Bi-presymplectic representation of Liouville integrable systems and related separability theory

Maciej Błaszak

September 2, 2010

Division of Mathematical Physics, A. Mickiewicz University, Umultowska 85 , 61-614 Poznan, Poland, e-mail blaszakm@amu.edu.pl

Abstract

Bi-presymplectic chains of one-forms of arbitrary co-rank are considered. The conditions in which such chains represent some Liouville integrable systems and the conditions in which there exist related bi-Hamiltonian chains of vector fields are presented. In order to derived the construction of bi-presymplectic chains, the notions of dual Poisson-presymplectic pair, d-compatibility of presymplectic forms and d-compatibility of Poisson bivectors is used. The completely algorithmic construction of separation coordinates is demonstrated. It is also proved that Stäckel separable systems have bi-inverse-Hamiltonian representation, i.e. are represented by bi-presymplectic chains of closed one-forms. The co-rank of related structures depends on the explicit form of separation relations.

1 Introduction

The theory of finite dimensional conservative integrable systems has a long history, starting from the works of Lagrange, Hamilton and Jacobi in the first half of XIX century. In fact the Hamilton-Jacobi (HJ) theory is one of the most powerful methods of integration by quadratures a wide class of systems described by nonlinear ordinary differential equations, with a long history as a part of analytical mechanics. The theory in question is closely related to the Liouville integrable Hamiltonian systems. The main difficulty of the HJ approach is that it demands a distinguished coordinates, so called *separation coordinates*, in order to work effectively.

There are two efficient and systematic methods of construction of separation variables for dynamical systems. The first one bases on Lax representation and r-matrix theory for derivation of separation coordinates [1]. In this approach the integrals of motion in involution appear as coefficients of characteristic equation (*spectral curve*) of the Lax matrix. This method was successfully applied for separating variables in many integrable systems [1]-[6]. The other one is a geometric separability theory on bi-Poissonian manifolds [7]-[14], related to the so-called Gel'fand-Zakharevich (GZ) bi-Hamiltonian systems [15, 16]. In this approach the constants of motion are closely related to the so-called *separation curve* which is intimately related to the Stäckel separation relations.

The bi-Poissonian formulation of finite dimensional integrable Hamiltonian systems has been systematically developed for the last two decades (see [17] and the literature quoted there). It has been found that most of the known Liouville integrable finite dimensional systems have more then one Hamiltonian representation. Moreover, in the majority of known cases, both Poisson structures of a given flow are degenerated. For such systems, related bi-Poissonian (bi-Hamiltonian) commuting vector fields belong to one or more bi-Hamiltonian chains starting and terminating with Casimirs of respective Poisson structures. The most important aspect of such a construction is its relation to the geometric separability theory. Having a bi-Hamiltonian representation of a given system, the sufficient condition for the existence of separation coordinates is the reducibility of one of the Poisson structures onto a symplectic leaf of the other one. Unfortunately, this procedure is non-algoritmic and has to be considered independently from case to case. Moreover, we do not have a proof that it is always possible for any GZ system. Anyway, once the reduction is done, the remaining procedure of the construction of separation coordinates is almost algoritmic. The relevance of bi-Hamiltonian formalism in separability theory was recently confirmed in [18], where it was proved that arbitrary Stäckel system, defined by an appropriate separation relations, has a bi-Hamiltonian extension.

On the other hand, it is well known from the classical mechanics, that if the Poisson structure is nondegenerate, i.e. if the rank of the Poisson tensor is equal to the dimension of a phase space, then the phase space becomes a symplectic manifold with a symplectic structure being just the inverse of the Poisson structure. In such a case there exists an alternative (dual) description of Hamiltonian vector fields in the language of symplectic geometry. So, a natural question arises, whether one can construct such a dual picture in the degenerated case, when there is no natural inverse of the Poisson tensor [19]. For such tensors the notion of dual presymplectic structures was developed in [20, 21].

The presymplectic picture is especially interesting in the case of Liouville integrable systems. As was mentioned above, there is well developed bi-Hamiltonian theory of such systems, based on Poisson pencils of the GZ type, with polynomial in pencil parameter Casimir functions and related separability theory. The important question is whether it is possible to formulate an independent, alternative bi-presymplectic (bi-inverse-Hamiltonian in particular) theory of Liouville integrable systems with related separability theory and how both theories are related to each other.

The advantage of formalism presented is as follows. In the bi-Hamiltonian approach the existence of bi-Hamiltonian representation of a given flow is a necessary condition of separability but not a sufficient one. Contrary, the existence of bi-presymplectic representation of a flow considered is a sufficient condition of separability. Moreover, the construction of separation coordinates is a fully algorithmic procedure (in a generic case obviously), as the restriction of both presymplectic structures to any leaf of a given foliation always exists and is a simple task. For this reason the new formalism presented in the paper seems to be relevant for the modern separability theory.

The present paper develops the general bi-presymplectic theory of Liouville integrable systems when the co-rank of presymplectic forms is arbitrary. The whole formalism is based on the notion of *d*-

compatibility of presymplectic forms and *d-compatibility* of Poisson bivectors. Some elements of that formalism was presented in papers [21, 22]. Here we present a complete picture. Finally it is shown that any Stäckel system, defined by an appropriate separation relations, has a bi-inverse-Hamiltonian representation, what confirms the relevance of presented formalism.

The paper is organized as follows. In section 2 we give some basic information on Poisson tensors, presymplectic two-forms, Hamiltonian and inverse Hamiltonian vector fields and dual Poisson-presymplectic pairs. In sections 3 the concept of d-compatibility of Poisson bivectors and d-compatibility of closed two-forms is developed. Then, in section 4, the main properties of bi-presymplectic chains of arbitrary co-rank are investigated. The conditions in which the bi-presymplectic chain is related to some Liouville integrable system and the conditions in which the chain is bi-inverse-Hamiltonian are presented. Moreover, the conditions in which Hamiltonian vector fields, constructed from a given bi-presymplectic chain, constitute a related bi-Hamiltonian chain are also found. In section 5 we prove that arbitrary Stäckel system, defined by an appropriate set of separation relations, has a bi-inverse-Hamiltonian formulation. Finally, in section 6, we illustrate presented theory by few representative examples.

Our treatment in this work is local. Thus, we always restrict our considerations to the domain $\mathcal{O}$ of manifold M where appropriate functions, vector fields and one-forms never vanish and respective Poisson tensors and presymplectic forms are of constant co-rank. In some examples we perform calculations in particular local chart from $\mathcal{O}$.

2 Preliminaries

Given a manifold $\mathcal{M}$ of $\dim \mathcal{M} = m$, a *Poisson operator* Π of co-rank r on $\mathcal{M}$ is a bivector $\Pi \in \Lambda^2(\mathcal{M})$ with vanishing Schouten bracket:

$$[\Pi, \Pi]_S = 0, \tag{2.1}$$

whose kernel is spanned by exact one-forms

$$\ker \Pi = Sp\{dc_i\}_{i=1,\dots,r}.$$

In a local coordinate system $(x^1, \dots, x^m)$ on $\mathcal{M}$ we have

$$\Pi = \sum_{i<j}^{m} \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

while the Poisson property (2.1) takes the form

$$\sum_l (\Pi^{lj}\partial_l \Pi^{ik} + \Pi^{il}\partial_l \Pi^{kj} + \Pi^{kl}\partial_l \Pi^{ji}) = 0, \quad \partial_i := \frac{\partial}{\partial x^i}.$$

Let $C(\mathcal{M})$ denote the space of all smooth real-valued functions on $\mathcal{M}$. A function $c \in C(\mathcal{M})$ is called the *Casimir function* of the Poisson operator Π if $\Pi dc = 0$. Having a Poisson tensor we can define a Hamiltonian vector fields on $\mathcal{M}$. A vector field X_F related to a function $F \in C(\mathcal{M})$ by the relation

$$X_F = \Pi dF, \tag{2.2}$$

is called the *Hamiltonian vector field* with respect to the Poisson operator Π.

A linear combination $\Pi_\lambda = \Pi_1 + \lambda \Pi_0$ $(\lambda \in \mathbb{R})$ of two Poisson operators Π_0 and Π_1 is called a *Poisson pencil* if the operator Π_λ is Poisson for any value of the parameter λ. In this case we say that Π_0 and Π_1 are *compatible*. When all Casimir functions of Π_λ are polynomials in parameter λ then we say that the pencil is of Gel'fand-Zakharevich (GZ) type.

Further, a *presymplectic operator* Ω on $\mathcal{M}$ is defined by a two-form that is closed, i.e. $d\Omega = 0$, degenerated in general. In the local coordinate system $(x^1, \dots, x^m)$ on $\mathcal{M}$ we can represent Ω as

$$\Omega = \sum_{i<j}^{m} \Omega_{ij} dx^i \wedge dx^j,$$

where the closeness condition takes the form

$$\partial_i \Omega_{jk} + \partial_k \Omega_{ij} + \partial_j \Omega_{ki} = 0.$$

Moreover, the kernel of any presymplectic form is an integrable distribution. A vector field X^F related to a function $F \in C(\mathcal{M})$ by the relation

$$\Omega X^F = dF \tag{2.3}$$

is called the *inverse Hamiltonian vector field* with respect to the presymplectic operator Ω.

As in the case of presymplectic forms their linear combination is always presymplectic, hence the notion of compatibility, as it was defined for Poisson tensors, does not make sense. We will come back to this problem in the next section.

Any non-degenerate closed two form on $\mathcal{M}$ is called a *symplectic* form. The inverse of a symplectic form is an *implectic* operator, i.e. invertible Poisson tensor on $\mathcal{M}$ and vice versa.

Definition 1 *A pair (Π, Ω) is called dual implectic-symplectic pair on $\mathcal{M}$ if Π is non-degenerate Poisson tensor, Ω is non-degenerate closed two-form and the following partition of unity holds on $T\mathcal{M}$, respectively on $T^*\mathcal{M}$: $I = \Pi\Omega$ and $I = \Omega\Pi$.*

So, in the non-degenerate case, dual implectic-symplectic pair is a pair of mutually inverse operators on $\mathcal{M}$. Moreover, the Hamiltonian and the inverse Hamiltonian representations are equivalent as for any implectic bivector Π there is a unique dual symplectic form $\Omega = \Pi^{-1}$ and hence a vector field Hamiltonian with respect to Π is an inverse Hamiltonian with respect to Ω.

Let us extend these considerations onto a degenerate case. In order to do it let us introduce the concept of dual pair as it was done in [20]. Consider a manifold $\mathcal{M}$ of an arbitrary dimension m.

Definition 2 *[20] A pair of tensor fields (Π, Ω) on $\mathcal{M}$ of co-rank r, where Π is a Poisson tensor and Ω is a closed two-form, is called a dual pair (Poisson-presymplectic pair) if there exists r one-forms dc_i and r linearly independent vector fields Z_i, such that the following conditions are satisfied:*
1. $\ker \Pi = Sp\{dc_i : i = 1, \dots r\}$.
2. $\ker \Omega = Sp\{Z_i : i = 1, \dots r\}$.
3. $Z_i(c_j) = \delta_{ij}$, $i = 1, 2 \dots r$.
4. The following partition of unity holds on $T\mathcal{M}$, respectively on $T^*\mathcal{M}$

$$I = \Pi\Omega + \sum_{i=1}^{r} Z_i \otimes dc_i, \qquad I = \Omega\Pi + \sum_{i=1}^{r} dc_i \otimes Z_i, \tag{2.4}$$

where $\otimes$ denotes the tensor product.

A presymplectic form Ω plays the role of an 'inverse' of Poisson bivector Π in the sense that on any symplectic leaf of the foliation defined by $\ker \Pi$, the restrictions of Ω and Π are inverses of each other. More information on geometric interpretation of dual pairs the reader can find in [20]. Contrary to the non-degenerated case, for a given Poisson tensor Π the choice of its dual is not unique. Also for a given presymplectic form Ω the choice of dual Poisson tensor is not unique. We will come back to that problem at the end of this section.

For the degenerate case the Hamiltonian and the inverse Hamiltonian vector fields are defined in the same way as for the non-degenerate case, but for degenerate structures the notion of Hamiltonian and inverse Hamiltonian vector fields do not coincide. For any degenerate dual pair it is possible to find a Hamiltonian vector field that is not inverse Hamiltonian and an inverse Hamiltonian vector field that is not Hamiltonian. Actually, assume that (Π, Ω) is a dual pair, $X_F = \Pi dF$ is a Hamiltonian vector field and $dF = \Omega X^F$ is an inverse Hamiltonian one-form, where X^F is an inverse Hamiltonian vector field. Having applied Ω to both sides of Hamiltonian vector field, Π to both sides of inverse Hamiltonian one-form and using the decomposition (5.3) we get

$$dF = \Omega(X_F) + \sum_{i=1}^{r} Z_i(F) dc_i, \qquad X_F = X^F - \sum_{i=1}^{r} X^F(c_i) Z_i. \tag{2.5}$$

It means that an inverse Hamiltonian vector field X^F is simultaneously a Hamiltonian vector field X_F, i.e. $X^F = X_F$, if dF is annihilated by $\ker(\Omega)$ and X^F is annihilated by $\ker(\Pi)$. Moreover, for any dual pair (Π, Ω), the following important relations hold [20]

$$[Z_i, Z_j] = 0, \quad L_{X_F}\Pi = 0, \quad L_{Z_i}\Pi = 0, \quad L_{X^F}\Omega = 0, \quad L_{Z_i}\Omega = 0, \tag{2.6}$$

for $i, j = 1, ..., r$, where L_X is the Lie-derivative operator in the direction of vector field X and $[.,.]$ is a commutator.

Let us return to a 'gauge freedom' for a duality property. In other words: given a dual pair (Π, Ω) how can we deform Ω to a new presymplectic form Ω' so that (Π, Ω') is again a dual pair, or how can we deform Π to a new Poisson operator Π' so that (Π', Ω) is also a dual pair?

Lemma 3 *[20] Let Π be a fixed Poisson tensor and Ω be a dual presympectic form. Assume that $dc_i \in \ker \Pi$, $Z_i \in \ker \Omega$ and $Z_i(c_j) = \delta_{ij}$. Define*

$$\Omega' = \Omega + \sum_i df_i \wedge dc_i,$$

where $f_i \in C(M)$. Then (Π, Ω') is a dual pair, with $\ker(\Omega') = Sp\{Z_i' = Z_i - \Pi\, df_i\}$, *provided that*

$$Z_i(f_j) - Z_j(f_i) + \Pi(df_i, df_j) = 0 \ \ \text{for all} \ \ i, j. \tag{2.7}$$

Lemma 4 *[20] Let Ω be a fixed presymplectic form and Π be a dual Poisson tensor. Assume that $Z_i \in \ker \Omega$, $dc_i \in \ker \Pi$ and $Z_i(c_j) = \delta_{ij}$. Define*

$$\Pi' = \Pi + \sum_i Z_i \wedge K_i, \tag{2.8}$$

where K_i are vector fields such that

$$K_i = \Pi dF_i, \quad dF_i = \Omega K_i \quad \Rightarrow \quad Z_j(F_i) = 0, \quad K_j(c_i) = 0, \quad i, j = 1, ..., r, \tag{2.9}$$

for some functions $F_i \in C(M)$. Then, (Π', Ω) is a dual pair, with $\ker(\Pi') = Sp\{dc_i'\}$, $c_i' = c_i + F_i$, *provided that*

$$\Omega(K_i, K_j) = 0 \quad \text{for all } i, j.$$

Poisson tensor Π, considered as the mapping $\Pi : T^*\mathcal{M} \to T\mathcal{M}$, induces a Lie bracket on the space $C(\mathcal{M})$

$$\{.,.\}_\Pi : C(\mathcal{M}) \times C(\mathcal{M}) \to C(\mathcal{M}), \quad \{F, G\}_\Pi \overset{\text{def}}{=} \langle dF, \Pi\, dG \rangle = \Pi(dF, dG), \tag{2.10}$$

(where $\langle .,. \rangle$ is the dual map between $T\mathcal{M}$ and $T^*\mathcal{M}$) which is skew-symmetric and satisfies Jacobi identity. It is called a *Poisson bracket*. Jacobi identity for (2.10) follows from the property (2.1) of Π.

When a Poisson operator Π is nondegenerate its dual Ω is its inverse $\Omega = \Pi^{-1}$. Moreover, any Hamiltonian vector field with respect to Π is simultaneously the inverse Hamiltonian with respect to Ω and $X_F = X^F$. Hence, a symplectic operator Ω defines the same Poisson bracket as the related Poisson operator Π

$$\begin{aligned} \{F, G\}^\Omega &\overset{\text{def}}{=} \Omega(X_F, X_G) = <\Omega X^F, X_G> = <dF, X_G> = <dF, \Pi dG> \\ &= \Pi(dF, dG) = \{F, G\}_\Pi. \end{aligned} \tag{2.11}$$

What is important, when Π is a degenerate Poisson tensor and Ω is its an arbitrary dual two-form, the formula (2.11) is still valid. It follows from the fact that although $X_F \neq X^F$, but $<\Omega X_F, \Pi dG> = <\Omega X^F, \Pi dG>$.

Finally, we remind the reader two identities important for further considerations. Let Π be a Poisson bivector and Ω be a closed two-form, then

$$L_{\Pi\gamma}\Pi + \Pi d\gamma \Pi = 0, \qquad L_\tau \Omega = d(\Omega\tau), \tag{2.12}$$

where $\tau \in T\mathcal{M}$ and $\gamma \in T^*\mathcal{M}$.

3 D-compatibility of closed two-forms and Poisson bivectors

In the following section we develope a concept of d-compatibility which is crutial for our further considerations. Let us start with a non degenerate case.

Definition 5 *We say that a closed two-form Ω_1 is d-compatible with a symplectic form Ω_0 if $\Pi_0\Omega_1\Pi_0$ is a Poisson tensor and $\Pi_0 = \Omega_0^{-1}$ is dual to Ω_0.*

Definition 6 *We say that a Poisson tensor Π_1 is d-compatible with an implectic tensor Π_0 if $\Omega_0\Pi_1\Omega_0$ is closed and $\Omega_0 = \Pi_0^{-1}$ is dual to Π_0.*

Now, the following lemma relates d-compatible Poisson structures, of which one is implectic, and d-compatible closed two-forms, of which one is symplectic.

Lemma 7 *[22]*

Let (Π_0, Ω_0) be a dual implectic-symplectic pair.

(i) Let a Poisson tensor Π_1 be d-compatible with Π_0. Then Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0$ are d-compatible closed two-forms.

(ii) Let a closed two-form Ω_1 be d-compatible with Ω_0. Then Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0$ are d-compatible Poisson tensors.

Let us extend the notion of d-compatibility onto the degenerate case.

Definition 8 *A closed two-form Ω_1 is d-compatible with a closed two-form Ω_0 if there exists a Poisson tensor Π_0, dual to Ω_0, such that $\Pi_0\Omega_1\Pi_0$ is Poisson. Then we say that the pair (Ω_0, Ω_1) is d-compatible with respect to Π_0.*

Definition 9 *A Poisson tensor Π_1 is d-compatible with a Poisson tensor Π_0 if there exists a presymplectic form Ω_0, dual to Π_0, such that $\Omega_0\Pi_1\Omega_0$ is closed. Then we say that the pair (Π_0, Π_1) is d-compatible with respect to Ω_0.*

Comparing the notions of compatibility and d-compatibility for Poisson pair (Π_0, Π_1) we will show that when Π_0 is non-degenerated both notions are equivalent, but for a degenerate case the notion of d-compatibility is the stronger one. Actually, let us consider the following identity, proved in [21],

$$\begin{aligned}
&L_{(\Pi_1+\lambda\Pi_0)\gamma}(\Pi_1 + \lambda\Pi_0) + (\Pi_1 + \lambda\Pi_0)d\gamma(\Pi_1 + \lambda\Pi_0)\\
&= \lambda\{L_\tau(\Omega_0\Pi_1\Omega_0) - d(\Omega_0\Pi_1\Omega_0\tau) - \textstyle\sum_i[\Omega_0(L_{Z_i}\Pi_1)\Omega_0]\tau \wedge dc_i\\
&\quad - \textstyle\sum_i\tau(c_i)\Omega_0(L_{Z_i}\Pi_1)\Omega_0\},
\end{aligned} \tag{3.1}$$

where Π_0, Π_1 are Poisson tensors, (Π_0, Ω_0) is a dual pair, where $dc_i \in \ker \Pi_0$, $Z_i \in \ker \Omega_0$, $\tau \in T\mathcal{M}$ and $\gamma = \Omega_0\tau \in T^*\mathcal{M}$. Assume first that Π_0 and Π_1 are d-compatible with respect to Ω_0. Then $\Omega_0\Pi_1\Omega_0$ is closed and

$$L_\tau(\Omega_0\Pi_1\Omega_0) - d(\Omega_0\Pi_1\Omega_0\tau) = 0, \qquad \tau \in T\mathcal{M}. \tag{3.2}$$

In particular, for $\tau = Z_i$, relation (3.2) gives

$$\Omega_0(L_{Z_i}\Pi_1)\Omega_0 = 0, \qquad i = 1, ..., r. \tag{3.3}$$

Hence

$$L_{(\Pi_1+\lambda\Pi_0)\gamma}(\Pi_1 + \lambda\Pi_0) + (\Pi_1 + \lambda\Pi_0)d\gamma(\Pi_1 + \lambda\Pi_0) = 0 \tag{3.4}$$

and $\Pi_1 + \lambda\Pi_0$ is Poisson. On the other hand, from the compatibility relation (3.4) the d-compatibility (3.2) follows under additional conditions (3.3).

Theorem 10 *Let a Poisson tensor Π_0 and a closed two-form Ω_0 form a dual pair, where $Y_0^{(k)} \in \ker \Omega_0$, $dH_0^{(k)} \in \ker \Pi_0$ and $Y_0^{(k)}(H_0^{(m)}) = \delta_{km}$, $k, m = 1, ..., r$.*

(i) If Π_1 is a Poisson tensor d-compatible with Π_0 with respect to Ω_0, then forms Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0$ are d-compatible with respect to Π_0.

(ii) If Ω_1 is a closed two-form d-compatible with Ω_0 with respect to Π_0, then Poisson tensors Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0$ are d-compatible with respect to Ω_0, provided that

$$\Pi_0\Omega_1 Y_0^{(k)} = \Pi_0 dF^{(k)}, \qquad k = 1, ..., r \tag{3.5}$$

for some functions $F^{(k)} \in C(\mathcal{M})$ and

$$\Omega_1(Y_0^{(k)}, Y_0^{(m)}) + Y_0^{(k)}(F^{(m)}) - Y_0^{(m)}(F^{(k)}) = const, \qquad k, m = 1, ..., r. \tag{3.6}$$

Proof.
(i) Ω_1 is closed as Π_1 is d-compatible with Π_0. Then, $\Pi_0\Omega_1\Pi_0 = \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0$ is Poisson (as was shown in [21]).

(ii) From the d-compatibility of Ω_0 and Ω_1 it follows that Π_1 is Poisson. Then,

$$\begin{aligned}\Omega_0\Pi_1\Omega_0 &= \Omega_0\Pi_0\Omega_1\Pi_0\Omega_0 = (I - \textstyle\sum_k dH_0^{(k)} \otimes Y_0^{(k)})\Omega_1(I - \textstyle\sum_m Y_0^{(m)} \otimes dH_0^{(m)}) \\ &= \Omega_1 + \textstyle\sum_k dH_0^{(k)} \wedge \Omega_1(Y_0^{(k)}) + \frac{1}{2}\textstyle\sum_{k,m} \Omega_1(Y_0^{(m)}, Y_0^{(k)}) dH_0^{(k)} \wedge dH_0^{(m)}.\end{aligned}$$

From the assumption (3.5) and decompositions (5.3) it follows that

$$\Omega_1 Y_0^{(k)} = dF^{(k)} + \textstyle\sum_m \left[\Omega_1(Y_0^{(k)}, Y_0^{(m)}) - Y_0^{(m)}(F^{(k)})\right] dH_0^{(m)},$$

hence,

$$\begin{aligned}\Omega_0\Pi_1\Omega_0 &= \Omega_1 + \textstyle\sum_k dH_0^{(k)} \wedge dF^{(k)} \\ &\quad + \textstyle\sum_{k,m} \left[\frac{1}{2}\Omega_1(Y_0^{(k)}, Y_0^{(m)}) - Y_0^{(m)}(F^{(k)})\right] dH_0^{(k)} \wedge dH_0^{(m)} \\ &= \Omega_1 + \textstyle\sum_k dH_0^{(k)} \wedge dF^{(k)} \\ &\quad + \frac{1}{2}\textstyle\sum_{k,m} \left[\Omega_1(Y_0^{(k)}, Y_0^{(m)}) - Y_0^{(m)}(F^{(k)}) + Y_0^{(k)}(F^{(m)})\right] dH_0^{(k)} \wedge dH_0^{(m)}\end{aligned}$$

and under condition (3.6) $\Omega_0\Pi_1\Omega_0$ is closed. □

The important for further considerations special case occurs when

$$\Omega_1(Y_0^{(k)}, Y_0^{(m)}) = 0, \qquad Y_0^{(k)}(F^{(m)}) = Y_0^{(m)}(F^{(k)}), \qquad k, m = 1, ..., r. \tag{3.7}$$

Theorem 11 *Let a Poisson tensor Π_0 and a closed two-form Ω_0 form a dual pair, where $Y_0^{(k)} \in \ker \Omega_0$, $dH_0^{(k)} \in \ker \Pi_0$ and $Y_0^{(k)}(H_0^{(m)}) = \delta_{km}$, $k, m = 1, ..., r$.*

(i)If Π_1 is a Poisson tensor d-compatible with Π_0 with respect to Ω_0 and

$$X^{(k)} = \Pi_1 dH_0^{(k)} = \Pi_0 dH_1^{(k)}, \qquad k = 1, ..., r \tag{3.8}$$

are bi-Hamiltonian vector fields for some functions $H_1^{(k)}$, then Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0 + \sum_k dH_1^{(k)} \wedge dH_0^{(k)}$ is d-compatible pair of presymplectic forms with respect to Π_0.

(ii) If Ω_1 is a presymplectic form d-compatible with Ω_0 with respect to Π_0 and

$$\beta^{(k)} = \Omega_0 Y_1^{(k)} = \Omega_1 Y_0^{(k)}, \qquad k = 1, ..., r \tag{3.9}$$

are bi-presymplectic one-forms, then Poisson tensors Π_0 *and* $\Pi_1 = \Pi_0\Omega_1\Pi_0 + \sum_k X^{(k)} \wedge Y_0^{(k)}$*, are d-compatible with respect to* Ω_0 *if*

$$X^{(k)} = \Pi_0\Omega_1 Y_0^{(k)} = \Pi_0 dF^{(k)}, \quad \Pi_0\Omega_1 Y_1^{(k)} = \Pi_0 dG^{(k)}, \qquad k = 1, ..., r, \tag{3.10}$$

for some functions $F^{(k)}, G^{(k)} \in C(\mathcal{M})$ *and*

$$\Omega_0(Y_1^{(k)}, Y_1^{(m)}) = 0, \qquad k, m = 1, ..., r, \tag{3.11}$$

$$Y_0^{(m)}(F^{(k)}) = Y_0^{(k)}(F^{(m)}), \qquad k, m = 1, ..., r, \tag{3.12}$$

$$Y_1^{(m)}(H_0^{(k)}) = Y_0^{(k)}(F^{(m)}), \qquad k, m = 1, ..., r. \tag{3.13}$$

Proof.
(i) Ω_1 is closed as Π_1 is d-compatible with Π_0. Then, $\Pi_0\Omega_1\Pi_0 = \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0$ is Poisson (as was shown in [21]).

(ii) From (3.10) we have

$$\Omega_1(Y_0^{(k)}, Y_0^{(m)}) = 0 \tag{3.14}$$

and by previous theorem part (ii) the form $\Omega_0\Pi_1\Omega_0 = \Omega_0\Pi_0\Omega_1\Pi_0\Omega_0$ is closed under condition (3.12). Moreover, (3.14) yields

$$\Omega_0 Y_1^{(k)} = dF^{(k)} - \textstyle\sum_m Y_0^{(k)}(F^{(k)}) dH_0^{(m)},$$
$$\Omega_1 Y_1^{(k)} = dG^{(k)} - \textstyle\sum_m Y_0^{(k)}(G^{(k)}) dH_0^{(m)}.$$

Conditions (3.11) and (3.13) are sufficient for Π_1 to be a Poisson tensor. From (3.11) it follows that

$$\Omega_0(Y_1^{(k)}, Y_1^{(m)}) = 0 \Longrightarrow \Pi_0(dF^{(k)}, dF^{(m)}) = 0 \Longrightarrow [X^{(k)}, X^{(m)}] = 0. \tag{3.15}$$

Now we show that the Schouten bracket of Π_1 is zero. As $\Pi_0\Omega_1\Pi_0$ is Poisson (it follows from compatibility of Ω_0 and Ω_1), we have

$$[\Pi_1, \Pi_1]_S = 2\textstyle\sum_k [\Pi_0\Omega_1\Pi_0, X^{(k)} \wedge Y_0^{(k)}]_S + \sum_{k,m} [X^{(k)} \wedge Y_0^{(k)}, X^{(m)} \wedge Y_0^{(m)}]_S,$$

$$[\Pi_0\Omega_1\Pi_0, X^{(k)} \wedge Y_0^{(k)}]_S = Y_0^{(k)} \wedge \Pi_0 d(\Omega_1 X^{(k)})\Pi_0 - X^{(k)} \wedge \Pi_0 d(\Omega_1 Y_0^{(k)})\Pi_0,$$
$$[X^{(k)} \wedge Y_0^{(k)}, X^{(m)} \wedge Y_0^{(m)}]_S = 2X^{(k)} \wedge Y_0^{(m)} \wedge [Y_0^{(k)}, X^{(m)}].$$

In last equality we used the fact that $[Y_0^{(k)}, Y_0^{(m)}] = 0$ and relation (3.15). Now,

$$\begin{aligned}[Y_0^{(k)}, X^{(m)}] &= [Y_0^{(k)}, \Pi_0\Omega_1 Y_0^{(m)}] = L_{Y_0^{(k)}}(\Pi_0\Omega_1) Y_0^{(m)} = \Pi_0(L_{Y_0^{(k)}}\Omega_1) Y_0^{(m)} \\ &= \Pi_0 d(\Omega_1 \mu Y_0^{(k)}) Y_0^{(m)} = \Pi_0(d\beta^{(k)}) Y_0^{(m)} \\ &= \Pi_0 \left[-\textstyle\sum_i d(Y_0^{(k)}(F^{(i)})) \wedge dH_0^{(i)}\right] Y_0^{(m)} \\ &= -\Pi_0 d(Y_0^{(k)}(F^{(m)})).\end{aligned}$$

From

$$Y_1^{(k)} = X^{(k)} + \textstyle\sum_m Y_1^{(k)}(H_0^{(m)}) Y_0^{(m)}$$

we have

$$\Omega_1 Y_1^{(k)} = \Omega_1 X^{(k)} + \textstyle\sum_m Y_1^{(k)}(H_0^{(m)}) \Omega_1 Y_0^{(m)}$$

and

$$\Omega_1 X^{(k)} = dG^{(k)} - \textstyle\sum_i Y_0^{(k)}(G^{(i)}) dH_0^{(i)} - \sum_i Y_1^{(k)}(H_0^{(i)}) dF^{(i)} + \sum_{i,j} Y_1^{(k)}(H_0^{(i)}) Y_0^{(i)}(F^{(j)}) dH_0^{(j)}.$$

Hence,

$$\begin{aligned}\Pi_0 d(\Omega_1 X^{(k)})\Pi_0 &= -\Pi_0 \left[\textstyle\sum_m d(Y_1^{(k)}(H_0^{(m)})) \wedge dF^{(m)}\right] \Pi_0 \\ &= \textstyle\sum_m \Pi_0 d(Y_1^{(k)}(H_0^{(m)})) \wedge X^{(m)}\end{aligned}$$

and then $[\Pi_1, \Pi_1]_S = 0$ under condition (3.13).□

4 Bi-presymplectic chains

Now we are ready to investigate main properties of bi-presymplectic chains.

Theorem 12 *Assume we have a pair of presymplectic forms (Ω_0, Ω_1), d-compatible with respect to some Π_0 dual to Ω_0, both of rank $2n$ and co-rank r on $\mathcal{O} \subset \mathcal{M}$. Assume further, that they form bi-presymplectic chains of one-forms*

$$\beta_i^{(k)} = \Omega_0 Y_i^{(k)} = \Omega_1 Y_{i-1}^{(k)}, \quad i = 1, 2, \ldots, n_k \tag{4.1}$$

where $k = 1, ..., r$, $n_1 + ... + n_r = n$ and each chain starts with a kernel vector field $Y_0^{(k)}$ of Ω_0 and terminates with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1. Then

(i)

$$\Omega_0(Y_i^{(k)}, Y_j^{(m)}) = \Omega_1(Y_i^{(k)}, Y_j^{(m)}) = 0, \tag{4.2}$$

for $k, m = 1, ..., r$, $i = 1, 2, \ldots, n_k$, $j = 1, 2, \ldots, n_m$.

Moreover, let us assume that

$$X_i^{(k)} = \Pi_0 \beta_i^{(k)} = \Pi_0 dH_i^{(k)}, \tag{4.3}$$

for $k = 1, ..., r$, $i = 1, 2, \ldots, n_k$, which implies

$$\beta_i^{(k)} = dH_i^{(k)} - \textstyle\sum_m Y_0^{(m)}(H_i^{(k)}) dH_0^{(m)}, \tag{4.4}$$

$$Y_i^{(k)} = X_i^{(k)} + \textstyle\sum_m Y_i^{(k)}(H_0^{(m)}) Y_0^{(m)}, \tag{4.5}$$

where $\Pi_0 dH_0 = 0$. Then,

(ii)

$$\Pi_0(dH_i^{(k)}, dH_j^{(m)}) = 0, \quad [X_i^{(k)}, X_j^{(m)}] = 0 \tag{4.6}$$

and equations (4.1) define a Liouville integrable system.

Additionally, if

$$Y_0^{(k)}(H_1^{(m)}) = Y_0^{(m)}(H_1^{(k)}) \tag{4.7}$$

and

$$Y_0^{(k)}(H_i^{(m)}) = Y_i^{(m)}(H_0^{(k)}), \tag{4.8}$$

where $k, m = 1, ..., r$, $i = 1, 2, \ldots, n_m$, then

(iii) vector fields $X_i^{(k)}$ (4.3) form bi-Hamiltonian chains

$$X_i^{(k)} = \Pi_0 dH_i^{(k)} = \Pi_1 dH_{i-1}^{(k)}, \quad i = 1, 2, \ldots, n \tag{4.9}$$

where

$$\Pi_1 = \Pi_0 \Omega_1 \Pi_0 + \textstyle\sum_m X_1^{(m)} \wedge Y_0^{(m)}, \tag{4.10}$$

$k, m = 1, ..., r$, $i = 1, 2, \ldots, n_k$ and $n_1 + ... + n_r = n$. The chain starts with $H_0^{(k)}$, a Casimir of Π_0, and terminates with $H_{n_k}^{(k)}$, a Casimir of Π_1. Moreover the Poisson pair (Π_0, Π_1) is d-compatible with respect to Ω_0.

Proof.

(i) From (4.1) we have

$$\begin{aligned}
\Omega_0(Y_i^{(k)}, Y_j^{(m)}) &= \Omega_0(Y_{i-1}^{(k)}, Y_{j+1}^{(m)}), \\
\Omega_1(Y_i^{(k)}, Y_j^{(m)}) &= \Omega_1(Y_{i+1}^{(k)}, Y_{j-1}^{(m)}) \\
\Omega_0(Y_i^{(k)}, Y_j^{(m)}) &= \Omega_1(Y_{i-1}^{(k)}, Y_j^{(m)}).
\end{aligned}$$

Then, (4.2) follows from

$$\Omega_0(Y_0^{(k)}, Y_i^{(m)}) = 0, \quad \Omega_1(Y_{n_k}^{(k)}, Y_i^{(m)}) = 0.$$

(ii) From properties of dual pair (Π_0, Ω_0), if $X_i^{(k)} = \Pi_0 dH_i^{(k)}$ then

$$\Pi_0(dH_i^{(k)}, dH_j^{(m)}) = \Omega_0(X_i^{(k)}, X_j^{(m)}).$$

On the other hand as $X_i^{(k)} = Y_i^{(k)} + \sum_m \alpha_m^{(k)} Y_0^{(m)}$, where $\alpha_m^{(k)}$ are an appropriate functions, it follows that

$$\Omega_0(X_i^{(k)}, X_j^{(m)}) = \Omega_0(Y_i^{(k)}, Y_j^{(m)}).$$

(iii) We have

$$\begin{aligned}
X_i^{(k)} &= \Pi_0 dH_i^{(k)} \\
&= \Pi_0\Omega_1 Y_{i-1}^{(k)} = \Pi_0\Omega_1(X_{i-1}^{(k)} + \textstyle\sum_m Y_{i-1}^{(k)}(H_0^{(m)})Y_0^{(m)}) \\
&= \Pi_0\Omega_1\Pi_0 dH_{i-1}^{(k)} + \textstyle\sum_m Y_{i-1}^{(k)}(H_0^{(m)})X_1^{(m)}) \\
&\overset{(4.8)}{=} (\Pi_0\Omega_1\Pi_0 + \textstyle\sum_m X_1^{(m)} \wedge Y_0^{(m)})dH_{i-1}^{(k)} \\
&= \Pi_1 dH_{i-1}^{(k)}.
\end{aligned}$$

Moreover, Π_0 and Π_1 are d-compatible Poisson tensors provided that (4.7) is fulfilled. We also have

$$\begin{aligned}
\Pi_1 dH_{n_k}^{(k)} &= (\Pi_0\Omega_1\Pi_0 + \textstyle\sum_m X_1^{(m)} \wedge Y_0^{(m)})dH_{n_k}^{(k)} = \Pi_0\Omega_1 X_{n_k}^{(k)} + \textstyle\sum_m Y_0^{(m)}(H_{n_k}^{(k)})X_1^{(m)} \\
&\overset{(4.5)}{=} \Pi_0\Omega_1(Y_{n_k}^{(k)} - \textstyle\sum_m Y_{n_k}^{(k)}(H_0^{(m)})Y_0^{(m)}) + \textstyle\sum_m Y_0^{(m)}(H_{n_k}^{(k)})X_1^{(m)} \\
&= -\textstyle\sum_m Y_{n_k}^{(k)}(H_0^{(m)})X_1^{(m)} + \textstyle\sum_m Y_0^{(m)}(H_{n_k}^{(k)})X_1^{(m)} \overset{(4.8)}{=} 0.
\end{aligned}$$

□

Notice, that in a special case, when

$$Y_0^{(k)}(H_i^{(m)}) = 0, \tag{4.11}$$

for all admissible values of k, m and i, chains (4.1) are bi-inverse-Hamiltonian as $\beta_i^{(k)} = dH_i^{(k)}$. Obviouslu $X_i^{(k)}$ are not bi-Hamiltonian until $Y_i^{(k)}(H_0^{(m)}) \neq 0$.

Finally we show that arbitrary Liouville integrable system which has a bi-presymplectic representation on $(2n+r)$-dimensional phase space, has also quasi-bi-Hamiltonian representation on any symplectic leaf of its Hamiltonian structure Π_0. Actually, from (4.1), (4.3) and (4.4) follows that

$$\begin{aligned}
\Pi_0 dH_i^{(k)} &= \Pi_0\Omega_1\left(Y_{i-1}^{(k)} + \textstyle\sum_m Y_0^{(m)}(H_i^{(k)})dH_0^{(m)}\right) \\
&= \Pi_0\left[\Omega_1 X_{i-1}^{(k)} + \textstyle\sum_m \left(Y_{i-1}^{(k)}(H_0^{(m)})\Omega_1 Y_0^{(m)} + Y_0^{(m)}(H_i^{(k)})dH_0^{(m)}\right)\right] \\
&= \Pi_0\Omega_1\Pi_0 dH_{i-1}^{(k)} + \textstyle\sum_m Y_{i-1}^{(k)}(H_0^{(m)})\Pi_0 dH_1^{(m)},
\end{aligned}$$

hence on $(2n+r)$-dimensional phase space we have quasi-bi-Hamiltonian representation

$$\Pi_1 dH_{i-1}^{(k)} = \Pi_0 dH_i^{(k)} + \sum_{m=1}^{r} F_{i-1}^{(k,m)}\Pi_0 dH_1^{(m)}, \tag{4.12}$$

where

$$\Pi_1 = \Pi_0\Omega_1\Pi_0, \qquad F_i^{(k,m)} = -Y_i^{(k)}(H_0^{(m)}).$$

Notice that both Poisson structures Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0$ share the same Casimirs $H_0^{(k)}$, so the quasi-bi-Hamiltonian dynamics can be restricted immediately to any common leaf of dimension $2n$

$$\pi_1 dh_{i-1}^{(k)} = \pi_0 dh_i^{(k)} + \sum_{m=1}^{r} F_{i-1}^{(k,m)}\pi_0 dh_1^{(m)}, \qquad i = 1, ..., n, \tag{4.13}$$

where π_i and $h_i^{(k)}$are restrictions of Π_i and $H_i^{(k)}$, respectively. Hence we deal with a Stäckel system whose separation coordinates are eigenvalues of the recursion operator $N = \pi_1 \pi_0^{-1}$ [23], provided that N has n distinct and functionally independent eigenvalues at any point of $\mathcal{O} \subset \mathcal{M}$, i.e. we are in a generic case. We will come back to separable systems in next sections.

The advantage of bi-presymplectic representation of Liouville integrable system, when compared to bi-Hamiltonian ones, is that the existence of the first guarantees that the system is separable and the construction of separation coordinates is purely algorithmic (in a generic case), while the bi-Hamiltonian representation does not guarantee the existence of quasi-bi-Hamiltonian representation and hence separability of the system in question. Moreover, the projection of the second Poisson structure onto the symplectic foliation of the first one, in order to construct a quasi-bi-Hamiltonian representation, necessary for separability, is far from being trivial non-algorithmic procedure [24].

5 Separable Stäckel systems

Consider a Liouville integrable system on a $2n$-dimensional phase space M, i.e. a set of $h_i \in C(M)$, $i = 1, ..., n$ which are in involution with respect to some Poisson tensor π_0, and related n Hamiltonian dynamic systems

$$u_{t_i} = \pi \, dh_i = x_{h_i}, \qquad i = 1, \ldots, n, \tag{5.1}$$

where $u \in M$ and x_{h_i} are respective Hamiltonian vector fields.

The Hamilton-Jacobi (HJ) method for solving (5.1) essentially amounts to the linearization of the latter via a canonical transformation (when $u \in \mathcal{O} \subset \mathcal{M}$ is some local canonical chart)

$$u = (q, p) \rightarrow \; (b, a), \quad a_i = h_i, \qquad i = 1, \ldots, n. \tag{5.2}$$

In order to find the conjugate coordinates b^i it is necessary to construct a generating function $W(q, a)$ of the transformation (5.2) such that

$$b^j = \tfrac{\partial W}{\partial a_j}, \quad p_j = \tfrac{\partial W}{\partial q^j}.$$

The function $W(q, a)$ is a complete integral of the associated *Hamilton-Jacobi equations*

$$h_i \left(q^1, \ldots, q^n, \tfrac{\partial W}{\partial q^1}, \ldots, \tfrac{\partial W}{\partial q^n} \right) = a_i, \qquad i = 1, \ldots, n. \tag{5.3}$$

In the (b, a) representation the t_i-dynamics is trivial:

$$(a_j)_{t_i} = 0, \quad (b^j)_{t_i} = \delta_{ij},$$

whence

$$b^j(q, a) = \tfrac{\partial W}{\partial a_j} = t_j + \gamma_j, \qquad j = 1, \ldots, n, \tag{5.4}$$

where γ_j are arbitrary constants.

Equations (5.4) provide implicit solutions for (5.1). Solving them for q_j is known as the *inverse Jacobi problem*. The reconstruction in explicit form of trajectories $q^j = q^j(t_i)$ is itself a highly nontrivial problem from algebraic geometry.

To solve the system (5.3) for W in a given local coordinate system is a hopeless task, as (5.3) is a system of nonlinear coupled partial differential equations. In essence, the only hitherto known way of overcoming this difficulty is to find distinguished canonical coordinates, denoted here by (λ, μ), for which there exist n relations

$$\varphi_i(\lambda^i, \mu_i; a_1, \ldots, a_n) = 0, \qquad i = 1, \ldots, n, \; a_i \in \mathbb{R}, \quad \det \left[\tfrac{\partial \varphi_i}{\partial a_j} \right] \neq 0, \tag{5.5}$$

such that each of these relations involves only a single pair of canonical coordinates [1]. The determinant condition in (5.5) means that we can solve the equations (5.5) for a_i and express a_i in the form $a_i = h_i(\lambda, \mu)$, $i = 1, \ldots, n$.

If the functions $W_i(\lambda^i, a)$ are solutions of a system of n *decoupled* ODEs obtained from (5.5) by substituting $\mu_i = \frac{dW_i(\lambda^i,a)}{d\lambda^i}$

$$\varphi_i\left(\lambda^i, \mu_i = \tfrac{dW_i(\lambda^i,a)}{d\lambda^i}, a_1, \dots, a_n\right) = 0, \quad i = 1, \dots, n, \tag{5.6}$$

then the function

$$W(\lambda, a) = \sum\nolimits_{i=1}^{n} W_i(\lambda^i, a)$$

is an additively separable solution of *all* the equations (5.6), and *simultaneously* it is a solution of all Hamilton-Jacobi equations (5.3). The distinguished coordinates (λ, μ) for which the original Hamilton-Jacobi equations (5.3) are equivalent to a set of separation relations (5.6) are called the *separation coordinates.*

In what follows we restrict ourselves to considering a special case of (5.5) when all separation relations are affine in h_i:

$$\sum_{k=1}^{n} S_i^k(\lambda^i, \mu_i) h_k = \psi_i(\lambda^i, \mu_i), \qquad i = 1, \dots, n, \tag{5.7}$$

where S_i^k and ψ_i are arbitrary smooth functions of their arguments. The relations (5.7) are called the generalized *Stäckel separation relations* and the related dynamical systems are called the *Stäckel separable* ones. The matrix $S = (S_i^k)$ will be called a *generalized Stäckel matrix.* To recover the explicit Stäckel form of the Hamiltonians it suffices to solve the linear system (5.7) with respect to h_i.

If in (5.7) we further have $S_i^k(\lambda^i, \mu_i) = S^k(\lambda^i, \mu_i)$ and $\psi_i(\lambda^i, \mu_i) = \psi(\lambda^i, \mu_i)$ then the separation conditions can be represented by n copies of the curve

$$\sum_{k=1}^{n} S^k(\lambda, \mu) h_k = \psi(\lambda, \mu) \tag{5.8}$$

in (λ, μ) plane, called a *separation curve.* The copies in question are obtained by setting $\lambda = \lambda^i$ and $\mu = \mu_i$ for $i = 1, \dots, n$.

For further convenience, let us collect the terms from the l.h.s. of (5.7) as follows:

$$\sum_{k=1}^{r} \varphi_i^k(\lambda^i, \mu_i) h^{(k)}(\lambda^i) = \psi_i(\lambda^i, \mu_i), \qquad i = 1, \dots, n, \tag{5.9}$$

where

$$h^{(k)}(\lambda) = \sum_{i=1}^{n_k} \lambda^{n_k - i} h_i^{(k)}, \qquad n_1 + \dots + n_r = n$$

and impose the normalization $\varphi_i^r(\lambda^i, \mu_i) = 1$.

Some informations about the classification of Stäckel systems (5.9) the reader can find in [18].

6 Bi-inverse-Hamiltonian representation of Stäckel systems

As recently proved in [18], the Stäckel Hamiltonians defined by separation relations (5.9) admit on M the following quasi bi-Hamiltonian chains in (λ, μ) representation

$$\pi_1 dh_i^{(k)} = \pi_0\, dh_{i+1}^{(k)} + \sum_{l=1}^{r} F_i^{(k,l)}\, \pi_0\, dh_1^{(l)}, \quad h_{n_k+1}^{(k)} = 0, \quad k = 1, \dots, r, \quad i = 1, \dots, n_k, \tag{6.1}$$

where π_0 is a canonical Poisson tensor

$$\pi_0 = \sum_i \frac{\partial}{\partial \lambda^i} \wedge \frac{\partial}{\partial \mu_i},$$

π_1 is a noncanonical Poisson tensor of the form

$$\pi_1 = \sum_i \lambda^i \frac{\partial}{\partial \lambda^i} \wedge \frac{\partial}{\partial \mu_i},$$

compatible with π_0, and the *expantion cefficients* $F_i^{(k,l)}$ are solutions of the set of linear algebraic equations

$$\sum_{k=1}^{r} \varphi_j^k(\lambda^j, \mu_j) F^{(k,l)}(\lambda^j) = \varphi_j^l(\lambda^j, \mu_j)(\lambda^j)^{n_l}, \qquad j = 1, \dots, n, \quad l = 1, ..., r, \tag{6.2}$$

where

$$F^{(k,l)}(\lambda) = \sum_{i=1}^{n_k} \lambda^{n_k - i} F_i^{(k,l)}, \qquad n_1 + \cdots + n_r = n.$$

Let us consider the following symplectic forms on M

$$\omega_0 = -\sum_i d\lambda^i \wedge d\mu_i, \qquad \omega_1 = -\sum_i \lambda^i d\lambda^i \wedge d\mu_i.$$

Observe that (π_0, ω_0) constituts non degenerate dual implectic-symplectic pair as $\omega_0 = \pi_0^{-1}$, π_0 and $\pi_1 = \pi_0\omega_1\pi_0$ are d-compatible with respect to ω_0 and ω_0 and $\omega_1 = \omega_0\pi_1\omega_0$ are d-compatible with respect to π_0. Besides, quasi bi-Hamiltonian chains (6.1) have equivalent quasi bi-inverse-Hamiltonian representations

$$\omega_1 x_i^{(k)} = \omega_0\, x_{i+1}^{(k)} + \sum_{l=1}^{r} F_i^{(k,l)}\, \omega_0\, x_1^{(l)}, \quad x_{n_k+1}^{(k)} = 0, \quad k = 1, \dots, r, \quad i = 1, \dots, n_k, \tag{6.3}$$

where

$$x_i^{(k)} = \pi_0 dh_i^{(k)}, \qquad dh_i^{(k)} = \omega_0 x_i^{(k)}.$$

Let us lift the whole construction to the extended phase space $M \to \mathcal{M} : (\lambda, \mu) \to (\lambda, \mu, c)$, where $\dim \mathcal{M} = 2n + r$. Then, on $\mathcal{M} : \omega_0 \to \Omega_0$, $\pi_0 \to \Pi_0$, both degenerated, where

$$\ker \Omega_0 = Sp\{Y_0^{(k)}\}, \quad k = 1, ..., r, \quad Y_0^{(k)} = \frac{\partial}{\partial c_k}, \quad \Omega_0 Y_0^{(k)} = 0$$

and

$$\ker \Pi_0 = Sp\{dc_k\}, \quad k = 1, ..., r, \qquad \Pi_0 dc_k = 0, \quad Y_0^{(k)}(c_j) = \delta_j^k.$$

Obviously, (Π_0, Ω_0) is a dual Poisson-presymplectic pair on $\mathcal{M}$. In the same fasion we lift

$$\omega_1 \to \Omega_{1D}, \qquad \pi_1 \to \Pi_{1D}, \qquad x_i^{(k)} \to X_i^{(k)},$$

where $\ker \Omega_{1D} = \ker \Omega_0$ and $\ker \Pi_{1D} = \ker \Pi_0$. On $\mathcal{M}$ quasi bi-inverse-Hamiltonian chains (6.3) take the form

$$\Omega_{1D} X_i^{(k)} = \Omega_0\, X_{i+1}^{(k)} + \sum_{l=1}^{r} F_i^{(k,l)}\, \Omega_0\, X_1^{(l)}, \quad k = 1, \dots, r, \quad i = 1, \dots, n_k. \tag{6.4}$$

Let us define the following presymplectic two-form

$$\Omega_1 = \Omega_{1D} + \sum_{k=1}^{r} dh_1^{(k)} \wedge dc_k \tag{6.5}$$

and the set of vector fields

$$Y_i^{(k)} = X_i^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)} Y_0^{(l)}. \tag{6.6}$$

Then, we have

$$
\begin{aligned}
\Omega_0 Y_{i+1}^{(k)} &= dh_{i+1}^{(k)} \\
&= \Omega_0 X_{i+1}^{(k)} = \Omega_{1D} X_i^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)}\, \Omega_0\, X_1^{(l)} \\
&= (\Omega_1 - \sum_{l=1}^{r} dh_1^{(l)} \wedge dc_l)(Y_i^{(k)} + \sum_{l=1}^{r} F_i^{(k,l)} Y_0^{(l)}) - \sum_{l=1}^{r} F_i^{(k,l)} dh_1^{(l)} \\
&= \Omega_1 Y_i^{(k)} + \sum_{l=1}^{r} F_i^{(k,l)} \Omega_1 Y_0^{(l)} - \sum_{l=1}^{r} Y_i^{(k)}(c_l) dh_1^{(l)} + \sum_{l=1}^{r} Y_i^{(k)}(h_1^{(l)}) dc_l \\
&\quad - \sum_{l=1}^{r} F_i^{(k,l)} dh_1^{(l)} + \sum_{l,m=1}^{r} F_i^{(k,m)} Y_0^{(m)}(h_1^{(l)}) dc_l - \sum_{l=1}^{r} F_i^{(k,l)} dh_1^{(l)} \\
&= \Omega_1 Y_i^{(k)},
\end{aligned}
$$

as

$$
\Omega_1 Y_0^{(l)} = \sum_{k=1}^{r} (dh_1^{(k)} \wedge dc_k) Y_0^{(l)} = dh_1^{(l)},
$$

$$
Y_i^{(k)}(h_1^{(l)}) = 0, \quad Y_i^{(k)}(c_l) = -F_i^{(k,l)}, \quad Y_0^{(m)}(c_k) = \delta_{mk.}
$$

Hence, on $\mathcal{M}$, differentials $dh_i^{(k)}$ form a bi-inverse-Hamiltonian hierarchies

$$
\Omega_0 Y_{i+1}^{(k)} = dh_{i+1}^{(k)} = \Omega_1 Y_i^{(k)}, \quad i = 0, 1, 2, \ldots, n_k, \quad k = 1, ..., r, \tag{6.7}
$$

which starts with a kernel vector field $Y_0^{(k)}$ of Ω_0 and terminates with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1. Indeed

$$
\begin{aligned}
\Omega_1 Y_{n_k}^{(k)} &= (\Omega_{1D} + \sum_{m=1}^{r} dh_1^{(m)} \wedge dc_m)(X_{n_k}^{(k)} - \sum_{m=1}^{r} F_{n_k}^{(k,m)} Y_0^{(m)}) \\
&= \sum_{m=1}^{r} F_{n_k}^{(k,m)} dh_1^{(m)} - \sum_{m=1}^{r} F_{n_k}^{(k,m)} dh_1^{(m)} = 0.
\end{aligned}
$$

Moreover, Ω_0 and Ω_1 are d-compatible with respect to Π_0, as

$$
\Pi_0 \Omega_1 \Pi_0 = \Pi_0 \Omega_{1D} \Pi_0 = \Pi_{1D}
$$

which is Poisson. According to theorem 12 vector fields $X_i^{(k)}$ are not bi-Hamiltonian as $Y_i^{(k)}(h_0^{(l)}) = -F_i^{(k,l)} \neq 0$.

In order to construct on $\mathcal{M}$ bi-Hamiltonian representation of considered Stäckel systems, one has to extend the original Hamiltonians

$$
h_i^{(k)} \to H_i^{(k)} = h_i^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)} c_l, \quad i = 1, ..., n. \tag{6.8}
$$

Then, on $\mathcal{M}$, vector fields

$$
K_i^{(k)} = X_i^{(k)} - \Pi_0 d(\sum_{l=1}^{r} F_i^{(k,l)} c_l) \tag{6.9}
$$

form a bi-Hamiltonian chains

$$
\Pi_0 dH_{i+1}^{(k)} = K_{i+1}^{(k)} = \Pi_1 dH_i^{(k)}, \quad i = 0, 1, \ldots, n_k, \quad k = 1, ..., r, \tag{6.10}
$$

where

$$\Pi_1 = \Pi_{1D} + \sum_{m=1}^{r} K_1^{(m)} \wedge Y_0^{(m)} \tag{6.11}$$

is a Poisson tensor compatible with Π_0 one. Each chain starts with the Casimir of Π_0, i.e. $H_0^{(k)} = c_k$, and terminates with the Casimir of Π_1, i.e. $H_{n_k}^{(k)}$. The details of the construction the reader finds in [18]. Poisson tensors Π_0 and Π_1 are d-compatible with respect to Ω_0 as

$$\Omega_0 \Pi_1 \Omega_0 = \Omega_0 \Pi_{1D} \Omega_0 = \Omega_{1D}$$

is closed. As was proved in [21], bi-Hamiltonian chains (6.10) have no bi-presymplectic counterparts as the conditions (4.7) are not satisfied (see also theorem 11). Indeed

$$Y_0^{(k)}(H_1^{(m)}) = -F_1^{(m,k)} \neq -F_1^{(k,m)} = Y_0^{(m)}(H_1^{(k)}).$$

The only exception is the case of co-rank one ($r = 1$), as then (4.7) is trivially fulfilled.

7 Examples

Here we illustrate the presented theory with three examples of separable systems, each of three degrees of freedom. Two of them are classical Stäckel systems with separation relations being quadratic in momenta, while the third example has separation relations cubic in momenta.

Example 1.

Consider the separation relations on a six-dimensional phase space M given by the following bare (potential-free) separation curve

$$h_1\lambda^2 + h_2\lambda + h_3 = \tfrac{1}{2}\mu^2.$$

This curve corresponds to geodesic motion for a classical Stäckel system (of Benenti type [25]). As in this example $k = 1$, we use the notation $h_i^{(1)} \equiv h_i$. The transformation $(\lambda, \mu) \to (q, p)$ to the flat coordinates of associated metric follows from the point transformation

$$\begin{aligned}
\sigma_1(q) &= q_1 = -\lambda^1 - \lambda^2 - \lambda^3, \\
\sigma_2(q) &= \tfrac{1}{4}q_1^2 + q_2 = \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3, \\
\sigma_3(q) &= \tfrac{1}{2}q_1 q_2 + q_3 = -\lambda^1\lambda^2\lambda^3.
\end{aligned}$$

In the flat coordinates the Hamiltonians take the form

$$\begin{aligned}
&E_1 = p_1p_3 + \tfrac{1}{2}p_2^2, \\
&E_2 = p_1p_2 + \tfrac{1}{2}q_1p_2^2 + \tfrac{1}{2}q_1p_1p_3 - \tfrac{1}{2}q_2p_2p_3 - \tfrac{1}{2}q_3p_3^2, \\
&E_3 = \tfrac{1}{2}p_1^2 + \tfrac{1}{8}q_1^2p_2^2 + \tfrac{1}{8}q_2^2p_3^2 + \tfrac{1}{2}q_1p_1p_2 + \tfrac{1}{2}q_2p_1p_3 \\
&\quad - (\tfrac{1}{4}q_1q_2 + q_3)p_2p_3,
\end{aligned}$$

and admit a quasi bi-inverse-Hamiltonian representation (6.3)

$$\begin{aligned}
&\omega_1 x_1 = \omega_0 x_2 + F_1\omega_0 x_1, \\
&\omega_1 x_2 = \omega_0 x_3 + F_2\omega_0 x_1, \\
&\omega_1 x_3 = F_3\omega_0 x_1,
\end{aligned}$$

with the operators ω_0 and ω_1 of the form

$$\omega_0 = \pi_0^{-1} = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}, \tag{7.1}$$

$$
\omega_1 = \begin{pmatrix} 0 & \frac{1}{2}p_2 & \frac{1}{2}p_3 & \frac{1}{2}q_1 & \frac{1}{2}q_2 & q_3 \\ -\frac{1}{2}p_2 & 0 & 0 & -1 & 0 & \frac{1}{2}q_2 \\ -\frac{1}{2}p_3 & 0 & 0 & 0 & -1 & \frac{1}{2}q_1 \\ -\frac{1}{2}q_1 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}q_2 & 0 & 1 & 0 & 0 & 0 \\ -q_3 & -\frac{1}{2}q_2 & -\frac{1}{2}q_1 & 0 & 0 & 0 \end{pmatrix}, \tag{7.2}
$$

where I_3 is an 3×3 unit matrix, the expansion coefficients $F_i^{(1,1)} \equiv F_i$:

$$
F_1 = -q_1, \quad F_2 = -\tfrac{1}{4}q_1^2 - q_2, \quad F_3 = -\tfrac{1}{2}q_1 q_2 - q_3
$$

and Hamiltonian vector fields $x_i = \pi_0 dh_i,\ i = 1,2,3.$

On the extended phase space $\mathcal{M}$ of dimension seven, with an additional coordinate c, the differentials dh_i form a bi-inverse-Hamiltonian chain

$$
\begin{aligned}
\Omega_0 Y_0 &= 0 \\
\Omega_0 Y_1 &= dh_1 = \Omega_1 Y_0 \\
\Omega_0 Y_2 &= dh_2 = \Omega_1 Y_1 \\
\Omega_0 Y_3 &= dh_3 = \Omega_1 Y_2 \\
0 &= \Omega_1 Y_3,
\end{aligned}
$$

with presymplectic forms

$$
\Omega_0 = \left(\begin{array}{c|c} \omega_0 & 0 \\ \hline 0 & 0 \end{array}\right) \quad , \quad \Omega_1 = \left(\begin{array}{c|c} \omega_1 & dh_1 \\ \hline -dh_1^T & 0 \end{array}\right)
$$

d-compatible with respect to Π_0 and vector fields

$$
Y_0 = (0, ..., 0, 1)^T, \quad Y_i = X_i - F_i Y_0, \quad i = 1,2,3,
$$

where $X_i = \Pi_0 dh_i$.

Example 2.
Consider the separation relations on a six-dimensional phase space given by the following bare separation curve

$$
\lambda^2(h_1^{(1)}\lambda + h_2^{(1)}) + h_1^{(2)} = \tfrac{1}{2}\mu^2
$$

representing geodesic motion for a classical Stäckel system (this time of non-Benenti type [25]). Using the coordinates, the Hamiltonians, and the functions σ_i from the previous example we find that

$$
\begin{aligned}
h_1^{(1)} &= -\tfrac{1}{\sigma_2}h_2, \\
h_2^{(1)} &= h_1 - \tfrac{\sigma_1}{\sigma_2}h_2, \\
h_1^{(2)} &= h_3 - \tfrac{\sigma_3}{\sigma_2}h_2
\end{aligned}
$$

and thus we see that the Hamiltonians $h_i^{(k)}$ are related to h_j through the so-called generalized Stäckel transform (see [26] for further details on the latter). They admit a quasi bi-inverse-Hamiltonian representation (6.3)

$$
\begin{aligned}
\omega_1 x_1^{(1)} &= \omega_0 x_2^{(1)} + F_1^{(1,1)}\omega_0 x_1^{(1)} + F_1^{(1,2)}\omega_0 x_1^{(2)}, \\
\omega_1 x_2^{(1)} &= F_2^{(1,1)}\omega_0 x_1^{(1)} + F_2^{(1,2)}\omega_0 x_1^{(2)}, \\
\omega_1 x_1^{(2)} &= F_1^{(2,1)}\omega_0 x_1^{(1)} + F_1^{(2,2)}\omega_0 x_1^{(2)}
\end{aligned}
$$

with the presymplectic forms (7.1), (7.2), the expansion coefficients

$$
F_1^{(1,1)} = -\sigma_1 + \tfrac{\sigma_3}{\sigma_2}, \quad F_2^{(1,1)} = -\sigma_2 + \tfrac{\sigma_1\sigma_3}{\sigma_2}, \quad F_1^{(2,1)} = \tfrac{\sigma_3^2}{\sigma_2},
$$

$$F_1^{(1,2)} = -\tfrac{1}{\sigma_2}, \quad F_2^{(1,2)} = -\tfrac{\sigma_1}{\sigma_2}, \quad F_1^{(2,2)} = -\tfrac{\sigma_3}{\sigma_2}$$

and Hamiltonian vector fields $x_i^{(k)} = \pi_0 dh_i^{(k)}$.

On the extended phase space $\mathcal{M}$ of dimension eight, with an additional coordinates c_1 and c_2, the differentials $dh_i^{(k)}$ form a bi–inverse-Hamiltonian chains (6.7)

$$\begin{array}{ll} \Omega_0 Y_0^{(1)} = 0 & \\ \Omega_0 Y_1^{(1)} = dh_1^{(1)} = \Omega_1 Y_0^{(1)} & \Omega_0 Y_0^{(2)} = 0 \\ \Omega_0 Y_2^{(1)} = dh_2^{(1)} = \Omega_1 Y_1^{(1)} & \Omega_0 Y_1^{(2)} = dh_1^{(2)} = \Omega_1 Y_0^{(2)} \\ \qquad\qquad 0 = \Omega_1 Y_2^{(1)} & \qquad\qquad 0 = \Omega_1 Y_1^{(2)}, \end{array}$$

with the presymplectic forms

$$\Omega_0 = \left(\begin{array}{c|c} \omega_0 & 0 \;\; 0 \\ \hline 0 & \\ 0 & 0 \end{array}\right) \quad , \quad \Omega_1 = \left(\begin{array}{c|c} \omega_1 & dh_1^{(1)} \quad dh_1^{(2)} \\ \hline -(dh_1^{(1)})^T & \\ -(dh_1^{(2)})^T & 0 \end{array}\right)$$

d-compatible with respect to Π_0 and vector fields

$$Y_0^{(1)} = (0, ..., 0, 1, 0)^T, \quad Y_0^{(2)} = (0, ..., 0, 0, 1)^T, \quad Y_1^{(1)} = X_1^{(1)} - F_1^{(1,1)} Y_0^{(1)} - F_1^{(1,2)} Y_0^{(2)},$$

$$Y_2^{(1)} = X_2^{(1)} - F_2^{(1,1)} Y_0^{(1)} - F_2^{(1,2)} Y_0^{(2)}, \quad Y_1^{(2)} = X_1^{(2)} - F_1^{(2,1)} Y_0^{(1)} - F_1^{(2,2)} Y_0^{(2)}.$$

Example 3.

Consider separation relations on a six-dimensional phase space given by the following bare separation curve, cubic in momenta,

$$h_1^{(1)} \mu + h_1^{(2)} \lambda + h_2^{(2)} = \mu^3.$$

The transformation $(\lambda, \mu) \to (q, p)$ to new canonical coordinates in which all Hamiltonians are of a polynomial form is obtained from the following two transformations:

$$\begin{aligned} u_1 &= 3q_2 - 3q_3, \\ u_2 &= -q_1 p_2 - q_1 p_3 + 3q_3^2 + 5q_1^3 - 6q_2 q_3, \\ u_3 &= -q_3^3 - 9q_1^3 q_3 + q_1 q_3 p_2 + q_1 q_3 p_3 - 6q_1^3 q_2 \\ &\quad + q_1^2 p_1 + 3q_2 q_3^2, \\ v_1 &= -\frac{1}{q_1}, \\ v_2 &= \frac{3q_2 - 2q_3}{q_1}, \\ v_3 &= p_3 + \tfrac{2}{3} p_2 - \frac{q_3^2}{q_1} + 3\frac{q_2 q_3}{q_1} - 4q_1^2, \end{aligned}$$

and

$$\begin{aligned} u_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ u_2 &= \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3, \\ u_3 &= \lambda_1 \lambda_2 \lambda_3, \\ \mu_i &= v_1 \lambda_i^2 + v_2 \lambda_i + v_3, \qquad i = 1, 2, 3. \end{aligned}$$

In the (q,p)-coordinates the Hamiltonians take the form

$$
\begin{aligned}
h_1^{(1)} &= p_2p_3 + \tfrac{1}{3}p_2^2 + p_3^2 - 7q_1^2p_3 - 4q_1^2p_2 - 3q_2p_1 + 18q_1q_2^2 + 13q_1^4 + 12q_3q_1q_2,\\
h_1^{(2)} &= 12q_1^3q_2 + 8q_1^3q_3 - 2q_1^2p_1 + (-6q_1q_2 - 4q_1q_3)p_3 + p_1p_3,\\
h_2^{(2)} &= \tfrac{1}{3}p_2p_3^2 + \tfrac{1}{3}p_2^2p_3 + \tfrac{2}{27}p_2^3 - q_1^2p_3^2 - \tfrac{4}{3}q_1^2p_2^2 - q_2p_1p_2 - q_1p_1^2 - \tfrac{10}{3}q_1^2p_3p_2\\
&\quad + (q_3 - 3q_2)p_1p_3 + (21q_1^2q_2 + 6q_3q_1^2)p_1 + (4q_3q_1q_2 + 6q_1q_2^2 + \tfrac{22}{3}q_1^4)p_2\\
&\quad + (7q_1^4 + 18q_1q_2^2 + 6q_3q_1q_2 - 4q_1q_3^2)p_3 - 8q_1^3q_3^2 - 72q_3q_1^3q_2 - 90q_1^3q_2^2 - 12q_1^6.
\end{aligned}
$$

They form a quasi bi-inverse-Hamiltonian chain (6.3)

$$
\begin{aligned}
\omega_1 x_1^{(1)} &= F_1^{(1,1)}\omega_0 x_1^{(1)} + F_1^{(1,2)}\omega_0 x_1^{(2)},\\
\omega_1 x_1^{(2)} &= \omega_0 x_2^{(2)} + F_1^{(2,1)}\omega_0 x_1^{(1)} + F_1^{(2,2)}\omega_0 x_1^{(2)},\\
\omega_1 x_2^{(2)} &= F_2^{(2,1)}\omega_0 x_1^{(1)} + F_2^{(2,2)}\omega_0 x_1^{(2)},
\end{aligned}
$$

with the non-canonical symplectic form

$$
\omega_1 = \begin{pmatrix}
0 & -B & -C & q_3 & -A & -2q_1^2\\
B & 0 & 24q_1^2 & -3q_1 & -3q_2+q_3 & 0\\
C & -24q_1^2 & 0 & 2q_1 & q_2 & q_3\\
-q_3 & 3q_1 & -2q_1 & 0 & 0 & 0\\
A & 3q_2-q_3 & -q_2 & 0 & 0 & \tfrac{1}{3}q_1\\
2q_1^2 & 0 & -q_3 & 0 & -\tfrac{1}{3}q_1 & 0
\end{pmatrix},
$$

where $A = \frac{1}{3}p_2 + \frac{1}{3}p_3 - 3q_1^2$, $B = 54q_1q_2 + 24q_1q_3 - 3p_1$, $C = -24q_1q_2 - 12q_1q_3 + p_1$ and the expansion coefficients

$$
F_1^{(1,1)} = -q_3,\quad F_1^{(1,2)} = -q_1,\quad F_1^{(2,1)} = -\tfrac{1}{3}p_2 + q_1^2,\quad F_1^{(2,2)} = -2q_3 + 3q_2,
$$

$$
F_2^{(2,1)} = 5q_3q_1^2 + 6q_1^2q_2 - q_1p_1 - \tfrac{1}{3}q_3p_2,\quad F_2^{(2,2)} = -4q_1^3 - q_3^2 + 3q_2q_3 + \tfrac{2}{3}q_1p_2 + q_1p_3.
$$

On the extended phase space $\mathcal{M}$ of dimension eight, with additional coordinates c_1 and c_2, the differentials $dh_i^{(k)}$ form a bi–inverse-Hamiltonian chains (6.7)

$$
\begin{aligned}
\Omega_0 Y_0^{(1)} &= 0\\
\Omega_0 Y_1^{(1)} &= dh_1^{(1)} = \Omega_1 Y_0^{(1)}\\
0 &= \Omega_1 Y_1^{(1)},
\end{aligned}
\qquad
\begin{aligned}
\Omega_0 Y_0^{(2)} &= 0\\
\Omega_0 Y_1^{(2)} &= dh_1^{(2)} = \Omega_1 Y_0^{(2)}\\
\Omega_0 Y_2^{(2)} &= dh_2^{(2)} = \Omega_1 Y_1^{(2)}\\
0 &= \Omega_1 Y_2^{(2)}
\end{aligned}
$$

with the presymplectic forms

$$
\Omega_0 = \left(\begin{array}{c|cc} \omega_0 & 0 & 0\\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{0} \end{array}\right), \quad \Omega_1 = \left(\begin{array}{c|cc} \omega_1 & dh_1^{(1)} & dh_1^{(2)}\\ \hline -(dh_1^{(1)})^T & & \\ -(dh_1^{(2)})^T & \multicolumn{2}{c}{0} \end{array}\right)
$$

d-compatible with respect to Π_0 and vector fields

$$
Y_0^{(1)} = (0,...,0,1,0)^T,\quad Y_0^{(2)} = (0,...,0,0,1)^T,\quad Y_1^{(1)} = X_1^{(1)} - F_1^{(1,1)}Y_0^{(1)} - F_1^{(1,2)}Y_0^{(2)},
$$

$$
Y_1^{(2)} = X_1^{(2)} - F_1^{(2,1)}Y_0^{(1)} - F_1^{(2,2)}Y_0^{(2)},\quad Y_2^{(2)} = X_2^{(2)} - F_2^{(2,1)}Y_0^{(1)} - F_2^{(2,2)}Y_0^{(2)}.
$$

The bi-Hamiltonian extensions of systems from presented examples the reader can find in [18].

Acknowledgement

The work is partially supported by Polish MNiSW research grant no. N N202 404933

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