_{t^*}}^\mu$ is for sufficiently large C. Hence all the energy quantities in the Theorem are positive.*

Remark 2. *Since we have the improved decay of the non-degenerate energy, the above Theorem can be extended beyond the event horizon. More precisely, for any $r_b \in (r_-, r_+)$, where r_- is the smaller root of $\Delta = r^2 - 2Mr - a^2$, the theorem holds up to $r \ge r_b$ for D understood as a regular derivative inside the black hole, and with the constant depending also on r_b. The proof would be similar to that in [19].*

7 Vector Field Multiplier N_e and Mild Growth of Non-degenerate Energy

Kerr spacetime has a Killing vector field ∂_t. The conservation law gives that

$$\int_{\Sigma_\tau} J_\mu^T(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^T(\Phi) n_{\Sigma_\tau}^\mu = \int_{\Sigma_{\tau_0}} J_\mu^T(\Phi) n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)} \partial_t \Phi G.$$

We add to the Killing vector field ∂_t a red-shift vector field. Here, we use the "non-regular" red-shift vector field as in [8]. Under this construction, N_e is C^0 but not C^1 at the event horizon $\mathcal{H}^+$. Compared to the smooth construction in [7], this construction would provide extra control for some derivatives near $\mathcal{H}^+$.

Define
$$Y = y_1(r) \hat{Y} + y_2(r) \hat{V},$$

where
$$y_1(r) = 1 + \frac{1}{(\log(r-r_+))^3},$$

$$y_2(r) = \frac{1}{(\log(r-r_+))^3}.$$

Notice that by this definition Y is compactly supported in $\{r \le r_Y^+\}$ and is invariant under the isomorphisms generated by ∂_{t^*} and ∂_{ϕ^*}.

In $r \le r_Y^-$, i.e., where $\hat{Y}$ is null and obeys (4),

$$\nabla_Y Y = \left(1 + \frac{1}{(\log(r-r_+))^3}\right) \left(\frac{-\nabla_{\hat{Y}} r}{(r-r_+)(\log(r-r_+))^4}\right) \left(\hat{Y} + \hat{V}\right).$$

Notice that $0 < c < \nabla_{\hat{Y}} r < C$ on the event horizon with c, C independent of t^*. Let $N_e = \partial_{t^*} + eY$..

Proposition 6. *For any e, there is a corresponding choice of $\epsilon \ll e$ and r_Y^- such that*

$$J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu \sim \sum_\alpha (\nabla_\alpha \Phi)^2,$$

$$K^{N_e}(\Phi) \geq c_p e\left(|\log(r - r_+)|^p \left((\nabla_{\hat{V}}\Phi)^2 + \sum_A (\nabla_{E_A}\Phi)^2\right) + (\nabla_{\hat{Y}}\Phi)^2\right), \quad \text{for } r \leq r_Y^-,$$

$$K^{N_e}(\Phi) \leq Ce J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu, \quad \text{for } r_Y^- \leq r \leq r_Y^+.$$

Proof. It is obvious that Y is timelike and future oriented for $r \leq r_Y^-$. Since ∂_{t^*} is casual in the exterior region of Schwarzschild spacetime and is null only on the event horizon, for every small $e > 0$, there exists sufficiently small $\epsilon > 0$ such that N_e is timelike and future directed on Kerr spacetimes. Hence we have

$$J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu \sim \sum_\alpha (\nabla_\alpha \Phi)^2,$$

To show that $K^{N_e}(\Phi)$ has the required positivity near the event horizon, we notice that

$$\nabla_{\hat{V}} Y = -\kappa \hat{V} + O(\epsilon) E_\alpha,$$

and

$$\nabla_{\hat{Y}} Y \sim -\frac{1}{(r - r_+)|\log(r - r_+)|^7}\left(\hat{Y} + \hat{V}\right) + O(\epsilon)E_\alpha.$$

Therefore, for $r \leq r_Y^-$,

$$K^Y(\Phi)$$

$$\geq \kappa \left(\nabla_{\hat{Y}}\Phi\right)^2 + \frac{c}{(r - r_+)|\log(r - r_+)|^7}\left((\nabla_{\hat{V}}\Phi)^2 + \sum_A (\nabla_{E_A}\Phi)^2\right) - O(\epsilon)(\nabla\Phi)^2 - O(1)(\nabla_{E_A}\Phi\nabla\Phi)$$

Since ∂_{t^*} is Killing, $K^{N_e}(\Phi) = eK^Y(\Phi)$, we have

$$K^{N_e}(\Phi) \geq c_p e\left(|\log(r - r_+)|^p \left((\nabla_{\hat{V}}\Phi)^2 + \sum_A (\nabla_{E_A}\Phi)^2\right) + (\nabla_{\hat{Y}}\Phi)^2\right), \quad \text{for } r \leq r_Y^-,$$

Finally, since in the region $r_Y^- \leq r \leq r_Y^+$, $J_\mu^{\partial_{t^*}} n_{\Sigma_{t^*}}^\mu$ controls all derivatives, we have

$$K^{N_e}(\Phi) \leq Ce J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu, \quad \text{for } r_Y^- \leq r \leq r_Y^+.$$

$\square$

Definition 2. *We call the positive quantity $\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu$ the non-degenerate energy.*

The following identity determines how the non-degenerate energy changes with τ.

Proposition 7. *Let Φ satisfy $\Box_{g_K}\Phi = G$. Then*

$$\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r \leq r_Y^-\}} K_\mu^{N_e}(\Phi)$$

$$= \int_{\Sigma_{\tau_0}} J_\mu^{N_e}(\Phi) \, n_{\Sigma_{\tau_0}}^\mu + e\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^- \leq r \leq r_Y^+\}} K^Y(\Phi) + \int_{\mathcal{R}(\tau_0,\tau)} (\partial_{t^*}\Phi + eY\Phi) \, G.$$

The estimates given by the vector field N is sufficient to show that modulo inhomogeneous terms, the quantity $\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) \, n_{\Sigma_\tau}^\mu$ cannot grow too much in a short time interval:

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Proposition 8. *Let Φ satisfy $\square_{g_K}\Phi = G$. For e sufficiently small, $\epsilon \ll e$ and $0 \leq \tau - \tau' \leq 1$, we have*

$$\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_\tau}^\mu \leq 2 \int_{\Sigma_{\tau'}} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + \iint_{\mathcal{R}(\tau',\tau)} G^2.$$

Proof. We first note that $\iint_{\mathcal{R}(\tau',\tau) \cap \{r_Y^- \leq r \leq r_Y^+\}} K^Y\left(\Phi\right) \leq C \int_{\tau'}^\tau \int_{\Sigma_{\bar\tau}} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_{\bar\tau}}^\mu d\bar\tau$, with C independent of e and ϵ whenever $\epsilon \ll e < 1$. With this estimate, we will prove the proposition by a bootstrap argument. We want to prove the conclusion by assuming

$$\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_\tau}^\mu \leq 4 \int_{\Sigma_{\tau'}} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + 2 \iint_{\mathcal{R}(\tau',\tau)} G^2.$$

Then, by Proposition 7,

$$\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}\left(\Phi\right) n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r_Y^-\}} K_\mu^{N_e}\left(\Phi\right)$$

$$= \int_{\Sigma_{\tau'}} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + e \iint_{\mathcal{R}(\tau',\tau) \cap \{r_Y^- \leq r \leq r_Y^+\}} K^Y\left(\Phi\right) + \int_{\mathcal{R}(\tau',\tau)} \left(\partial_{t^*}\Phi + eY\Phi\right) G$$

$$\leq \int_{\Sigma_{\tau'}} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + Ce \int_{\tau'}^\tau \int_{\Sigma_{\bar\tau}} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_{\bar\tau}}^\mu d\bar\tau + \frac{1}{2} \int_{\mathcal{R}(\tau',\tau)} \left(\left(\partial_{t^*}\Phi + eY\Phi\right)\right)^2 + \frac{1}{2} \int_{\mathcal{R}(\tau',\tau)} G^2$$

$$\leq \left(\frac{3}{2} + 4Ce\right) \int_{\Sigma_{\tau'}'} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + \left(\frac{1}{2} + 2Ce\right) \int_{\mathcal{R}(\tau',\tau)} G^2.$$

The proposition is proved by choosing $e \leq \frac{1}{8C}$. $\qquad\square$

8 Integrated Decay Estimates and Boundedness of Non-degenerate Energy

In this section we would like to show an integrated decay estimate. We first follow [19] to construct a vector field and prove an integrated decay estimates for the terms near spatial infinity. That construction is in turn inspired by [23]. In [19], the decay rate in r of this integrated decay estimate is crucial to control the error terms arising from the vector field commutator S. In the sequel, such estimate would also facilitate many computations as we prove the full integrated decay estimate.

In view of the red-shift, all derivatives of Φ can be controlled near the event horizon. However, we would also like to prove an integrated decay estimate that controls Φ itself near the event horizon. This is in contrast to the integrated decay estimate in [8] which degenerates near the event horizon. This extra control is useful as we are considering the inhomogeneous problem.

The proof of the integrated decay estimate for a finite region of r away from the horizon follows that in [8]. We would like to remark that one difference here is that we do not assume the boundedness of $\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_\tau}^\mu$ (even after ignoring inhomogeneous terms). We would instead like to prove the boundedness of $\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right) n_{\Sigma_\tau}^\mu$ using the integrated decay estimates. We will, however, use Proposition 8.

The reader should think of this integrated decay estimates as analogous to the estimates associated to the vector field X in [10],[7],[19]. However, it is impossible to obtain such estimates using a vector field in Kerr spacetimes and we therefore resort to a phase space analysis (see [1]).

To perform the phase space analysis, we will take the Fourier transform in the variable t^*, take the Fourier series in the variable ϕ^* and express the dependence on the θ variable in oblate spheroidal harmonics. Carter [5] discovered that with this decomposition, the wave equation can be separated. However, in order to take the Fourier transform in the variable t^*, we need Φ to be at least in L^2. To this end, we perform a cutoff in the variable t^*.

8.1 Estimates near Spatial Infinity

In this subsection, we follow [19] to construct a vector field $\tilde{X} = \tilde{f}\left(r^*\right) \partial_{r^*}$ so that the spacetime integral that can be controlled with a good weight in r. We refer the reader to [19] for the following:

Proposition 9. *In Schwarzschild spacetimes, using (t, r^*, x^A, x^B) coordinates, there exists $\tilde{X}_S = \tilde{f}(r^*)\, \partial_{r^*}$ and $w_S^{\tilde{X}}$ supported in $r \geq \frac{13M}{4}$ such that*

$$K^{\tilde{X}, w^{\tilde{X}}}(\Phi) \geq c\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-1}|\not{\nabla}\Phi|^2 + r^{-3-\delta}\Phi^2\right),$$

for $r^* \geq \max\{100, 100M\}$ and

$$|\int_{\Sigma_\tau} J_\mu^{\tilde{X}, w^{\tilde{X}}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C \int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.$$

This implies via stability (since the vector field is supported away from the event horizon) the following:

Proposition 10. *In Kerr spacetimes, using (t^*, r, x^A, x^B) coordinates, there exists $\tilde{X}$ and $w^{\tilde{X}}$ supported in $r \geq \frac{25M}{8}$ and $w^{\tilde{X}}$ such that for some large R,*

$$K^{\tilde{X}, w^{\tilde{X}}}(\Phi) \geq c_{\tilde{X}}\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-1}|\not{\nabla}\Phi|^2 + r^{-3-\delta}\Phi^2\right) - C_{\tilde{X}}\epsilon r^{-2}(\partial_{t^*}\Phi)^2,$$

for $r^* \geq R$ and

$$|\int_{\Sigma_\tau} J_\mu^{\tilde{X}, w^{\tilde{X}}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C \int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Now it is easy to construct the following vector field on Schwarzschild spacetimes:

Proposition 11. *In Schwarzschild spacetimes, using (t, r^*, x^A, x^B) coordinates, there exists $\tilde{\tilde{X}}_S = \tilde{\tilde{f}}(r^*)\, \partial_{r^*}$ supported in $r \geq \frac{13M}{4}$ such that*

$$K^{\tilde{\tilde{X}}}(\Phi) \geq cr^{-2}(\partial_{t^*}\Phi)^2 - C\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-1}|\not{\nabla}\Phi|^2 + r^{-3-\delta}\Phi^2\right),$$

for $r^* \geq \max\{100, 100M\}$ and

$$|\int_{\Sigma_\tau} J_\mu^{\tilde{\tilde{X}}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C \int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Proof. Let $\tilde{\tilde{f}}$ be supported appropriately and $\tilde{\tilde{f}}(r^*) = \frac{1}{(1+r^*)^\delta}$ whenever r^* is large. $\qquad\square$

As before, a stability argument would give:

Proposition 12. *In Kerr spacetimes, using (t^*, r, x^A, x^B) coordinates, there exists $\tilde{\tilde{X}}$ supported in $r \geq \frac{25M}{8}$ such that for some large R,*

$$K^{\tilde{\tilde{X}}}(\Phi) \geq cr^{-2}(\partial_{t^*}\Phi)^2 - C_{\tilde{\tilde{X}}}\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-1}|\not{\nabla}\Phi|^2 + r^{-3-\delta}\Phi^2\right),$$

for $r^* \geq R$ and

$$|\int_{\Sigma_\tau} J_\mu^{\tilde{\tilde{X}}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C \int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Now using the vector field $\tilde{X} + \frac{1}{2}\frac{c_{\tilde{X}}}{C_{\tilde{\tilde{X}}}}\tilde{\tilde{X}}$ and modifying function $w^{\tilde{X}}$, we get the following estimate for ϵ sufficiently small:

Proposition 13.

$$\iint_{\mathcal{R}(\tau_0, \tau) \cap \{r \geq R\}} \left(r^{-1-\delta} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu + r^{-3-\delta}\Phi^2\right)$$

$$\leq C \left(\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau_0}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau_0, \tau) \cap \{\frac{25M}{8} \leq r \leq R\}} \left(J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu + \Phi^2\right)\right.$$

$$\left. + \iint_{\mathcal{R}(\tau_0, \tau)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G|\right).$$

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8.2 Estimates near the Event Horizon

The integrated decay estimates shown in [8] degenerate around the event horizon. Here we will prove the corresponding estimates near the event horizon. In view of the availability of the red-shift estimate K^{N_e}, we will focus on the zeroth order term Φ^2. It turns out that we can use a construction in [19].

Proposition 14. *In Schwarzschild spacetimes, using (t, r^*, x^A, x^B) coordinates, there exists $X_h = f_h(r^*)\partial_{r^*}$ and w^{X_h} supported in $r \leq \frac{23M}{8}$ such that*

$$K^{X_h, w^{X_h}}(\Phi) \geq c\left((\partial_r \Phi)^2 + |\nabla \Phi|^2 + \Phi^2\right),$$

for $r \leq r_Y^-$ and

$$|\int_{\Sigma_\tau} J_\mu^{X_h, w^{X_h}}(\Phi) n_{\Sigma_\tau}^\mu| \leq C \int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu.$$

Proof. Let

$$X_h = f_h(r_S^*)\partial_{r_S^*} = -\chi(r)\frac{M^3}{(1 + 4\mu^{-2})}\partial_{r_S^*} = -\chi(r)\frac{\mu^3 r^3}{8(1 + 4\mu^{-2})}\partial_{r_S^*},$$

where $\chi(r)$ is a cutoff function that is compactly supported in $r \leq \frac{23M}{8}$ and is identically 1 for $r \leq r_Y^-$. Also, let

$$w^{X_h} = 2f_h'(r^*) + \frac{4(1 - \mu)}{r}f_h(r^*).$$

From now on, we will focus on the behavior when $r \leq r_Y^-$ and treat the terms in $\{r_Y^- \leq r \leq \frac{23M}{8}\}$ as errors. Recall that on Schwarzschild spacetime:

$$K^{X_h, w^{X_h}}(\Phi) = \frac{f'(r^*)}{1 - \mu}(\partial_{r^*}\Phi)^2 + \frac{(2 - 3\mu)f(r^*)}{2r}|\nabla\Phi|^2$$
$$- \frac{1}{4}\left(\frac{1}{1 - \mu}f'''(r^*) + \frac{4}{r}f''(r^*) + \frac{\mu}{r^2}f'(r^*) - \frac{2\mu}{r^3}(3 - 4\mu)f(r^*)\right)\Phi^2$$

We now look at the sign of this expression for $r \leq r_Y^-$. It is easy to see that the coefficient for $(\partial_{r^*}\Phi)^2$ is positive:

$$f'(r^*) = (1 - \mu)\partial_r f_0(r)$$
$$= \frac{\mu r^2(1 - \mu)}{(1 + 4\mu^{-2})^2} \geq \frac{c(1 - \mu)}{r^3},$$

The coefficient of $|\nabla\Phi|^2$ is also clearly positive. A computation shows that

$$\frac{1}{1 - \mu}f''' + \frac{4}{r}f'' + \frac{\mu}{r^2}f' - \frac{2\mu}{r^3}(3 - 4\mu)f$$
$$= -\frac{\mu^6(192 + \mu(128 + \mu(-784 + \mu(464 + \mu(-28 + \mu(52 + \mu(-3 + 4\mu)))))))}{4(4 + \mu^2)^4}$$

We want to show that $P(\mu) = 192 + \mu(128 + \mu(-784 + \mu(464 + \mu(-28 + \mu(52 + \mu(-3 + 4\mu)))))) \geq \frac{1}{7}$ for $\frac{16}{23} \leq \mu \leq 1$.
$192 + 128\mu - 784\mu^2 + 464\mu^3 = 16(-12 - 20\mu + 29\mu^2)(\mu - 1) \geq 0$.
$52 - 3\mu + 4\mu^2$ reaches its minimum at $\frac{3}{8}$. Hence, $52 - 3\mu + 4\mu^2 \geq \frac{823}{16}$.
$-28 + \mu(52 - 3\mu + 4\mu^2) \geq -28 + \frac{11}{20}\frac{823}{16} \geq \frac{93}{320}$.
Therefore, $P(\mu) \geq \frac{1023}{6400} \geq \frac{1}{7}$ for $\frac{16}{23} \leq \mu \leq 1$. Therefore, for $r \leq r_Y^-$,

$$K^{X_h, w^{X_h}}(\Phi) \geq c\left((\partial_r \Phi)^2 + |\nabla\Phi|^2 + \Phi^2\right).$$

The second statement

$$|\int_{\Sigma_\tau} J_\mu^{X_h, w^{X_h}}(\Phi) n_{\Sigma_\tau}^\mu| \leq C \int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu$$

follows from the boundedness of f_h and w^{X_h}. $\qquad\square$

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This implies via a stability argument:

Proposition 15. *In Kerr spacetimes, using $(t_S, r_S^*, x_S^1, x_S^2)$ coordinates, there exists X_h and w^{X_h} supported in $r \leq \frac{23M}{8}$ such that*

$$K^{X_h, w^{X_h}}(\Phi) \geq c\Phi^2 - C\epsilon(\partial_{t^*}\Phi)^2,$$

for $r \leq r_Y^-$ and

$$|\int_{\Sigma_\tau} J_\mu^{X_h, w^{X_h}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.$$

We therefore have the following integrated decay estimate near the event horizon. Note that the error, while containing only the $\partial_{r_S^*}$ derivative on Schwarzschild spacetimes, now has potentially all the derivatives.

Proposition 16.

$$\iint_{\mathcal{R}(\tau_0, \tau) \cap \{r \leq r_Y^-\}} \Phi^2$$

$$\leq C\left(\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau_0}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau_0, \tau) \cap \{r_Y^- \leq r \leq \frac{23M}{8}\}} \Phi^2 + \iint_{\mathcal{R}(\tau_0, \tau) \cap \{r \leq \frac{23M}{8}\}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu \right.$$

$$\left. + \iint_{\mathcal{R}(\tau_0, \tau) \cap \{r \leq \frac{23M}{8}\}} \left(|\partial\Phi| + r^{-1}|\Phi| \right) |G| \right).$$

8.3 Cutoff, Decomposition and Separation

Following [8], we define the cutoff

$$\Phi_{\tau'}^\tau = \xi\Phi,$$

where

$$\xi = \chi(t^* - 1 - \tau)\chi(-t^* - 1 + \tau'),$$

for some smooth cutoff function $\chi(x)$ identically 1 for $x \leq -1$ and support on $\{x \leq 0\}$. Then

$$\Box_g \Phi_{\tau'}^\tau = \xi G + 2\nabla^\alpha\Phi\nabla_\alpha\xi + \Phi\Box_g\xi =: F.$$

We then decompose in frequency. We decompose the Fourier transform in t of Φ into Fourier series in ϕ and oblate spheroidal harmonics:

$$\hat{\Phi}_{\tau'}^\tau = \sum_{m,\ell} R_{m\ell}^\omega(r) S_{m\ell}(a\omega, \cos\theta)e^{im\phi}.$$

We also decompose the inhomogeneous term F (which comes both from the original inhomogeneous term G and the cutoff):

$$\hat{F} = \sum_{m,\ell} F_{m\ell}^\omega(r) S_{m\ell}(a\omega, \cos\theta)e^{im\phi}.$$

Letting ζ be a sharp cutoff with such that $\zeta = 1$ for $|x| \leq 1$ and $\zeta = 0$ for $|x| > 0$, we define

$$\Phi_\flat = \int_{-\infty}^\infty \zeta(\frac{\omega}{\omega_1}) \sum_{m,l:\lambda_{ml}(\omega) \leq \lambda_1} R_{ml}^\omega(r) S_{ml}(a\omega, \cos\theta)e^{im\phi}e^{i\omega t}d\omega$$

$$\Phi_\natural = \int_{-\infty}^\infty \zeta(\frac{\omega}{\omega_1}) \sum_{m,l:\lambda_{ml}(\omega) > \lambda_1} R_{ml}^\omega(r) S_{ml}(a\omega, \cos\theta)e^{im\phi}e^{i\omega t}d\omega$$

$$\Phi_\natural = \int_{-\infty}^\infty (1 - \zeta(\frac{\omega}{\omega_1})) \sum_{m,l:\lambda_{ml}(\omega) \geq \lambda_2} R_{ml}^\omega(r) S_{ml}(a\omega, \cos\theta)e^{im\phi}e^{i\omega t}d\omega$$

$$\Phi_\sharp = \int_{-\infty}^\infty (1 - \zeta(\frac{\omega}{\omega_1})) \sum_{m,l:\lambda_{ml}(\omega) < \lambda_2} R_{ml}^\omega(r) S_{ml}(a\omega, \cos\theta)e^{im\phi}e^{i\omega t}d\omega.$$

In this decomposition, we think of ω_1 as large and λ_2 as small.

8.4 The Trapped Frequencies

Trapping occurs for $\Phi_\natural$. An integrated decay estimate is proved in detail in [8]. We refer the readers to Section 5.3.3 of [8].

Proposition 17.

$$\iint_{\mathcal{R}(-\infty,\infty)} \left(\chi\Phi_\natural^2 + \chi(\partial_r\Phi_\natural)^2 + \chi\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}} J_\mu^N\left(\Phi_\natural\right) n_{\Sigma_{t^*}}^\mu \right)$$

$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N\left(\Phi_\natural\right) n_{\Sigma_{\tau'}}^\mu + C\epsilon \int_{\mathcal{H}(\tau',\tau)} \left(\partial_{\phi^*}\Phi_\natural\right)^2 + \int_{-\infty}^\infty dt^* \int_{r\geq R} \left(2f(r^2+a^2)^{1/2} F_\natural\partial_{r^*}((r^2+a^2)^{1/2}\Phi_{\tau'}^\tau) \right.$$

$$\left. + f'(r^2+a^2)F_\natural\Phi_{\tau'}^\tau \right) \frac{\Delta}{r^2+a^2} \sin\theta \, d\phi \, d\theta \, dr^*,$$

where χ is a weight that degenerates at infinity and f is increasing and $f = \tan^{-1}\frac{r^*-\alpha-\sqrt{\alpha}}{\alpha} - \tan^{-1}(-1-\alpha)^{-\frac{1}{2}}$, for $r > R$ for some fixed α.

8.5 The Untrapped Frequencies

For each of the pieces that are untrapped, i.e., $\Phi_\natural$ for $\natural = \flat, \natural$ or $\sharp$, a vector field $X_\natural$ is constructed in [8] so that

$$\iint_{\mathcal{R}(-\infty,\infty)} \left(\chi J_\mu^N\left(\Phi_\natural\right) n_{\Sigma_{t^*}}^\mu + \tilde{\chi}\Phi_\natural^2 \right) \leq C \iint_{\mathcal{R}(-\infty,\infty)} K^{X_\natural}\left(\Phi_\natural\right),$$

where χ is a weight function that degenerates at infinity and $\tilde{\chi}$ is a weight function that both degenerates at infinity and vanishes around the event horizon. Using this vector field and the conservation identity, it is shown in Section 5.3.4 in [8] that

Proposition 18.

$$\iint_{\mathcal{R}(-\infty,\infty)} \left(\chi(J_\mu^N(\Phi_\flat) + J_\mu^N(\Phi_\natural) + J_\mu^N(\Phi_\sharp)) n_{\Sigma_\tau}^\mu + \tilde{\chi}\left(\Phi_\flat^2 + \Phi_\natural^2 + \Phi_\sharp^2\right) \right)$$

$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n^\mu + C\delta^{-1} \iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}} F^2$$

$$+ C\delta \iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}} (\Phi_{\tau'}^\tau)^2 + (\partial_r\Phi_{\tau'}^\tau)^2 + \mathbb{1}_{\{r\leq\frac{23M}{8}\}} J_\mu^N(\Phi_{\tau'}^\tau)N^\mu$$

$$+ \int_{-\infty}^\infty dt^* \int_{r\geq R} \left(2f(r^2+a^2)^{1/2}(F_\flat + F_\natural + F_\sharp)\partial_{r^*}((r^2+a^2)^{1/2}\Phi_{\tau'}^\tau) \right.$$

$$\left. + f'(r^2+a^2)(F_\flat + F_\natural + F_\sharp)\Phi_{\tau'}^\tau \right) \frac{\Delta}{r^2+a^2} \sin\theta \, d\phi \, d\theta \, dr^*,$$

where χ and f are exactly as in Proposition 17.

8.6 The Integrated Decay Estimates

In order to add up the estimates in the previous sections, we need a Hardy-type inequality:

Proposition 19.

$$\int_{\Sigma_\tau} r^{\alpha-2}\Phi^2 \leq C \int_{\Sigma_\tau} r^\alpha J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu.$$

Proof. Let $k(r)$ be defined by solving

$$k'(r,\theta,\phi) = r^{\alpha-2}vol,$$

where $vol = vol\,(r,\theta,\phi)$ is the volume density on Σ_τ with r,θ,ϕ coordinates, with boundary condition $k(r_+,\theta,\phi)=0$. Now

$$\int_{\Sigma_\tau} r^{\alpha-2}\Phi^2 = \iiint_{r_+}^{\infty} k'(r)\Phi^2 dr d\theta d\phi$$

$$= -2\iiint k(r)\Phi\partial_r\Phi dr d\theta d\phi$$

$$\leq 2\left(\iiint \frac{k(r)^2}{k'(r)}\left(\partial_r\Phi\right)^2 dr d\theta d\phi\right)^{\frac{1}{2}}\left(\iiint k'(r)\Phi^2 dr d\theta d\phi\right)^{\frac{1}{2}}$$

Notice that $vol \sim r^2$, $k(r) \sim r^{\alpha+1}$ and $k'(r) \sim r^\alpha$. Hence $\frac{k(r)^2}{k'(r)} \sim r^\alpha vol$. The lemma follows. $\qquad\square$

We now add up the estimates for $\Phi_\flat$, $\Phi_\natural$, $\Phi_\sharp$ and $\Phi_\sharp$.

Proposition 20.

$$\iint_{\mathcal{R}(\tau',\tau)}\left(r^{-1-\delta}\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^N\left(\Phi\right)n_{\Sigma_\tau}^\mu + r^{-1-\delta}\left(\partial_r\Phi\right)^2 + r^{-3-\delta}\Phi^2\right)$$

$$\leq C\left(\int_{\Sigma_\tau}J_\mu^N\left(\Phi\right)n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau'}}J_\mu^N\left(\Phi\right)n_{\Sigma_{\tau'}}^\mu + \int_{\mathcal{H}(\tau',\tau)}J_\mu^N\left(\Phi\right)n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau'-1,\tau+1)}\left(|\partial_r\Phi|+r^{-1}|\Phi|\right)|G|\right)$$

$$\iint_{\mathcal{R}(\tau'-1,\tau+1)}G^2\Bigg).$$

Proof. Since the function f appears identically in Propositions 17 and 18, we can add up the estimates to obtain:

$$\iint_{\mathcal{R}(-\infty,\infty)}\left(\chi(J_\mu^N(\Phi_\flat)+J_\mu^N(\Phi_\natural)+\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^N(\Phi_\sharp)+J_\mu^N(\Phi_\sharp))n_{\Sigma_\tau}^\mu + \tilde{\chi}\left(\Phi_\flat^2+\Phi_\natural^2+\Phi_\sharp^2+\Phi_\sharp^2\right)\right)$$

$$\leq C\int_{\Sigma_{\tau'}}J_\mu^N(\Phi)n^\mu + C\epsilon\int_{\mathcal{H}(\tau',\tau)}\left(\partial_{\phi^*}\Phi_\sharp\right)^2 + C\delta^{-1}\int_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}}F^2$$

$$+ C\delta\int_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}}(\Phi_{\tau'}^\tau)^2+(\partial_r\Phi_{\tau'}^\tau)^2+\mathbb{1}_{\{r\leq\frac{23M}{8}\}}J_\mu^N(\Phi_{\tau'}^\tau)n_{\Sigma_{t^*}}^\mu$$

$$+ \int_{-\infty}^\infty dt^*\int_{r\geq R}\left(2f(r^2+a^2)^{1/2}(F_\flat+F_\natural+F_\sharp+F_\sharp)\partial_{r^*}((r^2+a^2)^{1/2}\Phi_{\tau'}^\tau)\right.$$

$$\left.+f'(r^2+a^2)(F_\flat+F_\natural+F_\sharp+F_\sharp)\Phi_{\tau'}^\tau\right)\frac{\Delta}{r^2+a^2}\sin\theta\,d\phi\,d\theta\,dr^*.$$

By the definition of the cutoff, we have the pointwise equalities

$$\Phi = \Phi_\flat+\Phi_\natural+\Phi_\sharp+\Phi_\sharp,$$

$$F = F_\flat+F_\natural+F_\sharp+F_\sharp.$$

Therefore, we have

$$\iint_{\mathcal{R}(-\infty,\infty)}\chi\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^N(\Phi_{\tau'}^\tau)n_{\Sigma_{t^*}}^\mu + \tilde{\chi}(\Phi_{\tau'}^\tau)^2$$

$$\leq C\int_{\Sigma_{\tau'}}J_\mu^N(\Phi)n_{\Sigma_{\tau'}}^\mu + C\epsilon\int_{\mathcal{H}(\tau',\tau)}J_\mu^N\left(\Phi\right)n_{\mathcal{H}}^\mu + C\delta^{-1}\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}}F^2$$

$$+ C\delta\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}}(\Phi_{\tau'}^\tau)^2+(\partial_r\Phi_{\tau'}^\tau)^2+\mathbb{1}_{\{r\leq\frac{23M}{8}\}}J_\mu^N(\Phi_{\tau'}^\tau)n_{\Sigma_{t^*}}^\mu$$

$$+ \int_{-\infty}^\infty dt^*\int_{r\geq R}\left(2f(r^2+a^2)^{1/2}F\partial_{r^*}((r^2+a^2)^{1/2}\Phi_{\tau'}^\tau)+f'(r^2+a^2)F\Phi_{\tau'}^\tau\right)\frac{\Delta}{r^2+a^2}\sin\theta\,d\phi\,d\theta\,dr^*.$$

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We want to estimate the expression in the last line. Recall that

$$F = \xi G + 2\nabla^\alpha \Phi \nabla_\alpha \xi + \Phi \square_{g_K} \xi.$$

Notice that by the definition of ξ, the last two terms are supported in the t^* range $(\tau'-1,\tau') \cup (\tau,\tau+1)$. Using this, we immediately have, using the trivial bounds:

$$C\delta^{-1} \iint_{\mathcal{R}(-\infty,\infty) \cap \{r \leq R\}} F^2$$

$$\leq C\delta^{-1} \left(\iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq R\}} G^2 + \iint_{\mathcal{R}(\tau'-1,\tau') \cup \mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu \right) \right).$$

Similarly, we have

$$\int_{-\infty}^\infty dt^* \int_{r \geq R} f'(r^2 + a^2) F \Phi_{\tau'}^\tau \frac{\Delta}{r^2 + a^2} \sin\theta \, d\phi \, d\theta \, dr^*$$

$$\leq C \left(\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{-1}|\Phi||G| + \iint_{\mathcal{R}(\tau'-1,\tau') \cup \mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu \right) \right).$$

The other term with F is more delicate to estimate. One of the terms in the expansion does not have sufficient decay in r:

$$\int_{-\infty}^\infty dt^* \int_{r \geq R} 2f(r^2 + a^2)^{1/2} F \partial_{r^*}((r^2 + a^2)^{1/2} \Phi_{\tau'}^\tau) \frac{\Delta}{r^2 + a^2} \sin\theta \, d\phi \, d\theta \, dr^*$$

$$\leq C \left(\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{-1}|\Phi||G| + \iint_{\mathcal{R}(\tau'-1,\tau') \cup \mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu \right) \right)$$

$$+ \int_{-\infty}^\infty dt^* \int_{r \geq R} 2f(r^2 + a^2)^{1/2} \Phi \square_{g_K} \xi \partial_{r^*}((r^2 + a^2)^{1/2} \Phi) \xi \frac{\Delta}{r^2 + a^2} \sin\theta \, d\phi \, d\theta \, dr^*$$

Nevertheless, noting that ξ is independent of r^*, an integration by parts in r^* would give

$$\int_{-\infty}^\infty dt^* \int_{r \geq R} 2f(r^2 + a^2)^{1/2} \Phi \square_{g_K} \xi \partial_{r^*}((r^2 + a^2)^{1/2} \Phi) \xi \frac{\Delta}{r^2 + a^2} \sin\theta \, d\phi \, d\theta \, dr^*$$

$$= -\int_{-\infty}^\infty dt^* \int_{r \geq R} (r^2 + a^2) \Phi^2 \xi \square_{g_K} \xi \partial_{r^*} \left(f \frac{\Delta}{r^2 + a^2} \right) \sin\theta \, d\phi \, d\theta \, dr^* + \text{boundary terms}$$

$$\leq C \iint_{\mathcal{R}(\tau'-1,\tau') \cup \mathcal{R}(\tau,\tau+1)} r^{-2}\Phi^2,$$

where the boundary terms can be controlled (after possibly changing R) by pigeonholing in $r \in [R, R+1]$. By the mild growth estimate of Proposition 8 and the Hardy inequality of Proposition 19,

$$\iint_{\mathcal{R}(\tau'-1,\tau') \cup \mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu \right)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + \iint_{\mathcal{R}(\tau'-1,\tau') \cup \mathcal{R}(\tau,\tau+1)} G^2 \right).$$

Therefore, using all the above estimates and noticing the support of ξ, we have

$$\iint_{\mathcal{R}(\tau',\tau)} \chi \mathbb{1}_{\{|r-3M| \geq \frac{M}{8}\}} J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu + \tilde{\chi} \Phi^2$$

$$\leq C\delta^{-1} \int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + C\delta^{-1} \int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + C\epsilon \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu$$

$$+ C \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r \Phi| + r^{-1}|\Phi| \right) |G| + C\delta^{-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2$$

$$+ C\delta \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq R\}} \Phi^2 + (\partial_r \Phi)^2 + \mathbb{1}_{\{r \leq \frac{23M}{8}\}} J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu.$$

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We add to this to the estimates near spatial infinity and the event horizon, i.e., Propositions 13 and 16, to get

$$\iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta} \mathbb{1}_{\{|r-3M|\geq \frac{M}{8}\}} J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu + r^{-1-\delta} (\partial_r \Phi)^2 + r^{-3-\delta} \Phi^2$$

$$\leq C\delta^{-1} \int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + C\delta^{-1} \int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + C\epsilon \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu$$

$$+ C \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r \Phi| + r^{-1}|\Phi|\right) |G| + C\delta^{-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2$$

$$+ C\delta \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq R\}} \Phi^2 + (\partial_r \Phi)^2 + \mathbb{1}_{\{r\leq \frac{23M}{8}\}} J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu.$$

By choosing δ sufficiently small, we can absorb the last term restricted to $\mathcal{R}(\tau',\tau)$:

$$\iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta} \mathbb{1}_{\{|r-3M|\geq \frac{M}{8}\}} J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu + r^{-1-\delta} (\partial_r \Phi)^2 + r^{-3-\delta} \Phi^2$$

$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + C \int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + C\epsilon \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu$$

$$+ C \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r \Phi| + r^{-1}|\Phi|\right) |G| + C \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2$$

$$+ C \iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)\cap\{r\leq R\}} \Phi^2 + (\partial_r \Phi)^2 + \mathbb{1}_{\{r\leq \frac{23M}{8}\}} J_\mu^N(\Phi) n_{\Sigma_{t^*}}^\mu$$

$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + C \int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + C\epsilon \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu$$

$$+ C \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r \Phi| + r^{-1}|\Phi|\right) |G| + C \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2,$$

using Proposition 8 and 19 at the last step. $\qquad\square$

Definition 3. *From now on, denote*

$$K^{X_0}(\Phi) = r^{-1-\delta} \mathbb{1}_{\{|r-3M|\geq \frac{M}{8}\}} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + r^{-1-\delta} (\partial_r \Phi)^2 + r^{-3-\delta} \Phi^2, \quad and$$

$$K^{X_1}(\Phi) = r^{-1-\delta} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + r^{-3-\delta} \Phi^2.$$

We remark that this is a slight abuse of notation because these "currents" do not arise directly from a vector field.

8.7 Boundedness of the Non-degenerate Energy

Proposition 21. *Let Φ satisfy $\Box_{g_K}\Phi = G$. For e sufficiently small and $\epsilon \ll e$, we have*

$$\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi) n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K_\mu^{N_e}(\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi) n_{\Sigma_{\tau'}}^\mu + \left| \iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*}\Phi G \right| + \left| \iint_{\mathcal{R}(\tau'-1,\tau+1)} eY\Phi G \right| \right.$$

$$\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r \Phi| + r^{-1}|\Phi|\right) |G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2 \right).$$

Proof. We recall that

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\leq r\leq r_Y^+\}} K^Y(\Phi) \leq C \int_{\tau'}^\tau \int_{\Sigma_{\bar\tau}} J_\mu^{N_e}(\Phi) n_{\Sigma_{t^*}}^\mu dt^*,$$

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with C independent of e and ϵ whenever $\epsilon \ll e < 1$. At this point, we choose $r_Y^+ < \frac{11M}{4} < \frac{23M}{8}$. Hence this term can be controlled by the integrated decay estimates. Then, by Proposition 7,

$$
\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K_\mu^{N_e}(\Phi)
$$

$$
= \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + e\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\leq r\leq r_Y^+\}} K^Y(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} (\partial_{t^*}\Phi + eY\Phi)\, G
$$

$$
\leq \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + Ce\int_{\tau'}^\tau \int_{\Sigma_{\bar\tau}\cap\{r_Y^-\leq r\leq r_Y^+\}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\bar\tau}}^\mu\, d\bar\tau + |\iint_{\mathcal{R}(\tau',\tau)} (\partial_{t^*}\Phi + eY\Phi)\, G|
$$

$$
\leq \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + Ce\left(\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G|\right.
$$

$$
\left. + |\iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2|\right) + |\iint_{\mathcal{R}(\tau',\tau)} (\partial_{t^*}\Phi + eY\Phi)\, G|.
$$

Hence, the Proposition holds if e is chosen to be sufficiently small. $\qquad\square$

Remark 3. *From this point on, we will consider r_Y^+ and e to be fixed. After e is fixed, the vector field N_e will be written simply as N.*

We now estimate the inhomogeneous terms in Proposition 21:

Proposition 22.

$$
\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi)
$$

$$
\leq C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} G^2\right)^{\frac12} dt^*\right)^2 + \iint_{\mathcal{R}(\tau',\tau)} G^2\right).
$$

Proof. Adding the estimates in Propositions 20 and δ times the estimates in Proposition 21,

$$
\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \delta\iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi)
$$

$$
\leq C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial\Phi| + r^{-1}|\Phi|\right)|G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right)
$$

$$
+ C\delta\left(\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu\right)
$$

$$
\leq C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \sup_{t^*\in[\tau'-1,\tau+1]} \left(\int_{\Sigma_{t^*}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu\right)^{\frac12} \int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} G^2\right)^{\frac12} dt^*\right.
$$

$$
\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right) + C\delta\left(\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu\right),
$$

where at the last step we have used Proposition 19. Choosing $C\delta \leq \frac12$, we can absorb the last term to the left hand side to get

$$
\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \delta\iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi)
$$

$$
\leq C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \sup_{t^*\in[\tau'-1,\tau+1]} \left(\int_{\Sigma_{t^*}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu\right)^{\frac12} \int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} G^2\right)^{\frac12} dt^*\right.
\tag{7}
$$

$$
\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right).
$$

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By considering the above estimate on $[\tau', \tilde{\tau}]$, where $\tilde{\tau}$ is when the supremum on the right hand side is achieved, and using Proposition 8, we get

$$\int_{\Sigma_{\tilde{\tau}}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tilde{\tau}}} + \int_{\mathcal{H}(\tau',\tilde{\tau})} J^N_\mu (\Phi) n^\mu_{\mathcal{H}^+} + \iint_{\mathcal{R}(\tau',\tilde{\tau}) \cap \{r \leq r^-_Y\}} K^N (\Phi) + \delta \iint_{\mathcal{R}(\tau',\tilde{\tau})} K^{X_0} (\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} G^2 \right)^{\frac{1}{2}} dt^* \right)^2 \right).$$

We plug this into (7) and apply Cauchy-Schwarz to prove the proposition. □

We can also estimate the inhomogeneous terms not in $L^1 L^2$ but in $L^2 L^2$, provided that we allow some extra factors of r and some loss of derivatives in G. This is especially useful for estimating the commutator terms from S, which do not have sufficient decay in t^* in the interior to be estimated in $L^1 L^2$. More precisely, we have

Proposition 23.

$$\int_{\Sigma_\tau} J^N_\mu (\Phi) n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu (\Phi) n^\mu_{\mathcal{H}^+} + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r^-_Y\}} K^N (\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} (\partial^m_{t^*} G)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2 \right).$$

Proof. By Propositions 20 and 21,

$$\int_{\Sigma_\tau} J^N_\mu (\Phi) n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu (\Phi) n^\mu_{\mathcal{H}^+} + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r^-_Y\}} K^N (\Phi) + \delta \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J^{N_e}_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + | \iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*} \Phi G | + | \iint_{\mathcal{R}(\tau'-1,\tau+1)} e Y \Phi G | \right.$$

$$\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} (|\partial_r \Phi| + r^{-1} |\Phi|) |G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2 \right)$$

$$+ C\delta \left(\int_{\Sigma_\tau} J^N_\mu (\Phi) n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu (\Phi) n^\mu_{\mathcal{H}^+} \right).$$

Choosing $C\delta \leq \frac{1}{2}$, we can absorb the last term to the left hand side to get

$$\int_{\Sigma_\tau} J^N_\mu (\Phi) n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu (\Phi) n^\mu_{\mathcal{H}^+} + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r^-_Y\}} K^N (\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + | \iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*} \Phi G | + | \iint_{\mathcal{R}(\tau'-1,\tau+1)} e Y \Phi G | \right.$$

$$\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} (|\partial_r \Phi| + r^{-1} |\Phi|) |G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2 \right).$$

For the bulk error term, we focus at the region $\{|r - 3M| \leq \frac{M}{8}\}$ and integrate by parts.

$$\left| \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} \partial_{t^*} \Phi G \right|$$

$$\leq \delta \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} \Phi^2 + C\delta^{-1} \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} (\partial_{t^*} G)^2$$

$$+ \left| \int_{\Sigma_{\tau+1} \cap \{|r-3M| \leq \frac{M}{8}\}} \Phi G \right| + \left| \int_{\Sigma_{\tau'-1} \cap \{|r-3M| \leq \frac{M}{8}\}} \Phi G \right|$$

$$\leq \delta \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} r^{-3-\delta} \Phi^2 + C\delta^{-1} \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} (\partial_{t^*} G)^2$$

$$+ \sup_{t^* \in [\tau'-1,\tau+1]} \left(\delta \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} r^{-2} \Phi^2 + C\delta^{-1} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2 \right)$$

$$\leq \delta \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} r^{-3-\delta} \Phi^2 + C\delta^{-1} \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} (\partial_{t^*} G)^2$$

$$+ \sup_{t^* \in [\tau'-1,\tau+1]} \left(\delta \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} J^N_\mu (\Phi) n^\mu_{\Sigma_{t^*}} + C\delta^{-1} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2 \right),$$

where at the last step we used Proposition 19.

Therefore,

$$\int_{\Sigma_\tau} J^N_\mu (\Phi) n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu (\Phi) n^\mu_{\mathcal{H}^+} + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r^-_Y\}} K^N (\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + \left| \iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*} \Phi G \right| + \left| \iint_{\mathcal{R}(\tau'-1,\tau+1)} eY \Phi G \right| \right.$$

$$+ \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r \Phi| + r^{-1} |\Phi| \right) |G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2 \Bigg)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + \left| \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{|r-3M| \leq \frac{M}{8}\}} \partial_{t^*} \Phi G \right| \right) + C\delta^{-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G^2 \qquad (8)$$

$$+ \delta \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(r^{-3-\delta} \Phi^2 + r^{-1-\delta} (\partial_r \Phi)^2 + \mathbb{1}_{\{r \leq r^+_Y\}} J^N_\mu (\Phi) n^\mu_{\Sigma_{t^*}} \right)$$

$$\leq C \int_{\Sigma_{\tau'}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + C\delta^{-1} \sum_{m=0}^1 \int_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} (\partial^m_{t^*} G)^2 + \delta \iint_{\mathcal{R}(\tau'-1,\tau+1)} K^{X_0} (\Phi)$$

$$+ \sup_{t^* \in [\tau'-1,\tau+1]} \left(\delta \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} J^N_\mu (\Phi) n^\mu_{\Sigma_{t^*}} + C\delta^{-1} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2 \right).$$

where at the last step we have used Propositions 8, 19. Suppose $\displaystyle \sup_{t^* \in [\tau'-1,\tau+1]} \delta \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} J^N_\mu (\Phi) n^\mu_{\Sigma_{t^*}}$ is achieved by $t^* = \tilde{\tau}$. Apply (8) on $[\tau', \tilde{\tau}]$, we get

$$\int_{\Sigma_{\tilde{\tau}}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tilde{\tau}}}$$

$$\leq C \int_{\Sigma_{\tau'}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tau'}} + C\delta^{-1} \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} (\partial^m_{t^*} G)^2 + \delta \iint_{\mathcal{R}(\tau'-1,\tau+1)} K^{X_0} (\Phi)$$

$$+ \delta \int_{\Sigma_{\tilde{\tau}} \cap \{|r-3M| \leq \frac{M}{8}\}} J^N_\mu (\Phi) n^\mu_{\Sigma_{\tilde{\tau}}} + C \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2,$$

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which, upon choosing $\delta \leq \frac{1}{2}$ and subtracting the small term on both sides, gives

$$\int_{\Sigma_{\tilde{\tau}}} J_\mu^N (\Phi) \, n_{\Sigma_{\tilde{\tau}}}^\mu$$

$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N (\Phi) \, n_{\Sigma_{\tau'}}^\mu + C\delta^{-1} \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m G\right)^2 + \delta \iint_{\mathcal{R}(\tau'-1,\tau+1)} K^{X_0} (\Phi)$$

$$+ C \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2,$$

Therefore, plugging this back into (8)

$$\int_{\Sigma_\tau} J_\mu^N (\Phi) \, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N (\Phi) \, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r_Y^-\}} K^N (\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi)$$

$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N (\Phi) \, n_{\Sigma_{\tau'}}^\mu + \delta \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi) + \delta \iint_{\mathcal{R}(\tau'-1,\tau') \cup \mathcal{R}(\tau,\tau+1)} K^{X_0} (\Phi)$$

$$+ C \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m G\right)^2 + C \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2$$

$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N (\Phi) \, n_{\Sigma_{\tau'}}^\mu + \delta \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi) + C\delta \int_{\Sigma_\tau} J_\mu^N (\Phi) \, n_{\Sigma_\tau}^\mu$$

$$+ C \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m G\right)^2 + C \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2,$$

where at the last step we have used Propositions 8 and 19. Finally, by choosing $C\delta \leq \frac{1}{2}$, we can absorb the small terms to the left hand side and achieve the conclusion of the Proposition. $\qquad\square$

Notice that in the proof of Proposition 23, there is a loss in derivative for G because we have to integrate by parts in the region $\{|r - 3M| \leq \frac{M}{8}\}$. Therefore, if G is supported away from this region, we can repeat the proof without this loss. In other words, we have

Proposition 24. *Suppose G is supported away from $\{|r - 3M| \leq \frac{M}{8}\}$. Then*

$$\int_{\Sigma_\tau} J_\mu^N (\Phi) \, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N (\Phi) \, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r_Y^-\}} K^N (\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J_\mu^N (\Phi) \, n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G^2 \right).$$

This will be useful in Section 13.

In applications, it is useful to have both ways of estimating G.

Proposition 25. *Let $G = G_1 + G_2$ be any way to decompose the function G. Then*

$$\int_{\Sigma_\tau} J_\mu^N (\Phi) \, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N (\Phi) \, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r_Y^-\}} K^N (\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0} (\Phi)$$

$$\leq C \left(\int_{\Sigma_{\tau'}} J_\mu^N (\Phi) \, n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} G_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G_1^2 \right.$$

$$\left. + \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m G_2\right)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G_2^2 \right).$$

In the above estimates, only the function Φ and its ∂_r derivative can be estimated without a loss around the trapped set. To estimate the other derivatives, we need to commute with the Killing vector field ∂_{t^*}.

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Proposition 26.

$$
\iint_{\mathcal{R}(\tau',\tau)} K^{X_1}(\Phi)
$$

$$
\leq C \left(\sum_{m=0}^{1} \int_{\Sigma_{\tau'}} J_\mu^N (\partial_{t^*}^m \Phi) \, n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^{1} \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} (\partial_{t^*}^m G_1)^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} (\partial_{t^*}^m G_1)^2 \right.
$$

$$
\left. + \sum_{m=0}^{2} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} (\partial_{t^*}^m G_2)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \sum_{m=0}^{1} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} (\partial_{t^*}^m G_2)^2 \right).
$$

Proof. Using Proposition 25 and the fact that ∂_{t^*} is Killing, we immediately have the following estimate for $\partial_{t^*} \Phi$:

$$
\iint_{\mathcal{R}(\tau',\tau)} r^{-3-\delta} (\partial_{t^*} \Phi)^2
$$

$$
\leq C \left(\int_{\Sigma_{\tau'}} J_\mu^N (\partial_{t^*} \Phi) \, n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} (\partial_{t^*} G_1)^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} (\partial_{t^*} G_1)^2 \right.
$$

$$
\left. + \sum_{m=1}^{2} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} (\partial_{t^*}^m G_2)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} (\partial_{t^*} G_2)^2 \right).
$$

This would allow us to estimate all derivatives of Φ except for the fact that the estimates for the angular derivatives of Φ degenerate around $r = 3M$:

$$
\iint_{\mathcal{R}(\tau',\tau)} \left(r^{-1-\delta} \mathbb{1}_{\{|r-3M| \geq \frac{M}{8}\}} |\nabla \!\!\!\!/ \, \Phi|^2 + r^{-1-\delta} (\partial_r \Phi)^2 + r^{-1-\delta} (\partial_{t^*} \Phi)^2 + r^{-3-\delta} \Phi^2 \right)
$$

$$
\leq C \left(\sum_{m=0}^{1} \int_{\Sigma_{\tau'}} J_\mu^N (\partial_{t^*}^m \Phi) \, n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^{1} \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} (\partial_{t^*}^m G_1)^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} (\partial_{t^*}^m G_1)^2 \right.
$$

$$
\left. + \sum_{m=0}^{2} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} (\partial_{t^*}^m G_2)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \sum_{m=0}^{1} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} (\partial_{t^*}^m G_2)^2 \right).
$$

We now use this known estimate and construct another vector field to control the angular derivatives in the region $r \sim 3M$. The argument is simple because the estimate is only local. Take $f_{an}(r)$ to be compactly support in $3M - \frac{M}{4} \leq r \leq 3M + \frac{M}{4}$ and identically equal to -1 in $3M - \frac{M}{8} \leq r \leq 3M + \frac{M}{8}$. If we consider $X_{an} = f_{an}(r) \partial_{r^*}$ in Schwarzschild spacetime, we get that the coefficient in front of the terms with angular derivatives is $\frac{\mu}{2r}$, which is bounded below in $3M - \frac{M}{8} \leq r \leq 3M + \frac{M}{8}$. In other words, one gets an estimate of the following form:

$$
\iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta} \mathbb{1}_{\{|r-3M| \leq \frac{M}{8}\}} |\nabla \!\!\!\!/ \, \Phi|^2
$$

$$
\leq C \left(\iint_{\mathcal{R}(\tau',\tau)} \left(r^{-1-\delta} \mathbb{1}_{\{|r-3M| \geq \frac{M}{8}\}} |\nabla \!\!\!\!/ \, \Phi|^2 + r^{-1-\delta} (\partial_r \Phi)^2 + r^{-1-\delta} (\partial_{t^*} \Phi)^2 + r^{-3-\delta} \Phi^2 \right) \right.
$$

$$
+ \int_{\Sigma_\tau} J_\mu^N (\Phi) \, n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau'}} J_\mu^N (\Phi) \, n_{\Sigma_{\tau'}}^\mu + \iint_{\mathcal{R}(\tau',\tau)} \left(|\partial_r \Phi| + |r^{-1} \Phi| \right) |G|
$$

$$
\left. + \iint_{\mathcal{R}(\tau',\tau)} G^2 \right). \tag{9}
$$

Using a stability argument, (9) would hold also on Kerr spacetimes. One easily checks that the terms with G on the right hand side can be estimated in the same manner as before. Hence, the Proposition can be proved by applying Proposition 25. $\qquad\square$

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9 Vector Field Multiplier Z and Decay of Non-degenerate Energy

We follow the definition of Z in [8]. Let $Z = u^2\underline{L} + v^2 L$, where u, v are the Schwarzschild coordinates $u = \frac{1}{2}\left(t - r_S^*\right)$, $v = \frac{1}{2}\left(t + r_S^*\right)$, $\underline{L} = \partial_u$ and $L = V - \underline{L}$, where $V = \partial_{t^*} + \chi(r)\frac{a}{2Mr_+}\partial_{\phi^*}$ with χ being a cutoff function that is identically 1 for $r \leq r_Y^- - \frac{r_Y^- - r_+}{2}$ and is compactly supported in $\{r \leq r_Y^- - \frac{r_Y^- - r_+}{4}\}$. Notice that with this definition, V is Killing except in the set $\{r_Y^- - \frac{r_Y^- - r_+}{2} \leq r \leq r_Y^- - \frac{r_Y^- - r_+}{4}\}$. Let $w^Z = \frac{8tr_S^*(1-\mu)}{r}$. Notice also that while $u \to \infty$ as one approaches the event horizon, Z is continuous up to the event horizon due to the following (However, Z is not C^1 and hence its deformation tensor is not continuous up to the event horizon):

Proposition 27. *In the Kerr (t^*, r, θ, ϕ^*) coordinates,*

$$\underline{L} = (1 - \mu)\,\partial_{t^*} - (1 - \mu)\left(\frac{2r_s - 2M}{2r - 2M}\right)\partial_r.$$

In the null frame near the event horizon in Section 2.3, we can write

$$\underline{L} = \underline{L}^{\hat{V}}\hat{V} + \underline{L}^{\hat{Y}}\hat{Y} + \underline{L}^A E_A, \quad where\ |\underline{L}^\alpha| \leq C(1 - \mu).$$

Heuristically, we want to show that in the region $\{r \geq r_Y^-\}$,

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} J_\mu^{Z,w^Z}(\Phi)\,n_{\Sigma_\tau}^\mu \geq 0.$$

Moreover, we would like to have

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} J_\mu^{Z,w^Z}(\Phi)\,n_{\Sigma_\tau}^\mu \geq \int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} u^2\left(\underline{L}\Phi\right)^2 + v^2\left(L\Phi\right)^2 + \left(u^2 + v^2\right)|\nabla\!\!\!/\,\Phi|^2 + \left(\frac{u^2 + v^2}{r^2}\right)\Phi^2$$

These are true modulo some error terms that can be controlled:

Proposition 28.

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} u^2\left(\underline{L}\Phi\right)^2 + v^2\left(L\Phi\right)^2 + \left(u^2 + v^2\right)|\nabla\!\!\!/\,\Phi|^2 + \left(\frac{u^2 + v^2}{r^2}\right)\Phi^2$$

$$\leq C\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\,n_{\Sigma_\tau}^\mu + C\int_{\Sigma_\tau} J_\mu^N(\Phi)\,n_{\Sigma_\tau}^\mu + C^2\tau^2\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N(\Phi)\,n_{\Sigma_\tau}^\mu.$$

Proof. The proof is analogous to that in Minkowski spacetime (see [21]) and Schwarzschild spacetime (see [10]). Recall that on Schwarzschild spacetime, on a t slice [10]:

$$\left(J_S^{Z,w^Z}\right)_\mu (\Phi)\,n_{\Sigma_t}^\mu$$

$$= \frac{1}{\sqrt{1-\mu}}\left(v^2\left(L\Phi\right)^2 + u^2\left(\underline{L}\Phi\right)^2 + \left(u^2 + v^2\right)|\nabla\!\!\!/\,\Phi|^2 + \frac{2tr^*\left(1-\mu\right)}{r}\Phi\partial_t\Phi - \frac{r^*\left(1-\mu\right)}{r}\Phi^2\right).$$

Now, since t, r^* are stable under perturbation on $\{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}$, we have, on this set:

$$\left(J_K^{Z,w^Z}\right)_\mu (\Phi)\,n_{\Sigma_\tau}^\mu$$

$$\geq \frac{1}{\sqrt{1-\mu}}\left(v^2\left(L\Phi\right)^2 + u^2\left(\underline{L}\Phi\right)^2 + \left(u^2 + v^2\right)|\nabla\!\!\!/\,\Phi|^2 + \frac{2tr^*\left(1-\mu\right)}{r}\Phi\partial_{t^*}\Phi - \frac{r^*\left(1-\mu\right)}{r}\Phi^2\right)$$

$$- C\epsilon r^{-2}\left(\left(u^2 + v^2\right)\left(\nabla\Phi\right)^2 + t^*\Phi^2\right).$$

We now cutoff Φ: define $\hat{\Phi}$ so that it is supported in $\{r \geq r_Y^- - (r_Y^- - r_+)/4\}$ and equals Φ in $\{r \geq r_Y^-\}$. All the error terms arising from the cutoff will be controlled using the red-shift vector field.

$$\int_{\Sigma_\tau} J_\mu^{Z, w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu$$

$$\geq \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{1}{\sqrt{1-\mu}}\left(v^2\left(L\hat{\Phi}\right)^2 + u^2\left(\underline{L}\hat{\Phi}\right)^2 + (u^2 + v^2)|\nabla\!\!\!\!/\,\hat{\Phi}|^2\right)$$

$$+ \frac{2tr_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi} - \frac{r_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}^2 - C\epsilon r^{-2}\left((u^2+v^2)\left(\nabla\!\!\!\!/\,\hat{\Phi}\right)^2 + t^*\hat{\Phi}^2\right).$$

The term

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{2tr_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}$$

is to be handled by two different integration by parts. Recall that [10] on Schwarzschild spacetimes we have

$$t\partial_t\hat{\Phi} = vL\hat{\Phi} + u\underline{L}\hat{\Phi} - r_S^*\partial_{r_S^*}\hat{\Phi}, \quad \text{and}$$

$$t\partial_t\hat{\Phi} = \frac{t}{r_S^*}\left(vL\hat{\Phi} - u\underline{L}\hat{\Phi}\right) - \frac{t^2}{r_S^*}\partial_{r_S^*}\hat{\Phi}.$$

Therefore, upon integrating by parts, we have on Schwarzschild spacetimes that

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{tr_S^*(1-\mu)}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}$$

$$= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \left((1-\mu)r^2\frac{r_S^*}{r}\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi}\right)\hat{\Phi} + \frac{1}{2}\partial_{r_S^*}\left((1-\mu)r(r^*)^2\right)\hat{\Phi}^2\right)dr^*$$

$$= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} (1-\mu)r^2\left(\frac{r_S^*}{r}\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi}\right)\hat{\Phi} + \left(\frac{1}{2}\frac{(r_S^*)^2}{r^2} + \frac{r_S^*}{r}\right)\hat{\Phi}^2\right).$$

Alternatively, we have

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{tr_S^*(1-\mu)}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}$$

$$= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \left((1-\mu)r^2\frac{t^*}{r}\left(vL\hat{\Phi} - u\underline{L}\hat{\Phi}\right)\hat{\Phi} + \frac{1}{2}\partial_{r^*}\left((1-\mu)r(t^*)^2\right)\hat{\Phi}^2\right)$$

$$= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} (1-\mu)r^2\left(\frac{t^*}{r}\left(vL\hat{\Phi} - u\underline{L}\hat{\Phi}\right)\hat{\Phi} + \frac{1}{2}\frac{(t^*)^2}{r^2}\hat{\Phi}^2\right).$$

We would like to imitate this integration by parts on Kerr spacetimes. We notice that we have

$$t\partial_t\hat{\Phi} = vL\hat{\Phi} + u\underline{L}\hat{\Phi} - r_S^*\partial_{r_S^*}\hat{\Phi}, \quad \text{and} \tag{10}$$

$$t\partial_t\hat{\Phi} = \frac{t}{r_S^*}\left(vL\hat{\Phi} - u\underline{L}\hat{\Phi}\right) - \frac{t^2}{r_S^*}\partial_{r_S^*}\hat{\Phi}. \tag{11}$$

The volume form on a constant t^* slice on a Kerr spacetime is close to that on a Schwarzschild spacetime, including in the region being considered. In other words, for $r \geq r_Y^- - (r_Y^- - r_+)/4$,

$$dVol_{\Sigma_\tau} = \left(r^2(1-\mu)^{-\frac{1}{2}} + O_1(\epsilon)\right)drdx^Adx^B.$$

Moreover, for $r \geq r_Y^- - (r_Y^- - r_+)/4$,

$$\partial_{r_S^*} = \left((1-\mu) + O_1(\epsilon r^{-2})\right)\partial_r.$$

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Therefore, using (10), we have

$$
\int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{tr_S^* (1-\mu)^{\frac{1}{2}}}{r} \hat{\Phi} \partial_{t^*} \hat{\Phi}
$$
$$
= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \left((rr_S^* + O(\epsilon)) \left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} \right) \hat{\Phi} + \left(\frac{1}{2} \partial_r \left((1-\mu)r \, (r_S^*)^2 \right) + O(\epsilon) \right) \hat{\Phi}^2 \right) dr dx^A dx^B
$$
$$
= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} (1-\mu)^{\frac{1}{2}} \left(\left(\frac{r_S^*}{r} + O(\epsilon r^{-2}) \right) \left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} \right) \hat{\Phi} + \left(\frac{1}{2} \frac{(r_S^*)^2}{r^2} + \frac{r_S^*}{r} + O(\epsilon r^{-2}) \right) \hat{\Phi}^2 \right).
$$

Alternatively, we can integrate by parts after using (11):

$$
\int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{tr_S^* (1-\mu)^{\frac{1}{2}}}{r} \hat{\Phi} \partial_{t^*} \hat{\Phi}
$$
$$
= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \left((rt^* + O(\epsilon)) \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} \right) \hat{\Phi} + \left(\frac{1}{2} \partial_r \left((1-\mu)r(t^*)^2 \right) + O(\epsilon) \right) \hat{\Phi}^2 \right)
$$
$$
= \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} (1-\mu)^{\frac{1}{2}} \left(\left(\frac{t^*}{r} + O(\epsilon r^{-2}) \right) \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} \right) \hat{\Phi} + \left(\frac{1}{2} \frac{(t^*)^2}{r^2} + O(\epsilon r^{-2}) \right) \hat{\Phi}^2 \right).
$$

Therefore, we have

$$
\int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu
$$
$$
\geq \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{1}{\sqrt{1-\mu}} \left(v^2 \left(L\hat{\Phi} \right)^2 + u^2 \left(\underline{L}\hat{\Phi} \right)^2 + (u^2 + v^2) |\slashed{\nabla}\hat{\Phi}|^2 \right)
$$
$$
+ \frac{2tr_S^* (1-\mu)^{\frac{1}{2}}}{r} \hat{\Phi} \partial_{t^*} \hat{\Phi} - \frac{r_S^* (1-\mu)^{\frac{1}{2}}}{r} \hat{\Phi}^2 - C\epsilon r^{-2} \left((u^2 + v^2) \left(D\hat{\Phi} \right)^2 + t^* \hat{\Phi}^2 \right)
$$
$$
\geq \int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{1}{\sqrt{1-\mu}} \left(v^2 \left(L\hat{\Phi} \right)^2 + u^2 \left(\underline{L}\hat{\Phi} \right)^2 + (u^2 + v^2) |\slashed{\nabla}\hat{\Phi}|^2 \right)
$$
$$
+ \frac{r_S^* (1-\mu)^{\frac{1}{2}}}{r} \left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} \right) \hat{\Phi} + \frac{1}{2} \frac{(r_S^*)^2 (1-\mu)^{\frac{1}{2}}}{r^2} \hat{\Phi}^2 \tag{12}
$$
$$
+ \frac{t^* (1-\mu)^{\frac{1}{2}}}{r} \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} \right) \hat{\Phi} + \frac{1}{2} \frac{(t^*)^2}{r^2} \hat{\Phi}^2 - C\epsilon r^{-2} \left((u^2 + v^2) \left(D\hat{\Phi} \right)^2 + t^* \hat{\Phi}^2 \right)
$$
$$
\geq c \left(\int_{\Sigma_\tau \cap \{r \geq r_Y^- - (r_Y^- - r_+)/4\}} \mu \left(\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} \right)^2 + \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} \right)^2 \right) \right.
$$
$$
+ (1-\mu) \left(\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} + \frac{r_S^*}{r}\hat{\Phi} \right)^2 + \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} + \frac{t^*}{r}\hat{\Phi} \right)^2 + 2 (u^2 + v^2) |\slashed{\nabla}\hat{\Phi}|^2 \right)
$$
$$
\left. - C\epsilon r^{-2} \left((u^2 + v^2) \left(D\hat{\Phi} \right)^2 + t^* \hat{\Phi}^2 \right) \right),
$$

where the last line is obtained by first completing the square and using $c \leq 1 - \mu \leq C$ in this region of r. Let us for now ignore the error term and look at the other terms (which are manifestly positive). With exactly

the same argument as in [10], we have that these positive terms provide good estimates:

$$
\left(\int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} \mu \left(\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} \right)^2 + \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} \right)^2 \right) \right.
$$

$$
\left. + (1-\mu) \left(\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} + \frac{r_S^*}{r}\hat{\Phi} \right)^2 + \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} + \frac{t^*}{r}\hat{\Phi} \right)^2 + 2\left(u^2 + v^2\right) |\nabla\!\!\!/\,\hat{\Phi}|^2 \right) \right)
$$

$$
\geq c \int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} \left(v^2 \left(L\hat{\Phi} \right)^2 + u^2 \left(\underline{L}\hat{\Phi} \right)^2 + \left(u^2 + v^2\right) |\nabla\!\!\!/\,\hat{\Phi}|^2 + \frac{u^2 + v^2}{r^2}\hat{\Phi}^2 \right)
$$

$$
+ C\epsilon \int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} r^{-2} \left(\left(u^2 + v^2\right) \left(D\hat{\Phi} \right)^2 + t^*\hat{\Phi}^2 \right).
$$

We refer the reader to [10] for the proof. This together with $J_\mu^N \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu$ bound the error term in (12):

$$
\int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} r^{-2} \left(\left(u^2 + v^2\right) \left(D\hat{\Phi} \right)^2 + \tau\Phi^2 \right)
$$

$$
\leq C \int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} \left(v^2 \left(L\hat{\Phi} \right)^2 + u^2 \left(\underline{L}\hat{\Phi} \right)^2 + \left(u^2 + v^2\right) |\nabla\!\!\!/\,\hat{\Phi}|^2 + \frac{u^2 + v^2}{r^2}\hat{\Phi}^2 \right)
$$

$$
+ C \int_{\Sigma_\tau \cap \{ r \geq \frac{\tau}{4} \}} \left(\underline{L}\hat{\Phi} \right)^2
$$

$$
\leq C \int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} \left(v^2 \left(L\hat{\Phi} \right)^2 + u^2 \left(\underline{L}\hat{\Phi} \right)^2 + \left(u^2 + v^2\right) |\nabla\!\!\!/\,\hat{\Phi}|^2 + \frac{u^2 + v^2}{r^2}\hat{\Phi}^2 \right)
$$

$$
+ C \int_{\Sigma_\tau} J_\mu^N \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu
$$

$$
\leq C \left(\int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} \mu \left(\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} \right)^2 + \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} \right)^2 \right) \right.
$$

$$
\left. + (1-\mu) \left(\left(vL\hat{\Phi} + u\underline{L}\hat{\Phi} + \frac{r_S^*}{r}\hat{\Phi} \right)^2 + \left(vL\hat{\Phi} - u\underline{L}\hat{\Phi} + \frac{t^*}{r}\hat{\Phi} \right)^2 + 2\left(u^2 + v^2\right) |\nabla\!\!\!/\,\hat{\Phi}|^2 \right) \right)
$$

$$
+ C \int_{\Sigma_\tau} J_\mu^N \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu.
$$

Therefore, if ϵ is chosen to be small enough, then (12) implies that

$$
\int_{\Sigma_\tau} J_\mu^{Z, w^Z} \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu
$$

$$
\geq c \int_{\Sigma_\tau \cap \{ r \geq r_Y^- - (r_Y^- - r_+)/4 \}} \left(v^2 \left(L\hat{\Phi} \right)^2 + u^2 \left(\underline{L}\hat{\Phi} \right)^2 + \left(u^2 + v^2\right) |\nabla\!\!\!/\,\hat{\Phi}|^2 + \frac{u^2 + v^2}{r^2}\hat{\Phi}^2 \right). \tag{13}
$$

We note that c here is independent of the choice of r_Y^-. With this bound we would like to estimate $\int_{\mathbb{S}^2} \Phi\left(\tau, r\right)^2$. Using (13), there exists a $\tilde{r} \in [r_Y^-, r_Y^- + 1]$ such that

$$
\int_{\mathbb{S}^2} \Phi\left(\tau, \tilde{r}\right)^2 = \int_{\mathbb{S}^2} \hat{\Phi}\left(\tau, \tilde{r}\right)^2 \leq C\tau^{-2} \left(\int_{\Sigma_\tau} J_\mu^{Z, w^Z} \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N \left(\hat{\Phi} \right) n_{\Sigma_\tau}^\mu \right).
$$

Then for every $r \in [r_+, r_Y^-]$, since

$$
\Phi\left(\tau, \tilde{r}\right) - \Phi\left(\tau, r\right) = \int_r^{\tilde{r}} \partial_r \Phi dr,
$$

we have

$$\int_{\mathbb{S}^2} \Phi\,(\tau, r)^2 \le \int_{\mathbb{S}^2} \Phi\,(\tau, \tilde{r})^2 + (\tilde{r} - r) \int_{\Sigma_\tau \cap [r,\tilde{r}]} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau}$$

$$\le C \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + C\tau^{-2} \left(\int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} \right) \tag{14}$$

Now we need to obtain estimates for Φ from that for $\hat{\Phi}$. It is obvious that

$$\int_{\Sigma_\tau \cap \{r \ge r_Y^-\}} \left(v^2\,(L\Phi)^2 + u^2\,(\underline{L}\Phi)^2 + (u^2 + v^2)\,|\slashed{\nabla}\Phi|^2 + \frac{u^2 + v^2}{r^2}\Phi^2 \right)$$

$$\le \int_{\Sigma_\tau \cap \{r \ge r_Y^-\}} \left(v^2\,\big(L\hat{\Phi}\big)^2 + u^2\,\big(\underline{L}\hat{\Phi}\big)^2 + (u^2 + v^2)\,|\slashed{\nabla}\hat{\Phi}|^2 + \frac{u^2 + v^2}{r^2}\hat{\Phi}^2 \right)$$

$$\le \int_{\Sigma_\tau \cap \{r \ge r_Y^- - (r_Y^- - r_+)/4\}} \left(v^2\,\big(L\hat{\Phi}\big)^2 + u^2\,\big(\underline{L}\hat{\Phi}\big)^2 + (u^2 + v^2)\,|\slashed{\nabla}\hat{\Phi}|^2 + \frac{u^2 + v^2}{r^2}\hat{\Phi}^2 \right),$$

and

$$\int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau}$$

$$\le \int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} \Phi^2$$

where we have used Proposition 27 to show that the u^2 factor comes with a factor of $1 - \mu$

$$\le \int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} \tag{15}$$

$$+ C\,(r_Y^- - r_+) \left(\int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} \right)$$

$$\le \int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau}$$

$$+ \frac{1}{2} \left(\int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} \right),$$

for r_Y^- chosen to be sufficiently close to r_+. Then

$$\int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,\big(\hat{\Phi}\big)\, n^\mu_{\Sigma_\tau}$$

$$\le \int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_\tau} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau}.$$

$\square$

Remark 4. *From this point onward, we consider r_Y^- to be fixed. We note again that r_Y^- is chosen so that (5) and (15) hold.*

Remark 5. *The proof of the above proposition in particular shows that*

$$\int_{\Sigma_\tau} J^{Z,w^Z}_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} J^N_\mu\,(\Phi)\, n^\mu_{\Sigma_\tau} \ge 0$$

In order to use this Proposition, it is helpful to have a localized version of Φ. This follows [10], [8]. The idea is to use the finite speed of propagation and cutoff Φ outside the domain of dependence. Suppose we now focus on the time interval $[\tau', \tau]$. Take $\tilde{G}$ to be any smooth function agreeing with G on the domain of dependence of the region $(t^* = \tau, r \le \frac{\tau}{2})$. Let $\tilde{\Phi}\,(\tau') = \chi\Phi\,(\tau')$, $\partial_{t^*}\tilde{\Phi}\,(\tau') = \chi\partial_{t^*}\Phi\,(\tau')$, where χ is a cutoff

function identically equals to 1 for $r \leq \frac{7\tau'}{10}$ and compactly supported in $r \leq \frac{9\tau'}{10}$. Notice that the region for which χ is one is inside the domain of dependence of the region $(t^* = \tau, r \leq \frac{\tau}{2})$ if $\tau' \leq \tau \leq (1.1)\tau'$. We solve for $\Box_{g_K} \tilde{\Phi} = \tilde{G}$.

With this definition of $\tilde{\Phi}$, we have two ways to estimate the non-degenerate energy of $\tilde{\Phi}$:

Proposition 29.

$$\int_{\Sigma_{\tau'}} J_\mu^N \left(\tilde{\Phi} \right) n_{\Sigma_{\tau'}}^\mu \leq C \int_{\Sigma_{\tau'}} J_\mu^N \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu.$$

$$\int_{\Sigma_{\tau'}} J_\mu^N \left(\tilde{\Phi} \right) n_{\Sigma_{\tau'}}^\mu$$
$$\leq C^2 \int_{\Sigma_{\tau'} \cap \{r \leq r_Y^-\}} J_\mu^N \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu + C(\tau')^{-2} \left(\int_{\Sigma_{\tau'}} J_\mu^N \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z} \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu \right).$$

Proof. The first part is an easy application of Proposition 19

$$\int_{\Sigma_{\tau'}} J_\mu^N \left(\tilde{\Phi} \right) n_{\Sigma_{\tau'}}^\mu \leq C \int_{\Sigma_{\tau'} \cap \{r \leq \frac{9\tau'}{10}\}} \left((D\Phi)^2 + (\tau')^{-2}\Phi^2 \right)$$
$$\leq C \int_{\Sigma_{\tau'} \cap \{r \leq \frac{9\tau'}{10}\}} \left((D\Phi)^2 + r^{-2}\Phi^2 \right)$$
$$\leq C \int_{\Sigma_{\tau'}} J_\mu^N \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu.$$

Following (14), we have

$$\int_{\Sigma_{\tau'} \cap \{r \leq r_Y^-\}} \Phi^2 \leq C \left(\int_{\Sigma_{\tau'} \cap \{r \leq r_Y^-\}} J_\mu^N \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_{\tau'} \cap \{r_Y^- \leq r \leq r_Y^+\}} \Phi^2 \right).$$

Using this and Proposition 28, we have

$$\int_{\Sigma_{\tau'}} J_\mu^N \left(\tilde{\Phi} \right) n_{\Sigma_{\tau'}}^\mu$$
$$\leq C \int_{\Sigma_{\tau'} \cap \{r \leq \frac{9\tau'}{10}\}} \left((D\Phi)^2 + (\tau')^{-2}\Phi^2 \right)$$
$$\leq C \int_{\Sigma_{\tau'} \cap \{r \leq r_Y^-\}} \left((D\Phi)^2 + \Phi^2 \right)$$
$$+ C(\tau')^{-2} \int_{\Sigma_{\tau'} \cap \{r_Y^- \leq r \leq \frac{9\tau'}{10}\}} \left(u^2 (\underline{L}\Phi)^2 + v^2 (L\Phi)^2 + (u^2 + v^2) |\nabla \!\!\!\! / \Phi|^2 + \left(\frac{u^2 + v^2}{r^2} \right) \Phi^2 \right)$$
$$\leq C^2 \int_{\Sigma_{\tau'} \cap \{r \leq r_Y^-\}} J_\mu^N \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu + C(\tau')^{-2} \left(\int_{\Sigma_{\tau'}} J_\mu^N \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z} \left(\Phi \right) n_{\Sigma_{\tau'}}^\mu \right).$$

$\square$

The cutoff procedure above would also allow us to localized the estimates for the bulk term:

Proposition 30. *Let $G = G_1 + G_2$ be any way to decompose the function G. Then for $\tau' \leq \tau \leq (1.1)\tau'$, we have*

1. Localized Boundedness Estimate

$$\int_{\Sigma_\tau \cap \{r \le \frac{\tau}{2}\}} J^N_\mu(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu(\Phi)\, n^\mu_{\mathcal{H}^+}$$

$$+ \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{t^*}{2}\}} K^{X_0}(\Phi)$$

$$\le C \left(\int_{\Sigma_{\tau'}} J^N_\mu(\Phi)\, n^\mu_{\Sigma_{\tau'}} + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*} \cap \{r \le \frac{9t^*}{10}\}} G_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 \right.$$

$$+ \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \le \frac{9t^*}{10}\}} G_1^2 + \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \le \frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G_2 \right)^2$$

$$+ \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\} \cap \{r \le \frac{9t^*}{10}\}} G_2^2 \bigg).$$

2. Localized Decay Estimate

$$\int_{\Sigma_\tau \cap \{r \le \frac{\tau}{2}\}} J^N_\mu(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu(\Phi)\, n^\mu_{\mathcal{H}^+}$$

$$+ \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{t^*}{2}\}} K^{X_0}(\Phi)$$

$$\le C \left(\tau^{-2} \int_{\Sigma_{\tau'}} J^{Z+N,w^Z}_\mu(\Phi)\, n^\mu_{\Sigma_{\tau'}} + C \int_{\Sigma_{\tau'} \cap \{r \le r_Y^-\}} J^N_\mu(\Phi)\, n^\mu_{\Sigma_{\tau'}} \right)$$

$$+ C \left(\left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*} \cap \{r \le \frac{9t^*}{10}\}} G_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \le \frac{9t^*}{10}\}} G_1^2 \right.$$

$$+ \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \le \frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G_2 \right)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\} \cap \{r \le \frac{9t^*}{10}\}} G_2^2 \bigg).$$

Proof. Applying Proposition 25 to the equation $\Box_{g_K} \tilde{\Phi} = \tilde{G}$, we have

$$\int_{\Sigma_\tau \cap \{r \le \frac{\tau}{2}\}} J^N_\mu(\tilde{\Phi})\, n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu(\tilde{\Phi})\, n^\mu_{\mathcal{H}^+} + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} K^N(\tilde{\Phi}) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{t^*}{2}\}} K^{X_0}(\tilde{\Phi})$$

$$\le C \left(\int_{\Sigma_{\tau'}} J^N_\mu(\tilde{\Phi})\, n^\mu_{\Sigma_{\tau'}} + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} \tilde{G}_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \tilde{G}_1^2 \right.$$

$$+ \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m \tilde{G}_2 \right)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\}} \tilde{G}_2^2 \bigg).$$

Since by the finite speed of propagation, $\tilde{\Phi} = \Phi$ in $\{r \le \frac{t^*}{2}\}$, we have

$$\int_{\Sigma_\tau \cap \{r \le \frac{\tau}{2}\}} J^N_\mu(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu(\Phi)\, n^\mu_{\mathcal{H}^+} + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{t^*}{2}\}} K^{X_0}(\Phi)$$

$$\le C \left(\int_{\Sigma_{\tau'}} J^N_\mu(\tilde{\Phi})\, n^\mu_{\Sigma_{\tau'}} + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} \tilde{G}_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \tilde{G}_1^2 \right.$$

$$+ \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m \tilde{G}_2 \right)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\}} \tilde{G}_2^2 \bigg).$$

Now, we choose a particular $\tilde{G}$. Define $\tilde{G}$ to be G for $r \leq \frac{7t^*}{10}$, and 0 for $r \geq \frac{9t^*}{10}$. It can be easily shown that one can have the bounds $|\partial_{t^*}^m \tilde{G}| \leq C \sum_{k=0}^m |\left(\frac{r^*}{(t^*)^2}\right)^k \partial_{t^*}^{m-k} G| \leq C \sum_{k=0}^m |(t^*)^{-k} \partial_{t^*}^{m-k} G|$ for $\frac{7t^*}{10} \leq r \leq \frac{9t^*}{10}$. Therefore, we have

$$
\int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{2}\}} J_\mu^N (\Phi) \, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N (\Phi) \, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r_Y^-\}} K^N (\Phi) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq \frac{t^*}{2}\}} K^{X_0} (\Phi)
$$

$$
\leq C \left(\int_{\Sigma_{\tau'}} J_\mu^N \left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*} \cap \{r \leq \frac{9t^*}{10}\}} G_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} G_1^2 \right.
$$

$$
\left. + \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G_2\right)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G_2^2 \right).
$$

We can now conclude the Proposition using Proposition 29. $\qquad\square$

We can remove the degeneracy around $r \sim 3M$ using an extra derivative.

Proposition 31. *Let $G = G_1 + G_2$ be any way to decompose the function G. Then for $\tau' \leq \tau \leq (1.1)\tau'$, we have*

1. *Localized Boundedness Estimate*

$$
\iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq \frac{t^*}{2}\}} K^{X_1} (\Phi)
$$

$$
\leq C \left(\sum_{m=0}^1 \int_{\Sigma_{\tau'}} J_\mu^N \left(\partial_{t^*}^m \Phi\right) n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^1 \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*} \cap \{r \leq \frac{9t^*}{10}\}} \left(\partial_{t^*}^m G_1\right)^2 \right)^{\frac{1}{2}} dt^* \right)^2 \right.
$$

$$
+ \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} \left(\partial_{t^*}^m G_1\right)^2 + \sum_{m=0}^2 \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G_2\right)^2
$$

$$
\left. + \sup_{t^* \in [\tau'-1,\tau+1]} \sum_{m=0}^1 \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\} \cap \{r \leq \frac{9t^*}{10}\}} \left(\partial_{t^*}^m G_2\right)^2 \right).
$$

2. *Localized Decay Estimate*

$$
\iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq \frac{t^*}{2}\}} K^{X_1} (\Phi)
$$

$$
\leq C \left(\tau^{-2} \sum_{m=0}^1 \int_{\Sigma_{\tau'}} J_\mu^{Z+N,w^Z} \left(\partial_{t^*}^m \Phi\right) n_{\Sigma_{\tau'}}^\mu + C \sum_{m=0}^1 \int_{\Sigma_{\tau'} \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^m \Phi\right) n_{\Sigma_{\tau'}}^\mu \right)
$$

$$
+ C \left(\sum_{m=0}^1 \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*} \cap \{r \leq \frac{9t^*}{10}\}} \left(\partial_{t^*}^m G_1\right)^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} \left(\partial_{t^*}^m G_1\right)^2 \right.
$$

$$
+ \sum_{m=0}^2 \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G_2\right)^2
$$

$$
\left. + \sup_{t^* \in [\tau'-1,\tau+1]} \sum_{m=0}^1 \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\} \cap \{r \leq \frac{9t^*}{10}\}} \left(\partial_{t^*}^m G_2\right)^2 \right).
$$

Proof. We repeat the argument in Proposition 30, using Proposition 26 instead of 25. $\qquad\square$

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When using the conservation law for Z, we can ignore the part of the bulk term that has a good sign.

Definition 4. *Let* $K_+^{Z,w^Z}(\Phi) = \max\{K^{Z,w^Z}(\Phi), 0\}$.

Using the conservation law for the modified vector field, we have a one-sided bound:

Proposition 32.

$$\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{Z,w^Z}(\Phi)\, n_{\mathcal{H}^+}^\mu$$

$$\leq C(\tau')^2 \int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{\tau'}}^\mu$$

$$+ \iint_{\mathcal{R}(\tau',\tau)} K_+^{Z,w^Z}(\Phi) + |\iint_{\mathcal{R}(\tau',\tau)} \left(u^2 L\Phi + v^2 \underline{L}\Phi - \frac{1}{4}w\Phi\right) G|.$$

Remark 6. *In the above Proposition, the left hand side is not claimed to be positive. Note, however, that the right hand side is positive by Remark 5.*

Remark 7. *We note also that* $\int_{\mathcal{H}(\tau',\tau)} J_\mu^{Z,w^Z}(\Phi)\, n_{\mathcal{H}^+}^\mu \geq 0$ *because Z and $n_{\mathcal{H}^+}^\mu$ are both null and future directed and* $w^Z = 0$ *on the event horizon.*

To show that $\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu$ is almost bounded, we would have to show that $\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu$ in fact decays. This is given by the following Proposition:

Proposition 33.

$$\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu$$

$$\leq C^2\tau^{-2} \int_{\Sigma_{(1.1)^{-2}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu + C\tau^{-2} \int_{\Sigma_{(1.1)^{-2}\tau}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu$$

$$+ C\tau^{-2} \iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} K_+^{Z,w^Z}(\Phi) + C\tau^{-2}|\iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} \left(u^2 L\Phi + v^2 \underline{L}\Phi - \frac{1}{4}w\Phi\right) G|$$

$$+ C\left(\left(\int_{(1.1)^{-2}\tau-1}^{\tau+1} \left(\int_{\Sigma_{t^*}\cap\{r\leq\frac{9t^*}{10}\}} G_1^2\right)^{\frac{1}{2}} dt^*\right)^2 + \iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} G_1^2\right)$$

$$+ C\left(\sum_{m=0}^1 \iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}(\partial_{t^*}^m G_2)^2 + \sup_{t^*\in[(1.1)^{-2}\tau-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G_2^2\right).$$

Proof. By Proposition 30.2 applied to the t^* interval $[(1.1)^{-1}\tau,\tau]$, we have

$$\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi)$$

$$\leq C\tau^{-2} \int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu + C^2 \int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu$$

$$+ C\left(\sum_{m=0}^1 \iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}(\partial_{t^*}^m G)^2 + \sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2\right).$$

By taking the infimum there exists $\tilde{\tau} \in [(1.1)^{-1}\tau,\tau]$ such that

$$\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu \leq C\tau^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi).$$

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Hence,

$$
\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu
$$

$$
\leq C\tau^{-2}\int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu + C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu
$$

$$
+ C\tau^{-1}\left(\sum_{m=0}^{1}\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 + \sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2\right).
$$

Apply Proposition 30.2 to the t^* interval $[\tilde{\tau},\tau]$ and use Proposition 25 and 32

$$
\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu
$$

$$
\leq C\left(\tau^{-2}\int_{\Sigma_{\tilde{\tau}}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu + C\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu\right)
$$

$$
+ C\left(\sum_{m=0}^{1}\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 + \sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)
$$

$$
\leq C\tau^{-2}\left(\int_{\Sigma_{\tilde{\tau}}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu + \int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu\right)
$$

$$
+ C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu
$$

$$
+ C\left(\sum_{m=0}^{1}\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 + \sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)
$$

$$
\leq C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu + C\tau^{-2}\int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu
$$

$$
+ C\tau^{-2}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} K_+^{Z,w^Z}(\Phi) + C\tau^{-2}\left|\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)}\left(u^2\underline{L}\Phi + v^2 L\Phi - \frac{1}{4}w\Phi\right)G\right|
$$

$$
+ C\left(\sum_{m=0}^{1}\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 + \sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)
\tag{16}
$$

Replacing $[(1.1)^{-1}\tau,\tau]$ with $[(1.1)^{-2}\tau,(1.1)^{-1}\tau]$, we get also

$$
\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu
$$

$$
\leq C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-2}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu + C\tau^{-2}\int_{\Sigma_{(1.1)^{-2}\tau}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu
$$

$$
+ C\tau^{-2}\iint_{\mathcal{R}((1.1)^{-2}\tau,(1.1)^{-1}\tau)} K_+^{Z,w^Z}C(\Phi) + C\tau^{-2}\left|\iint_{\mathcal{R}((1.1)^{-2}\tau,(1.1)^{-1}\tau)}\left(u^2 L\Phi + v^2\underline{L}\Phi - \frac{1}{4}w\Phi\right)G\right|
$$

$$
+ C\left(\sum_{m=0}^{1}\iint_{\mathcal{R}((1.1)^{-2}\tau-1,(1.1)^{-1}\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right.
$$

$$
\left. + \sup_{t^*\in[(1.1)^{-2}\tau-1,(1.1)^{-1}\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)
\tag{17}
$$

Therefore, plugging (17) into (16),

$$
\int_{\Sigma_\tau \cap \{r \le \frac{t^*}{2}\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau}
$$

$$
\le C^2 \tau^{-2} \int_{\Sigma_{(1.1)^{-2}\tau} \cap \{r \le r_Y^-\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_{(1.1)^{-2}\tau}} + C\tau^{-2} \int_{\Sigma_{(1.1)^{-2}\tau}} J^{Z,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_{(1.1)^{-1}\tau}}
$$

$$
+ C\tau^{-2} \iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} K^{Z,w^Z}_+ (\Phi) + C\tau^{-2} | \iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} \left(u^2 L\Phi + v^2 \underline{L}\Phi - \frac{1}{4} w\Phi \right) G|
$$

$$
+ C \left(\sum_{m=0}^1 \iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1) \cap \{r \le \frac{9t^*}{10}\}} r^{1+\delta} (\partial^m_{t^*} G)^2 + \sup_{t^* \in [(1.1)^{-2}\tau-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\} \cap \{r \le \frac{9t^*}{10}\}} G^2 \right)
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 33 immediately gives control over the non-degenerate energy and conformal energy using Propositions 28 and 32 respectively:

Corollary 34. *For any $\gamma < 1$,*

$$
\int_{\Sigma_\tau} J^{Z,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le \gamma\tau\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau}
$$

$$
\le C \left(\int_{\Sigma_{\tau_0}} J^{Z,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_{\tau_0}} + C \int_{\Sigma_{\tau_0}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_{\tau_0}} \right.
$$

$$
+ \iint_{\mathcal{R}(\tau_0,\tau)} K^{Z,w^Z}_+ (\Phi) + | \iint_{\mathcal{R}(\tau_0,\tau)} \left(u^2 \underline{L}\Phi + v^2 L\Phi - \frac{1}{4} w\Phi \right) G| \right)
$$

$$
+ C \left(\sum_{m=0}^1 \iint_{\mathcal{R}(\tau_0-1,\tau+1) \cap \{r \le \frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} (\partial^m_{t^*} G)^2 + \sup_{t^* \in [\tau_0-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\} \cap \{r \le \frac{9t^*}{10}\}} (t^*)^2 G^2 \right).
$$

Proof. By Proposition 28,

$$
\tau^2 \int_{\Sigma_\tau \cap \{r \le \gamma\tau\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau} \le C \int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_\tau} + C^2 \tau^2 \int_{\Sigma_\tau \cap \{r \le r_Y^-\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau}.
$$

Therefore, by Propositions 32 and 33,

$$
\int_{\Sigma_\tau} J^{Z,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau \cap \{r \le \gamma\tau\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau}
$$

$$
\le C \left(\int_{\Sigma_{(1.1)^{-2}\tau}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_{(1.1)^{-2}\tau}} + \int_{\Sigma_{(1.1)^{-2}\tau}} J^{Z,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_{(1.1)^{-2}\tau}} + \iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} K^{Z,w^Z}_+ (\Phi) \right.
$$

$$
+ | \iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} \left(u^2 \underline{L}\Phi + v^2 L\Phi - \frac{1}{4} w\Phi \right) G| \right)
$$

$$
+ C\tau^2 \left(\sum_{m=0}^1 \iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1)} r^{1+\delta} (\partial^m_{t^*} G)^2 + \sup_{t^* \in [(1.1)^{-2}\tau-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\} \cap \{r \le \frac{9t^*}{10}\}} G^2 \right)
$$

We then use the same estimate for $[(1.1)^{-4}\tau, (1.1)^{-2}\tau], [(1.1)^{-6}\tau, (1.1)^{-4}\tau], \dots$ $\qquad\qquad\qquad$ $\square$

The term $\iint_{\mathcal{R}(\tau_0,\tau)} K^{Z,w^Z}_+ (\Phi)$ can be controlled. Here is where the control of the logarithmic divergences from the red-shift vector field is crucially used.

Proposition 35.

$$\iint_{\mathcal{R}(\tau',\tau)} K_+^{Z,w^Z}\left(\Phi\right)$$

$$\leq C \iint_{\mathcal{R}(\tau',\tau)\cap\{r\geq r_Y^-\}} t^* \left(r^{-2}J_\mu^N\left(\Phi\right)n_{\Sigma_{\bar\tau}}^\mu + r^{-4}\Phi\right) + \epsilon \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} \left(t^*\right)^2 K^N\left(\Phi\right).$$

Proof. See [8]. $\qquad\square$

The bulk term arising from the inhomogeneous term G can also be controlled.

Proposition 36.

$$|\iint_{\mathcal{R}(\tau_0,\tau)} \left(u^2\underline{L}\Phi + v^2 L\Phi - \frac{1}{4}w\Phi\right) G|$$

$$\leq \delta' \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} \left(t^*\right)^2 K^{X_0}\left(\Phi\right) + \delta \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left(t^*\right)^2 K^N\left(\Phi\right)$$

$$+ \delta' \sup_{t^*\in[\tau_0,\tau]} \left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} J_\mu^{Z+N,w^Z}\left(\Phi\right)n_{\Sigma_{t^*}}^\mu + \left(t^*\right)^2 \int_{\Sigma_{t^*}\cap\{r\leq\frac{23M}{8}\}} J_\mu^N\left(\Phi\right)n_{\Sigma_{t^*}}^\mu\right)$$

$$+ C(\delta')^{-1} \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} \left(t^*\right)^2 r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 + C(\delta')^{-1} \left(\int_{\tau_0}^\tau \left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} r^2 G^2\right)^{\frac{1}{2}} dt^*\right)^2$$

$$+ C(\delta')^{-1} \sup_{t^*\in[\tau_0,\tau]} \int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} \left(t^*\right)^2 G^2.$$

Proof. Two regions require particular care to deal with. The first is the region $\{r\leq r_Y^-\}$, since the coefficients of the vector field Z are not bounded as $r\to r_+$. The other is the region $\{|r-3M|\leq\frac{M}{8}\}$. This is where trapping occurs and where the integrated decay estimate degenerates or loses derivatives. We first look at the region $\{r\leq r_+\}$ using the null frame:

$$|\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left(u^2\underline{L}\Phi + v^2 L\Phi - \frac{1}{4}w\Phi\right) G|$$

$$\leq C \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left((t^*)^2 + (r_S^*)^2\right)\left(|\nabla_{\hat V}\Phi G| + (1-\mu)|\nabla_{\hat Y}\Phi G| + (1-\mu)\sum_A |\nabla_{E_A}\Phi G|\right)$$

using Proposition 27

$$\leq C \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left(t^*\right)^2 \left(|\log|r-r_+||^2|\nabla_{\hat V}\Phi G| + |\nabla_{\hat Y}\Phi G| + \sum_A |\nabla_{E_A}\Phi G|\right)$$

$$\leq \delta' \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left(t^*\right)^2 \left(|\log|r-r_+||^4\left(\nabla_{\hat V}\Phi\right)^2 + \left(\nabla_{\hat Y}\Phi\right)^2 + \sum_A \left(\nabla_{E_A}\Phi\right)^2\right)$$

$$+ C(\delta')^{-1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left(t^*\right)^2 G^2$$

$$\leq \delta' \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left(t^*\right)^2 K^N\left(\Phi\right) + C(\delta')^{-1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} \left(t^*\right)^2 G^2.$$

For the region $\{r_Y^-\leq r\leq\frac{25M}{8}\}$, where trapping occurs, we integrate by parts in t^* so that the bulk term

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does not have $\partial_{t^*}\Phi$, which cannot be controlled by the integrated decay estimate.

$$|\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\le r\le\frac{25M}{8}\}}\left(u^2\underline{L}\Phi+v^2L\Phi-\frac{1}{4}w\Phi\right)G|$$

$$\le C\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\le r\le\frac{25M}{8}\}}(t^*)^2|\partial_r\Phi G|+(t^*)^2|\Phi\partial_{t^*}G|+t^*|\Phi G|+\int_{\Sigma_\tau}\tau^2|\Phi G|+\int_{\Sigma_{\tau_0}}\tau_0^2|\Phi G|$$

$$\le C\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\le r\le\frac{25M}{8}\}}(t^*)^2\left(\Phi^2+(\partial_r\Phi)^2\right)\right)^{\frac{1}{2}}\left(\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\le r\le\frac{25M}{8}\}}(t^*)^2\left(\partial_{t^*}^m G\right)^2\right)^{\frac{1}{2}}$$

$$+\delta'\int_{\Sigma_\tau\cap\{r_Y^-\le r\le\frac{25M}{8}\}}\tau^2 J_\mu^N(\Phi)n_{\Sigma_\tau}^\mu+\delta'\int_{\Sigma_{\tau_0}\cap\{r_Y^-\le r\le\frac{25M}{8}\}}\tau_0^2 J_\mu^N(\Phi)n_{\Sigma_{\tau_0}}^\mu$$

$$+C(\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le\frac{25M}{8}\}}(t^*)^2 G^2,$$

using Proposition 19. We then move to the region $\{\frac{25M}{8}\le r\le\frac{t^*}{2}\}$:

$$|\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\le r\le\frac{t^*}{2}\}}\left(u^2\underline{L}\Phi+v^2L\Phi-\frac{1}{4}w\Phi\right)G|$$

$$\le C\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\le r\le\frac{t^*}{2}\}}\left((t^*)^2|\partial\Phi|+t^*|\Phi|\right)|G|$$

$$\le C\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\le r\le\frac{t^*}{2}\}}(t^*)^2\left(r^{-3-\delta}\Phi^2+r^{-1-\delta}J_\mu^N(\Phi)n_{\Sigma_{t^*}}^\mu\right)\right)^{\frac{1}{2}}$$

$$\times\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\le r\le\frac{t^*}{2}\}}\left(r^{3+\delta}+(t^*)^2 r^{1+\delta}\right)G^2\right)^{\frac{1}{2}}$$

$$\le C\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\le r\le\frac{t^*}{2}\}}(t^*)^2\left(r^{-3-\delta}\Phi^2+r^{-1-\delta}J_\mu^N(\Phi)n_{\Sigma_{t^*}}^\mu\right)\right)^{\frac{1}{2}}\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\le r\le\frac{t^*}{2}\}}(t^*)^2 r^{1+\delta}G^2\right)^{\frac{1}{2}}$$

Finally, we estimate in the region $\{r\ge\frac{t^*}{2}\}$:

$$|\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\ge\frac{t^*}{2}\}}\left(u^2\underline{L}\Phi+v^2L\Phi-\frac{1}{4}w\Phi\right)G|$$

$$\le C\sup_{t^*\in[\tau_0,\tau]}\left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}}J_\mu^{Z,w^Z}(\Phi)n_{\Sigma_{t^*}}^\mu+(t^*)^2\int_{\Sigma_{t^*}\cap\{r\le r_Y^-\}}J_\mu^N(\Phi)n_{\Sigma_{t^*}}^\mu\right)^{\frac{1}{2}}\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}}r^2 G^2\right)^{\frac{1}{2}}dt^*,$$

where we have used Proposition 28.The Proposition follows from Cauchy-Schwarz. $\qquad\square$

We have therefore proved the following decay result associated to the vector field Z.

Proposition 37. *For $\delta,\delta'>0$ sufficiently small and $0\le\gamma<1$, there exist $c=c(\delta,\gamma)$ and $C=C(\delta,\gamma)$ such*

that the following estimate holds for any solution to $\Box_{g_K} \Phi = G$:

$$c \int_{\Sigma_\tau} J^{Z,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_\tau} + \tau^2 \int_{\Sigma_\tau \cap \{r \leq \gamma\tau\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau}$$

$$\leq C \int_{\Sigma_{\tau_0}} J^{Z+CN,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_{\tau_0}} + C \iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1} (\Phi)$$

$$+ C\delta' \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq \frac{t^*}{2}\}} (t^*)^2 K^{X_0} (\Phi) + C(\delta' + \epsilon) \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq r_Y^-\}} (t^*)^2 K^N (\Phi)$$

$$+ C(\delta')^{-1} \left(\int_{\tau_0}^\tau \left(\int_{\Sigma_{t^*} \cap \{r \geq \frac{t^*}{2}\}} r^2 G^2 \right)^{\frac{1}{2}} dt^* \right)^2 + C\delta^{-1} \sum_{m=0}^1 \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq \frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} (\partial^m_{t^*} G)^2$$

$$+ C(\delta')^{-1} \sup_{t^* \in [\tau_0,\tau]} \int_{\Sigma_{t^*} \cap \{r_Y^- \leq r \leq \frac{25M}{8}\}} (t^*)^2 G^2.$$

10   Estimates for Solutions to $\Box_{g_K} \Phi = 0$

From this point onwards, we consider $\Box_{g_K} \Phi = 0$. In this section, we write down the energy estimates derived by Dafermos-Rodnianski [8]. These will be used in later sections.

Proposition 38.

$$\tau^2 \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{2}\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau} + c \int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_\tau}$$

$$\leq C\tau^\eta \sum_{m=0}^2 \left(\int_{\Sigma_{\tau_0}} J^{Z,w^Z}_\mu (\partial^m_{t^*} \Phi) \, n^\mu_{\Sigma_{\tau_0}} + \int_{\Sigma_{\tau_0}} J^N_\mu (\partial^m_{t^*} \Phi) \, n^\mu_{\Sigma_{\tau_0}} \right).$$

Proof. We introduce the bootstrap assumptions:

$$\tau^2 \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{2}\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_\tau} + c \int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_\tau}$$

$$\leq A^2 \tau^\eta \sum_{m=0}^2 \int_{\Sigma_{\tau_0}} J^{Z+CN,w^Z}_\mu (\partial^m_{t^*} \Phi) \, n^\mu_{\Sigma_{\tau_0}}. \tag{18}$$

$$\tau^2 \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{2}\}} J^N_\mu (\partial_{t^*} \Phi) \, n^\mu_{\Sigma_\tau} + c \int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_\tau}$$

$$\leq A \tau^{1+\eta} \sum_{m=0}^2 \int_{\Sigma_{\tau_0}} J^{Z+CN,w^Z}_\mu (\partial^m_{t^*} \Phi) \, n^\mu_{\Sigma_{\tau_0}}. \tag{19}$$

Here we think of η as a small positive number. We divide the interval $[\tau_0,\tau]$ dyadically into $\tau_0 \leq \tau_1 \leq \ldots \leq \tau_{n-1} \leq \tau_n = \tau$ with $\tau_{i+1} \leq (1.1)\tau_i$ and n the smallest integer for doing such division. We then have $n \sim \log |\tau - \tau'|$. We can now apply Proposition 30 on the intervals $[\tau_{i-1},\tau_i]$ and use the bootstrap assumption (18):

$$\iint_{\mathcal{R}(\tau_{i-1},\tau_i) \cap \{r \leq \frac{t^*}{2}\}} K^{X_0} (\Phi) + \iint_{\mathcal{R}(\tau_{i-1},\tau_i) \cap \{r \leq r_Y^-\}} K^N (\Phi)$$

$$\leq C \left(\tau_i^{-2} \int_{\Sigma_{\tau_{i-1}}} J^{Z,w^Z}_\mu (\Phi) \, n^\mu_{\Sigma_{\tau_{i-1}}} + C \int_{\Sigma_{\tau_{i-1}} \cap \{r \leq r_Y^-\}} J^N_\mu (\Phi) \, n^\mu_{\Sigma_{\tau_{i-1}}} \right)$$

$$\leq C A^2 \tau_i^{-2+\eta} \sum_{m=0}^2 \int_{\Sigma_{\tau_0}} J^{Z+CN,w^Z}_\mu (\partial^m_{t^*} \Phi) \, n^\mu_{\Sigma_{\tau_0}}.$$

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Similarly, we can apply Proposition 30 on the intervals $[\tau_{i-1}, \tau_i]$ for $\partial_{t^*}\Phi$ and use the bootstrap assumption (19):

$$\iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(\partial_{t^*}\Phi\right) + \iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*}\Phi\right)$$

$$\leq C\left(\tau_i^{-2}\int_{\Sigma_{\tau_{i-1}}} J_\mu^{Z,w^Z}\left(\partial_{t^*}\Phi\right) n_{\Sigma_{\tau_{i-1}}}^\mu + C\int_{\Sigma_{\tau_{i-1}}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}\Phi\right) n_{\Sigma_{\tau_{i-1}}}^\mu\right)$$

$$\leq CA\tau_i^{-1+\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

By Proposition 26, we have

$$\iint_{\mathcal{R}(\tau_{i-1},\tau_i)} r^{-1+\delta}K^{X_1}\left(\partial_{t^*}\Phi\right) \leq C\sum_{m=0}^{1}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

By Propositions 26 and 31, we have

$$\iint_{\mathcal{R}(\tau_{i-1},\tau_i)} r^{-1+\delta}K^{X_1}\left(\Phi\right)$$

$$\leq C\iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(\Phi\right) + C\tau_i^{-1+\delta}\iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\geq\frac{t^*}{2}\}} K^{X_1}\left(\Phi\right)$$

$$\leq CA\tau_i^{-1+\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Apply Proposition 37, we get

$$c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\leq C\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\iint_{\mathcal{R}(\tau_0,\tau)} t^*r^{-1+\delta}K^{X_1}\left(\Phi\right)$$

$$+ C\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} (t^*)^2 K^{X_0}\left(\Phi\right) + C\left(\delta'+\epsilon\right)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} (t^*)^2 K^N\left(\Phi\right)$$

$$\leq \left(C + \left(C+CA+CA^2(2\delta'+\epsilon)\right)\sum_{i=0}^{n-1}\tau_i^\eta\right)\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu$$

$$\leq \left(C+\eta^{-1}\left(C+CA+CA^2(2\delta'+\epsilon)\right)\tau^\eta\right)\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Now take A large, $\epsilon = \frac{\eta}{4C}$ and $\delta' = \frac{\epsilon}{2}$, we improve (18). Apply Proposition 37 again, this time to $\partial_{t^*}\Phi$, we have

$$c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\partial_{t^*}\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N\left(\partial_{t^*}\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\leq C\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\iint_{\mathcal{R}(\tau_0,\tau)} t^*r^{-1+\delta}K^{X_1}\left(\partial_{t^*}\Phi\right)$$

$$+ C\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} (t^*)^2 K^{X_0}\left(\partial_{t^*}\Phi\right) + C\left(\delta'+\epsilon\right)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} (t^*)^2 K^N\left(\partial_{t^*}\Phi\right)$$

$$\leq \left(C + C\sum_{i=0}^{n-1}\tau_i + CA(2\delta'+\epsilon)\sum_{i=0}^{n-1}\tau_i^{1+\eta}\right)\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu$$

$$\leq \left(C+C\tau+CA(2\delta'+\epsilon)\tau^{1+\eta}\right)\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

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Now take A large, $\delta' = \epsilon$ and ϵ sufficiently small, we also improve (19). $\qquad\square$

In particular, the Theorem of Dafermos-Rodnianski [8] is retrieved.

Corollary 39 (Dafermos-Rodnianski). *Suppose $\Box_{g_K}\Phi = 0$. Then for all $\eta > 0$ and all $M > 0$ there exists a_0 such that the following estimates hold on Kerr spacetimes with (M, a) for which $a \leq a_0$:*

1. *Boundedness of Non-degenerate Energy*

$$\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi)$$
$$\leq C\int_{\Sigma_{\tau_0}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau_0}}^\mu.$$

2. *Decay of Non-degenerate Energy*

$$\tau^2 \int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \leq C\tau^{1+\eta} \sum_{m=0}^{1} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu.$$

and

$$\tau^2 \int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \leq C\tau^{\eta} \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu.$$

3. *Decay of Local Integrated Energy*
   For $\tau' \leq \tau \leq (1.1)\tau'$,

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}(\Phi) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) \leq C\tau^{-2+\eta} \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu.$$

and

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}(\Phi) \leq C\tau^{-2+\eta} \sum_{m=0}^{3} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu.$$

Proof. 1 follows directly from Proposition 22. 2 contains two statements. The second one is a restatement of Proposition 38. The first one is evident from the proof of Proposition 38. 3 again has two statements. For the first statement, we revisit the proof of Proposition 38. Notice that the bootstrap assumptions are true. Hence it holds. For the second statement, we note by comparing Propositions 30 and 31 that K^{X_1} can be estimated in the same way as K^{X_0} except for an extra derivative. The second statement in 3 can then be proved by re-running the argument in Proposition 38 with an extra derivative. $\qquad\square$

11 Estimates for $\hat{Y}\Phi$ and Elliptic Estimates

Away from the event horizon, we can control all higher order derivatives simply by commuting with ∂_{t^*} and using standard elliptic estimates. We write down a general version of the estimates in which we have inhomogeneous terms.

Proposition 40. *Suppose $\Box_{g_K}\Phi = G$. For $m \geq 1$ and for any α,*

1. *Boundedness of Weighted Energy*

$$\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}} r^\alpha (D^m\Phi)^2 \leq C_\alpha \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau} r^\alpha J_\mu^N(\partial_{t^*}^j\Phi)\, n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha (D^j G)^2 \right).$$

2. *Boundedness of Local Energy*
   For any $0 < \gamma < \gamma'$,

$$\int_{\Sigma_\tau \cap \{r_Y^- \leq r \leq \gamma t^*\}} r^\alpha \left(D^m \Phi\right)^2$$

$$\leq C_{\alpha,\gamma,\gamma'} \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + \tau^{\alpha-\beta-2} \int_{\Sigma_\tau} r^\beta J_\mu^N \left(\Phi\right) n_{\Sigma_\tau}^\mu \right.$$

$$\left. + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha \left(D^j G\right)^2 \right).$$

Proof. This is obvious for $m = 1$ (even without the restriction $r \geq r_Y^-$). We will proceed by induction. Take $\delta \ll \frac{r_Y^- - r_+}{4}$. Assume

$$\sum_{j=1}^{m-1} \int_{\Sigma_\tau \cap \{r \geq r_Y^- - 2\delta\}} r^\alpha \left(D^j \Phi\right)^2 \leq C \left(\sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-3} \int_{\Sigma_\tau} r^\alpha \left(D^j G\right)^2 \right).$$

We want to show

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^- - \delta\}} r^\alpha \left(D^m \Phi\right)^2 \leq C \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau} r^\alpha J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha \left(D^j G\right)^2 \right),$$

which would then imply the conclusion. Denote by Δ_{g_K} the Laplace-Beltrami operator for the metric g_K restricted on the spacelike hypersurface $t^* = $ constant. Notice that since ∂_{t^*} is Killing, the operator is defined independent of t^*. Then we have

$$[\Delta_{g_K}, D^k] \Phi \leq C \sum_{j=1}^{k+1} D^j \Phi.$$

Denote by ∇ the spatial derivatives with respect to the spatial coordinate variables in the Schwarzschild $\left(t_S^*, r_S, x_S^1, x_S^2\right)$ coordinate system. On the set $\{r \geq r_Y^- - \frac{r_Y^- - r_+}{4}\}$, Δ_{g_K} is elliptic and therefore controls all spatial derivatives:

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^- - \delta\}} r^\alpha \left(D^m \Phi\right)^2$$

$$\leq C \int_{\Sigma_\tau \cap \{r \geq r_Y^- - 2\delta\}} r^\alpha \left(\left(\Delta_{g_K} D^{m-2} \Phi\right)^2 + \left(D^{m-1} \Phi\right)^2 + \left(\partial_{t^*}^{m-1} \nabla \Phi\right)^2 + \left(\partial_{t^*}^m \Phi\right)^2 \right)$$

$$\leq C \int_{\Sigma_\tau \cap \{r \geq r_Y^- - 2\delta\}} r^\alpha \left(\left(D^{m-2} \Delta_{g_K} \Phi\right)^2 + \sum_{j=1}^{m-1} \left(D^j \Phi\right)^2 + r^{-2} \Phi^2 + \left(\partial_{t^*}^{m-1} \nabla \Phi\right)^2 + \left(\partial_{t^*}^m \Phi\right)^2 \right)$$

The last two terms are obviously bounded by $C \int_{\Sigma_\tau} J_\mu^N \left(\partial_{t^*}^{m-1} \Phi\right) n_{\Sigma_\tau}^\mu$. The second term can be bounded using the induction hypothesis. The third term can be bounded using the Hardy inequality in Proposition 19. Finally, to estimate the first term we use the equation $\Box_{g_K} \Phi = G$. Then, by the form of the Kerr metric, $\Delta_{g_K} \Phi = G - g^{t^* t^*} \partial_{t^*}^2 \Phi - 2 g^{t^* \phi^*} \partial_{t^*} \partial_{\phi^*} \Phi$. Therefore,

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^- - 2\delta\}} r^\alpha \left(D^{m-2} \Delta_{g_K} \Phi\right)^2$$

$$\leq C \int_{\Sigma_\tau \cap \{r \geq r_Y^- - 2\delta\}} r^\alpha \left(\left(D^{m-1} \partial_{t^*} \Phi\right)^2 + \left(D^{m-2} G\right)^2 \right)$$

$$\leq C \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau} r^\alpha J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha \left(D^j G\right)^2 \right),$$

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where at the last step we have used the induction hypothesis for $\partial_{t^*}\Phi$. We have thus proved the boundedness of weighted energy. To prove the second part of the Proposition, consider the function $\chi(\frac{r}{\tau})\Phi(\tau)$ for a fixed time $t^* = \tau$, where $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is supported in $\{x \leq \gamma'\}$ and is identically 1 in $\{x \leq \gamma\}$. Now

$$\Box_{g_K}\Phi = \chi G + \tau^{-1}\tilde{\chi}\partial_r\Phi + \tau^{-2}\tilde{\tilde{\chi}}\Phi,$$

where $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ are supported in $\{\gamma \leq \frac{t^*}{r} \leq \gamma'\}$. Thus, by the estimate just proved,

$$\int_{\Sigma_\tau \cap \{r_Y^- \leq r \leq \gamma t^*\}} r^\alpha \left(D^m\Phi\right)^2$$

$$\leq C_\alpha \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha J_\mu^N \left(\partial_{t^*}^j\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau \cap \{\gamma t \leq r \leq \gamma' t^*\}} r^\alpha \tau^{-4}\Phi^2 + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha \left(D^j G\right)^2 \right)$$

$$\leq C_\alpha \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha J_\mu^N \left(\partial_{t^*}^j\Phi\right) n_{\Sigma_\tau}^\mu + \tau^{\alpha-\beta-2} \int_{\Sigma_\tau} r^\beta J_\mu^N \left(\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha \left(D^j G\right)^2 \right),$$

by Hardy inequality in Proposition 19. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 8. *The boundedness of local energy should be seen as a decay result because for example for the homogeneous equation, the right hand side of the inequality decays.*

Near the event horizon, higher order derivatives can be controlled by commuting with the red-shift vector field as in [7], [8]. The computation here will be completely local, i.e., only in the region $\{r \leq r_Y^-\}$.

We have the following estimate for higher order derivatives:

Proposition 41. *Suppose $\Box_{g_K}\Phi = G$. For every $m \geq 1$,*

$$\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^m\Phi\right)^2 \leq C \left(\sum_{j+k \leq m-1} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^j G\right)^2 \right).$$

Proof. This is obvious for $m = 1$. We will proceed by induction. Suppose, for some $m \geq 2$ that

$$\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^j\Phi\right)^2 \leq C \left(\sum_{j+k \leq m-2} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-3} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^j G\right)^2 \right). \quad (20)$$

Since $\Box_{g_K}\left(\partial_{t^*}\Phi\right) = \partial_{t^*}G$, this immediately implies

$$\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(\partial_{t^*}D^{m-1}\Phi\right)^2 \leq C \left(\sum_{j+k \leq m-1} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^j G\right)^2 \right).$$
$$(21)$$

Since $\Box_{g_K}\Phi = G$, we have $\Box_{g_K}\left(\hat{Y}\Phi\right) = \hat{Y}G + O(1)\left(D^2\Phi + D\Phi\right)$. Then using the induction hypothesis (20) (both on $\hat{Y}\Phi$ and Φ), we have

$$\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^j\hat{Y}\Phi\right)^2$$

$$\leq C \left(\sum_{j+k \leq m-2} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^j\hat{Y}^{k+1}\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-3} \int_{\Sigma_\tau} \left(D^j\hat{Y}G\right)^2 + \sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^j\Phi\right)^2 \right) \quad (22)$$

$$\leq C \left(\sum_{j+k \leq m-1} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \left(D^j G\right)^2 \right).$$

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Using the null frame $\{\hat{V}, \hat{Y}, E_1, E_2\}$,

$$\Box_{g_K}\left(D^{m-2}\Phi\right) = -4\nabla_{\hat{Y}}\nabla_{\hat{V}}D^{m-2}\Phi + \slashed{\Delta}D^{m-2}\Phi + P_1 D^{m-2}\Phi,$$

where P_1 denotes a first order differential operator. Notice that we also have

$$|\Box_{g_K}\left(D^{m-2}\Phi\right)| = |[\Box_{g_K}, D^{m-2}]\Phi + D^{m-2}G| \leq C\left(\sum_{j=0}^{m-1}|D^j\Phi| + |D^{m-2}G|\right).$$

Now using a standard L^2 elliptic estimate on the sphere,

$$\int_{\mathbb{S}^2}|\slashed{\nabla}^2 D^{m-2}\Phi|^2 dA \leq C\int_{\mathbb{S}^2}\left(\left(D^{m-2}G\right)^2 + \sum_{j=0}^{m-1}\left(D^j\Phi\right)^2 + \left(D^{m-1}\nabla_{\hat{Y}}\Phi\right)^2\right)dA,$$

where we notice that the constant can be chosen uniformly because the metric on the sphere is everywhere close to that of the standard metric. Therefore, after integrate over $\{r_+ \leq r \leq r_Y^-\}$ and applying (20) and (22), we have

$$\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}}|\slashed{\nabla}^2 D^{m-2}\Phi|^2$$

$$\leq C\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}}\left(\left(D^{m-2}G\right)^2 + \sum_{j=0}^{m-1}\left(D^j\Phi\right)^2 + \left(D^{m-1}\nabla_{\hat{Y}}\Phi\right)^2\right) \tag{23}$$

$$\leq C\left(\sum_{j+k\leq m-1}\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2}\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}}\left(D^j G\right)^2\right).$$

Combining (21), (22) and (23), we have

$$\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}}\left(D^m\Phi\right)^2 \leq C\left(\sum_{j+k\leq m}\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2}\int_{\Sigma_\tau \cap \{r \leq r_Y^-\}}\left(D^j G\right)^2\right).$$

$$\square$$

We show that the currents associated to $\hat{Y}^k\Phi$ can actually be controlled. Again, in view of the nonlinear problem, we work in the setting of an inhomogeneous equation.

Proposition 42. *Suppose* $\Box_{g_K}\Phi = G$*. For every* $k \geq 0$*,*

$$\int_{\Sigma_\tau \cap \{r \leq r_Y^+\}}J_\mu^N\left(\hat{Y}^k\Phi\right)n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)}J_\mu^N\left(\hat{Y}^k\Phi\right)n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}}K^N\left(\hat{Y}^k\Phi\right)$$

$$\leq C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^+\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau'}}^\mu + \sum_{j=0}^k\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}}J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu\right.$$

$$\left.+\sum_{j=0}^k\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{23M}{8}\}}\left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_{t^*}}^\mu\right) + \sum_{j=0}^k\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{23M}{8}\}}\left(D^j G\right)^2\right).$$

Proof. We prove the Proposition by induction on k. The $k = 0$ case is trivial because the right hand side simply contains more terms than the left hand side. We suppose the Proposition is true for $k \leq k_0 - 1$ for some $k_0 \geq 1$. Commuting $\Box_{g_K}$ with $\hat{Y}$ for k_0 times, we get

$$\Box_{g_K}\hat{Y}^{k_0}\Phi = \kappa^{k_0}\hat{Y}^{k_0+1}\Phi + O(1)\hat{Y}^{k_0}\partial_{t^*}\Phi + O(\epsilon)D^{k_0+1}\Phi + O(1)\sum_{j=1}^{k_0}D^j\Phi + \hat{Y}^{k_0}G.$$

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We now use the energy identify for the vector field N, i.e., Proposition 7 for $\hat{Y}^k\Phi$. Notice that $\hat{Y}$ is supported in $\{r \le r_Y^+\}$ and therefore each term is supported in the same set.

$$\int_{\Sigma_\tau \cap \{r \le r_Y^+\}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \le r_Y^-\}} K^N\left(\hat{Y}^{k_0}\Phi\right)$$

$$= \int_{\Sigma_{\tau_0}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + e \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r_Y^- \le r \le r_Y^+\}} K^Y\left(\hat{Y}^{k_0}\Phi\right)$$

$$+ \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \le r_Y^+\}} \left(\partial_{t^*}\hat{Y}^{k_0}\Phi + e\hat{Y}^{k_0+1}\Phi\right)\left(-\kappa^{k_0}\hat{Y}^{k_0+1}\Phi + O(1)\hat{Y}^{k_0}\partial_{t^*}\Phi\right.$$

$$\left. + O(\epsilon)D^{k_0+1}\Phi + O(1)\sum_{j=1}^{k_0} D^j\Phi + \hat{Y}^k G\right).$$

The crucial observation in [7] is that one of the inhomogeneous terms has a good sign and thus gives

$$\int_{\Sigma_\tau} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} K^N\left(\hat{Y}^{k_0}\Phi\right)$$

$$+ \iint_{\mathcal{R}(\tau',\tau)} \left(\hat{Y}^{k_0+1}\Phi\right)^2$$

$$\le C\left(\int_{\Sigma_{\tau'} \cap \{r \le r_Y^+\}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau',\tau) \cap \{r_Y^- \le r \le r_Y^+\}} K^N\left(\hat{Y}^{k_0}\Phi\right) + \epsilon \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^+\}} \left(D^{k_0+1}\Phi\right)^2\right.$$

$$\left. + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^+\}} J_\mu^N\left(\partial_{t^*}\hat{Y}^{k_0-1}\Phi\right) n_{\Sigma_{t^*}}^\mu + \sum_{j=1}^{k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^+\}} \left(D^j\Phi\right)^2 + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^+\}} \left(\hat{Y}^{k_0}G\right)^2\right)$$

$$\le C\left(\int_{\Sigma_{\tau'} \cap \{r \le r_Y^+\}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{j=1}^{k_0+1} \iint_{\mathcal{R}(\tau',\tau) \cap \{r_Y^- \le r \le r_Y^+\}} \left(D^j\Phi\right)^2 + \epsilon \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} \left(D^{k_0+1}\Phi\right)^2\right.$$

$$\left. + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} J_\mu^N\left(\partial_{t^*}\hat{Y}^{k_0-1}\Phi\right) n_{\Sigma_{t^*}}^\mu + \sum_{j=1}^{k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} \left(D^j\Phi\right)^2 + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^+\}} \left(\hat{Y}^{k_0}G\right)^2\right).$$

Using Proposition 40 with a appropriate cutoff,

$$\sum_{j=1}^{k_0+1} \iint_{\mathcal{R}(\tau',\tau) \cap \{r_Y^- \le r \le r_Y^+\}} \left(D^j\Phi\right)^2$$

$$\le C\left(\sum_{j=0}^{k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} \left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right) n_{\Sigma_{t^*}}^\mu\right) + \sum_{j=0}^{k_0-1} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} \left(D^j G\right)^2\right).$$

Using Proposition 41,

$$\sum_{j=1}^{k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} \left(D^j\Phi\right)^2$$

$$\le C\left(\sum_{j+m \le k_0-1} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right) n_{\Sigma_{t^*}}^\mu + \sum_{j=0}^{k_0-2} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} \left(D^j G\right)^2\right)$$

$$\le C\left(\sum_{j+m \le k_0-1} \int_{\Sigma_{\tau'} \cap \{r \le r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right) n_{\Sigma_{\tau'}}^\mu + \sum_{j=0}^{k-1} \int_{\Sigma_\tau \cap \{r \le r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right) n_{\Sigma_\tau}^\mu\right.$$

$$\left. + \sum_{j+m \le k_0-1} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right) n_{\Sigma_{t^*}}^\mu + \sum_{j=0}^{k_0-1} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} \left(D^j G\right)^2\right),$$

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using the induction hypothesis (on $\partial_{t^*}^m \Phi$ instead of Φ) at the last step. Similarly, using Proposition 41,

$$\epsilon \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^+\}} \left(D^{k_0+1}\Phi \right)^2$$

$$\le C\epsilon \left(\sum_{j+m \le k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) n_{\Sigma_{t^*}}^\mu + \sum_{j=0}^{k-1} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} \left(D^j G \right)^2 \right)$$

$$\le C\epsilon \left(\iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^+\}} J_\mu^N \left(\hat{Y}^{k_0} \Phi \right) n_{\Sigma_{t^*}}^\mu + \sum_{j=0}^{k_0} \int_{\Sigma_{\tau'} \cap \{r \le r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \Phi \right) n_{\Sigma_{\tau'}}^\mu \right.$$

$$\left. + \sum_{j=0}^{k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} J_\mu^N \left(\partial_{t^*}^j \Phi \right) n_{\Sigma_{t^*}}^\mu + \sum_{j=0}^{k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} \left(D^j G \right)^2 \right),$$

where again the induction hypotheses is used at the last step. All these together give

$$\int_{\Sigma_\tau} J_\mu^N \left(\hat{Y}^{k_0} \Phi \right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N \left(\hat{Y}^{k_0} \Phi \right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \le r_Y^-\}} K^N \left(\hat{Y}^{k_0} \Phi \right)$$

$$\le C \left(\sum_{j+m \le k_0} \int_{\Sigma_{\tau'} \cap \{r \le r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) n_{\Sigma_{\tau'}}^\mu + \sum_{j=0}^{k_0} \int_{\Sigma_{\tau'} \cap \{r \le r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \Phi \right) n_{\Sigma_{\tau'}}^\mu \right.$$

$$+ \sum_{j=0}^{k_0} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} \left(\Phi^2 + J_\mu^N \left(\partial_{t^*}^j \Phi \right) n_{\Sigma_{t^*}}^\mu \right) + \sum_{j=0}^{k_0} \int_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{23M}{8}\}} \left(D^j G \right)^2$$

$$\left. + \epsilon \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} J_\mu^N \left(\hat{Y}^{k_0} \Phi \right) n_{\Sigma_{t^*}}^\mu \right).$$

The Proposition can be proved by noticing that

$$\iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} J_\mu^N \left(\hat{Y}^{k_0} \Phi \right) n_{\Sigma_{t^*}}^\mu \le C \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \le r_Y^-\}} K^N \left(\hat{Y}^{k_0} \Phi \right).$$

and absorbing the small term to the left hand side. □

We now specialize to the case $\Box_{g_K} \Phi = 0$. The above Proposition implies that the behavior of $\hat{Y}^k \Phi$ is determined by the behavior of $\partial_{t^*}^m \Phi$ in the region $\{r \le \frac{23M}{8}\}$.

Proposition 43. *Fix $k \ge 0$. Suppose $\Box_{g_K} \Phi = 0$ and suppose for some constants $\alpha, B > 0$ (independents of τ),*

$$\sum_{j=0}^k \int_{\Sigma_\tau \cap \{r \le \frac{23M}{8}\}} \left(\Phi^2 + J_\mu^N \left(\partial_{t^*}^j \Phi \right) n_{\Sigma_\tau}^\mu \right) \le CB\tau^{-\alpha}.$$

Then

$$\sum_{j+m \le k} \int_{\Sigma_\tau} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) n_{\Sigma_\tau}^\mu \le C\tau^{-\alpha} \left(\sum_{j+m \le k} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) n_{\Sigma_{\tau_0}}^\mu + B \right),$$

and

$$\sum_{j+m \le k} \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} K^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) \le C(\tau')^{-\alpha} \left(\sum_{j+m \le k} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) n_{\Sigma_{\tau_0}}^\mu + B \right).$$

Remark 9. *In the applications, we will apply this Proposition with B being some energy quantity of the initial condition.*

Proof. We will proof this with a bootstrap argument. Suppose for all τ that

$$\sum_{j+m\leq k}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu \leq A\tau^{-\alpha}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + B\right). \tag{24}$$

This obviously holds initially for any $A\geq 1$ (and in particular independent of Φ). By taking $\tau'=\tau-K$, for some (large and to be chosen) constant K and $\tau\geq 2K$, Proposition 42 implies

$$\sum_{j+m\leq k}\left(\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau-K,\tau)\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{t^*}}^\mu\right)$$

$$\leq C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau-K}\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau-K}}^\mu + \sum_{j=0}^k\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu\right.$$

$$\left.+\sum_{j=0}^k\iint_{\mathcal{R}(\tau-K,\tau)\cap\{r\leq\frac{23M}{8}\}}\left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_{t^*}}^\mu\right)\right)$$

$$\leq C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau-K}\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau-K}}^\mu + KB\tau^{-\alpha}\right)$$

using the assumption of the Proposition and using Proposition 40

$$\leq C\left(A(\tau-K)^{-\alpha}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + B\right) + KB\tau^{-\alpha}\right)$$

using the bootstrap assumption

$$\leq C\tau^{-\alpha}\left(\sum_{j+m\leq k} A\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + AB + KB\right).$$

Notice that C is independent of K. By selecting a t^* slice, we have that for some $\tilde{\tau}$,

$$\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\hat{Y}^k\Phi\right)n_{\Sigma_{\tilde{\tau}}}^\mu$$

$$\leq CK^{-1}\tau^{-\alpha}\left(\sum_{j+m\leq k} A\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + AB + KB\right).$$

Now apply Proposition 42 on $[\tilde{\tau},\tau]$ to get

$$\sum_{j+m\leq k}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu$$

$$\leq C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tilde{\tau}}}^\mu + \sum_{j=0}^k\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu\right.$$

$$\left.+\sum_{j=0}^k\iint_{\mathcal{R}(\tilde{\tau},\tau)\cap\{r\leq\frac{23M}{8}\}}\left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_{t^*}}^\mu\right)\right)$$

$$\leq CK^{-1}\tau^{-\alpha}\left(\sum_{j+m\leq k} A\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + AB + BK\right) + CB(K+1)\tau^{-\alpha}$$

$$\leq CAK^{-1}\tau^{-\alpha}\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + \left(CAK^{-1} + CK + C\right)B\tau^{-\alpha}.$$

This would improve (24) if we choose $K = 4C$ and A sufficiently large. Hence we have proved

$$\sum_{j+m\leq k} \int_{\Sigma_\tau \cap \{r\leq r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi\right) n_{\Sigma_\tau}^\mu \leq C\tau^{-\alpha} \left(\sum_{j+m\leq k} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi\right) n_{\Sigma_{\tau_0}}^\mu + B\right).$$

To prove the second statement of the Proposition, we simply use the first statement and Proposition 42. □

We can use Corollary 39 to show the decay of $\hat{Y}^k \Phi$.

Corollary 44. *Suppose* $\Box_{g_K} \Phi = 0$. *Then for* $\tau' \leq \tau \leq (1.1)\tau'$,

$$\sum_{j+m\leq k} \int_{\Sigma_\tau \cap \{r\leq r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi\right) n_{\Sigma_\tau}^\mu$$

$$\leq C\tau^{-2+\eta} \left(\sum_{j+m\leq k} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{j=0}^{k+2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN} \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu\right),$$

and

$$\sum_{j+m\leq k} \iint_{\mathcal{R}(\tau',\tau) \cap \{r\leq r_Y^-\}} K^N \left(\partial_{t^*}^j \hat{Y}^m \Phi\right)$$

$$\leq C\tau^{-2+\eta} \left(\sum_{j+m\leq k} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{j=0}^{k+2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN} \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).$$

12 Estimates for $\tilde{\Omega}\Phi$

In this section, we would like to prove estimates for $\tilde{\Omega}^\ell \Phi$. The estimates for $\tilde{\Omega}\Phi$ are useful to provide an extra factor of r in the energy estimates.

Proposition 45.

$$\int_{\Sigma_\tau \cap \{r\geq r_Y^-\}} r^2 |\slashed{\nabla}^2 \Phi|^2 \leq C \int_{\Sigma_\tau} J_\mu^N \left(\Phi, \partial_{t^*}\Phi, \tilde{\Omega}\Phi\right) n_{\Sigma_\tau}^\mu.$$

$$\iint_{\mathcal{R}(\tau',\tau) \cap \{r\geq r_Y^-\}} r^{1-\delta} |\slashed{\nabla}^2 \Phi|^2 \leq C \iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta} J_\mu^N \left(\Phi, \partial_{t^*}\Phi, \tilde{\Omega}\Phi\right) n_{\Sigma_{t^*}}^\mu.$$

$$\iint_{\mathcal{R}(\tau',\tau) \cap \{r\geq r_Y^-\}} r^{1-\delta} |\slashed{\nabla}^2 \Phi|^2 \leq C \iint_{\mathcal{R}(\tau',\tau)} K^{X_1} \left(\Phi, \partial_{t^*}\Phi\right) + K^{X_0} \left(\tilde{\Omega}\Phi\right).$$

In order to prove such estimates, we commute $\Box_{g_K}$ with $\tilde{\Omega}$. Recall from Section 4.3 that

$$|[\Box_{g_K}, \tilde{\Omega}]\Phi| \leq Cr^{-2} \left(|D^2\Phi| + |D\Phi|\right) \quad \text{everywhere, and}$$

$$[\Box_{g_K}, \tilde{\Omega}]\Phi = 0, \quad \text{for } r < R_\Omega.$$

Now suppose $\Box_{g_K} \Phi = 0$. We have

$$\Box_{g_K} \left(\tilde{\Omega}^\ell \Phi\right) = \sum_{j=0}^{\ell-1} \tilde{\Omega}^j [\Box_{g_K}, \tilde{\Omega}] \tilde{\Omega}^{\ell-j-1} \Phi =: G_{\Omega,\ell}.$$

Since $[D, \tilde{\Omega}] = D$, we have

$$|G_{\Omega,\ell}| \leq Cr^{-2} \left(\sum_{j=0}^{\ell-1} \left(|D^2\tilde{\Omega}^j \Phi| + |D\tilde{\Omega}^j \Phi|\right) + \sum_{j=0}^{\ell+1} |D^j \Phi|\right),$$

and $G_{\Omega,\ell}$ is supported in $\{r \geq R_\Omega\}$.

Definition 5.

$$E_{\Omega,\ell} = \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G_{\Omega,\ell}^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \le \frac{M}{8}\}} G_{\Omega,\ell}^2.$$

This is the error term for the energy estimates for $\tilde{\Omega}^\ell \Phi$. We show that this can be controlled.

Proposition 46.

$$E_{\Omega,\ell} \le C \sum_{m=0}^{1} \sum_{j=0}^{\ell-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) + C \sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m \Phi\right).$$

Proof.

$$\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G_{\Omega,\ell}^2$$

$$\le C \sum_{j=0}^{\ell-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} r^{-3+\delta}\left(\left(D^2 \tilde{\Omega}^j \Phi\right)^2 + \left(D\tilde{\Omega}^j \Phi\right)^2\right)$$

$$+ C \sum_{j=0}^{\ell+1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} r^{-3+\delta}\left(D^j \Phi\right)^2$$

$$\le C \sum_{m=0}^{1} \sum_{j=0}^{\ell-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} r^{-3+\delta} J_\mu^N\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) n_{\Sigma t^*}^\mu$$

$$+ C \sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} r^{-3+\delta} J_\mu^N\left(\partial_{t^*}^m \Phi\right) n_{\Sigma t^*}^\mu$$

$$\le C \sum_{m=0}^{1} \sum_{j=0}^{\ell-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) + C \sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\ge R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m \Phi\right).$$

By choosing R_Ω sufficiently large, the second term of $E_{\Omega,\ell}$ vanishes. $\qquad\square$

We can show that the non-degenerate energy of $\tilde{\Omega}^\ell \Phi$ is almost bounded.

Proposition 47. *Suppose $\Box_{g_K}\Phi = 0$. Then*

$$\int_{\Sigma_\tau} J_\mu^N\left(\tilde{\Omega}^\ell \Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N\left(\tilde{\Omega}^\ell \Phi\right) n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}} K^N\left(\tilde{\Omega}^\ell \Phi\right) + \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\tilde{\Omega}^\ell \Phi\right)$$

$$\le C \sum_{i+j\le\ell} \int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof. We prove this by induction on ℓ. The $\ell = 0$ case is true by setting $G = 0$ in Proposition 25. We assume that the Proposition is true for $\ell \le \ell_0 - 1$. This in particular implies, after a commutations with the Killing vector field ∂_{t^*}, that

$$\sum_{j=0}^{\ell_0-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) \le C \sum_{i+j\le m+\ell_0-1} \int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

By Propositions 24 and 46,

$$
\int_{\Sigma_\tau} J_\mu^N \left(\tilde{\Omega}^{\ell_0}\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N \left(\tilde{\Omega}^{\ell_0}\Phi\right) n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N \left(\tilde{\Omega}^{\ell_0}\Phi\right) + \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0} \left(\tilde{\Omega}^{\ell_0}\Phi\right)
$$

$$
\leq C \left(\int_{\Sigma_{\tau_0}} J_\mu^N \left(\tilde{\Omega}^{\ell_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G_{\Omega,\ell_0}^2 + \sup_{t^*\in[\tau'-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq \frac{M}{8}\}} G_{\Omega,\ell_0}^2 \right)
$$

$$
\leq C \left(\int_{\Sigma_{\tau_0}} J_\mu^N \left(\tilde{\Omega}^{\ell_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \sum_{m=0}^1 \sum_{j=0}^{\ell_0-1} \iint_{\mathcal{R}(\tau_0-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0} \left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) \right.
$$

$$
\left. +C \sum_{m=0}^{\ell_0} \iint_{\mathcal{R}(\tau_0-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0} \left(\partial_{t^*}^m \Phi\right) \right)
$$

$$
\leq C \sum_{i+j\leq \ell_0} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu .
$$

$\square$

Remark 10. *Only the $\ell = 1$ case will be used.*

13 Estimates for $S\Phi$

We will now use the energy estimates that we have derived to control $S\Phi$. In particular, we would like to prove a local integrated decay estimate for $S\Phi$. This will be used in the next section where we prove our main theorem. Recall from Section 4.2 that for r large

$$
|[\Box_{g_K}, S]\Phi - \left(2+\frac{r^*\mu}{r}\right)\Box_{g_K}\Phi - \frac{2}{r}\left(\frac{r^*}{r}-1-\frac{2r^*\mu}{r}\right)\partial_{r^*}\Phi - 2\left(\left(\frac{r^*}{r}-1\right)-\frac{3r^*\mu}{2r}\right)\slashed{\Delta}\Phi| \leq C\epsilon r^{-2}(\sum_{k=1}^2 |\partial^k\Phi|).
$$

From now on we will prove estimates for $S\Phi$ by considering the wave equation that it satisfies. We will assume, as before, $\Box_{g_K}\Phi = 0$ and let G denote the commutator term, i.e., $\Box_{g_K}(S\Phi) = G$. If we look at our estimates in the previous sections, we will need to control G in three different norms. We now consider them separately.

Proposition 48. *Let $\tau' \leq \tau \leq (1.1)\tau'$. Then*

$$
\sum_{m=0}^\ell \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m G\right)^2
$$

$$
\leq C\tau^{-1+\eta} \sum_{m+k+j\leq\ell+3} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z} \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).
$$

and

$$
\sum_{m=0}^\ell \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G\right)^2
$$

$$
\leq C\tau^{-2+\eta} \sum_{m+k+j\leq\ell+4} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z} \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).
$$

In other words, we can get more decay if we localize and allow an extra derivative.

Proof.

$$\sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m G\right)^2$$

$$\leq C \sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{-3+\delta} \left(\left(\partial_{t^*}^m D^2\Phi\right)^2 + \left(\partial_{t^*}^m D\Phi\right)^2 + \left(r\partial_{t^*}^m \slashed{\Delta}\Phi\right)^2 \right)$$

noting that the δ in the two lines are different

$$\leq C \sum_{m+k\leq\ell+1} \left(\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{t^*}{2}\}} r^{-1+\delta} J_\mu^N \left(\partial_{t^*}^m \hat{Y}^k \Phi\right) n_{\Sigma_t^*}^\mu \right.$$

$$\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq\frac{t^*}{2}\}} r^{-3+\delta} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^k \Phi\right) n_{\Sigma_t^*}^\mu \right)$$

by Proposition 40, 41 and 45

$$\leq C \sum_{m+k\leq\ell+3} \tau^{-1+\eta} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z} \left(\partial_{t^*}^m \Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m \hat{Y}^k \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right)$$

$$+ C \sum_{m+k\leq\ell+3} \int_{\tau'-1}^{\tau+1} (t^*)^{-3+\delta} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z} \left(\partial_{t^*}^m \Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m \hat{Y}^k \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right) dt^*$$

$$+ C \sum_{m\leq\ell+1} \int_{\tau'-1}^{\tau+1} (t^*)^{-3+\delta} \left(\int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}\Phi\right) n_{\Sigma_{\tau_0}}^\mu \right) dt^*$$

using Corollaries 39, 44 and Proposition 47

$$\leq C\tau^{-1+\eta} \sum_{m+k+j\leq\ell+3} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z} \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).$$

We then move on to the localized version:

$$\sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2$$

$$\leq C\sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{-3+\delta}\left(\left(\partial_{t^*}^m D^2\Phi\right)^2 + \left(\partial_{t^*}^m D\Phi\right)^2 + \left(r\partial_{t^*}^m \slashed{\Delta}\Phi\right)^2\right)$$

$$\leq C\sum_{m+k\leq\ell+1}\left(\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{t^*}{2}\}} r^{-1+\delta} J_\mu^N\left(\partial_{t^*}^m \hat{Y}\Phi\right) n_{\Sigma_t^*}^\mu\right.$$

$$+\left.\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{\frac{t^*}{2}\leq r\leq\frac{19t^*}{20}\}} r^{-3+\delta} J_\mu^N\left(\partial_{t^*}^m \hat{Y}\Phi\right) n_{\Sigma_t^*}^\mu\right)$$

$$+C\sum_{m=0}^{1}\left(\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{\frac{t^*}{2}\leq r\leq\frac{19t^*}{20}\}} r^{-3+\delta} J_\mu^N\left(\partial_{t^*}^m \tilde{\Omega}\Phi\right) n_{\Sigma_t^*}^\mu\right)$$

by Proposition 40, 41 and 45

$$\leq C\sum_{m+k\leq\ell+4}\tau^{-2+\eta}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m \hat{Y}\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right)$$

$$+C\sum_{m+k\leq\ell+3}\int_{\tau'-1}^{\tau+1}(t^*)^{-3+\delta}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m \hat{Y}\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right)dt^*$$

$$+C\sum_{m=0}^{\ell}\int_{\tau'-1}^{\tau+1}(t^*)^{-3+\delta}\left(\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m \tilde{\Omega}\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right)dt^*$$

using Corollaries 39, 44 and Proposition 47

$$\leq C\tau^{-2+\eta}\sum_{m+k+j\leq\ell+4}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m \hat{Y}^k\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m \hat{Y}^k\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).$$

$\square$

To estimate the inhomogeneous term in the region $r\leq\frac{t^*}{2}$, we would also need to estimate a term not integrated over t^* which arises from the integration by parts.

Proposition 49. *For* $\tau'\leq\tau\leq(1.1)\tau'$,

$$\sup_{t^*\in[\tau'-1,\tau+1]}\sum_{m=0}^{\ell}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}}\left(\partial_{t^*}^m G\right)^2$$

$$\leq C\tau^{-2+\eta}\sum_{m+j\leq\ell+3}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).$$

Proof.

$$\sup_{t^*\in[\tau'-1,\tau+1]} \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}\cap\{|r-3M|\le\frac{M}{8}\}} (\partial_{t^*}^m G)^2$$

$$\le C \sup_{t^*\in[\tau'-1,\tau+1]} \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le\frac{25M}{8}\}} \left((D^2\partial_{t^*}^m\Phi)^2 + (D\partial_{t^*}^m\Phi)^2 + (r\slashed{\Delta}\partial_{t^*}^m\Phi)^2\right)$$

$$\le C \sup_{t^*\in[\tau'-1,\tau+1]} \left(\sum_{m=0}^{\ell+1} \int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le\frac{25M}{8}\}} J_\mu^N (\partial_{t^*}^m\Phi)\, n_{\Sigma_t^*}^\mu + \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le\frac{25M}{8}\}} J_\mu^N \left(\tilde\Omega\partial_{t^*}^m\Phi\right) n_{\Sigma_t^*}^\mu\right)$$

by Proposition 40 and 45

$$\le C\tau^{-2+\eta} \sum_{m+j\le\ell+3} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z} \left(\partial_{t^*}^m\tilde\Omega^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m\tilde\Omega^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right),$$

using Corollary 39 and Proposition 47

$\square$

Finally, we estimate the third norm:

Proposition 50.

$$\sum_{m=0}^{\ell} \left(\int_{\tau_0}^{\tau} \left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}} r^2(\partial_{t^*}^m G)^2\right)^{\frac{1}{2}} dt^*\right)^2 \le C\tau^\eta \sum_{m+j\le\ell+3} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m\tilde\Omega^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof.

$$\sum_{m=0}^{\ell} \left(\int_{\tau_0}^{\tau} \left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}} r^2(\partial_{t^*}^m G)^2\right)^{\frac{1}{2}} dt^*\right)^2$$

$$\le C \sum_{m=0}^{\ell} \left(\int_{\tau_0}^{\tau} (t^*)^{-1+\delta} \left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}} \left((D^2\Phi)^2 + (D\partial_{t^*}^m\Phi)^2 + (r\slashed{\Delta}\partial_{t^*}^m\Phi)^2\right)\right)^{\frac{1}{2}} dt^*\right)^2$$

$$\le C \left(\int_{\tau_0}^{\tau} (t^*)^{-1+\delta} \left(\sum_{m=0}^{\ell+1} \int_{\Sigma_{t^*}} J_\mu^N (\partial_{t^*}^m\Phi)\, n_{\Sigma_{t^*}}^\mu + \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}} J_\mu^N \left(\tilde\Omega\partial_{t^*}^m\Phi\right) n_{\Sigma_{t^*}}^\mu\right)^{\frac{1}{2}} dt^*\right)^2$$

$$\le C\tau^\eta \sum_{m+j\le\ell+3} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^m\tilde\Omega^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

$\square$

Now that we have control of the inhomogeneous terms in the equation $\Box_{g_K}\Phi = G$, we can prove the decay of $S\Phi$. To this end, we will introduce the bootstrap assumptions:

$$c\int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(\partial_{t^*} S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau\cap\{r\le\gamma\tau\}} J_\mu^N \left(\partial_{t^*} S\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\le A\tau \sum_{m=1}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + A\tau^{1+\eta} \sum_{m+k+j\le5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m\hat Y^k\tilde\Omega^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu. \tag{25}$$

$$c\int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau\cap\{r\le\gamma\tau\}} J_\mu^N \left(S\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\le A^2\tau^\eta \left(\sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\le5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m\hat Y^k\tilde\Omega^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right). \tag{26}$$

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We think of A as some large constant to be chosen. We will improve the constants A and A^2 in the above assumptions. Under these two assumptions, we will get the following three estimates for the bulk terms:

Proposition 51.

$$\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(\partial_{t^*}S\Phi\right)$$

$$\leq C\left(\sum_{m=1}^{2}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).$$

Proof. By Proposition 26 for the equation $\Box_{g_K}\left(\partial_{t^*}S\Phi\right) = \partial_{t^*}G$, taking $\tau' = \tau_0$ and $G_1 = 0$, $G_2 = \partial_{t^*}G$. Then use Propositions 48 and 49 to estimate the terms with G. $\qquad\square$

Proposition 52. *For $\tau' \leq \tau \leq (1.1)\tau'$,*

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*}S\Phi\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(\partial_{t^*}S\Phi\right) + \iint_{\mathcal{R}(\tau_0,\tau)} r^{-1+\delta}K^{X_1}\left(S\Phi\right)$$

$$\leq CA\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu\right)$$

$$+ CA\tau^{-1+\eta}\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof. By Propositions 30 and 31, taking $G_1 = 0$ and $G_2 = G$, and using Propositions 48 and 49 to estimate the terms with G, we have

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*}S\Phi\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(\partial_{t^*}S\Phi\right) + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(S\Phi\right)$$

$$\leq CA\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu\right)$$

$$+ CA\tau^{-1+\eta}\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

It now remains to estimate $r^{-1+\delta}K^{X_1}$ in the region $r \geq \frac{t^*}{2}$. Here, we will use crucially the decay in r. Clearly,

$$\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\geq\frac{t^*}{2}\}} r^{-1+\delta}K^{X_1}\left(S\Phi\right) \leq C\tau^{-1+\delta}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(S\Phi\right).$$

Then we can estimate the right hand side by Proposition 26, taking $\tau' = \tau_0$ and $G_1 = 0$, $G_2 = \partial_{t^*}G$. Then use Propositions 48 and 49 to estimate the terms with G. $\qquad\square$

Proposition 53. *For $\tau' \leq \tau \leq (1.1)\tau'$,*

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(S\Phi\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(S\Phi\right)$$

$$\leq CA^2\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu\right)$$

$$+ CA^2\tau^{-2+\eta}\sum_{m+k+j\leq 4}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof. This follows from using Proposition 30, taking $G_1 = 0$ and $G_2 = G$, and using Propositions 48 and 49 to estimate the terms with G. $\qquad\square$

We are now ready to retrieve the bootstrap assumptions:

Proposition 54.

$$c \int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(\partial_{t^*} S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau \cap \{r \leq \gamma\tau\}} J_\mu^N \left(\partial_{t^*} S\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\leq \frac{A}{2}\tau \sum_{m=1}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \frac{A}{2}\tau^{1+\eta} \sum_{m+k+j \leq 5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof. By Proposition 37,

$$c \int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(\partial_{t^*} S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau \cap \{r \leq \gamma\tau\}} J_\mu^N \left(\partial_{t^*} S\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\leq C \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*} S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1} \left(\partial_{t^*} S\Phi\right)$$

$$+ C\delta' \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq \frac{t^*}{2}\}} (t^*)^2 K^{X_0} \left(\partial_{t^*} S\Phi\right) + C\left(\delta' + \epsilon\right) \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq r_Y^-\}} (t^*)^2 K^N \left(\partial_{t^*} S\Phi\right)$$

$$+ C(\delta')^{-1} \left(\int_{\tau_0}^{\tau} \left(\oint_{\Sigma_{t^*} \cap \{r \geq \frac{t^*}{2}\}} r^2 (\partial_{t^*} G)^2\right)^{\frac{1}{2}} dt^*\right)^2 + C(\delta')^{-1} \sum_{m=1}^{2} \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq \frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} \left(\partial_{t^*}^m G\right)^2$$

$$+ C(\delta')^{-1} \sup_{t^* \in [\tau_0,\tau]} \int_{\Sigma_{t^*} \cap \{r_Y^- \leq r \leq \frac{25M}{8}\}} (t^*)^2 (\partial_{t^*} G)^2.$$

It suffices to check that by Propositions 48, 49 and 50, all terms are acceptable. $\qquad\square$

We can now retrieve the bootstrap assumption (26).

Proposition 55.

$$c \int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau \cap \{r \leq \gamma\tau\}} J_\mu^N \left(S\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\leq A^2 \tau^\eta \left(\sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j \leq 5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).$$

Proof. By Proposition 37,

$$c \int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau \cap \{r \leq \gamma\tau\}} J_\mu^N \left(S\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\leq C \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C \iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1} \left(S\Phi\right)$$

$$+ C\delta' \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq \frac{t^*}{2}\}} (t^*)^2 K^{X_0} \left(S\Phi\right) + C\left(\delta' + \epsilon\right) \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq r_Y^-\}} (t^*)^2 K^N \left(S\Phi\right)$$

$$+ C(\delta')^{-1} \left(\int_{\tau_0}^{\tau} \left(\oint_{\Sigma_{t^*} \cap \{r \geq \frac{t^*}{2}\}} r^2 G^2\right)^{\frac{1}{2}} dt^*\right)^2 + C(\delta')^{-1} \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau) \cap \{r \leq \frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} \left(\partial_{t^*}^m G\right)^2$$

$$+ C(\delta')^{-1} \sup_{t^* \in [\tau_0,\tau]} \int_{\Sigma_{t^*} \cap \{r_Y^- \leq r \leq \frac{25M}{8}\}} (t^*)^2 G^2.$$

It suffices to check that by Propositions 48, 49 and 50, all terms are acceptable. $\qquad\square$

We have thus showed the following:

Proposition 56. *For all $\eta > 0$, there exists $\epsilon > 0$ small enough such that for Kerr spacetimes satisfying (3), the following estimates hold:*

$$c \int_{\Sigma_\tau} J_\mu^{Z,w^Z} \left(S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau \cap \{r \le \gamma\tau\}} J_\mu^N \left(S\Phi\right) n_{\Sigma_\tau}^\mu$$

$$\le C\tau^\eta \sum_{m=0}^2 \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\tau^\eta \sum_{m+k+j \le 5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Moreover, for $\tau' \le \tau \le (1.1)\tau'$,

$$\iint_{\mathcal{R}(\tau',\tau) \cap \{r \le r_Y^-\}} K^N \left(S\Phi\right) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{t^*}{2}\}} K^{X_0} \left(S\Phi\right)$$

$$\le C\tau^{-2+\eta} \sum_{m=0}^2 \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\tau^{-2+\eta} \sum_{m+k+j \le 5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

and

$$\iint_{\mathcal{R}(\tau',\tau) \cap \{r \le \frac{t^*}{2}\}} K^{X_1} \left(S\Phi\right)$$

$$\le C\tau^{-2+\eta} \sum_{m=0}^3 \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\tau^{-2+\eta} \sum_{m+k+j \le 6} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof. The first statement is proved by the bootstrap above. Since the bootstrap assumptions are true, the conclusion in Proposition 53 is also true, hence the second statement is true. The third statement makes use of the fact that K^{X_1} can be estimated in the same way as K^{X_0} with an extra derivative. $\qquad\square$

14 Improved Decay for the Linear Homogeneous Wave Equation

To use the estimates for $S\Phi$, we need to integrate along integral curves of S. We first find the integral curves by solving the ordinary differential equation

$$\frac{dr_S}{dt_S^*} = \frac{h(r_S)}{t_S^*}$$

where $h(r_S)$ is as in the definition of S. Hence the integral curves are given by

$$\frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr_S'}{h(r_S')}\right)}{t_S^*} = \text{constant},$$

where $r_0 > 2M$ can be chosen arbitrarily. Let $\sigma = t^*$, $\rho = \frac{\exp\left(\int_{r_0}^r \frac{dr'}{h(r')}\right)}{t^*}$ and consider (σ, ρ, x^A, x^B) as a new system of coordinates. Notice that

$$\partial_\sigma = \frac{h(r)}{t^*}\partial_r + \partial_{t^*} = \frac{1}{t^*}S.$$

Now for each fixed ρ, we have

$$\Phi^2(\tau) \le \Phi^2(\tau') + |\int_{\tau'}^\tau \frac{1}{\sigma} S(\Phi^2) d\sigma|.$$

Integrating along a finite region of ρ, we get:

$$\int_{\rho_1}^{\rho_2} \Phi^2(\tau) d\rho \le \int_{\rho_1}^{\rho_2} \Phi^2(\tau') d\rho + \int_{\rho_1}^{\rho_2} \int_{\tau'}^\tau |\frac{2}{\sigma} \Phi S\Phi| d\sigma d\rho.$$

We would like to change coordinates back to $(t_S^*, r_S, x_S^A, x_S^B)$. Notice that since $h(r_S)$ is everywhere positive, (ρ, τ) would correspond to a point with a larger value of r than (ρ, τ'). Therefore,

$$\int_{r_+}^{r_2} \Phi^2(\tau) \frac{\exp\left(\int_{r_0}^r \frac{dr'}{h(r')}\right)}{\tau h(r)} dr$$

$$\leq \int_{r_+}^{r_2} \Phi^2(\tau') \frac{\exp\left(\int_{r_0}^r \frac{dr'}{h(r')}\right)}{\tau' h(r)} dr + \int_{\tau'}^{\tau} \int_{r_+}^{r_2} |\frac{2}{\sigma} \Phi S\Phi| \frac{\exp\left(\int_{r_0}^r \frac{dr'}{h(r')}\right)}{t^* h(r)} dr dt^*.$$

We have to compare $\frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr_S'}{h(r_S')}\right)}{h(r_S)}$ with the volume form. Very close to the horizon, $h(r_S) = r_S - 2M$. Hence

$$\frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr_S'}{h(r_S')}\right)}{h(r_S)} = e^{\int_{(r_S)_0}^{r_Y^-} \frac{dr_S'}{h(r_S')}} \left(\frac{1}{r_Y^- - 2M}\right) \sim 1.$$

The corresponding expression on the compact set $[r_Y^-, R]$ is obviously bounded. Hence we have

$$\int_{\Sigma_\tau \cap \{r < r_2\}} \frac{\Phi^2(\tau)}{\tau} \leq C \left(\int_{\Sigma_{\tau'} \cap \{r < r_2\}} \frac{\Phi^2(\tau')}{\tau'} + \iint_{\mathcal{R}(\tau', \tau) \cap \{r < r_2\}} |\frac{2}{(t^*)^2} \Phi S\Phi|\right). \tag{27}$$

This easily implies the following improved decay for the non-degenerate energy:

Proposition 57.

$$\int_{\Sigma_\tau \cap \{r < R\}} \Phi^2 \leq C_R \tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} \Phi^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} (S\Phi)^2\right).$$

Proof. By choosing an appropriate $\tilde{\tau}$, we have

$$\int_{\Sigma_{\tilde{\tau}} \cap \{r < R\}} \Phi^2 \leq C \tau^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} \Phi^2.$$

Now, apply (27) with $\tau' = \tilde{\tau}$, we have

$$\int_{\Sigma_\tau \cap \{r < R\}} \Phi^2$$

$$\leq C \tau \left(\int_{\Sigma_{\tilde{\tau}} \cap \{r < R\}} \frac{\Phi^2}{\tilde{\tau}} + \iint_{\mathcal{R}(\tilde{\tau}, \tau) \cap \{r < R\}} |\frac{2}{(t^*)^2} \Phi S\Phi|\right)$$

$$\leq C \tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} \Phi^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} (S\Phi)^2\right),$$

using Cauchy-Schwarz for the second term. $\qquad\square$

We can now conclude with the improved decay for solutions to the homogeneous wave equation.

Proof of Main Theorem 1. By Proposition 39, 57 and 56, we have

$$\int_{\Sigma_\tau \cap \{r < R\}} \Phi^2$$

$$\leq C_R \tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} \Phi^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} (S\Phi)^2\right)$$

$$\leq C_R \tau^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r < R\}} \left(K^{X_0}(\Phi) + K^{X_0}(S\Phi)\right)$$

$$\leq C_R \tau^{-3+\eta} \left(\sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN, w^Z} (\partial_{t^*}^m S\Phi) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j \leq 5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN, w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu\right)$$

Similarly we can use Proposition 57 for the derivatives of Φ. By Proposition 39, 57 and 56, we have

$$\int_{\Sigma_\tau \cap \{r<R\}} (D\Phi)^2$$

$$\leq C_R \tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}} (D\Phi)^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}} (SD\Phi)^2 \right)$$

$$\leq C_R \tau^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}} \left(K^{X_1}(\Phi) + K^{X_1}(S\Phi) \right)$$

since we have the commutation $[D, S] = D$

$$\leq C_R \tau^{-3+\eta} \left(\sum_{m=0}^{3} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} (\partial_{t^*}^m S\Phi) \, n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq 6} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu \right)$$

By commuting with ∂_{t^*}, we get

$$\int_{\Sigma_\tau \cap \{r<R\}} \left(D\partial_{t^*}^\ell \Phi \right)^2$$

$$\leq C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+3} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} (\partial_{t^*}^m S\Phi) \, n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq \ell+6} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu \right).$$

Without loss of generality, we can take $R > \frac{23M}{8}$. Then, by Proposition 43,

$$\sum_{j+m\leq\ell} \int_{\Sigma_\tau \cap \{r\leq r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) n_{\Sigma_\tau}^\mu$$

$$\leq C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+3} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} (\partial_{t^*}^m S\Phi) \, n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq \ell+6} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu \right).$$

Hence, by Proposition 40 and 41,

$$\sum_{j=0}^{\ell} \int_{\Sigma_\tau \cap \{r\leq R\}} \left(D^j \Phi \right)^2$$

$$\leq C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} (\partial_{t^*}^m S\Phi) \, n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq \ell+5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu \right).$$

The pointwise decay statement follows from standard Sobolev Embedding. $\qquad\square$

15 Discussion

The Theorem that we proved in this paper holds in the set $\{r_+ \leq r \leq R\}$ for any fixed R. It is however interesting also to derive the same estimates, for example, in the set $\{r_+ \leq r \leq \frac{t^*}{2}\}$. This can be achieved by proving the full decay result when we commuted the equation with $\tilde{\Omega}^\ell$. Using this we can prove (with more loss in derivatives) that

$$|\Phi| \leq CE(t^*)^{-\frac{3}{2}+\eta} r^{\frac{1}{2}+\eta}, \quad |D\Phi| \leq CE(t^*)^{-\frac{3}{2}+\eta} r^{-\frac{1}{2}+\eta},$$

for $r \leq \frac{t^*}{2}$. This will be useful in studying nonlinear problems. This decay rate will be proved as a corollary in our forthcoming paper on the null condition.

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