Thermodynamic Geometry and Hawking Radiation

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This work explores the role of thermodynamic fluctuations in the two parameter Hawking radiating black hole configurations. The system is characterized by an ensemble of arbitrary mass and radiation frequency of the black holes. In the due course of the Hawking radiations, we find that the intrinsic geometric description exhibits an intriguing set of exact pair correction functions and global correlation lengths. We investigate the nature of the constant amplitude radiation and find that it's not stable under fluctuations of the mass and frequency. Subsequently, the consideration of the York model decreasing amplitude radiation demonstrates that thermodynamic fluctuations are globally stable in the small frequency region. In connection with quantum gravity refinements, we take an account of the logarithmic correction into the constant amplitude and York amplitude over the Hawking radiation. In both considerations, we notice that the nature of the possible parametric fluctuations may precisely be ascertained without any approximation. In the frequency domain $w \in (0, \infty)$, we observe that both the local and the global thermodynamic fluctuations of the radiation energy flux are stable in the s-channel. The intrinsic geometry exemplifies a definite stability character to the thermodynamic fluctuations, and up to finitely many topological defects on the parametric surface, the notion remains almost the same for both the constant amplitude and the York model. The Gaussian fluctuations over equilibrium radiation energy flux and fluctuating horizon configurations accomplish a well-defined, non-degenerate, curved and regular intrinsic Riemannian manifolds, for all the physically admissible domains of the radiation parameters.

Keywords: Quantum Gravity, Hawking Radiation; Horizon Perturbations; Vaidya Geometry; York Model; Statistical Fluctuations; Thermodynamic Configurations.

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Contents:

1 Introduction

Since the Hawking's discovery of black hole radiation and quantum physics [1], many subsequent aspects of the semiclassical analysis [2] have been investigated. From the viewpoint of the semiclassical theory of gravity, the present study explores the limiting nature of two parameter Hawking radiating black holes from the perspective of fluctuation theory. Importantly, the present anticipation shares the notion of a very controversial issue, i.e. that of determining the role of arbitrarily large transplanckian frequencies of vacuum fluctuations [3–7] and the gravitational back reaction due to a specific quantum [8]. Besides the 'spontaneous' metric fluctuations, such a discussion takes an account of the fact that there exist induced metric fluctuations, generated by the fluctuations of the other quantum fields interacting with the gravitational one. In the regime where the induced metric fluctuations dominate, the problem of black hole fluctuations and back reaction is consistently described by (i) Schwinger-Keldysh effective stochastic semiclassical theory of gravity [9] and (ii) Feynman-Vernon influence functional methods [10, 11]. Thus, the semiclassical approximation leads to the stochastic theory of gravity, extending Einstein equations to the generalized Einstein-Langevin equations. Such a stochastic stress-energy tensor describes the most promising theory of the quantum gravity [12, 13] from the phenomenon of the induced metric fluctuations.

Apart from the notion of the semiclassical gravity [2] and the subsequent understanding of black hole physics [14], the quantum mechanical questions make the validity doubtful of this semi-classical evolution, which has received so much attention in the past. Thus, the present paper sets-up an intrinsic geometric investigation, for analyzing the horizon properties of a class of two parameter family of Hawking radiation black hole solutions. An appropriate description is expected to stem from the background quantum fluctuations. Thus, in order to determine their effects on the Hawking radiation, one requires the full theory of the quantum gravity, which is far from the reach of the present understanding of the subject. It is worth mentioning further that the study of the black hole fluctuations problem is technically a complicated issue. However, their statistical effects connected with the limiting interactions can be explored from the intrinsic thermodynamic geometric perspective. Having been motivated from the fluctuating horizon geometry and Hawking radiation [15], we first consider the issue how the black hole horizon fluctuations affect the thermodynamic properties of underlying statistical configuration. Secondly, we analyze the effects arising from the energy flux fluctuations of an underlying system, containing an ensemble of Hawking radiated particles and the Hawking radiating black hole.

In this paper, we take an account of York model [16] and its possible alternatives arising from the consideration of fluctuating geometry near the horizon of the black hole. The configuration of interest is represented by a Vaidya-type metric with a fluctuating mass. From the viewpoint of the statistical fluctuation theory, we determine how these fluctuations modify the horizon configuration, thermodynamic properties, energy flux of the radiation and what is the asymptotic behavior of the most dominant s-wave spectrum, originating from the zero point fluctuations of the background quantum fields, causing the notion of the generalized uncertainties [17]. Thus, after neglecting the scattering of the gravitational potential arising from the 4-dimensional D'Alembertian profile, we may take an account of the spherically symmetric fluctuations. In order to do so, let us share a parallel view with a set of important recent studies, offering the thermodynamic nature of the horizon properties of diverse (rotating) black holes. This elucidates interesting aspects of phase transitions, if any, in the thermodynamic geometric framework and their associated relations with extremal black hole solutions in the context of $\mathcal{N} \geq 2$ compactifications [18]. It is worth mentioning further that the connection of such a geometric formulation to the thermodynamic fluctuation theory of black holes in general relativity and in string theory requires several modifications [19].

Such a geometric formulation has first been applied to $\mathcal{N} \geq 2$ supergravity extremal black holes in $D = 4$. These solutions are described as the low energy effective field theories, arising from the compactifications of Type II string theories on a compact manifold, *i.e.* K_3, Calabi-Yau and the other associated manifolds [20]. Since then, there have been numerous investigations and several authors have attempted to understand the possible connections and associated vacuum phase transitions, if any, which involve some change of the black hole horizon topology [21–24]. Intrinsic geometric modelings involving equilibrium configurations of the extremal and non-extremal black holes in string theory [25–32] and M-theory [33–40] possess rich intrinsic geometric structures [41–45]. Further interesting discussions [46, 47] are aimed to understand the case of supersymmetric and non-supersymmetric black holes in various spacetime dimensions and rotating black string and ring solutions in the five dimensional spacetimes.

We are motivated from Ruppenier's advocation "that all the statistical degrees of freedom of black hole live on the black hole event horizon". Thus, the scalar curvature signifies the average number of correlated Planck areas on the event horizon, *e.g.*, see for the Kerr-Newman black holes [48]. Besides several notions analyzed in condensed matter physics [48–50, 56–58], we consider specific Hawking radiating configurations with a finite set of radiation parameters and analyze possible parametric pair correlations and their correlation relations. While the Hawking radiation is going on, the intrinsic geometric computations turn out to be highly non-trivial, and thus this paper restricts the intrinsic geometric exploration for the two parameter Hawking radiating black hole configurations. Following Ruppenier's argument, we take an account of the fact that the zero scalar curvature indicates certain bits of information on the event horizon fluctuating independently of each other, while the diverging scalar curvature signals a phase transition, indicating highly correlated pixels of information. Bekenstein has introduced an elegant picture for the quantization of the area of the event horizon, being defined in terms of Planck areas [59]. Such issues serve as motivations for considering the quantum correction to the limiting thermodynamic geometric configuration.

Recently, the thermodynamic geometries of the equilibrium systems thus anticipated have extensively been explored to investigate the thermodynamic nature of the limiting configuration of a class of (rotating) black holes and field theory configurations [43–45, 47, 60]. It is worth mentioning that there exists an intriguing relationship between the scalar curvature of the thermodynamic intrinsic Riemannian geometry and the correlation volume of the corresponding phase-space configuration, which are both characterized by the parameters of the radiating black hole. Furthermore, the general coordinate transformations on the limiting thermodynamic manifolds thus considered expound to certain microscopic duality relations associated with the fundamental invariant quantities of the black hole configuration, *viz.*, mass, charges and chemical potentials describing the considered Hawking radiation. Additionally, Ruppeiner has revived the subject with the fact that the state-space scalar curvature remains proportional to the correlation volume, which reveals related information residing in the microscopic models [50]. Notice further that the state-space scalar curvature in general signifies a possible interaction in the underlying radiating configuration. The present paper is thus intended to analyze both pictures, *viz.*, horizon fluctuations and quantum gravity corrections.

From the perspective of an intrinsic Riemannian geometry, indeed our study strikingly provides a set of exact covariant thermodynamic geometric quantities ascribed to a Hawking radiating black hole. There exists an obvious mechanism on the black hole side, and it would be interesting to illuminate an associated quantum gravity notion [61] for the statistical correlations to the microstates, or an ensemble of gravity particles arising from the Hawking radiating systems and vice-versa. In the context of the Hawking radiation of a neutral black hole, a microscopic construction follows from the computation of the moments. In the framework of W-infinity algebra [51], the Hawking radiation arising from a neutral black hole are realized via the trace anomaly consideration. The finite W-algebras, super algebras and their explicit field realizations have been considered from the perspective of the W-strings, affine currents and conformal realizations [52–55]. In this concern, it turns out that the present construction elucidates certain fundamental issues, such as statistical interactions and the stability of the underlying black hole configurations, with fluctuating mass and radiation frequency. Nevertheless, one may arrive to a definite possible realization of the near equilibrium statistical structures, and thus the possibility to determine the behavior of the limiting thermodynamic fluctuations, in terms of the parameters of an ensemble of limiting equilibrium configurations. Our hope is that finding statistical mechanical models with similar behavior might yield further insight into the microscopic properties of the black hole, and thus leading to a conclusive physical interpretation of the thermodynamic curvatures and related intrinsic geometric invariants.

As a matter of fact, a few simple manipulations illustrate that the relation of a non-zero scalar curvature with an underlying interacting statistical system remains valid even for higher dimensional intrinsic Riemannian manifolds, and the connection of a divergent scalar curvature with phase transitions may accordingly be divulged from the Hessian matrix of the considered energy flux/ fluctuating horizon area. It is worth mentioning that our analysis takes an intriguing account of the scales that are larger than the correlation length and considers that only a few microstates do not dominate the whole macroscopic equilibrium intrinsic quantities. Specifically, we share the interpretation that the underlying horizon perturbations and energy fluxes include contributions from a large number of subensembles of the fluctuating horizon black holes. In this sense, such a consideration takes an account of the fact that the saddle point approximation leads to an appropriate physics of the fluctuating horizon. Thus, the present investigation characterizes the covariant intrinsic geometric description for understanding the quantum statistical physics [61] of a Hawking radiating black hole.

With this motivation, our geometric formulations tacitly involves unified statistical basis, in terms of the chosen ensemble. In particular, we have outlined some space-time implications of the thermodynamic geometry for the two parameter Hawking radiating configurations. The underlying motivations arising from the microscopic quantum gravity configuration turn out to have a bootstrapping significance. Although the analysis has only been considered in the limit of small statistical fluctuations, however the underlying correlation length takes an intriguing account upon the quartic corrections of the energy flux, or the horizon area of the fluctuating black hole. Herewith, we focus our attention on the interpretation that the underlying energy includes contributions from a large number of excited particles, described in the framework of quantum field theories on curved space-time geometry. Thus, our description of the geometric thermodynamics extends itself to all possible configurations. Some of these issues have been anticipated in the final section of the paper.

The present paper is organized as follows. In the first section, we have presented the motivations to study the intrinsic Riemannian surfaces obtained from the Gaussian fluctuations of a Hawking radiating black hole configuration. In section 2, we briefly explain what are the intrinsic thermodynamic geometries based on an ensemble of equilibrium statistical basis characterizing a two parameter Hawking radiating black hole. In section 3, we obtain the two parameter canonical configurations with a constant amplitude and with decreasing amplitude of background fluctuations. In section 4, we investigate the corrections to the above considered thermodynamic geometric configurations, arising from the quantum gravity fluctuations. In section 5, we focus our attention on the thermodynamic geometry of the aforementioned configurations, with a fluctuating radiation energy flux. In all the above cases, we have explained that such an intrinsic geometric configuration, obtained from an effective energy flux or fluctuating horizon area of the black holes, results to be well-defined and pertains to an interacting statistical system. Finally, section 6 contains a set of concluding issues and a possible perspective discussion of the Hawking radiating black holes and their thermodynamic geometries. The general implications thus obtained may divulge the intrinsic geometry of both the chemical and equilibrium microscopic acquisitions. These issues are the matter of a future investigation.

2 Thermodynamic Geometries

The present section presents a brief review of the essential features of thermodynamic geometries from the perspective of the application to the two parameter Hawking radiating configurations. From the viewpoint of the Hawking radiation, the present analysis considers two parameter radiating black holes. Subsequently, we divulge the most probable fluctuations around a chosen equilibrium black hole configuration. Thereby, we provide provide arguments to understand the most puzzling black hole, *viz.*, a radiating Schwarzschild black hole.

The underlying intrinsic configuration is parameterized by the mass M and frequency w of the radiation. The thermodynamic fluctuations around a minimum energy flux $F(M,w)$ (or maximum horizon area $A(M,w)$ are described as an intrinsic configuration. The concerned local pair correlations are characterized by the metric tensor of the associated Wienhold(/ Ruppenier) surface. Consequently, both of the above two dimensional intrinsic Riemannian geometry are conformally the same. Thus, up to a constant factor, the components of the thermodynamic metric tensor are given by

$$\begin{aligned} g_{MM} &= \frac{\partial^2 A}{\partial M^2} \\ g_{Mw} &= \frac{\partial^2 A}{\partial M \partial w} \\ g_{ww} &= \frac{\partial^2 A}{\partial w^2} \end{aligned} \tag{1}$$

In this case, it follows that the determinant of the metric tensor is

$$\|g(M,w)\| = A_{MM}A_{ww} - A_{Mw}^2 \tag{2}$$

Explicitly, we can calculate the Γ_{ijk}, R_{ijkl}, R_{ij} and R for the above two dimensional thermodynamic geometry (M_2, g). One may easily inspect that the scalar curvature is given by

$$\begin{aligned} R(M,w) = & -\frac{1}{2}(A_{MM}A_{ww} - A_{Mw}^2)^{-2}(A_{ww}A_{MMM}A_{Mww} \\ & + A_{Mw}A_{MMw}A_{Mww} + A_{MM}A_{MMw}A_{www} \\ & - A_{Mw}A_{MMM}A_{www} - A_{MM}A_{Mww}^2 - A_{ww}A_{MMw}^2) \end{aligned} \tag{3}$$

Furthermore, the relation between the intrinsic scalar curvature and the Riemann covariant curvature tensor for any two dimensional intrinsic geometry is given (see for details [43]) by

$$R(M,w) = \frac{2}{\|g(M,w)\|} R_{MwMw}(M,w) \tag{4}$$

The relation in Eqn.4 is standard for an arbitrary intrinsic surface $(M_2(R), g)$. Correspondingly, the Legendre transformed version of the energy fluctuation manifold is known as a state-space manifold with a fluctuating entropy configuration. Subsequently, we investigate the nature of the thermodynamic thus defined, as an intrinsic Riemannian manifold obtained from the horizon perturbations, with the leading order contributions being considered in an underlying ensemble. Such an analysis continues, even in the context of chemical reactions or in any closed system, and it strongly suggests that a non-zero scalar curvature might provide useful information regarding the range of underlying phase-space correlations between various components of an underlying Hawking radiating black hole configuration. Furthermore, our results are expected to be in a close connection with the microscopic implications, if any, arising from the underlying field theory theories. Thus, these results yield an anticipation to offer physically definite explanations of the Hawking radiating configurations.

3 Horizon Perturbation

With the general introduction of the thermodynamic geometric nature of Hawking radiating black holes, we shall now define the negative (positive) Hessian function of the horizon area (energy flux) and thereby proceed to investigate the thermodynamic behavior of the two parameter Hawking radiating black holes and their thermo-geometric structures.

3.1 Constant Amplitude Fluctuations

In order to begin the intrinsic geometric analysis of thermodynamic fluctuations, we consider the correlation in the configuration arising from the fluctuations of the radiation parameters. Thus, we introduce an ensemble of black holes fluctuating over the limiting Gaussian configuration. In this analysis, we consider that the black hole can have non-zero fluctuations due to the vibrations of frequencies, residual factors in vacuum, and possibly other quantum properties. This follows from the fact that we do not restrict ourselves to the specific domains of radiation and particle productions.

Consequently, we allow for an ensemble of limiting horizon configurations with finite fluctuations in an arbitrary non linear domain of the parameters and thereby analyze the nature of a class of generic limiting thermodynamic evolutions. We further take an account of the variable radiation frequency defining the speed response tuning parameter of the particles. Subsequently, the stability of the system can be analyzed in the entire mass and frequency domains of the radiation. This is observed, when there are back reactions in the black hole. In this framework, we find that the area of the fluctuating horizon, as the function of mass and radiation frequency, takes an intriguing expression

$$A(M,w) := 16\pi M^2 \Big(1 + \frac{u^2}{(2+16M^2w^2)}\Big) \tag{5}$$

The correlations are described by the Hessian matrix of the horizon area, defined with a set of desired corrections over a chosen black hole with adjustable frequency band. Following Eqn.(1), the components of the metric tensor, defined as the Hessian function of the horizon area $Hess(A(M,w))$, reduce to the following expressions

$$\begin{aligned} g_{MM} &= -16\pi \frac{(1024M^6w^6 + 384M^4w^4 + 28M^2w^2(2-u^2) + 2 + u^2)}{(1+8M^2w^2)^3} \\ g_{Mw} &= 512\pi \frac{M^3u^2w}{(1+8M^2w^2)^3} \\ g_{ww} &= -128\pi \frac{M^4u^2(24M^2w^2-1)}{(1+8M^2w^2)^3} \end{aligned} \tag{6}$$

In this framework, we observe that the geometric nature of parametric pair correlations divulges the notion of a fluctuating Hawking radiating black hole. The fluctuating black hole may thus be easily determined in terms of the intrinsic parameters of the underlying configurations. It is worth mentioning that the two parameter black hole turns out to be well-behaved for the generic values of the mass and radiation frequency. Over the domain of radiation parameters $\{M, w\}$, we notice that the Gaussian correlations form stable correlations, if the determinant of the metric tensor

$$\|g(M,w)\| = 2048\pi^2 \frac{M^4u^2}{(1+8M^2w^2)^5} g_c(M,w) \tag{7}$$

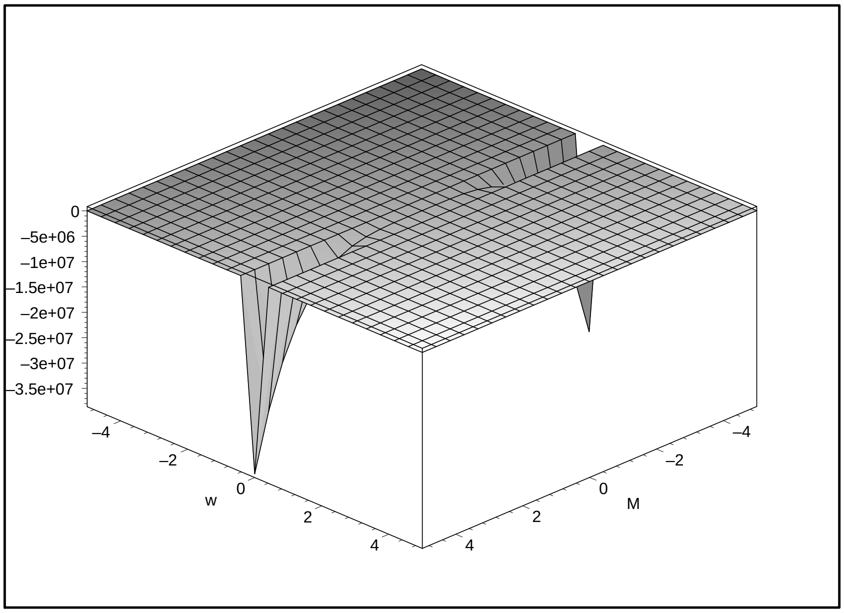

Figure 1: The determinant of the metric tensor plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the constant amplitude Hawking radiating black hole.

remains a positive function on the parametric surface $(M_2(R), g)$. The stability of the radiation is thus defined by the positivity of

$$g_c(M,w) := 3072M^6w^6 + 640M^4w^4 + 8w^2(2-9u^2)M^2 - 2 - u^2 \tag{8}$$

Thus, the cubic equation $g_c(M,w) = 0$ determines the thermodynamic stability structure of the constant amplitude Hawking radiating black holes. This follows simply from the fact that the Eqn.(8) is a cubic equation in M^2, and thus the positivity of $g_c(M,w)$ could be realized in a specific domain of the radiation amplitude. It is not difficult to compute an exact expression for the scalar curvature describing the global parametric intrinsic correlations. By defining a radiation function $R_c(M,w)$, we find that the most general scalar curvature can explicitly be presented as

$$R(M,w) = -\frac{1}{2\pi M^2}\frac{R_c(M,w)}{g_c(M,w)^2} \tag{9}$$

where the global interaction of the radiation is defined by the function

$$\begin{aligned} R_c(M,w) \quad := \quad & 589824M^{10}w^{10} + 204800M^8w^8 + (22528w^6 - 4608w^6u^2)M^6 \\ & +(-576u^2w^4 + 768w^4)M^4 + (8u^2w^2 + 16w^2)M + 2 + u^2 \end{aligned} \tag{10}$$

Thus, the present intrinsic geometric analysis anticipates that the constant amplitude Hawking radiating black hole is always an interacting statistical system over the surface of the fluctuating mass and radiation frequency, except on the real roots of the equation $R_c(M,w) = 0$. These points correspond to a non-equilibrium statistical system, and their numerical values may precisely be easily obtained from the roots of the following degree five equation

$$589824w^{10}p^5 + 204800w^8p^4 + (22528w^6 - 4608w^6u^2)p^6 + (-576u^2w^4 + 768w^4)p^2 + (8u^2w^2 + 16w^2)p + 2 + u^2 = 0, \tag{11}$$

where $p := M^2$. Notice further that the limit $M = 0$ is unstable for the Hawking radiating black hole. It is expected that an extremely small (large) mass black hole goes beyond the validity of the saddle point approximation of the Euclidean path integral. Thus, it may be anticipated that the higher order quantum gravity corrections would dominate the physical observations, in such extreme domains. Interestingly, the quantum notion may further be made obvious from the very basic addition of the background space-time fluctuations to the spectrum of the theory. Furthermore, such an initiation

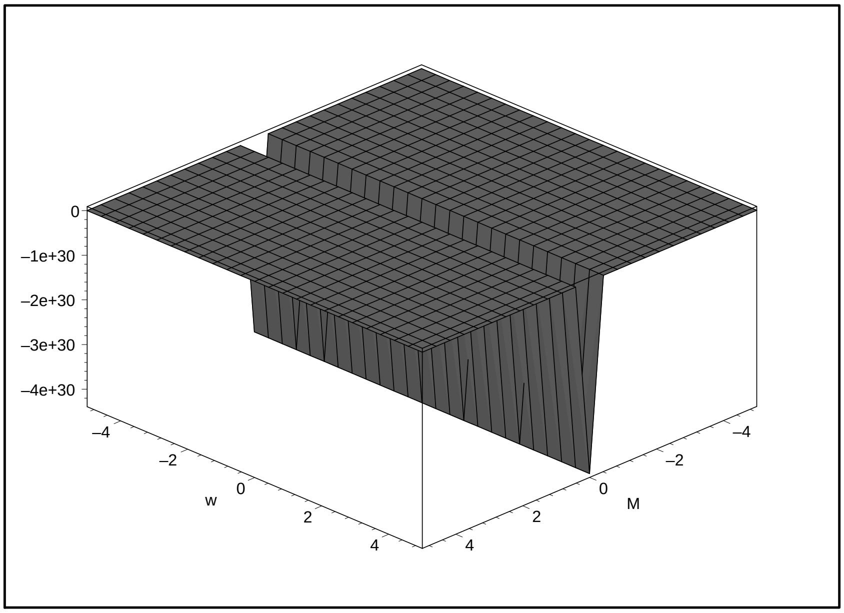

Figure 2: The curvature scalar plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the constant amplitude Hawking radiating black hole.

would be the matter of the next section of the present investigation. From the thermodynamic perspective, we would thereby explore what could statistically happen to a black hole, while the Hawking radiations are in the verge of an extinction.

In the present case the fluctuating horizon geometries, the thermodynamic stability hypothesis follows directly from the sign of the determinant of the metric tensor. The corresponding horizon analysis shows that the amplitude becomes a purely imaginary number *viz.*, $u = \sqrt{2}i$ in the limit of $M = 0$ or $w = 0$. What follows further that we specialize ourselves to the limiting value of the radiation frequency, and subsequently, we analyze the stability for $w = 0$ corresponding to the limiting equilibrium statistical basis. Physically, the limiting equilibrium horizon is realized with no Hawking radiation. Thus, for the black hole thus achieved, the zero frequency configuration has the following limiting local thermodynamic correlations

$$\begin{aligned} g_{MM} &= -16\pi(2+u^2) \\ g_{Mw} &= 0 \\ g_{ww} &= 128\pi M^4 u^2 \end{aligned} \tag{12}$$

Under such a limiting specification of the parameters, the global quantities, *viz.*, the determinant of the metric tensor and Ricci scalar are reduced to the following limiting values

$$\begin{aligned} \|g(M, w=0)\| &= -2048\pi^2 M^4 u^2 (2+u^2) \\ R(M, w=0) &= -\frac{1}{2\pi M^2(2+u^2)} \end{aligned} \tag{13}$$

Significantly, the behavior of the determinant of the metric tensor shows that such limiting black holes become unstable for all possible physical values of the mass. For $u = 1$, the pictorial nature of the determinant of the metric tensor is depicted in Fig.[1]. We observe that the system is stable in the zero mass limit, but acquires a large negative fluctuation of order 10^7. As expected on physical grounds, it is worth mentioning further that the limiting zero mass black hole becomes highly self-contracting. As a matter of fact, Fig.[2] shows that the limit of vanishing mass yields a negative strength of the thermodynamic interaction in the order of 10^{30}. In this process. we observe further that the system acquires a triangular wall of instability, in the limit of $|M| \to 0$.

Physically, if one shares the notion that the black hole has a remnant, then the above picture changes slightly, and we find that the small mass black holes are unstable with a high degree of self-interaction. A refined analysis is the matter of the next section, where we shall take an account of the quantum gravity corrections. Specifically, it is important to mention that the correlation length of the underlying nearly equilibrium system is globally characterized by the scalar curvature of (M_2, g). The present investigation shows that a typical Hawking radiating black hole is globally correlated over all possible Gaussian fluctuations of the frequency and a non-zero mass.

3.2 Decreasing Amplitude Fluctuations

In this section, we study the thermodynamic geometry arising from the horizon area of a Hawking radiating black hole. It is well known that the extremal black holes do not Hawking radiate, since the Hawking temperature is proportional to the difference of the inner and outer horizon radius of the hole configuration. Therefore, we focus our attention of the non-extremal configuration and analyze the thermodynamic properties of such black holes with a large mass $M \gg m$, where m is the Planck mass. Following York [16] we assume the dimensionless amplitude

$$u = a(\frac{m}{M}), \tag{14}$$

where a is a pure number. In particular, the assumption $a \ll 1$ means that the black hole is far away from the Planck size black holes. In order to get a more realistic result, it is assumed that one should average over the entire spectrum of the space-time metric fluctuations. This is realized with fluctuating the parameters $\{M, w\}$ of the black hole space-time. In this section, $A(M, w)$ denotes the horizon entropy of the black hole, since up to a normalization both quantities are the same for the present discussion. With the consideration of the York model decreasing amplitude Eqn.14, we find that the horizon entropy is given by the following expression

$$A(M, w) := 4\pi(\frac{M}{m})^2 - 4\pi\frac{a^2}{(1 + 16M^2w^2)} \tag{15}$$

While the Hawking radiation is going on, there exist non-trivial intrinsic correlations. The components of the thermodynamic metric tensor, defining fluctuation among the parameters, are given by

$$\begin{aligned}
g_{MM} &= 8\pi\frac{(-4096M^6w^6 - 768M^4w^4 + (768a^2w^2m^2 - 48)w^2M^2 - 1 - 16a^2w^2m^2)}{m^2(1 + 16M^2w^2)^3} \\
g_{Mw} &= 256\pi a^2 Mw\frac{(16M^2w^2 - 1)}{(1 + 16M^2w^2)^3} \\
g_{ww} &= 128\pi a^2 M^2\frac{(48M^2w^2 - 1)}{(1 + 16M^2w^2)^3}
\end{aligned} \tag{16}$$

For such Hawking radiating black holes, we observe that the principle components of the metric tensor $\{g_{MM}, g_{ww}\}$ signifying self pair correlations, are positive definite functions over a range of the parameters. Physically, this signifies a set of heat capacities against the intrinsic interactions, arising from a fictitious potential flowing on the surface $(M_2(R), g)$ of an ensemble of Hawking radiated particles. Moreover, it is evident that the determinant of the metric tensor reduces to the following simple expression

$$\|g(M, w)\| = 1024\pi^2\frac{a^2M^2}{m^2(1 + 16M^2w^2)^5}g_d(M, w) \tag{17}$$

where stability of the decreasing amplitude radiation is determined by the positivity of the function

$$g_d(M, w) := -12288M^6w^6 - 1280M^4w^4 + 16(80a^2w^2m^2 - 1)M^2 - 48a^2w^2m^2 + 1 \tag{18}$$

As in the case of a constant amplitude Hawking radiating black hole, we observe in the present case that the stability structure is again determined by a cubic equation. This follows from the fact that the sign is governed by the variable M^2. Furthermore, the configuration in which the cubic equation $g_d(M^2, w) = 0$ has a real positive value is stable. Thus, it is evident that the global stability of the fluctuating Hawking radiating decreasing amplitude configuration is determined by the Ricci scalar curvature of the parametric surface. For the above metric tensor Eqn.(16), we may easily obtain a compact formula for the scalar curvature. Exclusively, we find in the present case that the intrinsic geometric analysis assigns a compact expression to the invariant quantity. For an arbitrary mass and

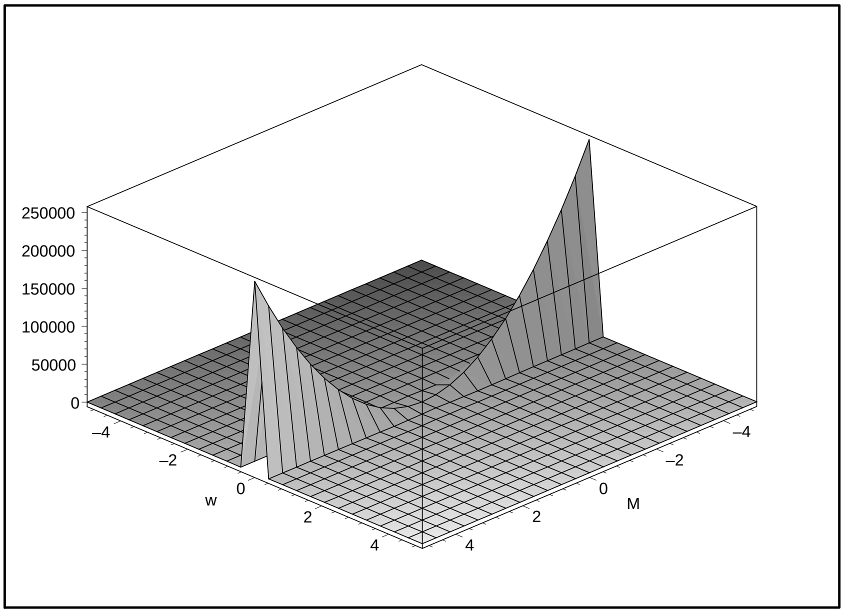

Figure 3: The determinant of the metric tensor plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the decreasing amplitude Hawking radiating black hole.

frequency of the Hawking radiation, a straightforward computation shows that the most general scalar curvature is given by the ratio of the two polynomials of the radiation parameters

$$R(M,w) = \frac{m^2}{4\pi M^2}\frac{R_d(M,w)}{g_d(M,w)^2} \tag{19}$$

where the global interaction of the decreasing amplitude fluctuating radiation is defined by the function

$$\begin{aligned} R_d(M,w) \quad := \quad & 150994944w^{12}M^{12} - 2818048w^8M^8 + (1572864a^2w^2m^2 - 163840)w^6M^6 \\ & +(65536a^2w^2m^2 + 768)w^4M^4 + (-2048a^2w^2m^2 + 128)w^2M^2 - 1 \end{aligned} \tag{20}$$

The notion of the fluctuations is easily determined in terms of the intrinsic parameters of the radiating black hole. It is worth mentioning that the thermodynamic equilibrium of the Hawking radiation could be anticipated to be reached in the limit of zero Hawking frequency, *viz.*, $w = 0$. In this limit, it follows that the underlying local thermodynamic pair correlations of the Hawking radiating black hole with the present type amplitude reduce to the following limiting values

$$\begin{aligned} g_{MM} &= -8\frac{\pi}{m^2} \\ g_{Mw} &= 0 \\ g_{ww} &= -128\pi a^2 M^2 \end{aligned} \tag{21}$$

As in the previous section, the global quantities of the fluctuating parametric surface of a decaying amplitude Hawking radiating black hole, *viz.*, the determinant of the metric tensor and the Ricci scalar curvature, respectively reduce to the following limiting values

$$\begin{aligned} \|g(M,w=0)\| &= 1024\pi^2a^2(\frac{M}{m})^2 \\ R(M,w=0) &= -\frac{1}{4\pi}(\frac{m}{M})^2 \end{aligned} \tag{22}$$

Herewith, the intrinsic geometric framework makes the nature of parametric pair correlations more clearly observable. In the limit of $w = 0$, we find that the determinant of the metric tensor is positive, signifying stable limiting thermodynamic fluctuations and thus a favorable condition for the stability of the radiating black hole. Similarly, the negativity of the scalar curvature indicates that

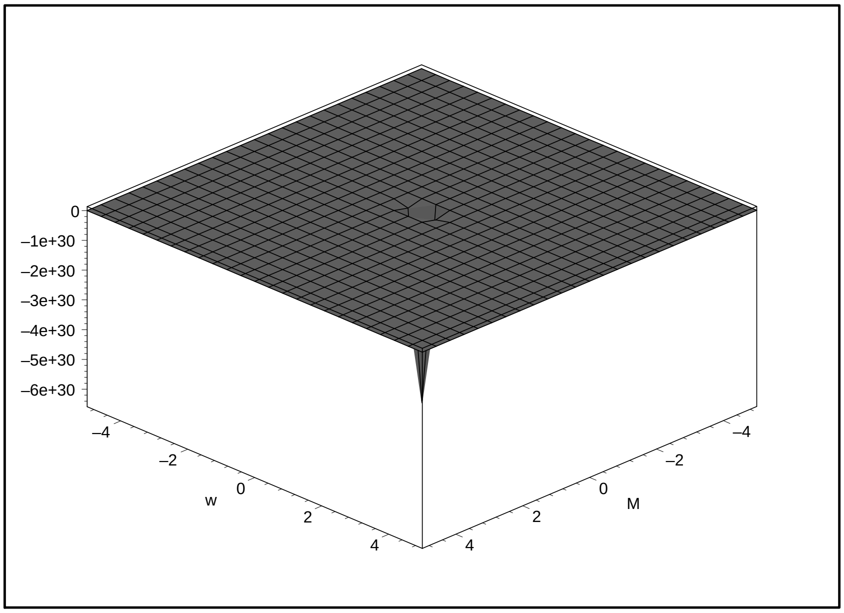

Figure 4: The curvature scalar plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the decreasing amplitude Hawking radiating black hole.

the thermodynamic correlations are attractive on $(M_2(R), g)$. Notice further that such a viewpoint suggests that the Planck size black holes would be self-attractive with a constant correlation length.

Graphically, the behavior of the determinant of the metric tensor shows that such limiting black holes become unstable in the limit of small frequency and large mass. For $a = 1, m = 1$, the pictorial nature of the determinant of the metric tensor is depicted in the Fig.[3]. Herewith, we notice that the system is stable in the zero mass limit, but acquires a large positive fluctuation of order 10^5. From the Fig.[3,4], we observe for $a = 1, m = 1$ that the nature of the decaying amplitude Hawking radiating black hole is precisely opposite to the constant amplitude Hawking radiating black hole. As per the anticipation of the present consideration, the Fig.[4] shows that the limiting zero mass and zero frequency black hole becomes highly self-contracting. It is worth mentioning further that the limit of vanishing mass and radiation frequency makes the negative strength of the thermodynamic global interaction to be of order 10^{30}. In this process. we observe from Fig.[4] that the system acquires a large conical wall of thermodynamic instability, in the limit of $|M| \to 0$. As a matter of fact, the existence of the black hole remnant would make the Fig.[4] to be capped-off. Thus, the underlying thermodynamic singularity would turn out to be a large finite asymmetric cylinder.

4 Quantum Gravity Correction

In the present section, we explore the statistical nature of an ensemble of generic particles generated from the Hawking radiation of a massive black hole. As in the previous section, we restrict our attention to the constant amplitude and decreasing amplitude radiation black holes. The main goal of the present section is to describe the thermodynamic behavior of the quantum gravity fluctuations over the constant amplitude and decreasing amplitude radiating configurations.

4.1 Constant Amplitude Fluctuations

To consider the most general case, we chose the horizon area as the function of the mass and fluctuation frequency, along with the incorporation of the non-perturbative quantum corrections to the horizon area of the black hole [61]. In this case, the bare horizon area is given by

$$B(M,w) := 16\pi M^2 (1 + \frac{u^2}{(2+16M^2w^2)}) \quad (23)$$

The quantum corrected horizon area [61] is expressed as

$$A(M,w) := 4\pi(\frac{M}{L})^2(1+\frac{u^2}{(2+16M^2w^2)}) + a\ln(\frac{16\pi M^2}{L^2}(1+\frac{u^2}{(2+16M^2w^2)}) \quad (24)$$

Let us restrict ourselves to a situation where the number of parameters remains unchanged under the quantum dynamics, producing an ensemble of particles. When the mass and radiation frequency is allowed to fluctuate, we can exploit the definition of the Hessian function $Hess(A(M,w))$ for the Eqn.(24). For the quantum gravity corrected Hawking radiating black hole, we see that there is the following set of parametric pair correlations

$$\begin{aligned}
g_{MM} &= \frac{2}{M^2L^2}\frac{(g_{12}^{cMM}M^{12}+g_{10}^{cMM}M^{10}+g_8^{cMM}M^8+g_6^{cMM}M^6+g_4^{cMM}M^4+g_2^{cMM}M^2+g_0^{cMM})}{(2+16M^2w^2+u^2)^2(1+8M^2w^2)^3}\\
g_{Mw} &= -\frac{32Mu^2w}{L^2}\frac{(g_6^{cMw}+M^6+g_4^{cMw}M^4+g_2^{cMw}M^2+g_0^{cMw})}{(2+16M^2w^2+u^2)^2(1+8M^2w^2)^3}\\
g_{ww} &= -\frac{16M^2u^2}{L^2}\frac{(g_8^{cww}M^8+g_6^{cww}M^6+g_4^{cww}M^4+g_2^{cww}M^2+g_0^{cww})}{(2+16M^2w^2+u^2)^2(1+8M^2w^2)^3} \quad (25)
\end{aligned}$$

where the functions $\{g_{2i}^{cab} |\ a,b \in \{M,w\}; 0 \leq i \leq 6\}$ are defined in the Appendix (A1). Subsequently, we see that the fluctuations of the radiating massive black hole systems comply with the physically expected conclusions. In particular, the self-pair correlations, defining a set of heat capacities, remain positive quantities for a domain of mass and frequency. A straightforward computation further demonstrates the over-all nature of the parametric fluctuations. A radiating black hole is stable under the set of Gaussian fluctuations, if the associated principle minors $\{g_{MM}, g\}$ remain positive functions on (M_2, g). Subsequently, an explicit computation shows that the stability constraint on the surface is given by the positivity of the determinant of the metric tensor

$$\|g(M,w)\| = -\frac{32u^2}{L^4}\frac{\tilde{g}(M,w)}{(2+16M^2w^2+u^2)^3(1+8M^2w^2)^5} \quad (26)$$

Notice that the co-ordinate charts on (M_2, g) are described by the parameters $\{M, w\}$ of the radiation. In the above Eqn.(26), the determinant of the metric tensor can be positive, if the function

$$\tilde{g}(M,w) := \sum_{n=0}^{8} g_{2n}^c M^{2n} \quad (27)$$

takes a positive value on the intrinsic thermodynamic surface. To be specific, we have defined the functions $\{g_{2n}^c |\ 0 \leq n \leq 8\}$ in the Appendix (A2). Notice further that it is not difficult to compute an exact expression for the scalar curvature describing the global parametric intrinsic correlations. By defining a set of functions $\{r_{2n}^c\}$ depending on the frequency of the Hawking radiation, it turns out that the most general scalar curvature can be expressed as

$$R(M,w) = (\frac{Lm}{M})^2\frac{\sum_{n=0}^{14} r_{2n}^c M^{2n}}{(\tilde{g}(M,w))^2} \quad (28)$$

where the radiation functions $\{r_{2n}^c |\ 0 \leq n \leq 14\}$ appearing in the numerator of the scalar curvature have been relegated to the Appendix (A3). Consequently, we notice that $\{r_{2n}^c\}$ can solely be expressed as functions of the radiating frequency of the black hole. In the limit $w = 0$, the local pair correlations reduce to the following values

$$\begin{aligned}
g_{MM} &= \frac{2}{M^2L^2}(\frac{(-2\pi u^6-12\pi u^4-24\pi u^2-16\pi)M^2+4aL^2+aL^2u^4+4aL^2u^2}{(2+u^2)^2})\\
g_{Mw} &= 0\\
g_{ww} &= \frac{16M^2u^2}{L^2}(\frac{(2\pi u^4+8\pi+8\pi u^2)M^2+aL^2u^2+2aL^2}{(2+u^2)^2}) \quad (29)
\end{aligned}$$

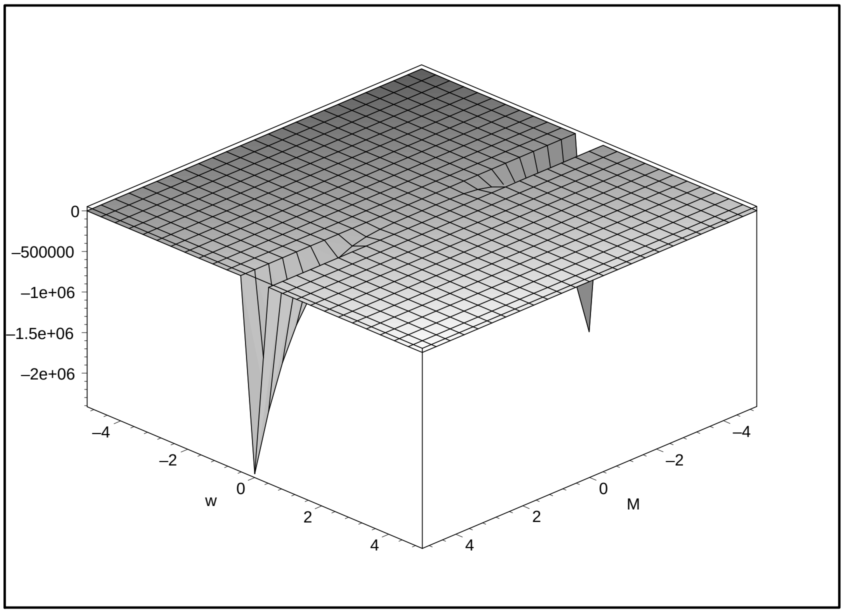

Figure 5: The determinant of the metric tensor plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the constant amplitude Hawking radiating black hole with the quantum gravity corrections.

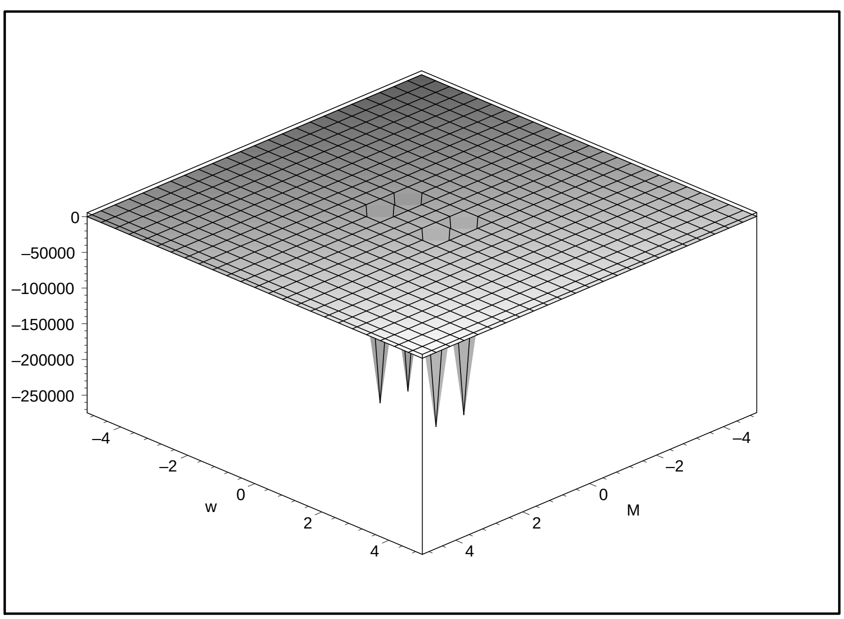

Figure 6: The scalar curvature plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the constant amplitude Hawking radiating black hole with the quantum gravity corrections.

The limit $w = 0$ reduces the determinant of the metric tensor to the following quantity

$$\|g(M, w = 0)\| = -\frac{32u^2}{L^4} \frac{d^c(M, w = 0)}{(2 + u^2)^3} \tag{30}$$

where the numerator of the limiting determinant of the metric tensor is given by the polynomial

$$\begin{aligned} d^c(M, w = 0) \quad := \quad & (64\pi^2 + 32\pi^2 u^6 + 128\pi^2 u^2 + 4\pi^2 u^8 + 96\pi^2 u^4)M^4 \\ & -4a^2 L^4 - a^2 L^4 u^4 - 4a^2 L^4 u^2 \end{aligned} \tag{31}$$

Notice further that the limiting small frequency configuration is stable over the Gaussian fluctuations, if the mass of the Hawking radiating black hole with small background fluctuations satisfies

$$|M| < \left(\frac{4a^2 L^4 + a^2 L^4 u^4 + 4a^2 L^4 u^2}{64\pi^2 + 32\pi^2 u^6 + 128\pi^2 u^2 + 4\pi^2 u^8 + 96\pi^2 u^4}\right)^{1/2} \tag{32}$$

This follows from the analysis of the algebraic equation $\alpha M^4 - \gamma = 0$, which has a real positive root in the domain $M < \sqrt{\gamma/\alpha}$ for $\alpha, \gamma \in R$. The global nature of phase transitions can be thus explored over the range of parameters describing the Hawking radiation of interest. For the specific Hawking radiations corresponding to the limiting frequency $w = 0$, the limiting intrinsic scalar curvature simplifies to the following ratio of the two polynomials

$$R(M, w = 0) = -2L^2 \frac{n^c(M, w = 0)}{d^c(M, w = 0)^2} \tag{33}$$

where the polynomial $n^c(M, w = 0)$ in the numerator is given by

$$\begin{aligned} n^c(M, w = 0) \quad := \quad & (224\pi^3 u^1 2 + 2048\pi^3 + 16\pi^3 u^1 4 + 4480\pi^3 u^8 \\ & +10752\pi^3 u^4 + 7168 u^2 \pi^3 + 1344\pi^3 u^1 0 + 8960\pi^3 u^6)M^6 \\ & +(-240aL^2\pi^2 u^8 - 48aL^2\pi^2 u^1 0 - 960aL^2\pi^2 u^4 \\ & -768aL^2\pi^2 u^2 - 640aL^2\pi^2 u^6 - 4aL^2\pi^2 u^1 2 - 256a\pi^2 L^2)M^4 \\ & +(-6a^2 L^4 \pi u^1 0 - 240a^2 L^4 \pi u^6 - 60a^2 L^4 \pi u^8 \\ & -192a^2 \pi L^4 - 480a^2 L^4 u^4 \pi - 480a^2 L^4 \pi u^2)M^2 \\ & -8a^3 L^6 u^6 - 24a^3 L^6 u^4 - 32a^3 L^6 u^2 - 16a^3 L^6 - a^3 L^6 u^8 \end{aligned} \tag{34}$$

Notice that the introduction of quantum gravity fluctuations makes the thermodynamic stability of the radiating black hole more complex. This is clear from the fact that the sign of the determinant of the thermodynamic metric tensor depends on the roots of $d^c(M, w = 0)$. The notion of the thermodynamic fluctuations is determined in terms of the intrinsic parameters, *viz.*, mass and radiation frequency of the constant amplitude Hawking radiating black hole.

The limit $w = 0$ shows that the sign of the determinant of the metric tensor is a (negative) positive, signifying (un)stable limiting thermodynamic fluctuations and thus a (un)favorable condition for the stability of the Hawking radiating black hole. Similarly, it is easy to notice that the sign of the scalar curvature depends on the relative signs of $n^c(M, w = 0)$ and $d^c(M, w = 0)$. More precisely, this indicates the nature of the thermodynamic correlations on the parametric manifold $(M_2(R), g)$. In the limit $w = 0$, we find that the quantum fluctuations make a complex behavior for an understanding of the Planck size black holes. The present analysis shows that the thermodynamic correlation length of such black holes is a nontrivial function of the mass and radiation frequency. In the limit $w = 0$, notice further that the behavior of the correlation length does not simplify sufficiently and it remains a non-trivial function of the mass of the black hole.

As mentioned earlier, this section offers a quantum gravity refined analysis of the fluctuation horizon analysis of the previous section. As per the name of the consideration, we find that the quantum gravity corrections indeed improve the configuration to be well-behaved. However, the behavior of the determinant of the metric tensor shows a similar qualitative feature. We find that such limiting black holes become unstable for small radiation frequency and large mass. For $u = 1, m = 1, L = 1$, the pictorial nature of the determinant of the metric tensor is depicted in the Fig.[5].

We observe that the present system is stable in the zero mass limit, but acquires a large negative fluctuation of order 10^6. As expected on physical ground, it is worth mentioning that the limiting zero mass black hole becomes highly self-contracting.

As a matter of fact, the corresponding Fig.[6] shows for $u = 1, m = 1, L = 1$ that there are four peaks of thermodynamic instability. For small mass and small radiation frequency, we see in the above limit of the parameters that the negative strength of the thermodynamic interactions is of order 10^5. Thus, the leading order quantum gravity corrections improve the stability of the radiation process. Fig.[6] shows further that the system acquires a four triangular wall of instability, in the limit $\{(M, w)|\ |M|, |w| \to 0\}$. Physically, if one shares the notion that the black hole has a remnant, then the above picture changes slightly. We find that the small mass and small frequency black holes have only four peaks of instability, and secondly they are unstable with a relatively smaller degree of self-interaction. Such a regulation of the instability is an outcome of the quantum gravity corrections to the horizon, which reduce the strength of the instability, when they are taken into account.

4.2 Decreasing Amplitude Fluctuations

In this subsection, we explore the nature of an ensemble of Hawking radiating decreasing amplitude black holes with the logarithmic quantum corrections to the horizon area. To consider the typical nature of the thermodynamic fluctuation, we choose the radiating ensemble as the function of mass and radiation frequency, taken as the system fluctuating parameters. The horizon area is given by the following function of the mass and radiation frequency

$$B(M, w) := 4\pi(\frac{M}{m})^2 - 4\pi\frac{a^2}{(1 + 16M^2w^2)} \tag{35}$$

Thus, the quantum gravity corrected horizon area [61] is given by

$$A(M, w) := (\frac{\pi}{L^2})\left((\frac{M}{m})^2 - \frac{a^2}{(1 + 16M^2w^2)}\right) + a\ln\left(\frac{4\pi}{L^2}(\frac{M}{m})^2 - \frac{4\pi}{L^2}\frac{a^2}{(1 + 16M^2w^2)}\right) \tag{36}$$

When both the parameters are allowed to fluctuate, we can exploit the definition of the Hessian function $Hess(A(M, w))$ of the Eqn.(1). Thus, we see for the decreasing amplitude radiating black holes that there is the following set of parametric pair correlations

$$\begin{aligned}
g_{MM} &= \frac{2}{m^2L^2}\frac{(g_{14}^{dMM}M^{14} + g_{12}^{dMM}M^{12} + g_{10}^{dMM}M^{10} + g_8^{dMM}M^8 + g_6^{dMM}M^6 + g_4^{dMM}M^4 + g_2^{dMM}M^2 + g_0^{dMM})}{(M^2 + 16M^4w^2 - a^2m^2)^2(1 + 16M^2w^2)^3} \\
g_{Mw} &= \frac{64wMa^2}{L^2}\frac{(g_{10}^{dMw}M^{10} + g_8^{dMw}M^8 + g_6^{dMw}M^6 + g_4^{dMw}M^4 + g_2^{dMw}M^2 + g_0^{dMw})}{(M^2 + 16M^4w^2 - a^2m^2)^2(1 + 16M^2w^2)^3} \\
g_{ww} &= \frac{32M^2a^2}{L^2}\frac{(g_{10}^{dww}M^{10} + g_8^{dww}M^8 + g_6^{dww}M^6 + g_4^{dww}M^4 + g_2^{dww}M^2 + g_0^{MM})}{(M^2 + 16M^4w^2 - a^2m^2)^2(1 + 16M^2w^2)^3}
\end{aligned} \tag{37}$$

where the functions $\{g_{2i}^{ab}|\ a, b \in \{M, w\}; i \in \{0, 1, 2, 3, 4, 5, 6, 7\}\}$ are defined in the Appendix (B1). Furthermore, we see that the thermodynamic fluctuations of the quantum corrected decreasing amplitude radiating black holes comply with the physical expectation that the system would approach a stable configuration. In the limit of $g_{2i}^{ab} \geq 0$, we notice that the heat capacities, defined as the self-pair correlations, remain positive quantities for a massive radiating black hole. Moreover, an easy analysis shows the global properties of the thermodynamic fluctuations. In the present case, we find that the determinant of the metric tensor is given by

$$\|g(M, w)\| = -\frac{64a^2M^2}{m^2L^4}\frac{\sum_{n=0}^{9} g_{2n}^{d}M^{2n}}{(M^2 + 16M^4w^2 - a^2m^2)^3(1 + 16M^2w^2)^5} \tag{38}$$

where the functions $\{g_{2n}^d|\ 0 \leq n \leq 9\}\}$ are defined in the Appendix (B2). As we have shown in the previous considerations, the determinant of the metric tensor is non-zero for non-zero mass and $\{g_{2n}^d\}$. It is worth mentioning that this configuration is globally unstable over a large part of the intrinsic parametric configurations. This is also intelligible from the fact that the responsible

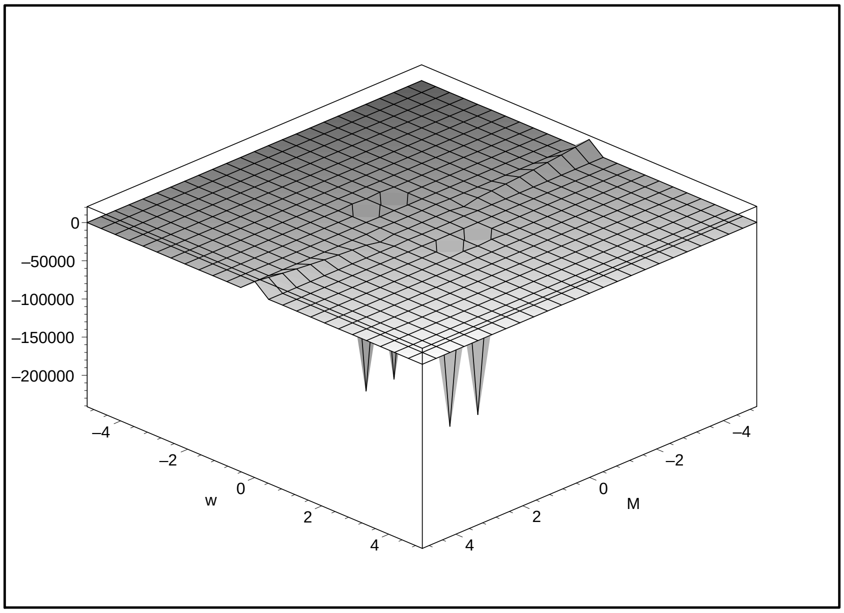

Figure 7: The determinant of the metric tensor plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the decreasing amplitude Hawking radiating black hole with the quantum gravity corrections.

quantum corrected entropy tends to its maximum value, while the same culmination does not remain valid over the entire domain of the Ruppenier's state-space manifold. For a class of g^d_{2n}, we find that the configuration is stable in the limit $M^2 + 16M^4w^2 < a^2m^2$. For a given Planck mass, we observe that the thermodynamic stability constraint requires a specific choice of one or more of the system parameters, *viz.*, mass, frequency and amplitude of the radiation.

In order to examine important global properties in the Hawking radiating black hole configurations, one is required to determine the associated invariants of the underlying thermodynamic geometry. As previously defined, the simplest invariant turns out to be the intrinsic scalar curvature, which may be easily computed by simply following our technology of the intrinsic Riemannian geometry. Our analysis further discovers that there exists an interacting thermodynamic configuration with the quantum gravity contributions. For the present configuration, a little involved computation shows that the Ricci scalar takes the following expression

$$R(M,w) = (\frac{Lm}{M})^2 \frac{\sum_{n=0}^{18} r^d_{2n} M^{2n}}{(\sum_{n=0}^{9} g^d_{2n} M^{2n})^2} \tag{39}$$

where the functions $\{r^d_{2n} | \; 0 \leq n \leq 18\}$ are defined in the Appendix (B3). Based on the Appendix (B2, B3), it is worth emphasizing further that the thermodynamic scalar curvature is non-vanishing, as long as the mass of the Hawking radiating black hole is non-zero. The argument follows from the fact that r^d_0 and g^d_0 are non-zero functions on the parametric thermodynamic surface, for the entire domain of a finite radiation frequency. In the limit $w = 0$, the local pair correlation functions reduce to a set of simple values. Subsequently, we find that the components of the covariant metric tensor are given by

$$\begin{aligned} g_{MM} &= \frac{2}{m^2L^2} \frac{(-\pi M^4 + (am^2L^2 + 2\pi a^2m^2)M^2 + a^3m^4L^2 - \pi a^4m^4)}{(M^2 - a^2m^2)^2} \\ g_{Mw} &= 0 \\ g_{ww} &= \frac{32M^2a^2}{L^2} \frac{(-\pi M^4 + (-am^2L^2 + 2\pi a^2m^2)M^2 + a^3m^4L^2 - \pi a^4m^4)}{(M^2 - a^2m^2)^2} \end{aligned} \tag{40}$$

It is easy to observe that the respective system becomes unstable in the vanishing frequency limit. This follows from the negativity of the limiting determinant of the metric tensor

$$\|g(M, w = 0)\| = -\frac{64a^2M^2}{m^2L^4} \frac{d^d(M, w = 0)}{(M^2 - a^2m^2)^3} \tag{41}$$

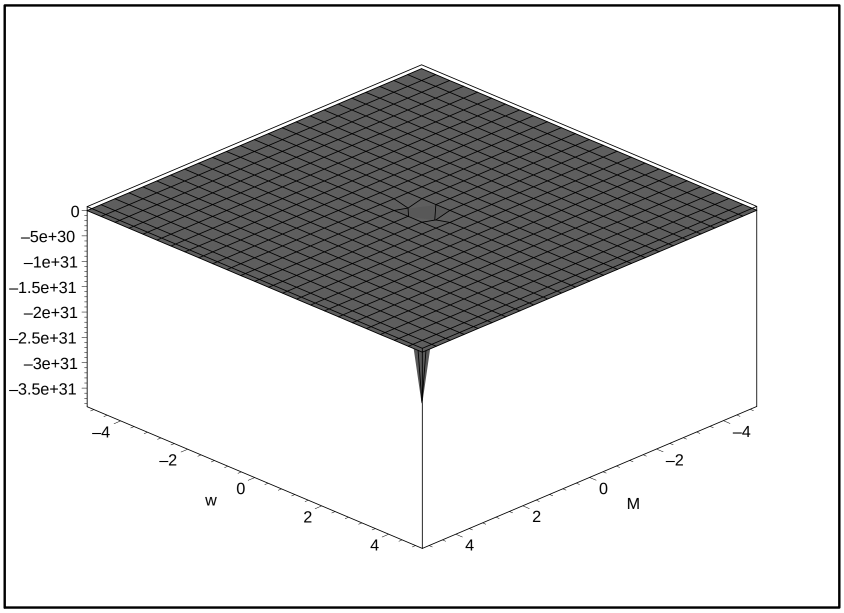

Figure 8: The curvature scalar plotted as the function of the radiation frequency w and mass M, describing the fluctuations in the decreasing amplitude Hawking radiating black hole with the quantum gravity corrections.

where the numerator $d^d(M, w = 0)$ is defined by the following polynomial

$$\begin{aligned} d^d(M, w = 0) \quad := \quad & -\pi^2 M^6 + \pi^2 a^6 m^6 + a^4 m^6 L^4 - 3\pi^2 a^4 m^4 M^2 + 3\pi^2 M^4 a^2 m^2 \\ & + a^2 m^4 L^4 M^2 + 2a^3 m^4 L^2 \pi M^2 - 2a^5 m^6 L^2 \pi \end{aligned} \tag{42}$$

It is worth mentioning that the thermodynamic behavior of the decreasing amplitude Hawking radiating black hole is very intriguing, even in the limit of the zero radiation frequency. This follows from the fact that the stability of the limiting small frequency configuration over the Gaussian fluctuations is determined by the following quadratic and cubic inequalities

$$\begin{aligned} d^d(M^2, w = 0) \quad & > 0 \quad \mathit{for}\ M^2 - a^2 m^2 < 0, \\ d^d(M^2, w = 0) \quad & < 0 \quad \mathit{for}\ M^2 - a^2 m^2 > 0 \end{aligned} \tag{43}$$

Thus, it is immediate to find the regions in which the small frequency decaying amplitude Hawking radiating black hole is (un)stable. Furthermore, the global nature of phase transitions, if any, could be observed from the singularities in the limiting scalar curvature. Specifically, in the limit $w = 0$, we find that the associated scalar curvature reduces to the following limiting value

$$R(M, w = 0) = \frac{L^2 m^2}{M^2} \frac{n^d(M, w = 0)}{d^d(M, w = 0)^2} \tag{44}$$

where the polynomial in the numerator is given by

$$\begin{aligned} n^d(M, w = 0) \quad := \quad & -\pi^3 M^{12} + (2a\pi^2 m^2 L^2 + 6a^2 \pi^3 m^2) M^{10} \\ & + (-15a^4 \pi^3 m^4 + a^2 \pi m^4 L^4 - a^3 \pi^2 m^4 L^2) M^8 \\ & + (-12a^5 \pi^2 m^6 L^2 - 2a^4 \pi m^6 L^4 + 20a^6 \pi^3 m^6) M^6 \\ & + (-a^5 m^8 L^6 - 6a^6 \pi m^8 L^4 - 15a^8 \pi^3 m^8 + 22a^7 \pi^2 m^8 L^2) M^4 \\ & + (6a^1 0\pi^3 m^1 0 + 10a^8 \pi m^1 0 L^4 - 14a^9 \pi^2 m^1 0 L^2 - 2a^7 m^1 0 L^6) M^2 \\ & + 3a^1 1\pi^2 m^1 2L^2 - 3a^1 0\pi m^1 2L^4 - a^1 2\pi^3 m^1 2 + a^9 m^1 2L^6 \end{aligned} \tag{45}$$

Herewith, we find that the decreasing amplitude radiation black holes with an incorporation of the quantum correction possess relatively distinct thermodynamic behavior over the parametric fluctuations. The small frequency decaying amplitude Hawking radiating black hole is non-interacting at the real roots of a six degree equation $n^d(M^2, w = 0) = 0$. As in the case of the constant amplitude Hawking radiating black hole, we observe that the introduction of quantum gravity fluctuations gives the decaying amplitude Hawking radiating black hole a more complex thermodynamic stability character. In fact, for such decaying amplitude Hawking radiating black holes, this is clear from the fact that the sign of the determinant of the thermodynamic metric tensor depends on the roots of $d^d(M^2, w = 0)$, even far way from the Planck size black holes. However, apart from the computational complexities, the conceptual consideration is very clear and lucid from the intrinsic geometric perspective. Thus, the notion of the thermodynamic fluctuations is determined in terms of the intrinsic parameters, *viz.*, mass and radiation frequency of the decaying amplitude Hawking radiating black hole with an account of the quantum gravity.

In the small frequency limit, *i.e.* $w = 0$, it is evident that the determinant of the metric tensor is negative (positive), signifying (un)stable limiting thermodynamic fluctuations and thus a (un)favorable condition for the stability of the decaying amplitude Hawking radiating black hole. Notice further that the sign of the scalar curvature depends on the relative signs of $n^d(M, w = 0)$ and $d^d(M, w = 0)$. In the limit $w = 0$, their identical signs indicate that the nature of the thermodynamic correlations is repulsive on the parametric manifold $(M_2(R), g)$. We find further that the quantum fluctuations produce a complex behavior of the decaying amplitude Hawking radiating black holes. Even in the limit $w = 0$, the correlation length of the decaying amplitude Hawking radiating black holes is non-trivially dependent on the mass of the black hole.

For $a = 1, m = 1, L = 1$, the graphical behavior of the determinant of the metric tensor is shown in the Fig.[7]. We observe that such limiting black holes become unstable in the limit of small frequency and small mass. For $a = 1, m = 1, L = 1$, the pictorial nature of the scalar curvature is depicted in the Fig.[8]. Herewith, we notice that the system is unstable in the zero mass limit, but acquires a large negative thermodynamic fluctuation of order 10^5 for the determinant of the metric tensor, and 10^{31}, for the thermodynamic scalar curvature. From the Fig.[7,8], we observe under quantum gravity contributions that the global nature of the decaying amplitude Hawking radiating black hole is very similar to the constant amplitude Hawking radiating black hole. As per the present consideration, the Fig.[8] shows that the limiting zero mass and zero frequency black hole becomes highly self-contracting as in the case of a constant amplitude black hole. It is worth mentioning that the limit of vanishing mass and radiation frequency makes the negative strength of the thermodynamic global interaction to be of higher order, in comparison with both the constant amplitude black hole, with and without the quantum gravity corrections, and only decreasing amplitude horizon fluctuating black hole.

In the process of the quantum gravity corrections to the horizon, we observe that the Fig.[8] has the same qualitative behavior as the corresponding Fig.[4] with $a = 1, m = 1$. This shows that the thermodynamic system is almost insensitive to the quantum gravity contributions, as they alter the global thermodynamic interaction from an order of 10^{30} to 10^{31}. Whilst, the behavior of the determinant of the metric tensor is quite different, and thus it depends significantly of the quantum parameter L. In fact, we find that the positive metric fluctuations are diminished by the quantum gravity fluctuations. However, they introduce four negative peaks of the thermodynamic conical singularities of the strength 10^5. Thus, in the limit $|M| \to 0$, the determinant of the metric tensor acquires four bumps of small strength thermodynamic instabilities, while the scalar curvature acquires a large conical thermodynamic instability of the strength 10^{31}. As a matter of fact, the existence of the black hole remnant would again make the Fig.[7,8] to be capped-off, and thus the local thermodynamic singularity would be regulated to be four large finite asymmetric cylinders, while the underlying global thermodynamic singularity would be regulated to a large finite asymmetric cylinder.

Our analysis illustrates that the physical properties of the specific local and global thermodynamic correlations may easily be exactly exploited in general, without any approximation. Within a small neighborhood of proclaimed statistical fluctuations over an equilibrium ensemble of configurations, the framework of the thermodynamic geometry explicates the functional nature of parameters without any surprise. As mentioned in the Appendix-I, the explicit forms of the thermodynamic correlations are well-behaved functions of the mass and the radiation frequency of the considered Hawking radiating black hole. The definite behavior of correlations, as accounted in the previous section, and the leading order quantum gravity logarithmic corrections in present section, suggest that the possible constant and decaying amplitude Hawking radiating black holes do have definite stability properties, except that the determinant may be non-positive definite in some specific limit of the mass and radiation frequency. As mentioned before, it is worth recalling that these black hole are generically well-defined from the thermodynamic perspective and indicate an interacting statistical basis in general. Herewith, we discover that the thermodynamic geometry indicate a promising nature of the Hawking radiating black hole configurations, with and without the leading order quantum gravity contributions.

5 Hawking Radiation Energy Flux

In the present section, we explore the nature of an ensemble of generic Hawking radiating black holes generated with the variable mass and variable radiation frequency of the black hole. To consider the most general case, we choose the energy flux as the function of mass and frequency. When both of them are allowed to fluctuate, we can exploit the definition of the Hessian function $Hess(F(M,w))$ of the energy flux of the Hawking radiation. As explored in the previous sections, we specialize ourselves to the constant amplitude and decaying amplitude radiating black holes.

5.1 Constant Amplitude Fluctuations

In this section, we study the case of a constant amplitude (u) Hawking radiation. Subsequently, the energy flux of the constant amplitude Hawking radiation is given by

$$F(M,w) = \frac{1}{768\pi M^2}(1 + \frac{4u^2\pi M w}{\exp(8\pi M w) - 1}) \tag{46}$$

As per the consideration of the present investigation, we treat the $x^a = (M,w) \in M_2$ to be fluctuating thermodynamic variables. The consideration of the intrinsic geometry is realized by defining the metric tensor as the Hessian matrix of the energy flux of the Hawking radiating configuration. In the present subsection, we treat the quantum gravity amplitude (u) to be a constant. From the viewpoint of the thermodynamic limit, our analysis shares the fact that the equilibrium of the Hawking radiated particles and remnant(s) black hole, if any, should be extracted from the limiting equilibrium statistical configuration(s). Of course, the procedure of the extraction could in general be highly non-trivial. However, the present consideration leads us to the conclusion that the energy flux of the radiation defines a set of radially computable expression against both the local and global thermodynamic correlations. First of all, such a conclusion arises from the Hessian matrix of the energy flux fluctuations. Explicitly, the components of covariant metric tensor are given by

$$\begin{aligned}
g_{MM} &= \frac{1}{384\pi M^4}\frac{(c_3^{cMM}M^3 + c_2^{cMM}M^2 + c_1^{cMM}M + c_0^{cMM})}{(\exp(8\pi M w) - 1)^3} \\
g_{Mw} &= \frac{u^2}{192M^2}\frac{(c_2^{cMw}M^2 + c_1^{cMw}M - c_0^{cMw})}{(\exp(8\pi M w) - 1)^3} \\
g_{ww} &= \frac{\pi u^2}{12}\exp(8\pi M w)\frac{(c_1^{cww}M - c_0^{cww})}{(\exp(8\pi M w) - 1)^3}
\end{aligned} \tag{47}$$

For the constant amplitude Hawking radiating black holes, the above behavior of the parametric pair correlations shows that the heat capacities, defined as the self-pair correlations, are positive in a domain of the functions $\{c_i^{cab} |\ a,b \in \{M,w\}; i \in \{0,1,2,3\}\}$. The precise functional nature of $\{c_i^{cab} |\ a,b \in \{M,w\}; i \in \{0,1,2,3\}\}$ is given in the Appendix (C1). Subsequently, we find that the determinant of the metric tensor is

$$\|g(M,w)\| = \frac{u^2}{36864M^4}\frac{(h_3^c M^3 + h_2^c M^2 + h_1^c M + h_0^c)}{(\exp(8\pi M w) - 1)^5} \tag{48}$$

where the functions $\{h_0^c, h_1^c, h_2^c, h_3^c\}$ are presented in the Appendix (C2). It is not difficult to compute

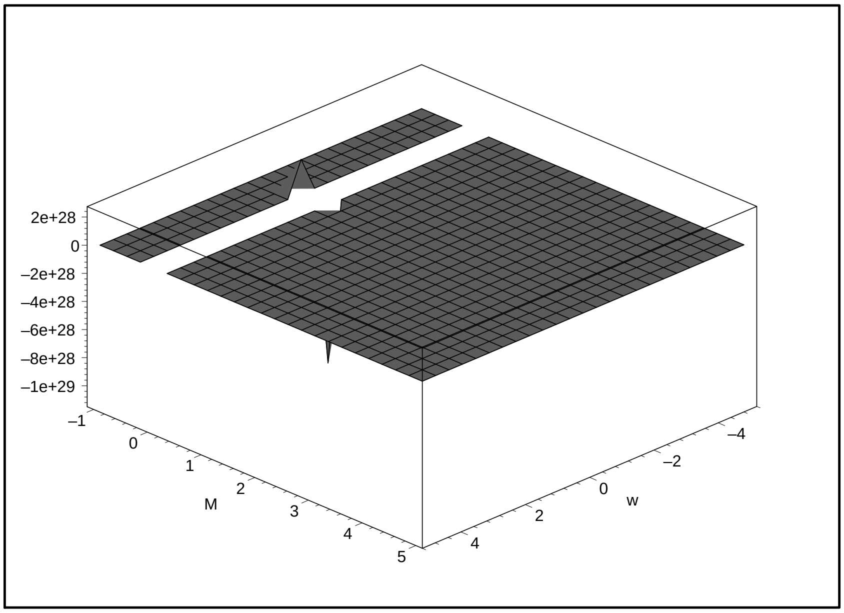

Figure 9: The determinant of the metric tensor plotted as the function of the radiation frequency w and mass M, describing the energy flux fluctuations in the constant amplitude Hawking radiating black hole.

an exact expression for the scalar curvature describing the global parametric intrinsic correlations. By defining a set of radiation functions, we find explicitly that the most general scalar curvature takes the following rational form

$$R(M,w) = 6144\pi M^2 \exp(8\pi M w) \frac{(l_5^c M^5 + l_4^c M^4 + l_3^c M^3 + l_2^c M^2 + l_1^c M + l_0^c)}{(h_3^c M^3 + h_2^c M^2 + h_1^c M + h_0^c)^2} \tag{49}$$

where the functions $\{l_0^c, l_1^c, l_2^c, l_3^c, l_4^c, l_5^c\}$ are defined in the Appendix (C3). Notice that all the functions $\{h_i^c\}$ vanish identically for $w = 0$. From the viewpoint of thermodynamic stability, the vanishing of the combination $h_3^c M^3 + h_2^c M^2 + h_1^c M + h_0^c$ makes the static frequency configuration ill-defined. The determinant of the thermodynamic metric tensor vanishes and the Ricci scalar diverges in the limit of zero frequency. This shows that the configuration may suffer certain phase transitions. Thus, the analysis of the constant amplitude configuration anticipates that the fluctuations of the energy flux would not stop, once the black hole starts Hawking radiating.

Pictorially, the behavior of the determinant of the metric tensor is shown in the Fig.[9]. We anticipate that the graph has a significant change in the regime of small mass and small radiation frequency. In the limit of small radiation frequency and large mass, we find that such a limiting black hole becomes highly unstable and shows a large negative strength thermodynamic instability of order 10^{21}. For a unit amplitude of the constant Hawking radiation $u = 1$, the pictorial nature of the scalar curvature is depicted in the Fig.[10]. Herewith, we notice that the system is unstable in the zero mass and zero frequency limit. Interestingly, the graphical analysis of the Fig.[9] shows that the constant amplitude Hawking radiating black hole acquires a region of small positive thermodynamic fluctuations in the limit $\{M \to 0^+, w \to 0^+$, while it acquires a region of negative fluctuation in the limit $\{M \to 0^-, w \to 0^-$. As expected from the physical basis of the Hawking radiating black hole, it is worth mentioning that the limiting zero mass and zero frequency black hole becomes highly self-contracting. As a matter of fact, the Fig.[10] shows that the limit of vanishing frequency yields a negative strength of the thermodynamic interaction of order 10^{21}. In this process. we observe further that the system acquires a spike of thermodynamic instability, in the limit $|w| \to 0$.

Physically, if one takes an account of the fact that the black holes have a remnant, then the above

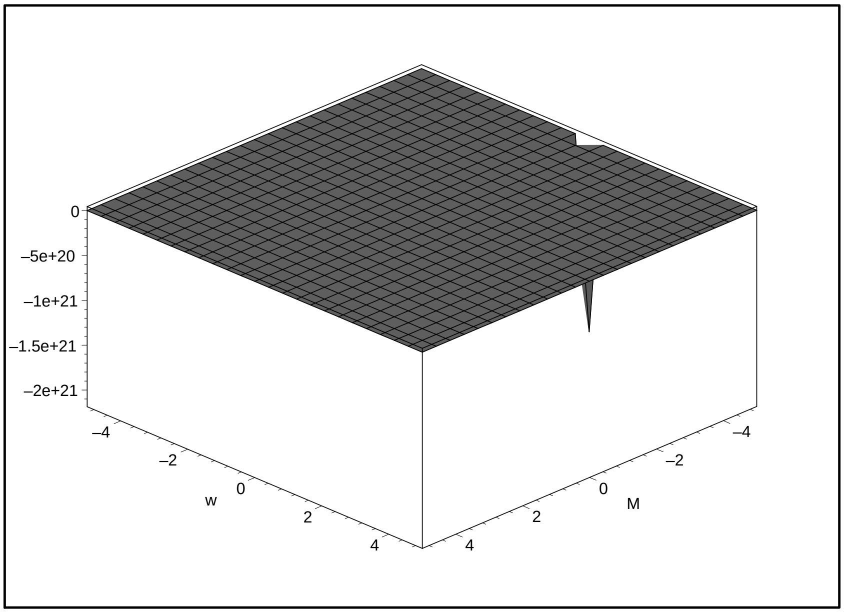

Figure 10: The curvature scalar plotted as the function of the radiation frequency w and mass M, describing the energy flux fluctuations in the constant amplitude Hawking radiating black hole.

picture changes slightly and we find herewith that the Hawking radiating black holes are unstable with a high degree of finite self-interaction. A refined analysis would be the matter of future investigation, where one might take an account of the higher loop quantum gravity corrections arising from the path integral formulation of the gravity, or the ones which are originating from the gravitational aspect of the string theory, with higher order correction in the string length and Planck length. Thus, an appropriate notion of string theory or quantum gravity theory may offer a better thermodynamic understanding of the Hawking radiating black holes. Specifically, it is important to mention that the underlying statistical correlation length of the underlying ensemble reducing to a nearly equilibrium system requires an understanding of the full theory of the quantum gravity. The present investigation is however premature, for showing such atypical qualification of the Hawking radiating black hole.

5.2 Decreasing Amplitude Fluctuations

In this section, the case of the decaying amplitude York model [16] is analyzed. In order to take a closer look at the thermodynamic limit of the energy flux of a Hawking radiating black hole, we consider the thermodynamic geometry of the limiting equilibrium ensemble of the black holes. To be precise, we consider the implications of the York model correction to the thermodynamic geometry. With such a consideration, the energy flux of the Hawking radiation with a decaying amplitude

$$u := a\frac{m}{M} \tag{50}$$

is given by

$$F(M, w) := \frac{1}{768\pi M^2}(1 + \frac{4\pi a^2 m^2 w}{M(\exp(8\pi M w) - 1)}) \tag{51}$$

The metric tensor of the thermodynamic geometry in the energy flux representation may be obtained as before from the Hessian matrix of the energy flux of the Hawking radiation. With respect to previously acclaimed thermodynamic variables, the thermodynamical behavior of the decaying amplitude radiation flux is not difficult to analyze. Furthermore, it is legitimate to expect that the local correlations would acquire a similar structure. In this subsection, we precisely show that the thermodynamic correlations take a set of the similar expressions as for the constant amplitude energy flux. Explicitly,

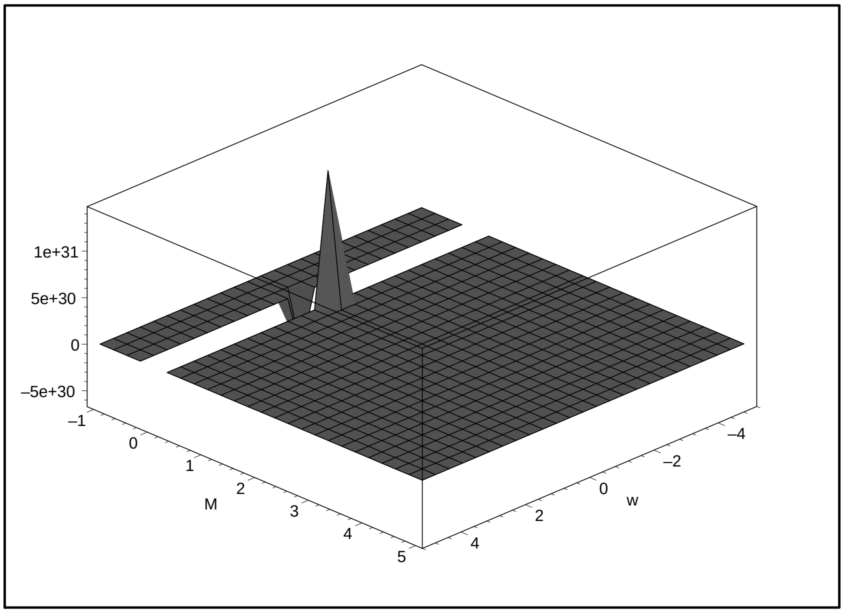

Figure 11: The determinant of the metric tensor plotted as the function of the radiation frequency w and mass M, describing the energy flux fluctuations in the decreasing amplitude Hawking radiating black hole.

as the function of the radiation frequency and mass of the black hole, we find that the components of the covariant metric tensor are given by

$$\begin{aligned} g_{MM} &= \frac{1}{384\pi M^5}\frac{(c_2^{dMM}M^2 + c_1^{dMM}M + c_0^{dMM})}{(\exp(8\pi Mw)-1)^3} \\ g_{Mw} &= \frac{a^2m^2}{192M^4}\frac{(c_2^{dMw}M^2 + c_1^{dMw}M - c_0^{dMw})}{(\exp(8\pi Mw)-1)^3} \\ g_{ww} &= \frac{\pi a^2m^2}{12M^2}\exp(8\pi Mw)\frac{(c_1^{dMw}M + c_0^{dMw})}{(\exp(8\pi Mw)-1)^3} \end{aligned} \tag{52}$$

where the functions $\{c_i^{dab} |\ a, b \in \{M, w\}; i \in \{0, 1, 2\}\}$ given in the Appendix (D1). Notice further that the principle components of the metric tensor signifying self pair correlations, are positive definite functions over a range of the parameters. Physically, this signifies a set of heat capacities against the intrinsic interactions arising from the mass and frequency of the radiation. Over the Gaussian limit such correlations form stable correlations, if the determinant of the metric tensor takes a positive value on the parametric surface of the Hawking radiation. For the given energy flux, it is evident that the determinant of the metric tensor may easily be presented in the following form

$$\|g(M,w)\| = \frac{a^2m^2}{36864M^8}\frac{(h_3^dM^3 + h_2^dM^2 + h_1^dM + h_0^d)}{(\exp(8\pi Mw)-1)^5} \tag{53}$$

where the functions $\{h_0^d, h_1^d, h_2^d, h_3^d\}$ are defined in the Appendix (D2). As a standard interpretation, the thermodynamic scalar curvature describes the nature of underlying statistical interactions of a possible particle on the horizon configurations of the black hole. In particular, it turns out to be non-zero and well defined for a wide range of the parameters of the Hawking radiating black hole solution. Furthermore, the intrinsic scalar curvature corresponding to the underlying thermodynamic geometry elucidates the typical feature of Gaussian fluctuations about the limiting equilibrium particles of the desired mass and radiation frequency. This picture is evident from the consideration of the quantum field theory on curved space-time, implying a modified asymptotic spectrum. More precisely, one can consider ingoing and outgoing waves and compute the properties of the Hawking radiated particles from the Bogoliubov coefficients and thus the spectrum of the Hawking radiation energy flux. As in the case of constant amplitude radiation, we find further for the decaying amplitude radiation that the Ricci scalar curvature reduces to the following specific expression

$$R(M,w) = 12288\pi M^4\exp(8\pi Mw)\frac{(l_5^dM^5 + l_4^dM^4 + l_3^dM^3 + l_2^dM^2 + l_1^dM + l_0^d)}{(h_3^dM^3 + h_2^dM^2 + h_1^dM + h_0^d)^2} \tag{54}$$

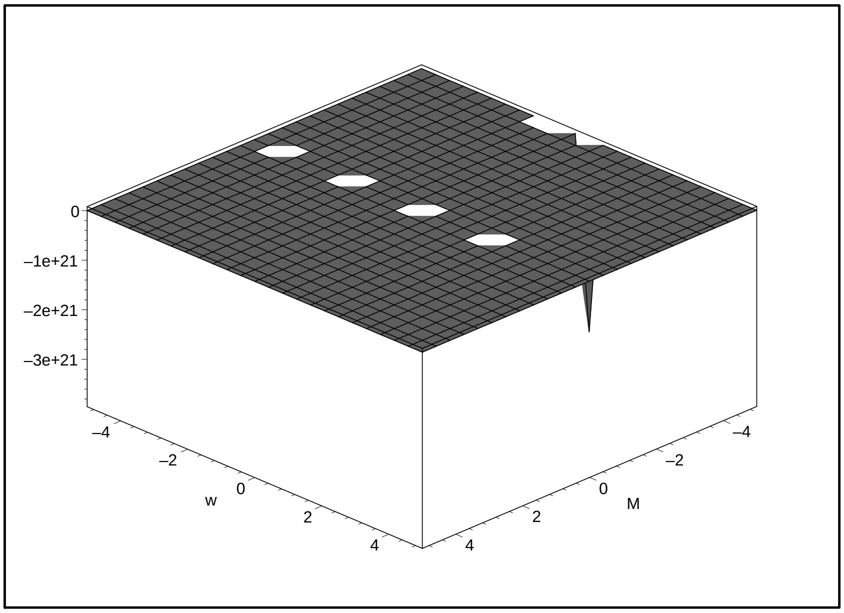

Figure 12: The curvature scalar plotted as the function of the radiation frequency w and mass M, describing the energy flux fluctuations in the decreasing amplitude Hawking radiating black hole.

where the functions $\{l_0^d, l_1^d, l_2^d, l_3^d, l_4^d, l_5^d\}$ are relegated to the Appendix (D3). In this case, it turns out that the observation made in the constant amplitude case continues to hold. Specifically, we find that all the functions $\{h_i\}$ vanish identically for $w = 0$. As mentioned before in the previous section, the vanishing of the combination $h_3^d M^3 + h_2^d M^2 + h_1^d M + h_0^d$ makes the static frequency configuration ill-defined, even for a decreasing amplitude Hawking radiating black hole.

From the viewpoint of thermodynamic stability, the determinant of the thermodynamic metric tensor vanishes and the Ricci scalar diverges in the limit of zero frequency, which implies that the system becomes unstable. In fact, such an analysis suggests that the configuration may suffer a phase transition, as well. Thus, as in the case of the constant amplitude configuration, the decreasing amplitude analysis anticipates that the fluctuations of the energy flux should not stop, once the black hole starts Hawking radiating. This opens an avenue to understand the nature of the interaction, while a black hole is approaching the limiting thermodynamic configuration.

As in the case of the constant amplitude Hawking radiation energy flux arising from the two parameter black hole, we observe in the case of a decreasing amplitude Hawking radiating energy flux that the behavior of the determinant of the metric tensor remains almost the same for $a = 1, m = 1$. Pictorially, this is shown in the Fig.[11], along with the fact that above observation follows from a direct comparison of the Fig.[11] with that of the Fig.[9]. For unit amplitude of decreasing Hawking radiating black hole $u = 1$ and decaying amplitude black hole, an observation follows from the anticipation of the qualitative properties of the Figs.[10] and Fig.[12]. We notice that both graphs of the scalar curvatures have a significant change along the line of $M = 0$. The regime of small radiation frequency remains almost the same, except for the fact that the small radiation frequency and large mass region becomes relatively narrower. In this regime, we find that the limiting black hole becomes highly unstable and it possesses a large negative strength thermodynamic instability of order 10^{21}, as per the case of the constant amplitude amplitude energy flux Hawking radiating black hole.

Globally speaking, the pictorial nature of the scalar curvature, as depicted in the Fig.[12], captures non-perturbative quantum gravity contributions, and thus the parametric surface has a different topology. Herewith, we notice that the system is unstable in the large mass and zero frequency limit. As in the case of the constant amplitude Hawking radiating black hole, the graphical analysis of the

Fig.[11] shows that the decreasing amplitude Hawking radiating black hole acquires a region small positive thermodynamic fluctuations in the limit of $\{M \to 0^+, w \to 0^+\}$, while it acquires a region of negative fluctuation in the limit of $\{M \to 0^-, w \to 0^-\}$. As expected from physical basis of the Hawking radiating black hole, the limiting zero mass and zero frequency black hole becomes highly self-contracting and much more intricate, for decreasing amplitude Hawking radiating black hole. As a matter of fact, the Fig.[12] shows that the limit of vanishing frequency makes a negative strength of the thermodynamic interaction of order 10^{21}. In this process. we observe further that the system acquires a double cut spike of thermodynamic instability, in the limit $|w| \to 0$.

As depicted in the Appendix-II, it is worth stating that the mathematical behavior of the thermodynamic metric tensor is given by the Hessian matrix of the concerned energy flux. In the case of the fluctuation of the energy flux, the mass and the radiation frequency of the Hawking radiating black hole form the co-ordinate chart on the intrinsic manifold of the parameters, which characterize the intrinsic geometric thermodynamic correlations. Our explicit computation shows that the exact set of correlations for both the constant amplitude and decaying amplitude energy flux of a Hawking radiating black holes is realized by employing the intrinsic geometric notion. Herewith, we have shown that the limiting thermodynamic correlations arising from the fluctuations of the energy flux demonstrate a simple structure in both cases, *viz,*, constant and decreasing amplitude Hawking radiating configurations. In a small neighborhood introduced over an ensemble of equilibrium configurations, the functional nature of parametric fluctuations may be exactly computed as the function of the radiation frequency and mass, with the form $\exp(8\pi M w)$.

Importantly, if one takes an account of the quantum gravity effects that the black holes would have remnant, and thus the above picture would change slightly, and one would have a rectified comprehension of the Hawking radiating black holes. This may offer a microscopic origin why such configurations are unstable with a high degree of finite self-attraction. A refined analysis would be the matter of future investigation, where one might take an account of the higher loop quantum gravity corrections arising from the path integral formulation of the gravity, beyond the saddle point, or the ones originating from the gravitational aspect of the string theory or D-brane ensemble, with higher order string length and Planck length contributions to the radiation energy flux. Thus, an appropriate notion of string theory or quantum gravity theory may offer a better thermodynamic understanding of the York type [16] decreasing amplitude Hawking radiating black holes. Specifically, it is important to mention that the underlying statistical correlation length of the underlying ensemble of decreasing amplitude Hawking radiating black holes might be anticipated to reduce to a nearly equilibrium system. Such unified issues are beyond the scope of the present investigation to fully anticipate the microscopic comprehension of the Hawking radiation, quantum gravity and black hole physics.

6 Conclusion and Outlook

Intrinsic geometric technology is well-suitable for analyzing the stability structure of Hawking radiation black holes, under the fluctuations of the radiation parameters. Such fluctuations are expected to arise due to non-zero heating effects, chemical reactions and possible conventional instabilities associated with the Hawking radiation, fluctuating horizon and quantum gravity corrections to the black hole configuration. The intrinsic geometric method is thus used to investigate the structure of the constant amplitude and decaying York amplitude ensembles, designed from the consideration of background space-time fluctuations. The present analysis is well suited for the statistical understanding of the formation of an equilibrium physical black hole. The quantum gravity characterization procedure is presented for both the constant amplitude radiation and decaying amplitude Hawking radiating black holes. The respective considerations are pictorially depicted for a black hole of (i) unit amplitude constant Hawking radiation and (ii) unit York strength decaying amplitude Hawking radiation.

In both types of horizon fluctuations, it turns out that the Hawking radiating black hole corresponds to an interacting statistical system. The global stability of the ensemble is determined from the positivity of the determinant of the metric tensor. In both of the above cases, this reduces to the positivity of an associated cubic equations for the entire domain of the mass and the frequency of Hawking radiation. In both cases, we find that the underlying configurations are globally non-interacting on the real roots of degree 10 and degree 12 equations in the mass, respectively. In the zero frequency limit, we observe further that both the constant amplitude and decaying amplitude Hawking radiating black holes are attractive from the thermodynamic perspective. However, the first one having a constant amplitude becomes unstable in the static frequency domain, while the second one does not. The above observation follows from the fact that the surface of the parameters has a negative determinant of the metric tensor for the first consideration, while it takes a positive definite value for the second case. Our observation thus supports that the decreasing amplitude York model [16] is a preferable choice, yielding an attractive and thermodynamically stable black hole.

Such an introduction is brought out from the consideration of background space-time fluctuations, yielding fluctuating mass and radiation frequency as the parameters for our intrinsic geometric analysis. In fact, the fluctuating horizon geometry shows an interesting property that the thermodynamic correlations are only the even function of the mass of the black hole. The conclusion is invariant, whether one considers the case of a constant amplitude radiation or the decreasing amplitude York type model. Our observation remains the same even after the incorporation of the non-perturbative logarithmic quantum gravity corrections to the chosen black hole horizon configurations. This shows the robustness of the present analysis. In the case of the fluctuating horizon configuration, the above robustness property follows from the Eqns.(6,7, 9,16,17,19). Interestingly, it is surprising to note that the corresponding observations remain intact under the quantum gravity logarithmic corrections in all possible domains of the mass and radiation frequency of the Hawking radiating black holes. It is evident from the corresponding quantum corrected thermodynamic metric tensors, determinants of the associated metric tensor and the respective scalar curvatures, *viz,*, Eqns.(25,26,28,37, 38,39).

In the dual energy flux representation, the nature of the generic thermodynamic correlations over an ensemble of Hawking radiating black hole, generated from the consideration of variable mass and variable radiation frequency, yields an expected statistical picture. Specifically, the present matter of intrinsic geometric affair demonstrates that the limiting thermodynamic correlations of a Hawking radiating black hole is well-defined in the frequency domain $w \in (0, \infty)$. To investigate the most general behavior of the fluctuations of the radiation energy flux, when both the mass and the frequency are allowed to fluctuate, the definition of Hessian function $Hess(F(M,w))$ of the energy flux $F(M,w)$ offers the limiting thermodynamic characterization of a Hawking radiating black hole. The exact local and global thermodynamic behavior of the parametric correlations are evident from the Eqns.(47,48, 49, 52, 53,54). Following these observations, the intrinsic geometric investigation concludes that the Hawking radiating black holes are thermodynamically stable in a specific domain of the radiating parameters and are interacting for both the constant and York type decaying amplitude. This offers an interesting platform to explore unified picture of the horizon fluctuations, background space-time quantum fluctuations and quantum statistical or limiting thermodynamic fluctuations.

With this understanding, our model is well suited and robust for the Hawking radiating black holes with and without quantum gravity effects. Such configurations are very popular nowadays because of their best compliance with the vacuum difficulties and renormalization group complications. These technicalities are prone to arise from the structure of the infalling and outgoing solutions of the D'Alembertian equations associated with the background fluctuations, *viz.* Vaidya space-time geometries. From the above viewpoints, our intrinsic geometric approach is thus very lucrative. It is worth mentioning that the intrinsic parametrization principle is rapidly growing in robust statistical theory and black hole physics, in the recent years. In fact, the present analysis is capable of accommodating the corresponding quantum considerations, as well. The present paper has explored these notions in detail and has additionally offered the associated pictorial depictions of the intrinsic geometric invariants for the unit Planck length, unit constant amplitude, unit York strength and unit logarithmic quantum gravity correction coefficient. In such considerations, the exact mathematical limiting formulae are obtained for the corresponding thermodynamic characterization of the black hole with and without the limiting zero Hawking radiation. From the viewpoints of the constant amplitude and decaying amplitude Hawking radiating configuration with and without the logarithmic quantum gravity correction, the present intrinsic geometric investigation shows that the nature of the stability structure is very illuminative for the other possible Hawking radiating black holes, and thus, our analysis offers a very prominent tool for understanding the limiting thermodynamic (un)stability structure, specifically for the most puzzling Schwarzschild black hole, since its discovery.

Applicative analysis is further in progress to validate the results with the conventional black holes arising from the modern gauge field theories, higher dimensional gravity and gravitational aspect of the (super)string theories. It is not surprising that the present perspective provides the limiting intrinsic geometric front to the stability structure of arbitrary finite parameter Hawking radiating black holes and their possible generalizations. Herewith, it may be anticipated that the nature of background disturbances, vacuum fluctuations, statistical fluctuations, limiting thermodynamic fluctuations and other perspective fluctuations can be suitably modeled in the diverse possible Hawking radiating black hole configurations. It follows from our intrinsic geometric method, that the present analysis offers perspective stability grounds, when applied to either the large mass or Planck size black holes. It is expected further that the present investigation would be an important factor in an appropriate understanding of the statistical indicators under the fluctuations of parameters, radiation frequency, quantum geometric invariants and the other possible higher loop components arising from the fluctuations of quantum gravity vacuum. Such explorations are the matter of a future investigation.

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Appendix-I: Coefficients in Quantum Corrections

In this appendix, we provide explicit forms of the thermodynamic correlation functions as the function of the radiation frequency for the Hawking radiating configuration. Our analysis illustrates that the physical properties of the specific correlations may easily be exactly exploited in general. The definite behavior of correlations as accounted in the concerned sections suggests that the various intriguing constant and decaying amplitude examples of radiating black hole solutions includes nice properties, i.e., that they do have a definite stability, except that the determinant may be non-positive definite in some cases at specific limit of the fluctuation and radiation frequency.

As mentioned in the main sections, these black hole configurations are generically well-defined and indicate an interacting statistical basis in general. We discover herewith that their thermodynamic geometries indicate the possible nature of general Hawking radiating black hole configurations. Significantly, we notice from the very definition of intrinsic metric tensor that the coefficients describing the thermodynamic correlations may be enlisted as follows.

Appendix (A): Constant Amplitude Horizon Fluctuations

Herewith, we provide the exact expressions for the constant amplitude thermodynamic correlations with an arbitrary fluctuating mass of the black hole. Explicitly, it turns out that the functional nature of parameters within a small neighborhood of statistical fluctuations introduced over an equilibrium ensemble of configurations may precisely be divulged. Surprisingly, we can expose in the framework of thermodynamic geometry that the coefficients of the constant amplitude thermodynamic correlations take the following exact and simple expressions.

Appendix (A1): Coefficients of the metric tensor

$$
\begin{aligned}
g_0^{cMM} &:= aL^2u^4 + 4aL^2u^2 + 4aL^2 \\
g_2^{cMM} &:= 144aL^2u^2w^2 - 24\pi u^2 - 2\pi u^6 - 12\pi u^4 + 32aL^2w^2u^4 - 16\pi + 160aL^2w^2 \\
g_4^{cMM} &:= -640\pi w^2 + 192aL^2u^4w^4 + 2560aL^2w^4 + 1408aL^2u^2w^4 + 48\pi u^6w^2 + 32\pi w^2u^4 - 448\pi u^2w^2 \\
g_6^{cMM} &:= 20480aL^2w^6 - 3584\pi u^2w^4 - 10240\pi w^4 + 768\pi u^4w^4 + 3072aL^2u^2w^6 \\
g_8^{cMM} &:= -20480\pi u^2w^6 - 8192aL^2u^2w^8 - 2048\pi w^6u^4 - 81920\pi w^6 + 81920aL^2w^8 \\
g_{10}^{cMM} &:= -327680\pi w^8 + 131072aL^2w^{10} - 65536\pi u^2w^8 \\
g_{12}^{cMM} &:= -524288\pi w^{10} \\
g_0^{cMw} &:= -aL^2u^2 - 2aL^2 \\
g_2^{cMw} &:= -4\pi u^4 - 16\pi - 8aL^2u^2w^2 - 16aL^2w^2 - 16\pi u^2 \\
g_4^{cMw} &:= 128aL^2w^4 - 128\pi u^2w^2 - 256\pi w^2 \\
g_6^{cMw} &:= 1024aL^2w^6 - 1024\pi w^4 \\
g_0^{cww} &:= -aL^2u^2 - 2aL^2 \\
g_2^{cww} &:= -2\pi u^4 - 8\pi + 16aL^2w^2 - 8\pi u^2 \\
g_4^{cww} &:= 128\pi u^2w^2 + 48\pi w^2u^4 + 64\pi w^2 + 64aL^2u^2w^4 + 640aL^2w^4 \\
g_6^{cww} &:= 3072aL^2w^6 + 2560\pi w^4 + 1536\pi u^2w^4 \\
g_8^{cww} &:= 12288\pi w^6
\end{aligned}
\tag{55}
$$

Appendix (A2): Coefficient of the determinant of the metric tensor

$$
\begin{aligned}
g_0^{c} &:= -a^2L^4u^4 - 4a^2L^4u^2 - 4a^2L^4 \\
g_2^{c} &:= -16a^2L^4u^2w^2 - 64a^2L^4w^2 + 8a^2L^4w^2u^4 \\
g_4^{c} &:= 320a^2L^4u^4w^4 + 192\pi u^6aL^2w^2 + 96\pi^2u^4 + 4\pi^2u^8 + 128\pi^2u^2 \\
&\quad +64\pi^2 + 768\pi u^4aL^2w^2 + 32\pi^2u^6 + 1280a^2L^4w^4 + 1024a^2L^4u^2w^4 \\
&\quad +768aL^2u^2\pi w^2 \\
g_6^{c} &:= 1024\pi^2w^2 + 11264\pi u^4aL^2w^4 + 1536\pi u^6aL^2w^4 + 1536a^2L^4w^6u^4 \\
&\quad +3840\pi^2u^2w^2 + 1856\pi^2u^6w^2 + 288\pi^2u^8w^2 + 10240a^2L^4u^2w^6 \\
&\quad +40960a^2L^4w^6 + 16384\pi aL^2u^2w^4 + 4224\pi^2u^4w^2 \\
g_8^{c} &:= 24576\pi^2u^2w^4 - 20480\pi^2w^4 + 39936\pi^2u^4w^4 + 16384a^2L^4u^2w^8 \\
&\quad +32768\pi u^4aL^2w^6 + 409600a^2L^4w^8 + 11264\pi^2u^6w^4 + 98304aL^2u^2\pi w^6 \\
g_{10}^{c} &:= -65536a^2L^4u^2w^{10} + 1835008a^2L^4w^{10} - 655360\pi^2w^6 + 24576\pi^2u^4w^6 \\
&\quad -65536aL^2u^4w^8\pi - 12288\pi^2u^6w^6 - 229376\pi^2u^2w^6 \\
g_{12}^{c} &:= -3145728\pi^2u^2w^8 - 6553600\pi^2w^8 + 3145728a^2L^4w^{12} \\
&\quad -589824\pi^2w^8u^4 - 1048576\pi w^{10}aL^2u^2 \\
g_{14}^{c} &:= -29360128\pi^2w^{10} - 9437184\pi^2w^{10}u^2 \\
g_{16}^{c} &:= -50331648\pi^2w^{12}
\end{aligned}
\tag{56}
$$

Appendix (A3): Coefficients of the scalar curvature

$$
\begin{aligned}
r_0^c &:= -8a^3L^6u^6 - 24a^3L^6u^4 - 16a^3L^6 - a^3L^6u^8 - 32a^3L^6u^2 \\
r_2^c &:= -6a^2L^4\pi u^{10} - 60a^2L^4\pi u^8 - 256a^3w^2L^6u^6 - 1024a^3w^2L^6u^2 \\
&\quad -480a^2L^4u^4\pi - 32a^3w^2L^6u^8 - 512a^3w^2L^6 - 192a^2\pi L^4 - 240a^2L^4\pi u^6 \\
&\quad -480a^2L^4\pi u^2 - 768a^3w^2L^6u^4 \\
r_4^c &:= -14336a^3w^4L^6u^2 - 12032a^3w^4L^6u^4 - 256a\pi^2L^2 - 2112a^2w^2L^4\pi u^8 \\
&\quad -15360a^2w^2\pi L^4 - 11136a^2w^2L^4\pi u^6 - 48aL^2\pi^2u^{10} - 6144a^3w^4L^6 \\
&\quad -144a^2w^2L^4\pi u^{10} - 640aL^2\pi^2u^6 - 768aL^2\pi^2u^2 - 4352a^3w^4L^6u^6 \\
&\quad -4aL^2\pi^2u^{12} - 960aL^2\pi^2u^4 - 33024a^2w^2L^4\pi u^2 - 576a^3w^4L^6u^8 \\
&\quad -240aL^2\pi^2u^8 - 27648a^2w^2L^4u^4\pi \\
r_6^c &:= 1344\pi^3u^10 + 8960\pi^3u^6 + 10752\pi^3u^4 + 224\pi^3u^12 + 4480\pi^3u^8 \\
&\quad -3584aw^2L^2\pi^2u^{10} - 1275904a^2w^4L^4\pi u^2 - 256aw^2L^2\pi^2u^{12} \\
&\quad -364032a^2w^4L^4\pi u^6 - 62720a^2w^4L^4\pi u^8 - 986112a^2w^4L^4u^4\pi \\
&\quad -20480aw^2L^2\pi^2u^8 - 61440aw^2L^2\pi^2u^6 + 16\pi^3u^{14} + 7168u^2\pi^3 \\
&\quad +2048\pi^3 - 32768aw^2\pi^2L^2 - 638976a^2w^4\pi L^4 - 77824a^3w^6L^6u^6 \\
&\quad -8192a^3w^6L^6u^8 - 278528a^3w^6L^6u^4 - 458752a^3w^6L^6u^2 \\
&\quad -102400aw^2L^2\pi^2u^4 - 90112aw^2L^2\pi^2u^2 - 3840a^2w^4L^4\pi u^{10} \\
&\quad -294912a^3w^6L^6 \\
r_8^c &:= 114688w^2\pi^3 - 185344aw^4L^2\pi^2u^{10} - 8646656a^2w^6L^4\pi u^6 \\
&\quad -980992aw^4L^2\pi^2u^8 - 4579328aw^4L^2\pi^2u^4 - 2809856aw^4L^2\pi^2u^6 \\
&\quad -67584a^2w^6L^4\pi u^{10} - 34226176a^2w^6L^4\pi u^2 - 4014080aw^4L^2\pi^2u^2 \\
&\quad -14848aw^4L^2\pi^2u^{12} - 1343488a^2w^6L^4\pi u^8 - 25100288a^2w^6L^4u^4\pi \\
&\quad -17891328a^2w^6\pi L^4 - 77824a^3w^8L^6u^8 - 16515072a^3w^8L^6u^2 \\
&\quad -1228800a^3w^8L^6u^6 - 1474560aw^4\pi^2L^2 - 6815744a^3w^8L^6u^4 \\
&\quad +352256w^2u^2\pi^3 + 128w^2\pi^3u^{14} + 454656w^2\pi^3u^4 + 3328w^2\pi^3u^{12} \\
&\quad +317440w^2\pi^3u^6 + 29184w^2\pi^3u^{10} + 128000w^2\pi^3u^8 - 14352384a^3w^8L^6 \\
r_{10}^c &:= 3538944w^4\pi^3 - 20709376aw^6L^2\pi^2u^8 - 424148992a^2w^8L^4u^4\pi \\
&\quad -637272064a^2w^8L^4\pi u^2 - 94896128aw^6L^2\pi^2u^2 - 107741184aw^6L^2\pi^2u^4 \\
&\quad -126976000a^2w^8L^4\pi u^6 - 15876096a^2w^8L^4\pi u^8 - 516096a^2w^8L^4\pi u^{10} \\
&\quad -34603008aw^6\pi^2L^2 - 99614720a^3w^{10}L^6u^4 - 12058624a^3w^{10}L^6u^6 \\
&\quad -393216a^3w^{10}L^6u^8 - 322961408a^3w^{10}L^6u^2 - 356253696a^2w^8\pi L^4 \\
&\quad -64094208aw^6L^2\pi^2u^6 - 3342336aw^6L^2\pi^2u^{10} - 196608aw^6L^2\pi^2u^{12} \\
&\quad -9216w^4\pi^3u^{14} - 86016w^4\pi^3u^{12} + 9043968w^4u^2\pi^3 - 196608w^4\pi^3u^{10} \\
&\quad +9043968w^4\pi^3u^4 + 614400w^4\pi^3u^8 + 4177920w^4\pi^3u^6 - 339738624a^3w^{10}L^6 \\
r_{12}^c &:= 97517568w^6\pi^3 - 1348468736aw^8L^2\pi^2u^2 - 1376256a^2w^{10}L^4\pi u^{10} \\
&\quad -1099694080a^2w^{10}L^4\pi u^6 - 24444928aw^8L^2\pi^2u^{10} - 4592762880a^2w^{10}L^4u^4\pi \\
&\quad -8088715264a^2w^{10}L^4\pi u^2 - 97779712a^2w^{10}L^4\pi u^8 - 5058330624a^2w^{10}\pi L^4 \\
&\quad -832569344a^3w^{12}L^6u^4 - 66060288a^3w^{12}L^6u^6 - 3640655872a^3w^{12}L^6u^2 \\
&\quad -786432a^3w^{12}L^6u^8 - 493879296aw^8\pi^2L^2 - 209977344aw^8L^2\pi^2u^8 \\
&\quad -737280aw^8L^2\pi^2u^{12} - 783810560aw^8L^2\pi^2u^6 - 1462239232aw^8L^2\pi^2u^4 \\
&\quad -4580179968a^3w^{12}L^6 - 73728w^6\pi^3u^{14} + 211812352w^6u^2\pi^3 + 166723584w^6\pi^3u^4 \\
&\quad -1409024w^6\pi^3u^{12} + 46530560w^6\pi^3u^6 - 7274496w^6\pi^3u^{10} - 7536640w^6\pi^3u^8 \\
r_{14}^c &:= -37849399296a^3w^{14}L^6 - 69172461568a^2w^{12}L^4\pi u^2 - 50734301184a^2w^{12}\pi L^4 \\
&\quad -5200936960aw^{10}L^2\pi^2u^6 - 1015021568aw^{10}L^2\pi^2u^8 - 12213813248aw^{10}L^2\pi^2u^2 \\
&\quad +3334471680w^8\pi^3u^4 - 3801088w^8\pi^3u^{12} - 5429526528a^2w^{12}L^4\pi u^6 + 969932800w^8\pi^3u^6 \\
&\quad -65011712aw^{10}L^2\pi^2u^{10} - 32243712w^8\pi^3u^{10} - 24964497408a^3w^{14}L^6u^2 \\
&\quad +2365587456w^8\pi^3 + 4638900224w^8u^2\pi^3 + 34078720w^8\pi^3u^8 - 4831838208aw^{10}\pi^2L^2 \\
&\quad -288358400a^2w^{12}L^4\pi u^8 - 184549376a^3w^{14}L^6u^6 - 11760828416aw^{10}L^2\pi^2u^4 \\
&\quad -31314673664a^2w^{12}L^4u^4\pi - 3959422976a^3w^{14}L^6u^4 \\
r_{16}^c &:= 42278584320w^{10}\pi^3 - 39460012032aw^{12}\pi^2L^2 - 190857609216a^3w^{16}L^6 \\
&\quad +144703488w^{10}\pi^3u^{10} - 201326592a^3w^{16}L^6u^6 + 74692165632w^{10}u^2\pi^3 \\
&\quad +2076180480w^{10}\pi^3u^8 + 49425678336w^{10}\pi^3u^4 + 9437184w^{10}\pi^3u^{12} \\
&\quad +14973665280w^{10}\pi^3u^6 - 301989888a^2w^{14}L^4\pi u^8 - 103616086016a^3w^{16}L^6u^2
\end{aligned}
$$

$$
\begin{aligned}
& -390439370752a^2w^{14}L^4\pi u^2 - 54358179840aw^{12}L^2\pi^2u^4 \\
& -1820327936aw^{12}L^2\pi^2u^8 - 12582912aw^{12}L^2\pi^2u^{10} \\
& -13774094336a^2w^{14}L^4\pi u^6 - 129117454336a^2w^{14}L^4u^4\pi \\
& -352724189184a^2w^{14}\pi L^4 - 74222403584aw^{12}L^2\pi^2u^2 \\
& -9797894144a^3w^{16}L^6u^4 - 16944988160aw^{12}L^2\pi^2u^6 \\
r^c_{18} \quad := \quad & 905969664w^{12}\pi^3u^10 + 121131499520w^{12}\pi^3u^6 - 343597383680aw^{14}L^2\pi^2u^2 \\
& -1384590082048a^2w^{16}L^4\pi u^2 + 15602810880w^{12}\pi^3u^8 + 796716433408w^{12}u^2\pi^3 \\
& +455803404288w^{12}\pi^3u^4 + 518617300992w^{12}\pi^3 - 1633161314304a^2w^{16}\pi L^4 \\
& -240518168576a^3w^{18}L^6u^2 + 536870912aw^{14}L^2\pi^2u^8 - 129922760704aw^{14}L^2\pi^2u^4 \\
& -15569256448aw^{14}L^2\pi^2u^6 - 288299679744a^2w^{16}L^4u^4\pi - 8589934592a^3w^{18}L^6u^4 \\
& -335007449088aw^{14}\pi^2L^2 - 12213813248a^2w^{16}L^4\pi u^6 - 523986010112a^3w^{18}L^6 \\
r^c_{20} \quad := \quad & 2486786064384w^{14}\pi^3u^4 - 377957122048a^3w^{20}L^6 - 4587025072128a^2w^{18}\pi L^4 \\
& +4294967296a^3w^{20}L^6u^4 - 240518168576a^2w^{18}L^4u^4\pi - 1541893259264aw^{16}L^2\pi^2u^2 \\
& -2735894167552a^2w^{18}L^4\pi u^2 + 4303557230592w^{14}\pi^3 + 805306368aw^{16}L^2\pi^2u^8 \\
& +5471788335104w^{14}u^2\pi^3 - 2744484102144aw^{16}\pi^2L^2 - 153545080832aw^{16}L^2\pi^2u^4 \\
& +36238786560w^{14}\pi^3u^8 + 25769803776aw^{16}L^2\pi^2u^6 - 240518168576a^3w^{20}L^6u^2 \\
& +485868175360w^{14}\pi^3u^6 + 5368709120a^2w^{18}L^4\pi u^6 \\
r^c_{22} \quad := \quad & -6734508720128aw^{18}L^2\pi^2u^2 + 773094113280w^{16}\pi^3u^6 \\
& -412316860416aw^{18}L^2\pi^2u^4 - 5772436045824a^2w^{20}\pi L^4 \\
& -2027224563712a^2w^{20}L^4\pi u^2 + 85899345920a^2w^{20}L^4u^4\pi \\
& -17042430230528aw^{18}\pi^2L^2 + 7395933683712w^{16}\pi^3u^4 \\
& +23811298689024w^{16}\pi^3 + 1649267441664a^3L^6w^{22} + 23347442221056w^{16}u^2\pi^3 \\
r^c_{24} \quad := \quad & 3298534883328a^3L^6w^{24} + 56487409876992w^{18}u^2\pi^3 \\
& -1236950581248aw^{20}L^2\pi^2u^4 + 824633720832a^2w^{22}L^4\pi u^2 \\
& -68719476736000aw^{20}\pi^2L^2 + 9277129359360w^{18}\pi^3u^4 \\
& +84387517431808w^{18}\pi^3 + 3298534883328a^2w^{22}\pi L^4 \\
& -20066087206912aw^{20}L^2\pi^2u^2 \\
r^c_{26} \quad := \quad & -26388279066624aw^{22}L^2\pi^2u^2 + 173722837188608w^{20}\pi^3 \\
& +13194139533312a^2\pi L^4w^{24} + 59373627899904w^{20}u^2\pi^3 \\
& -158329674399744aw^{22}\pi^2L^2 \\
r^c_{28} \quad := \quad & 158329674399744\pi^3w^{22} - 158329674399744a\pi^2L^2w^{24}
\end{aligned}
\tag{57}
$$

Appendix (B): Decaying Amplitude Horizon Fluctuations

In this case, we may again explicitly provide the exact expressions for the decaying amplitude thermodynamic correlations with an arbitrary fluctuating mass of the Hawking radiating black hole. It turns out that the functional nature of parameters, within a small neighborhood of statistical fluctuations introduced over an equilibrium ensemble of configurations, may precisely be divulged as in the case of the constant amplitude thermodynamic fluctuations. Similarly, we can expose in this framework that the coefficients of the decaying amplitude thermodynamic correlations take the following exact and simple expressions.

Appendix (B1): Coefficients of the metric tensor

$$
\begin{aligned}
g_0^{dMM} \quad := \quad & a^3m^4L^2 + 16a^5m^6L^2w^2 - \pi a^4m^4 - 16\pi a^6w^2m^6 \\
g_2^{dMM} \quad := \quad & am^2L^2 + 2\pi a^2m^2 + 112a^3m^4L^2w^2 + 768\pi a^6w^4m^6 - 16\pi w^2a^4m^4 \\
g_4^{dMM} \quad := \quad & -4096a^5m^6L^2w^6 + 4864a^3m^4L^2w^4 - \pi + 80am^2L^2w^2 - 1792\pi w^4a^4m^4 + 112\pi w^2a^2m^2 \\
g_6^{dMM} \quad := \quad & 3328\pi w^4a^2m^2 - 80\pi w^2 + 86016a^3m^4L^2w^6 + 2560am^2L^2w^4 - 28672\pi w^6a^4m^4 \\
g_8^{dMM} \quad := \quad & 40960am^2L^2w^6 + 524288a^3m^4L^2w^8 - 2560\pi w^4 + 53248\pi w^6a^2m^2 \\
g_{10}^{dMM} \quad := \quad & 327680\pi w^8a^2m^2 + 327680am^2L^2w^8 - 40960\pi w^6 \\
g_{12}^{dMM} \quad := \quad & -327680\pi w^8 + 1048576am^2L^2w^{10}
\end{aligned}
$$

$$
\begin{aligned}
g_{14}^{dMM} &:= -1048576\pi w^{10} \\
g_0^{dMw} &:= a^3m^4L^2 - \pi a^4m^4 \\
g_2^{dMw} &:= 16\pi w^2a^4m^4 + 2\pi a^2m^2 + 16a^3m^4L^2w^2 \\
g_4^{dMw} &:= -\pi + 32am^2L^2w^2 \\
g_6^{dMw} &:= -16\pi w^2 - 512\pi w^4a^2m^2 + 1024am^2L^2w^4 \\
g_8^{dMw} &:= 256\pi w^4 + 8192am^2L^2w^6 \\
g_{10}^{dMw} &:= 4096\pi w^6 \\
g_0^{dww} &:= a^3m^4L^2 - \pi a^4m^4 \\
g_2^{dww} &:= -am^2L^2 + 48\pi w^2a^4m^4 + 2\pi a^2m^2 \\
g_4^{dww} &:= -\pi - 256a^3m^4L^2w^4 + 16am^2L^2w^2 - 64\pi w^2a^2m^2 \\
g_6^{dww} &:= 1280am^2L^2w^4 + 16\pi w^2 - 1536\pi w^4a^2m^2 \\
g_8^{dww} &:= 1280\pi w^4 + 12288am^2L^2w^6 \\
g_{10}^{dww} &:= 12288\pi M^10w^6
\end{aligned} \tag{58}
$$

Appendix (B2): Coefficients of the determinant of the metric tensor

$$
\begin{aligned}
g_0^d &:= -2a^5m^6L^2\pi + \pi^2a^6m^6 + a^4m^6L^4 - 48a^8m^8\pi^2w^2 + 96a^7m^8\pi w^2L^2 - 48a^6m^8L^4w^2 \\
g_2^d &:= (-3a^4m^4\pi^2 + a^2m^4L^4 + 2a^3m^4L^2\pi + 128a^6m^6\pi^2w^2 + 32a^4m^6L^4w^2 \\
&\quad +1280a^8m^8\pi^2w^4 - 160a^5m^6L^2w^2\pi - 1280a^6m^8w^4L^4) \\
g_4^d &:= 2560a^5m^6L^2w^4\pi - 2560a^4m^6L^4w^4 + 3a^2m^2\pi^2 - 2816a^6m^6\pi^2w^4 - 144\pi^2w^2a^4m^4 \\
&\quad +32a^2m^4L^4w^2 + 96a^3m^4L^2w^2\pi - 4096a^6m^8L^4w^6 - 24576a^7m^8L^2w^6\pi) \\
g_6^d &:= (96a^2m^2\pi^2w^2 + 3840\pi^2w^4a^4m^4 - 1280a^2m^4L^4w^4 - 73728a^6m^6\pi^2w^6 \\
&\quad +65536a^6m^8w^8L^4 - 3072\pi w^4a^3m^4L^2 + 172032a^5m^6L^2w^6\pi - 131072a^4m^6L^4w^6 - \pi^2) \\
g_8^d &:= (-81920a^2m^4L^4w^6 - 212992\pi w^6a^3m^4L^2 - 3584a^2m^2\pi^2w^4 \\
&\quad +184320a^4m^4\pi^2w^6 - 2031616a^4m^6L^4w^8 - 32\pi^2w^2 + 1572864a^5m^6L^2w^8\pi) \\
g_{10}^d &:= (-3538944a^3m^4L^2w^8\pi - 10485760a^4m^6L^4w^10 - 1638400a^2m^4L^4w^8 \\
&\quad +1280\pi^2w^4 - 196608a^2m^2\pi^2w^6 + 1572864a^4m^4\pi^2w^8) \\
g_{12}^d &:= (-14680064a^2m^4L^4w^{10} - 2949120a^2m^2\pi^2w^8 - 18874368a^3m^4L^2w^{10}\pi + 81920\pi^2w^6) \\
g_{14}^d &:= (1638400\pi^2w^8 - 14680064a^2m^2\pi^2w^{10} - 50331648a^2m^4L^4w^{12}) \\
g_{16}^d &:= 14680064\pi^2w^{10} \\
g_{18}^d &:= 50331648\pi^2w^{12}
\end{aligned} \tag{59}
$$

Appendix (B3): Coefficients of the scalar curvature

$$
\begin{aligned}
r_0^d &:= 3a^{11}\pi^2m^{12}L^2 - 3a^{10}\pi m^{12}L^4 - a^{12}\pi^3m^{12} + a^9m^{12}L^6 \\
r_2^d &:= 128a^{12}\pi^3m^{12}w^2 + 6a^{10}\pi^3m^{10} + 10a^8\pi m^{10}L^4 - 14a^9\pi^2m^{10}L^2 \\
&\quad -288a^{11}\pi^2m^{12}L^2w^2 + 5120a^{13}\pi^2m^{14}L^2w^4 - 2048a^{14}\pi^3m^{14}w^4 \\
&\quad +1024a^{11}m^{14}L^6w^4 - 4096a^{12}\pi m^{14}L^4w^4 - 32a^9m^{12}L^6w^2 + 192a^{10}\pi m^{12}L^4w^2 \\
&\quad -2a^7m^{10}L^6 \\
r_4^d &:= -a^5m^8L^6 - 40704a^{11}\pi^2m^{12}L^2w^4 + 13056a^{12}\pi^3m^{12}w^4 - 384a^8\pi m^{10}L^4w^2 \\
&\quad +64a^7m^{10}L^6w^2 + 65536a^{14}\pi^3m^{14}w^6 + 992a^9\pi^2m^{10}L^2w^2 - 6a^6\pi m^8L^4 \\
&\quad -81920a^{12}\pi m^{14}L^4w^6 - 32768a^{13}\pi^2m^{14}L^2w^6 - 15a^8\pi^3m^8 - 12800a^9m^{12}L^6w^4 \\
&\quad +49152a^{11}m^{14}L^6w^6 - 672a^{10}\pi^3m^{10}w^2 + 22a^7\pi^2m^8L^2 + 40448a^{10}\pi m^{12}L^4w^4 \\
r_6^d &:= 1572864a^{14}\pi^3m^{14}w^8 + 2097152a^{12}\pi m^{14}L^4w^8 - 360448a^{12}\pi^3m^{12}w^6 \\
&\quad +26112a^7m^{10}L^6w^4 - 1280a^7\pi^2m^8L^2w^2 - 4194304a^{13}\pi^2m^{14}L^2w^8 \\
&\quad -12a^5\pi^2m^6L^2 - 102400a^8\pi m^{10}L^4w^4 + 32a^5m^8L^6w^2 - 2a^4\pi m^6L^4 \\
&\quad -47616a^{10}\pi^3m^{10}w^4 + 524288a^{11}m^{14}L^6w^8 + 128512a^9\pi^2m^{10}L^2w^4 \\
&\quad +20a^6\pi^3m^6 + 1081344a^{10}\pi m^{12}L^4w^6 + 1440a^8\pi^3m^8w^2 + 96a^6\pi m^8L^4w^2 \\
&\quad -49152a^{11}\pi^2m^{12}L^2w^6 - 647168a^9m^{12}L^6w^6
\end{aligned}
$$

$$
\begin{aligned}
r_8^d \quad := \quad & -15a^4\pi^3m^4 + 960a^5\pi^2m^6L^2w^2 + 909312a^{10}\pi^3m^{10}w^6 + 45416448a^{11}\pi^2m^{12}L^2w^8 \\
& +58720256a^{12}\pi m^{14}L^4w^{10} - 13893632a^{10}\pi m^{12}L^4w^8 + 1310720a^7m^{10}L^6w^6 + 110080a^8\pi^3m^8w^4 \\
& +64a^4\pi m^6L^4w^2 + 3840a^5m^8L^6w^4 + a^2\pi m^4L^4 + 876544a^9\pi^2m^{10}L^2w^6 - 96a^3m^6L^6w^2 \\
& -8388608a^{11}m^{14}L^6w^{10} - a^3\pi^2m^4L^2 - 3096576a^8\pi m^{10}L^4w^6 - 186368a^7\pi^2m^8L^2w^4 \\
& -18546688a^{12}\pi^3m^{12}w^8 + 64000a^6\pi m^8L^4w^4 - 1600a^6\pi^3m^6w^2 - 11010048a^9m^{12}L^6w^8 \\
& -25165824a^{13}\pi^2m^{14}L^2w^{10} \\
r_{10}^d \quad := \quad & 268435456a^{12}\pi m^{14}L^4w^{12} + 59506688a^8\pi m^{10}L^4w^8 - 150994944a^{12}\pi^3m^{12}w^{10} \\
& +125952a^5\pi^2m^6L^2w^4 - 960495616a^{10}\pi m^{12}L^4w^{10} + 2a\pi^2m^2L^2 + 79822848a^{10}\pi^3m^{10}w^8 \\
& +12320768a^7m^{10}L^6w^8 + 960a^4\pi^3m^4w^2 - 7680a^3m^6L^6w^4 + 11776a^4\pi m^6L^4w^4 \\
& -201326592a^{11}m^{14}L^6w^{12} + 937426944a^{11}\pi^2m^{12}L^2w^{10} - 32a^2\pi m^4L^4w^2 \\
& +48234496a^9m^{12}L^6w^{10} - 166592512a^9\pi^2m^{10}L^2w^8 + 2392064a^6\pi m^8L^4w^6 \\
& +6a^2\pi^3m^2 - 942080a^8\pi^3m^8w^6 - 163840a^5m^8L^6w^6 + 201326592a^{13}\pi^2m^{14}L^2w^{12} \\
& -608a^3\pi^2m^4L^2w^2 - 153600a^6\pi^3m^6w^4 - 1998848a^7\pi^2m^8L^2w^6 \\
r_{12}^d \quad := \quad & 425984a^4\pi m^6L^4w^6 - 1073741824a^{12}\pi m^{14}L^4w^{14} + 224a\pi^2m^2L^2w^2 \\
& -21102592a^5m^8L^6w^8 - 931135488a^7m^{10}L^6w^{10} + 5486149632a^{11}\pi^2m^{12}L^2w^{12} \\
& -37120a^3\pi^2m^4L^2w^4 - 327680a^6\pi^3m^6w^6 - 1073741824a^{11}m^{14}L^6w^{14} \\
& -14596177920a^{10}\pi m^{12}L^4w^{12} - 180879360a^8\pi^3m^8w^8 - 9216a^2\pi m^4L^4w^4 \\
& -56360960a^6\pi m^8L^4w^8 - 5475663872a^9\pi^2m^{10}L^2w^{10} + 282329088a^7\pi^2m^8L^2w^8 \\
& +123648a^4\pi^3m^4w^4 + 4328521728a^9m^{12}L^6w^{12} - 288a^2\pi^3m^2w^2 - 73728a^3m^6L^6w^6 \\
& +150994944a^{12}\pi^3m^{12}w^{12} + 1277165568a^{10}\pi^3m^{10}w^{10} + 1179648a^5\pi^2m^6L^2w^6 \\
& -\pi^3 + 5054136320a^8\pi m^{10}L^4w^{10} \\
r_{14}^d \quad := \quad & -38184943616a^7m^{10}L^6w^{12} + 61740154880a^9m^{12}L^6w^{14} + 4608a\pi^2m^2L^2w^4 \\
& +115695681536a^8\pi m^{10}L^4w^{12} - 792723456a^5m^8L^6w^{10} - 319488a^2\pi m^4L^4w^6 \\
& -4330618880a^8\pi^3m^8w^{10} + 14417920a^4\pi m^6L^4w^8 + 458752a^3\pi^2m^4L^2w^6 \\
& +32\pi^3w^2 - 68954357760a^9\pi^2m^{10}L^2w^{12} - 85899345920a^{10}\pi m^{12}L^4w^{14} \\
& +12322865152a^7\pi^2m^8L^2w^{10} + 1572864a^4\pi^3m^4w^6 + 12976128a^3m^6L^6w^8 \\
& +230686720a^6\pi^3m^6w^8 - 52736a^2\pi^3m^2w^4 - 5968494592a^6\pi m^8L^4w^{10} \\
& +5133828096a^{10}\pi^3m^{10}w^{1}2 - 246939648a^5\pi^2m^6L^2w^8 \\
r_{16}^d \quad := \quad & 117309440a^3\pi^2m^4L^2w^8 - 177905598464a^6\pi m^8L^4w^{12} + 6094848a^2\pi m^4L^4w^8 \\
& -38084280320a^8\pi^3m^8w^{12} + 9216\pi^3w^4 - 678604832768a^7m^{10}L^6w^{14} - 15737028608a^5m^8L^6w^{12} \\
& +373662154752a^9m^{12}L^6w^{16} + 223472517120a^7\pi^2m^8L^2w^{12} - 12255756288a^5\pi^2m^6L^2w^{10} \\
& -355945414656a^9\pi^2m^{10}L^2w^{14} + 763363328a^4\pi m^6L^4w^{10} + 7675576320a^6\pi^3m^6w^{10} \\
& +1293858897920a^8\pi m^{10}L^4w^{14} - 1236992a^2\pi^3m^2w^6 - 141733920768a^{10}\pi m^{12}L^4w^{16} \\
& -160825344a^4\pi^3m^4w^8 - 14495514624a^{10}\pi^3m^{10}w^{14} - 417792a\pi^2m^2L^2w^6 \\
& +641728512a^3m^6L^6w^{10} \\
r_{18}^d \quad := \quad & -56371445760a^8\pi^3m^8w^{14} + 824633720832a^9m^{12}L^6w^{18} + 717225984a^2\pi m^4L^4w^{10} \\
& +2037961981952a^7\pi^2m^8L^2w^{14} - 7423918080a^4\pi^3m^4w^{10} + 54263808a^2\pi^3m^2w^8 \\
& +319488\pi^3w^6 + 4961861632a^3\pi^2m^4L^2w^{10} + 12683575296a^3m^6L^6w^{12} - 26738688a\pi^2m^2L^2w^8 \\
& -6571299962880a^7m^{10}L^6w^{16} + 7189775253504a^8\pi m^{10}L^4w^{1}6 - 463856467968a^9\pi^2m^{10}L^2w^{16} \\
& -2780991324160a^6\pi m^8L^4w^{14} - 271790899200a^5\pi^2m^6L^2w^{12} - 182536110080a^5m^8L^6w^{14} \\
& +28655484928a^4\pi m^6L^4w^{12} + 104018739200a^6\pi^3m^6w^{12} \\
r_{20}^d \quad := \quad & -24592982736896a^6\pi m^8L^4w^{16} - 3298534883328a^5\pi^2m^6L^2w^{14} \\
& +579820584960a^8\pi^3m^8w^{16} + 15668040695808a^8\pi m^{10}L^4w^{1}8 + 536870912000a^6\pi^3m^6w^{14} \\
& -490733568a\pi^2m^2L^2w^{10} - 33809982554112a^7m^{10}L^6w^{18} + 104522055680a^3\pi^2m^4L^2w^{12} \\
& +8684423872512a^7\pi^2m^8L^2w^{16} + 3670016000a^2\pi^3m^2w^{10} - 6094848\pi^3w^8 \\
& +67645734912a^3m^6L^6w^{14} + 22548578304a^2\pi m^4L^4w^{12} + 627065225216a^4\pi m^6L^4w^{14} \\
& -1224065679360a^5m^8L^6w^{16} - 138412032000a^4\pi^3m^4w^{12} \\
r_{22}^d \quad := \quad & -116135915683840a^6\pi m^8L^4w^{18} + 11544872091648a^7\pi^2m^8L^2w^{18} \\
& +8134668058624a^4\pi m^6L^4w^{16} - 773094113280a^6\pi^3m^6w^{16} + 7046430720a\pi^2m^2L^2w^{12} \\
& +370440929280a^2\pi m^4L^4w^{14} + 89724551168a^2\pi^3m^2w^{12} - 717225984\pi^3w^{10} \\
& -1211180777472a^4\pi^3m^4w^{14} - 21698174779392a^5\pi^2m^6L^2w^{16} - 4260607557632a^5m^8L^6w^{18} \\
& -72567767433216a^7m^{10}L^6w^{20} + 1170378588160a^3\pi^2m^4L^2w^{14} - 2010044694528a^3m^6L^6w^{16} \\
r_{24}^d \quad := \quad & -22548578304\pi^3w^{12} - 226499395321856a^6\pi m^8L^4w^{20} + 537944653824a\pi^2m^2L^2w^{14} \\
& -12369505812480a^6\pi^3m^6w^{18} + 61847529062400a^4\pi m^6L^4w^{18} - 48241072668672a^3m^6L^6w^{18}
\end{aligned}
$$

$$
\begin{aligned}
& +5098126180352a^3\pi^2m^4L^2w^{16} - 2796023709696a^4\pi^3m^4w^{16} - 5497558138880a^5m^8L^6w^{20} \\
& +3208340570112a^2\pi m^4L^4w^{16} + 1117765238784a^2\pi^3m^2w^{14} - 65146063945728a^5\pi^2m^6L^2w^{18} \\
r^d_{26} \; := \; & 27212912787456a^4\pi^3m^4w^{18} - 370440929280\pi^3w^{14} + 252887674388480a^4\pi m^6L^4w^{20} \\
& -39582418599936a^5\pi^2m^6L^2w^{20} - 481586092965888a^3m^6L^6w^{20} + 6176162971648a^2\pi^3m^2w^{16} \\
& +8108898254848a^2\pi m^4L^4w^{18} - 26800595927040a^3\pi^2m^4L^2w^{18} + 12446815223808a\pi^2m^2L^2w^{16} \\
r^d_{28} \; := \; & 422212465065984a^4\pi m^6L^4w^{22} + 160941014515712a\pi^2m^2L^2w^{18} \\
& -2427721674129408a^3m^6L^6w^{22} - 3208340570112\pi^3w^{16} + 148434069749760a^4\pi^3m^4w^{20} \\
& -8383776161792a^2\pi^3m^2w^{18} - 392525651116032a^3\pi^2m^4L^2w^{20} \\
& -101155069755392a^2\pi m^4L^4w^{20} \\
r^d_{30} \; := \; & -949978046398464a^2\pi m^4L^4w^{22} - 8108898254848\pi^3w^{18} \\
& +1242448139386880a\pi^2m^2L^2w^{20} - 1266637395197952a^3\pi^2m^4L^2w^{22} \\
& -270479860432896a^2\pi^3m^2w^{20} - 5066549580791808a^3m^6L^6w^{24} \\
r^d_{32} \; := \; & 101155069755392\pi^3w^{20} + 5383208929591296a\pi^2m^2L^2w^{22} \\
& -949978046398464a^2\pi^3m^2w^{22} - 2533274790395904a^2\pi m^4L^4w^{24} \\
r^d_{34} \; := \; & 10133099161583616a\pi^2m^2L^2w^24 + 949978046398464\pi^3w^{22} \\
r^d_{36} \; := \; & 2533274790395904\pi^3w^{24}
\end{aligned}
\tag{60}
$$

Appendix-II: Coefficients in Hawking Radiation Flux

As stated earlier in the main section of the paper, the thermodynamic metric tensor in the case of the fluctuation of the energy flux is given by the Hessian matrix of the concerned energy flux. In this case, the mass and the radiation frequency of the Hawking radiating black hole configuration are respected to be the intrinsic variables. Herewith, the energy flux fluctuations indicate that the above two distinct parameters characterize the intrinsic thermodynamic correlation functions. The present computation shows that the exact set of correlations for both the constant and decaying amplitude Hawking radiating black holes are realized by employing the previously defined notations. Herewith, we may easily enlist the coefficients of the thermodynamic correlations arising from the fluctuations of the energy flux of a Hawking radiating black hole. Both of the cases, *viz,*, the constant amplitude and the decreasing amplitude configurations demonstrate that the coefficients of the thermodynamic correlations simplify to the following set of simple expressions.

Appendix (C): Constant Amplitude Flux Fluctuations

In this appendix, we enlist explicitly expressions for the constant amplitude thermodynamic correlations arising from the thermodynamic fluctuations of the energy flux with an arbitrary fluctuating mass and radiation frequency of a Hawking radiating black hole. The functional nature of parameters, within a small neighborhood of fluctuations introduced over an equilibrium ensemble of configurations, may precisely be divulged as the function of the radiation frequency and mass, with an exponential form $\exp(8\pi Mw)$. In the present framework, we can expose that the coefficients of the constant amplitude Hawking radiating black hole thermodynamic local and global correlations take the following set of exact and extremely simple expressions.

Appendix (C1): Coefficients of the metric tensor

$$
\begin{aligned}
c_0^{cMM} &:= 3\exp(24\pi Mw) - 9\exp(16\pi Mw) + 9\exp(8\pi Mw) - 3 \\
c_1^{cMM} &:= 4u^2\pi w\exp(16\pi Mw) - 8u^2\pi w\exp(8\pi Mw) + 4u^2\pi w \\
c_2^{cMM} &:= 32u^2\pi^2w^2\exp(16\pi Mw) - 32u^2\pi^2w^2\exp(8\pi Mw) \\
c_3^{cMM} &:= 128u^2\pi^3w^3\exp(16\pi Mw) + 128u^2\pi^3w^3\exp(8\pi Mw) \\
c_0^{cMw} &:= \exp(16\pi Mw) + 2\exp(8\pi Mw) - 1
\end{aligned}
$$

$$
\begin{aligned}
c_1^{cMw} &:= -8\pi w \exp(16\pi M w) + 8\pi w \exp(8\pi M w) \\
c_2^{cMw} &:= 64\pi^2 w^2 \exp(16\pi M w) + 64\pi^2 w^2 \exp(8\pi M w) \\
c_0^{cww} &:= \exp(8\pi M w) + 1) \\
c_1^{cww} &:= 4\pi w \exp(8\pi M w) + 4\pi w
\end{aligned} \tag{61}
$$

Appendix (C2): Coefficients of the determinant of the metric tensor

$$
\begin{aligned}
h_0^c &:= -24\exp(32\pi M w) - 3u^2\exp(8\pi M w) - u^2\exp(24\pi M w) + u^2 + 24\exp(8\pi M w) \\
&\quad +3u^2\exp(16\pi M w) + 72\exp(24\pi M w) - 72\exp(16\pi M w) \\
h_1^c &:= -48u^2\pi w\exp(24\pi M w) + 96\pi w\exp(8\pi M w) + 96u^2\pi w\exp(16\pi M w) \\
&\quad -96\pi w\exp(16\pi M w) - 96\exp(24\pi M w)\pi w + 96\exp(32\pi M w)\pi w - 48u^2\pi w\exp(8\pi M w) \\
h_2^c &:= -64u^2\pi^2 w^2\exp(24\pi M w) + 320u^2\pi^2 w^2\exp(16\pi M w) - 256u^2\pi^2 w^2\exp(8\pi M w) \\
h_3^c &:= 1024u^2\pi^3 w^3\exp(16\pi M w) + 1024u^2\pi^3 w^3\exp(24\pi M w)
\end{aligned} \tag{62}
$$

Appendix (C3): Coefficients of the scalar curvature

$$
\begin{aligned}
l_0^c &:= 9 - u^2 + 81\exp(40\pi M w) - 45\exp(48\pi M w) + 9\exp(56\pi M w) + 20u^2\exp(24\pi M w) \\
&\quad -45\exp(8\pi M w) - 15u^2\exp(16\pi M w) + 6u^2\exp(40\pi M w) - 45\exp(32\pi M w) \\
&\quad -\exp(48\pi M w)u^2 - 45\exp(24\pi M w) + 81\exp(16\pi M w) + 6u^2\exp(8\pi M w) \\
&\quad -15u^2\exp(32\pi M w) \\
l_1^c &:= 24\pi w - 16u^2\pi w + 888\exp(48\pi M w)\pi w + 1680\exp(24\pi M w)\pi w \\
&\quad +192\pi w\exp(8\pi M w) - 240\exp(56\pi M w)\pi w - 1080\pi w\exp(16\pi M w) \\
&\quad +160u^2\pi w\exp(24\pi M w) - 160u^2\pi w\exp(16\pi M w) - 864\exp(40\pi M w)\pi w \\
&\quad -80u^2\pi w\exp(32\pi M w) + 80u^2\pi w\exp(8\pi M w) - 600\exp(32\pi M w)\pi w \\
&\quad +16u^2\pi w\exp(40\pi M w) \\
l_2^c &:= 1152\pi^2 w^2\exp(8\pi M w) - 448u^2\pi^2 w^2\exp(24\pi M w) - 5760\exp(40\pi M w)\pi^2 w^2 \\
&\quad -32u^2\pi^2 w^2 + 11520\exp(32\pi M w)\pi^2 w^2 + 512u^2\pi^2 w^2\exp(16\pi M w) \\
&\quad -288u^2\pi^2 w^2\exp(32\pi M w) - 1152\pi^2 w^2\exp(48\pi M w) - 5760\exp(24\pi M w)\pi^2 w^2 \\
&\quad +1152\exp(56\pi M w)\pi^2 w^2 + 544u^2\pi^2 w^2\exp(40\pi M w) - 1152\pi^2 w^2\exp(16\pi M w) \\
&\quad -192u^2\pi^2 w^2\exp(48\pi M w) - 96u^2\pi^2 w^2\exp(8\pi M w) \\
l_3^c &:= 1536\pi^3 w^3\exp(8\pi M w) - 3072\exp(48\pi M w)\pi^3 w^3 + 3072\pi^3 w^3\exp(16\pi M w) \\
&\quad -1536\exp(56\pi M w)\pi^3 w^3 - 8192u^2\pi^3 w^3\exp(32\pi M w) - 10752\pi^3 w^3\exp(24\pi M w) \\
&\quad +2048u^2\pi^3 w^3\exp(16\pi M w) - 1792u^2\pi^3 w^3\exp(8\pi M w) + 3328u^2\exp(40\pi M w)\pi^3 w^3 \\
&\quad +4608u^2\pi^3 w^3\exp(24\pi M w) + 10752\exp(40\pi M w)\pi^3 w^3 \\
l_4^c &:= -6144u^2\pi^4 w^4\exp(16\pi M w) - 28672u^2\pi^4 w^4\exp(32\pi M w) + 4096u^2\pi^4 w^4\exp(40\pi M w) \\
&\quad -4096u^2\pi^4 w^4\exp(8\pi M w) + 2048u^2\pi^4 w^4\exp(48\pi M w) + 32768u^2\pi^4 w^4\exp(24\pi M w) \\
l_5^c &:= -32768\pi^5 w^5\exp(40\pi M w)u^2 + 32768\pi^5 w^5\exp(24\pi M w)u^2
\end{aligned} \tag{63}
$$

Appendix (D): Decaying Amplitude Horizon Fluctuations

In the framework of the intrinsic geometry, we notice that we can explicitly provide the exact expressions for the decaying amplitude thermodynamic correlations arising from the fluctuations of the energy flux with an arbitrary fluctuating mass of the Hawking radiating black hole. Within a small neighborhood of statistical fluctuations introduced over an equilibrium ensemble of configurations, the functional nature of parameters may precisely be computed as in the case of the constant amplitude thermodynamic fluctuations of the energy flux. In this appendix, we enlist the coefficients of the thermodynamic correlations arising from the decaying amplitude energy flux of a Hawking radiating black hole and thus show that they take the following exact and simple expressions.

Appendix (D1): Coefficients of the metric tensor

$$
\begin{aligned}
c_0^{dMM} &:= 24a^2m^2\pi w\exp(16\pi Mw) - 48a^2m^2\pi w\exp(8\pi Mw) + 24a^2m^2\pi w \\
c_1^{dMM} &:= -3 + 3\exp(24\pi Mw) - 9\exp(16\pi Mw) + 9\exp(8\pi Mw) \\
&\quad +96a^2m^2\pi^2w^2\exp(16\pi Mw) - 96a^2m^2\pi^2w^2\exp(8\pi Mw) \\
c_2^{dMM} &:= 128a^2m^2\pi^3w^3\exp(16\pi Mw) + 128a^2m^2\pi^3w^3\exp(8\pi Mw) \\
c_0^{dMw} &:= 3\exp(16\pi Mw) - 6\exp(8\pi Mw) + 3 \\
c_1^{dMw} &:= 8\pi w\exp(16\pi Mw) - 8\pi w\exp(8\pi Mw) \\
c_2^{dMw} &:= 64\pi^2w^2\exp(16\pi Mw) + 64\pi^2w^2\exp(8\pi Mw) \\
c_0^{dMw} &:= 1 - \exp(8\pi Mw) \\
c_1^{dww} &:= 4\pi w\exp(8\pi Mw) + 4\pi w
\end{aligned} \tag{64}
$$

Appendix (D2): Coefficients of the determinant of the metric tensor

$$
\begin{aligned}
h_0^d &:= -9a^2m^2\exp(24\pi Mw) - 27a^2m^2\exp(8\pi Mw) \\
&\quad +9a^2m^2 + 27a^2m^2\exp(16\pi Mw)) \\
h_1^d &:= 288\exp(16\pi Mw)a^2m^2\pi w - 144\exp(24\pi Mw)a^2m^2\pi w \\
&\quad -144\exp(8\pi Mw)a^2m^2\pi w \\
h_2^d &:= 832\exp(16\pi Mw)a^2m^2\pi^2w^2 + 72\exp(24\pi Mw) - 72\exp(16\pi Mw) \\
&\quad +320\exp(24\pi Mw)a^2m^2\pi^2w^2 - 1152\exp(8\pi Mw)a^2m^2\pi^2w^2 \\
&\quad -24\exp(32\pi Mw) + 24\exp(8\pi Mw) \\
h_3^d &:= 96\exp(32\pi Mw)\pi w - 96\pi w\exp(16\pi Mw) + 96\pi w\exp(8\pi Mw) \\
&\quad +1024\exp(16\pi Mw)a^2m^2\pi^3w^3 + 1024\exp(24\pi Mw)a^2m^2\pi^3w^3 \\
&\quad -96\exp(24\pi Mw)\pi w
\end{aligned} \tag{65}
$$

Appendix (D3): Coefficients of the scalar curvature

$$
\begin{aligned}
l_0^d &:= 54a^2m^2\exp(40\pi Mw) - 9a^2m^2 - 9a^2m^2\exp(48\pi Mw) + 180a^2m^2\exp(24\pi Mw) \\
&\quad -135a^2m^2\exp(16\pi Mw) - 135a^2m^2\exp(32\pi Mw) + 54a^2m^2\exp(8\pi Mw) \\
l_1^d &:= -144a^2m^2\pi w + 1680\exp(24\pi Mw)a^2m^2\pi w - 1560\exp(16\pi Mw)a^2m^2\pi w \\
&\quad -960\exp(32\pi Mw)a^2m^2\pi w + 744\exp(8\pi Mw)a^2m^2\pi w + 264\exp(40\pi Mw)a^2m^2\pi w \\
&\quad -24a^2m^2\exp(48\pi Mw)\pi w \\
l_2^d &:= -288a^2m^2\exp(48\pi Mw)\pi^2w^2 - 180\exp(24\pi Mw) - 12\exp(8\pi Mw) \\
&\quad +768\exp(8\pi Mw)a^2m^2\pi^2w^2 + 768\exp(40\pi Mw)a^2m^2\pi^2w^2 \\
&\quad -180\exp(40\pi Mw) + 72\exp(16\pi Mw) - 12\exp(56\pi Mw) + 72\exp(48\pi Mw) \\
&\quad -480\exp(32\pi Mw)a^2m^2\pi^2w^2 - 288a^2m^2\pi^2w^2 + 240\exp(32\pi Mw) \\
&\quad -480\exp(16\pi Mw)a^2m^2\pi^2w^2 \\
l_3^d &:= -1536a^2m^2\pi^3w^3\exp(8\pi Mw) - 21504\exp(32\pi Mw)a^2m^2\pi^3w^3 \\
&\quad +24\exp(56\pi Mw)\pi w + 96\pi w\exp(16\pi Mw) - 3584\exp(16\pi Mw)a^2m^2\pi^3w^3 \\
&\quad -120\exp(24\pi Mw)\pi w + 19456\exp(24\pi Mw)a^2m^2\pi^3w^3 - 96\exp(48\pi Mw)\pi w \\
&\quad +120\exp(40\pi Mw)\pi w + 6656\exp(40\pi Mw)a^2m^2\pi^3w^3 \\
&\quad +512a^2m^2\exp(48\pi Mw)\pi^3w^3 - 24\pi w\exp(8\pi Mw) \\
l_4^d &:= 3456\exp(32\pi Mw)\pi^2w^2 - 192\exp(48\pi Mw)\pi^2w^2 \\
&\quad -40960\exp(32\pi Mw)a^2m^2\pi^4w^4 + 65536\exp(24\pi Mw)a^2m^2\pi^4w^4 \\
&\quad +1024\exp(40\pi Mw)a^2m^2\pi^4w^4 + 288\pi^2w^2\exp(8\pi Mw) \\
&\quad -1824\exp(40\pi Mw)\pi^2w^2 + 288\exp(56\pi Mw)\pi^2w^2 \\
&\quad +1024\exp(48\pi Mw)a^2m^2\pi^4w^4 - 17408\exp(16\pi Mw)a^2m^2\pi^4w^4 \\
&\quad -1824\exp(24\pi Mw)\pi^2w^2 - 9216a^2m^2\pi^4w^4\exp(8\pi Mw) \\
&\quad -192\exp(16\pi Mw)\pi^2w^2
\end{aligned}
$$

$$
\begin{aligned}
l_5^d \quad := \quad & 16384\exp(24\pi Mw)\pi^5 w^5 a^2 m^2 - 768\exp(56\pi Mw)\pi^3 w^3 \\
& -16384\exp(40\pi Mw)\pi^5 w^5 a^2 m^2 + 5376\exp(40\pi Mw)\pi^3 w^3 \\
& -1536\exp(48\pi Mw)\pi^3 w^3 + 1536\exp(16\pi Mw)\pi^3 w^3 \\
& -5376\exp(24\pi Mw)\pi^3 w^3 + 768\pi^3 w^3 \exp(8\pi Mw)
\end{aligned} \tag{66}
$$

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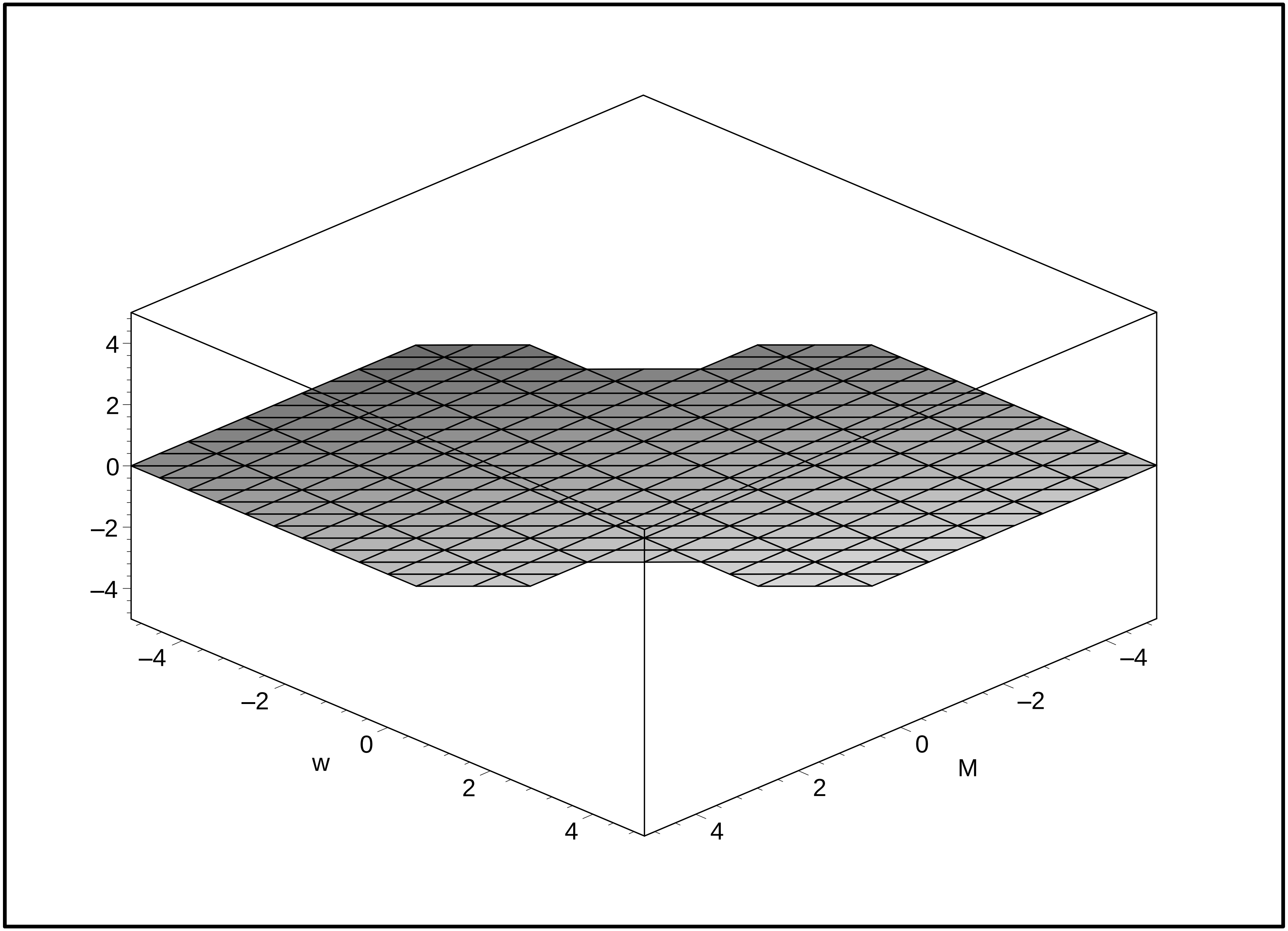
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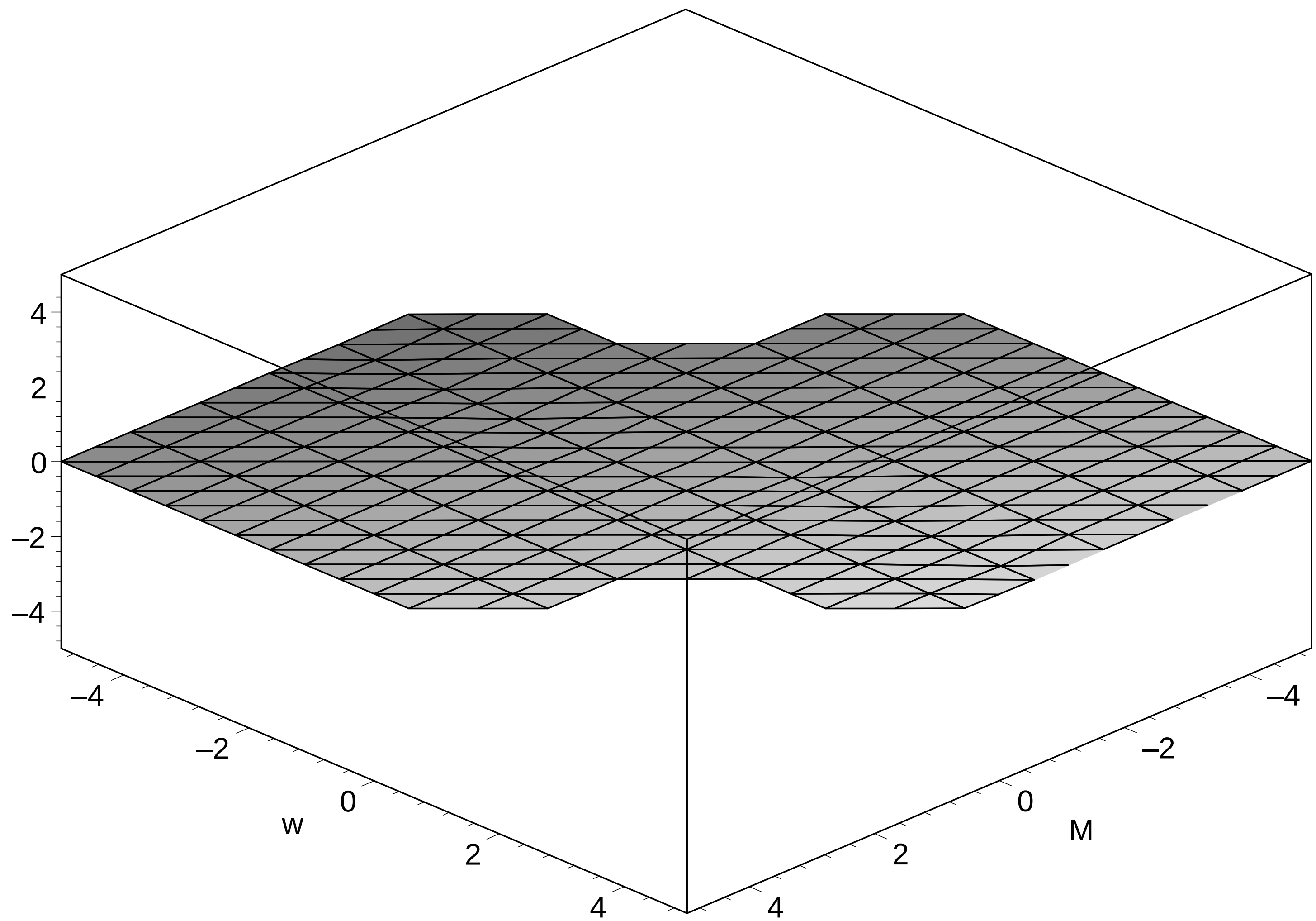

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