BALANCED METRICS AND CHOW STABILITY OF PROJECTIVE BUNDLES OVER RIEMANN SURFACES

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Abstract. In 1980, I. Morrison proved that slope stability of a vector bundle of rank 2 over a compact Riemann surface implies Chow stability of the projectivization of the bundle with respect to certain polarizations. We generalized Morrison's result to higher rank vector bundles over compact algebraic manifolds of arbitrary dimension that admit constant scalar curvature metric and have discrete automorphism group. In this article, we give a simple proof for polarizations $\mathcal{O}_{\mathbb{P}E^*}(d)\otimes\pi^*L^k$, where d is a positive integer, $k\gg 0$ and the base manifold is a compact Riemann surface of genus $g\geq 2$.

1. Introduction

In [M], Morrison proved that for the projectivization of a rank two holomorphic vector bundle over a compact Riemann surface, Chow stability is equivalent to the stability of the bundle. In [S], We generalized one direction of Morrison's result for higher rank vector bundles over compact algebraic manifolds of arbitrary dimension that admit constant scalar curvature metric and have discrete automorphism group ([S]).

Let X be a compact complex manifold of dimension m and $\pi : E \to X$ be a holomorphic vector bundle of rank r with dual bundle E^*. This gives a holomorphic fibre bundle $\mathbb{P}E^*$ over X with fibre $\mathbb{P}^{r-1}$. One can pull back the vector bundle E to $\mathbb{P}E^*$. We denote the tautological line bundle on $\mathbb{P}E^*$ by $\mathcal{O}_{\mathbb{P}E^*}(-1)$ and its dual by $\mathcal{O}_{\mathbb{P}E^*}(1)$. Let $L \to X$ be an ample line bundle on X and $\omega\in 2\pi c_1(L)$ be a Kähler form. Since L is ample, there is an integer k_0 so that for any $k\geq k_0$, $\mathcal{O}_{\mathbb{P}E_k^*}(1)$ is very ample over $\mathbb{P}E_k^*$, where $E_k = E\otimes L^{\otimes k}$. Note that there is a canonical isomorphism $\mathbb{P}E_k^*\cong\mathbb{P}E^*$ and $\mathcal{O}_{\mathbb{P}E_k^*}(1)\cong\mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k$. The main theorem of [S] is the following:

Theorem 1.1. *Suppose that $Aut(X)$ is discrete and X admits a constant scalar curvature Kähler metric in the class of $2\pi c_1(L)$. If E is Mumford stable, then*

$$(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k)$$

is Chow stable for $k\gg k_0$.

One of the earliest results in this spirit is the work of Burns and De Bartolomeis in [BD]. They constructed a ruled surface which does not admit any extremal metric in certain cohomology class. In [H1], Hong proved that there are constant scalar curvature Kähler metrics on the projectivization of stable bundles over curves. In [H2] and [H3], he generalized this result to higher dimensions with some extra assumptions. Combining Hong's results with Donaldson's, $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(n)\otimes\pi^*L^m)$

2000 *Mathematics Subject Classification.* Primary 32Q15; Secondary 53C07.

is Chow stable for $m, n \gg 0$ when the bundle E is stable. Note that it differs from our result, since it implies the Chow stability of $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*_m}(n))$ for n, m big enough.

In [RT], Ross and Thomas developed the notion of slope stability for polarized algebraic manifolds. As one of the applications of their theory, they proved that if $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$ is slope semi-stable for $k \gg 0$, then E is a slope semistable bundle and (X, L) is a slope semistable manifold.Again note that they look at stability of $\mathbb{P}E^*$ with respect to polarizations $\mathcal{O}_{\mathbb{P}E^*_m}(n)$ for n big enough. For the case of one dimensional base, however they showed stronger results. In this case they proved that if $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L)$ is slope (semi, poly) stable for any ample line bundle L, then E is a slope (semi, poly) stable bundle.

In order to prove Theorem 1.1 we used the concept of balanced metrics. Combining the results of Luo, Phong, Sturm, Wang and Zhang on the relation between balanced metrics and stability, we proved the following.

Theorem 1.2. *([S]) Suppose that $Aut(X)$ is discrete and X admits a constant scalar curvature Kähler metric in the class of $2\pi c_1(L)$. If E is Mumford stable, then*

$$(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$$

admits balanced metrics for $k \gg 0$.

In this paper, we give another proof of Theorem 1.1 in the case of one dimensional base X. The proof is simple in this case and can be generalized to polarizations $\mathcal{O}_{\mathbb{P}E^*}(d) \otimes \pi^* L^k$ for any positive integer d and $k \gg 0$. The main theorem of this paper is the following.

Theorem 1.3. *Let X be a compact Riemann surface of genus $g \geq 2$ and $E \to X$ be a holomorphic vector bundle on X. Let d be a positive integer. If E is Mumford stable, then*

$$(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(d) \otimes \pi^* L^k)$$

admits balanced metrics g_k for $k \gg 0$.

The Hitchin-Kobayashi correspondence implies that the stable bundle E admits a Hermitian-Einstein metric h_∞. A simple calculation shows that the Hermitian metric $\mathrm{Sym}^d h_\infty$ on $\mathrm{Sym}^d E$ is Hermitian-Einstein. Therefore the vector bundle $\mathrm{Sym}^d E$ is stable. By a theorem of Wang , we know that there exist balanced metrics $H^{(k)}$ on $\mathrm{Sym}^d E \otimes L^k$. This means that there exists a basis $s_1, ..., s_N$ for $H^0(X, \mathrm{Sym}^d E \otimes L^k)$ such that

$$\sum s_i \otimes s_i^{*_{H^{(k)}}} = I_{\mathrm{Sym}^d E}$$

$$\int_X \langle s_i, s_j \rangle_{H^{(k)}} \omega = \frac{rV}{N} \delta_{ij}.$$

Using the canonical isomorphism between $H^0(X, \mathrm{Sym}^d E \otimes L^k)$ and $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(d) \otimes L^k)$, we get a sequence of Hermitian metrics $\widehat{H^{(k)}}$ on $\mathcal{O}_{\mathbb{P}E^*}(d) \otimes L^k$. We prove that the sequence $\widehat{H^{(k)}}$ is "almost balanced", i.e.

$$\int \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{H^{(k)}}} dvol_{\widehat{h^{(k)}}} = D^{(k)} \delta_{ij} + M_{ij}^{(k)},$$

where $D^{(k)} \to C_{r,d}$ as $k \to \infty$ (see (2.2) for the definition of $C_{r,d}$) and $M^{(k)}$ is a trace-free Hermitian matrix such that $||M^{(k)}||_{op} = o(k^{-\infty})$ as $k \to \infty$.

The next step is to perturb these almost balanced metrics to get balanced metrics. As pointed out by Donaldson, the problem of finding balanced metric can be viewed also as a finite dimensional moment map problem solving the equation $M^{(k)} = 0$. Indeed, Donaldson shows that $M^{(k)}$ is the value of a moment map μ_D on the space of ordered bases with the obvious action of $SU(N)$. Now, the problem is to show that if for some ordered basis $\underline{s}$, the value of moment map is very small, then we can find a basis at which moment map is zero. The standard technique is flowing down $\underline{s}$ under the gradient flow of $|\mu_D|^2$ to reach a zero of μ_D. We need a Lojasiewicz type inequality to guarantee that the flow converges to a zero of the moment map. This was done in [S] by adapting Phong-Sturm proof to our situation. In [S2], we generalize Theorem 1.3 to higher dimensional base manifolds that admits cscK metrics and do not have any nonzero holomorphic vector fields. After this work was completed, we became aware of the preprint [DZ].

Acknowledgements: I am sincerely grateful to Richard Wentworth for many helpful discussions and suggestions and his continuous help, support and encouragement.

2. Preliminaries

Let V be a Hermitian vector space of dimension r. The projective space $\mathbb{P}V^*$ can be identified with the space of hyperplanes in V via

$$f \in V - \{0\} \to ker(f) = V_f \subset V.$$

There is a natural isomorphism between V and $H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$ which sends $v \in V$ to $\hat{v} \in H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$ such that for any $f \in V^*, \hat{v}(f) = f(v)$. Any Hermitian inner product h on V induces a Hermitian inner product $\widehat{h}$ on $\mathcal{O}_{\mathbb{P}V^*}(1)$ as follows:

$$\langle \hat{v}, \hat{w} \rangle_{\widehat{h}}(f) = \frac{f(v)\overline{f(w)}}{|f|^2},$$

for $v, w \in V$ and $f \in V^*$.

For any positive integer d, define an equivalence relation $\sim$ on $V^{\otimes d}$ by

$$v_1 \otimes \cdots \otimes v_d \sim v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \qquad \sigma \in S_d.$$

We define $\mathrm{Sym}^d V = V^{\otimes d}/\sim$ and simply denote the class of $v_1 \otimes \cdots \otimes v_d$ in $\mathrm{Sym}^d V$ by $v_1 \dots v_d$. Similar to the case of $d = 1$ any Hermitian inner product h on V induces a Hermitian inner product $\mathrm{Sym}^d h$ on $\mathrm{Sym}^d V$ by

$$\langle v_1 \dots v_d, w_1 \dots w_d \rangle_{\mathrm{Sym}^d h} = \frac{1}{d!} \sum_{\sigma \in S_d} \langle v_1, w_{\sigma(1)} \rangle_h \dots \langle v_d, w_{\sigma(d)} \rangle_h.$$

Remark 2.1. Let $e_1, \dots, e_r$ be an orthonormal basis for V with respect to h, then the set

$$\{ \Big(\frac{i_1! \dots i_r!}{d!} \Big)^{\frac{-1}{2}} e_1^{i_1} \dots e_r^{i_r} |\ 0 \le i_\alpha \le d,\ \sum_{\alpha=1}^{r} i_\alpha = d \}$$

forms an orthonormal basis for $\mathrm{Sym}^d V$ with respect to $\mathrm{Sym}^d h$.

There is a natural isomorphism between $\mathrm{Sym}^d V$ and $H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(d))$ which sends $v_1 \dots v_d \in \mathrm{Sym}^d V$ to $\widehat{v_1 \dots v_d} \in H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(d))$ defined by

$$\widehat{v_1 \dots v_d}([v^*])(w_1^* \otimes \cdots \otimes w_d^*) = w_1^*(v_1) \dots w_d^*(v_d), \tag{2.1}$$

where $v^* \in V^* - \{0\}$ and $w_i^* \in V^*$ are scalar multiple of v^*. There exist complex numbers $\lambda_1, \dots, \lambda_d$ such that $w_i^* = \lambda_i v^*$. Thus,

$$\widehat{v_1 \dots v_d}([v^*])(w_1^* \otimes \dots \otimes w_d^*) = \lambda_1 \dots \lambda_d v^*(v_1) \dots v^*(v_d)$$

and therefore (2.1) defines a well-defined section of $\mathcal{O}_{\mathbb{P}V^*}(d)$.

For any Hermitian inner product H on $\mathrm{Sym}^d V$, we define a metric $\hat{H}$ on $\mathcal{O}_{\mathbb{P}V^*}(d)$ by

$$\langle \hat{s}, \hat{t} \rangle_{\hat{H}}[v] = \frac{v^{\otimes d}(s)\overline{v^{\otimes d}(t)}}{|v \dots v|_H^2}.$$

In particular, we have

$$\langle \hat{s}, \hat{t} \rangle_{\widehat{\mathrm{Sym}^d h}}[v] = \frac{v^{\otimes d}(s)\overline{v^{\otimes d}(t)}}{|v|_h^{2d}}.$$

The following lemmas are straight forward.

Lemma 2.2. *For any Hermitian inner product h on V, we have*

$$\hat{h}^{\otimes d} = \widehat{Sym^d h}.$$

Lemma 2.3. *There exists a constant $C_{r,d}$ such that for any $v, w \in Sym^d V$ and any Hermitian inner product h on V,*

$$d^{r-1} \int_{\mathbb{P}V^*} \langle \hat{v}, \hat{w} \rangle_{\widehat{Sym^d h}} \frac{\omega_{FS,h}^{r-1}}{(r-1)!} = C_{r,d} \langle v, w \rangle_{Sym^d h}, \tag{2.2}$$

where $\omega_{FS,h} = i\bar{\partial}\partial \log \hat{h}$.

Remark 2.4. Let H be a Hermitian inner product on V. Suppose there exists a constant C such that

$$\int_{\mathbb{P}V^*} < \hat{v}, \hat{w} >_{\hat{H}} \frac{\omega_{\mathrm{FS},H}^{r-1}}{(r-1)!} = C < v, w >_H,$$

for any $v, w \in \mathrm{Sym}^d V$. Then $H = \mathrm{Sym}^d h$ for some Hermitian inner product h on V.

Lemma 2.5. *Let h_0 and h be Hermitian inner products on V. If $||h - h_0||_{h_0} \leq \epsilon$, then $||\hat{h} - \hat{h}_0||_{C^2(\hat{h}_0)} \leq C\epsilon$, for a constant C depends only on r and d.*

Lemma 2.6. *Let X be a Kähler manifold of dimension n and Ω_0 and Ω be two Kähler forms on X. There exists a constant C depends only on the dimension of X such that if $||\Omega - \Omega_0||_{C^0(\Omega_0)} \leq \epsilon$, then $\left|\frac{\Omega^n - \Omega_0^n}{\Omega_0^n}\right| \leq C\epsilon$.*

Proposition 2.7. *Let h be a Hermitian inner product on V and H be a Hermitian inner product on $Sym^d V$ such that $||H - Sym^d h||_{Sym^d h} < \min(\epsilon, \frac{1}{2})$. Then for any $v, w \in Sym^d V$, we have*

$$\left| d^{r-1} \int_{\mathbb{P}V^*} \langle \widehat{v}, \widehat{w} \rangle_{\widehat{H}} \frac{\omega_{FS,h}^{r-1}}{(r-1)!} - C_{r,d} \langle v, w \rangle_H \right| \leq C\epsilon |v|_H |w|_H,$$

where C is a constant depends only on r and d.

Proof. Let $e_1, \dots e_K$ be an orthonormal basis for $\mathrm{Sym}^d V$ with respect to $\mathrm{Sym}^d h$. Define $H_{ij} = H(e_i, e_j)$ and $\epsilon_{ij} = H_{ij} - \delta_{ij}$. We have $|\epsilon_{ij}| = |H_{ij} - \delta_{ij}| \le \epsilon$. Given $v, w \in \mathrm{Sym}^d V$, we can write $v = \sum a_i e_i$ and $w = \sum b_i e_i$. We have

$$\begin{aligned}\Big|\langle v,w\rangle_{\mathrm{Sym}^d h} - \langle v,w\rangle_H\Big| &= \Big|\sum a_i \bar{b_i} - \sum a_i \bar{b_j} H_{ij}\Big| = \Big|\sum a_i \bar{b_j}\epsilon_{ij}\Big| \\ &\le \epsilon \sum |a_i||\bar{b_j}| \le K\epsilon \sum |a_i|^2 \sum |b_j|^2 \\ &= \epsilon K |v|_{\mathrm{Sym}^d h} |w|_{\mathrm{Sym}^d h}.\end{aligned}$$

The last inequality follows from Cauchy-Shwartz inequality. By a unitary change of basis we may assume that $H_{ij} = 0$ if $i \neq j$. Therefore the basis $\{f_1 = H_{11}^{\frac{-1}{2}} e_1, \dots f_K = H_{11}^{\frac{-1}{2}} e_K\}$ is an orthonormal basis for $\mathrm{Sym}^d V$ with respect to H. We have $\frac{1}{2} \le H_{ii} \le \frac{3}{2}$ since $|H_{ii} - 1| \le \frac{1}{2}$. Thus,

$$\Big||f_i|^2_{\mathrm{Sym}^d h} - |f_i|^2_H\Big| = |1 - H_{ii}^{-1}| = \frac{|1 - H_{ii}|}{H_{ii}} \le 2\epsilon.$$

Therefore by the same argument, we conclude that

$$\Big|\langle v,w\rangle_{\mathrm{Sym}^d h} - \langle v,w\rangle_H\Big| \le 2\epsilon K |v|_H |w|_H. \tag{2.3}$$

Applying (2.3), Lemma 2.3, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}&\Big|d^{r-1}\int_{\mathbb{P}V^*} \langle \widehat{v}, \widehat{w}\rangle_{\widehat{H}} \frac{\omega_{FS,h}^{r-1}}{(r-1)!} - C_{r,d}\langle v,w\rangle_H\Big| \\ &\le \Big|d^{r-1}\int_{\mathbb{P}V^*} \langle \widehat{v}, \widehat{w}\rangle_{\widehat{H}} \frac{\omega_{FS,h}^{r-1}}{(r-1)!} - d^{r-1}\int_{\mathbb{P}V^*} \langle \widehat{v}, \widehat{w}\rangle_{\widehat{\mathrm{Sym}^d h}} \frac{\omega_{FS,h}^{r-1}}{(r-1)!}\Big| + C_{r,d}\Big|\langle v,w\rangle_{\mathrm{Sym}^d h} - \langle v,w\rangle_H\Big| \\ &\le d^{r-1}\int_{\mathbb{P}V^*} \Big|\langle \widehat{v}, \widehat{w}\rangle_{\widehat{H}} - \langle \widehat{v}, \widehat{w}\rangle_{\widehat{\mathrm{Sym}^d h}}\Big| \frac{\omega_{FS,h}^{r-1}}{(r-1)!} + C_{r,d}\Big|\langle v,w\rangle_{\mathrm{Sym}^d h} - \langle v,w\rangle_H\Big| \\ &\le C\epsilon d^{r-1}\int_{\mathbb{P}V^*} |\widehat{v}|_{\widehat{\mathrm{Sym}^d h}} |\widehat{w}|_{\widehat{\mathrm{Sym}^d h}} \frac{\omega_{FS,h}^{r-1}}{(r-1)!} + 2KC_{r,d}\epsilon |v|_H |w|_H \\ &\le \epsilon(Cd^{r-1}V + 2KC_{r,d})|v|_H|w|_H\end{aligned}$$

The last inequality follows from the fact that $\sup_{\mathbb{P}V^*} |\widehat{v}|_{\widehat{\mathrm{Sym}^d h}} = |v|_{\mathrm{Sym}^d h}$. □

3. Balanced Metrics On Holomorphic Vector Bundles

Let (X, ω_0) be a compact Kähler manifold of dimension n and (L, g) be an ample holomorphic Hermitian line bundle over X such that $i\bar{\partial}\partial \log g = \omega_0$. Let E be a holomorphic vector bundle of rank r over X. By possibly tensoring with high power of the ample line bundle L, we may assume that E is very ample. Therefore we can embed X into $G(r, H^0(X,E)^*)$, the Grassmanian of r-planes in $H^0(X,E)^*$. Indeed, for any $x \in X$, we have the evaluation map $H^0(X,E) \to E_x$, which sends s to $s(x)$. Since E is globally generated, this map is a surjection. So its dual is an inclusion of $E_x^* \hookrightarrow H^0(X,E)^*$, which determines an r-dimensional subspace of $H^0(X,E)^*$. Therefore we get a map $\iota : X \to G(r, H^0(X,E)^*)$. Since E is very ample, ι is an embedding. Clearly we have $\iota^* U_r = E^*$, where U_r is the tautological vector bundle on $G(r, H^0(X,E)^*)$, i.e. at any r-plane in $G(r, H^0(X,E)^*)$, the fibre of U_r is exactly that r-plane. A choice of basis for $H^0(X,E)$ gives an isomorphism between

$G(r, H^0(X,E)^*)$ and the standard Grassmanian $G(r,N)$, where $N = \dim H^0(X,E)$. We have the standard Fubini-Study Hermitian metric on U_r, so we can pull it back to E and get a Hermitian metric on E.

Definition 3.1. The embedding is called balanced if

$$\int_X \langle s_i, s_j\rangle_{\iota^* h_{\mathrm{FS}}} \frac{\omega^n}{n!} = C\delta_{ij}.$$

Notice that being balanced depends on the choice of the Kähler form. A Hermitian metric on E is called balanced (more precisely ω-balanced) if it is the pull back $\iota^* h_{\mathrm{FS}}$, where ι is a balanced embedding.

Equivalently, we can formulate the definition of balance metrics in terms of Bergman kernels.

Definition 3.2. Let h be a Hermitian metric on E and $s_1, ..., s_N$ be an orthonmal basis for $H^0(X,E)$ with respect to the inner product

$$\langle s,t\rangle = \int_X \langle s(x), t(x)\rangle_h \frac{\omega_0^n}{n!}$$

The Bergman kernel of (E,h) is an endomorhism of E defined by

$$B(h,\omega_0) = \sum_{i=1}^N s_i \otimes s_i^{*_h}.$$

Note that $B(h,\omega_0)$ does not depend on the choice of the orthonmal basis.

A Hermitian metric h on E is balanced if and only if $B(h,\omega_0) = CI_E$ for a positive constant C.

We recall Catlin-Tian-Yau-Zeldich asymptotic expansion of Bergman kernel.

Theorem 3.3. *([C], [Z]) Let (X,ω_0) be a compact Kähler manifold of dimension n and (L,g) be an ample holomorphic Hermitian line bundle over X such that $i\bar{\partial}\partial \log g = \omega_0$. For any Hermitian metric h on the vector bundle E, there exist smooth endomorphisms $A_i(h) \in \Gamma(X, End(E))$ such that the following asymptotic expansion holds as $k \to \infty$*

$$B(h \otimes g^{\otimes k}, \omega_0) \sim k^n + A_1(h)k^{n-1} + \dots. \tag{3.1}$$

There is a close relationship between stability of vector bundles and the existence of balanced metrics given by the following theorem of Wang.

Theorem 3.4. *([W],[W2, Theorem 1.2]) The bundle E is Gieseker stable if and only if there exist balanced metrics $h^{(k)}$ on $E \otimes L^k$ for $k \gg 0$. In addition if there exists a Hermitian metric h_∞ on E such that $h_k \to h_\infty$ in C^∞, then*

$$\frac{i}{2\pi}\Lambda F_{(E,h_\infty)} + \frac{1}{2}S(\omega_\infty)I_E = \Big(\frac{d}{Vr} + \frac{\overline{s}}{2}\Big)I_E, \tag{3.2}$$

where $h_k = h^{(k)} \otimes g_\infty^{\otimes(-k)}$, $S(\omega_\infty)$ is the scalar curvature of ω_∞ and $\overline{s}$ is the average of the scalar curvature. Conversely, if h_∞ solves (3.2), then there exists a sequence of balanced metrics $h^{(k)}$ on $E \otimes L^k$ for $k \gg 0$ and $h_k \to h_\infty$ in C^∞.

In the case that the base manifold X has dimension one and the Kähler metric ω_∞ has constant curvature, we prove that the rate of convergence of h_k to h_∞ is $O(k^{-\infty})$.

Theorem 3.5. *Let X be a compact Riemann surface and ω_∞ be a Kähler form of constant curvature on X. Let a be a positive integer. Suppose that the Hermitian metric h_∞ on E satisfies the Hermitian-Einstein equation*

$$\frac{i}{2\pi}F_{(E,h_\infty)} = \omega_\infty I_E.$$

Let $h^{(k)}$ be a sequence of balanced metric on $E\otimes L^k$ for $k \gg 0$ and $h_k = h^{(k)} \otimes g_\infty^{\otimes(-k)}$. If $h_k \to h_\infty$, then

$$||h_k - h_\infty||_{C^a(h_\infty)} = O(k^{-\infty}).$$

The proof follows from Theorem 3.4, lemma 3.6 and lemma 3.7.

Lemma 3.6. *Let h be a Hermitian metric on E. Suppose that E is stable and coefficients $A_1, \dots, A_q$ in the asymptotic expansion* (3.1) *are constant endomorphisms of E. If q is big enough, then there exists a sequence of balanced metrics $h^{(k)}$ on $E \otimes L^k$ for $k \gg 0$ such that*

$$||h - h^{(k)} \otimes g^{\otimes(-k)}||_{C^a(h)} = O(k^{3+\frac{13n}{2}+\frac{a}{2}-q}).$$

Proof. First we claim that

$$B_k(h) = \frac{\chi(k)}{rV}(I_E + \sigma_k),$$

where $||\sigma_k||_{C^a} = O(k^{n-q-1})$. In order to prove this, we observe that there exists a smooth section $A(x)$ of $End(E)$ such that

$$B_k(h) = k^n + A_1 k^{n-1} + ... + A_q k^{n-q} + A(x)k^{n-q-1}.$$

The bundle E is stable and A_j's are constant sections of $End(E)$. Therefore there exist numbers $a_1, ..., a_q$ such that $A_j = a_j I_E$. On the other hand

$$\int_X tr(B_k(h)\frac{\omega_\infty^n}{n!} = \chi(k)V,$$

where $V = \int_X \frac{\omega_\infty^n}{n!}$. Thus,

$$B_k(h) - \frac{\chi(k)}{rV}I_E = \Big(A(x) - \frac{1}{rV}\int_X A(x)I_E\Big)k^{n-q-1}.$$

Define $\sigma_k = \Big(A(x) - \frac{1}{rV}\int_X A(x)I_E\Big)k^{n-q-1}$, we have

$$B_k(h) = \frac{\chi(k)}{rV}(I + \sigma_k),$$

where $||\sigma_k||_{C^a} = O(k^{n-q-1})$. Now Wang's argument ([W2, page 276]) concludes the proof. □

Lemma 3.7. *In the situation of Theorem 3.5, all coefficients A_i's are constant.*

Proof. The coefficients of the asymptotic expansion of the Bergman kernel are polynomials of the curvature tensor on the base manifold, curvature tensor on the bundle and their covariant derivatives. The whole curvature tensors on the base manifold and on the bundle are constant. Therefore all coefficients are constant. □

4. Constructing Almost Balanced Metrics

The goal of this section is to prove Theorem 1.3. In order to prove Theorem 1.3, we construct a sequence of almost balanced metrics on $\mathcal{O}_{\mathbb{P}E^*}(d)\otimes L^k$(Theorem 4.5). We start with definition of balanced metrics on polarized manifolds.

Let (Y,ω) be a compact Kähler manifold of dimension n and $\mathcal{O}(1)\to Y$ be a very ample line bundle on Y equipped with a Hermitian metric σ such that $i\bar\partial\partial\log\sigma=\omega$. Since $\mathcal{O}(1)$ is very ample, using global sections of $\mathcal{O}(1)$, we can embed Y into $\mathbb{P}(H^0(Y,\mathcal{O}(1))^*)$. A choice of ordered basis $\underline{s}=(s_1,...,s_N)$ of $H^0(Y,\mathcal{O}(1))$ gives an isomorphism between $\mathbb{P}(H^0(Y,\mathcal{O}(1))^*)$ and $\mathbb{P}^{N-1}$. Hence for any such $\underline{s}$, we have an embedding $\iota_{\underline{s}}:Y\hookrightarrow\mathbb{P}^{N-1}$ such that $\iota_{\underline{s}}^*\mathcal{O}_{\mathbb{P}^N}(1)=\mathcal{O}(1)$. Using $\iota_{\underline{s}}$, we can pull back the Fubini-Study metric and Kähler form of the projective space to $\mathcal{O}(1)$ and Y respectively.

Definition 4.1. An embedding $\iota_{\underline{s}}$ is called balanced if

$$\int_Y\langle s_i,s_j\rangle_{\iota_{\underline{s}}^*h_{\mathrm{FS}}}\frac{\iota_{\underline{s}}^*\omega_{\mathrm{FS}}^n}{n!}=\frac{V}{N}\delta_{ij},$$

where $V=\int_Y\omega^n/n!$. A Hermitian metric (resp. a Kähler form) is called balanced if it is the pull back $\iota_{\underline{s}}^*h_{\mathrm{FS}}$ (resp. $\iota_{\underline{s}}^*\omega_{\mathrm{FS}}$) where $\iota_{\underline{s}}$ is a balanced embedding.

Remark 4.2. The concepts of balanced metric on holomorphic vector bundles (Definition 3.1) and balanced metric on polarized manifolds (Definition 4.1) are different. In order to find a balanced metric on a holomorphic vector bundle $E\to X$, we need to fix a Kähler form ω_0 on X. A Hermitian metric h on E is balanced (more precisely ω_0-balanced) if $B(h,\omega_0)=CI_E$, where C is a constant. But in order to find a balanced metric on a polarized manifold $(Y,\mathcal{O}(1))$, we do not need to fix a Kähler form. A positive Hermitian metric g on $\mathcal{O}(1)$ is balanced if $B(g,i\bar\partial\partial\log g)$ is constant.

Definition 4.3. A sequence of Hermitian metrics h_k on $\mathcal{O}(1)\otimes L^k$ and ordered bases $\underline{s}^{(k)}=(s_1^{(k)},...,s_N^{(k)})$ for $H^0(Y,\mathcal{O}(1)\otimes L^k)$ is called *almost balanced of order* q if for any k

$$\sum|s_i^{(k)}|^2_{h_k}=1$$

and

$$\int_Y\langle s_i^{(k)},s_j^{(k)}\rangle_{h_k}dvol_{h_k}=D^{(k)}\delta_{ij}+M_{ij}^{(k)},$$

where $D^{(k)}$ is a scalar so that $D^{(k)}\to C$ as $k\to\infty$, where C is a constant and $M^{(k)}$ is a trace-free Hermitian matrix such that $||M^{(k)}||_{\mathrm{op}}=O(k^{-q-1})$. Here $||M^{(k)}||_{op}$ is the operator norm of the matrix $M^{(k)}$.

For the rest of this section, let X be a compact Riemann surface and L be an ample line bundle on X. Let g be a positive Hermitian metric on L and $\omega_\infty=i\bar\partial\partial\log g$ be a Kähler form on X. Let E be a holomorphic vector bundle on X of rank r and slope μ. The slope of E is defined by $\mu=\frac{\deg(E)}{r}$.

Similar to the case of vector spaces, we have the natural isomorphism $H^0(\mathbb{P}E^*,\mathcal{O}_{\mathbb{P}E^*}(d)\otimes L^k)=H^0(X,\mathrm{Sym}^dE\otimes L^k)$. Also, any Hermitian metric H on Sym^dE induces a Hermitian metric $\widehat{H}$ on $\mathcal{O}_{\mathbb{P}E^*}(d)\otimes L^k$.

Suppose that H is a Hermitian metric on Sym^dE and $s_1,...,s_N$ is an orthonormal basis for $H^0(X,\mathrm{Sym}^dE\otimes L^k)$ with respect to $L^2(H_k,\omega_\infty)$, where $H_k=H\otimes g^{\otimes k}$. Let $\hat{s_1},...,\hat{s_N}$ be the corresponding basis for $H^0(\mathbb{P}E^*,\mathcal{O}_{\mathbb{P}E^*}(d))$.

We prove that the matrix $[\int_{\mathbb{P}E^*}\langle \widehat{s_i}, \widehat{s_j}\rangle_{\widehat{H}} dvol_{\widehat{H}}]$ is close to a scalar matrix. More precisely, we prove the following.

Proposition 4.4. *Let h_∞ be a Hermitian-Eienstein metric on E, i.e.*

$$iF_{(\bar{\partial}_E, h_\infty)} = \mu\omega_\infty I_E, \tag{4.1}$$

where $F_{(\bar{\partial}_E, h_\infty)}$ is the curvature of the chern connection of h_∞ and μ is the slope of E. Then there exists a constant C depends only on r and d such that if

$$||H - Sym^d h_\infty||_{C^2(Sym^d h_\infty)} \leq \min(\epsilon, \frac{1}{2}), \tag{4.2}$$

then

$$\Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j}\rangle_{\widehat{H_k}} dvol_{\widehat{H_k}} - C_{r,d}(d\mu + k)\delta_{ij}\Big| \leq C\epsilon(d\mu + k).$$

Here $H_k = H \otimes g^{\otimes k}$.

Proof. In this proof C denotes a constant depends only on r and d that might change from line to line. Define $H_\infty = \mathrm{Sym}^d h_\infty$, $\omega_0 = i\bar{\partial}\partial \log \widehat{H_\infty}$ and $\omega_k = \omega_0 + k\omega_\infty$. Lemma 2.2 implies that $\widehat{H_\infty} = \widehat{h_\infty}^{\otimes d}$ and therefore $\omega_0 = di\bar{\partial}\partial \log \widehat{h_\infty} = d\omega_{\widehat{h_\infty}}$. A simple calculation shows that $\omega^r_{\widehat{h_\infty}} = r\mu\omega^{r-1}_{\widehat{h_\infty}} \wedge \omega_\infty$, since h_∞ satisfies the Hermitian-Einstein equation (4.1). Thus,

$$\omega_k^r = \omega_0^r + rk\omega_0^{r-1} \wedge \omega_\infty = r(d\mu + k)\omega_0^{r-1} \wedge \omega_\infty. \tag{4.3}$$

Therefore,

$$\begin{aligned}\Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j}\rangle_{\widehat{H_k}} \frac{\omega_k^r}{r!} - C_{r,d}(d\mu + k)\delta_{ij}\Big| &= \Big|(d\mu + k) \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j}\rangle_{\widehat{H_k}} \frac{\omega_0^{r-1}}{(r-1)!} \wedge \omega_\infty - C_{r,d}(d\mu + k)\delta_{ij}\Big| \\ &= (d\mu + k)\Big| \int_X \Big(d^{r-1} \int_{\mathbb{P}E^*_x} \langle \widehat{s_i}, \widehat{s_j}\rangle_{\widehat{H_k}} \frac{\omega^{r-1}_{\widehat{h_\infty}}}{(r-1)!} - C_{r,d}\delta_{ij}\Big) \wedge \omega_\infty\Big| \\ &\leq C(d\mu + k)\epsilon \int_X |s_i|_{H_k} |s_j|_{H_k} \omega_\infty.\end{aligned}$$

The last inequality follows from Proposition 2.7. Hence Cauchy-Scwarz inequality implies that

$$\Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j}\rangle_{\widehat{H_k}} \frac{\omega_k^r}{r!} - C_{r,d}(d\mu + k)\delta_{ij}\Big| \leq C(d\mu + k)\epsilon\Big(\int_X |s_i|^2_{H_k}\omega_\infty\Big)^{\frac{1}{2}}\Big(\int_X |s_j|^2_{H_k}\omega_\infty\Big)^{\frac{1}{2}} = C(d\mu + k)\epsilon,$$

since $s_1, ..., s_N$ is an orthonormal basis for $H^0(X, \mathrm{Sym}^d E \otimes L^k)$ with respect to $L^2(H_k, \omega_\infty)$.

On the other hand, Lemma 2.5 implies that $||\omega - \omega_0||_{C^0(\omega_0)} \leq C\epsilon$. Therefore,

$$||(\omega + k\omega_\infty) - \omega_k||_{C^0(\omega_k)} = ||(\omega + k\omega_\infty) - (\omega_0 + k\omega_\infty)||_{C^0(\omega_k)} \leq ||\omega - \omega_0||_{C^0(\omega_0)} \leq C\epsilon,$$

since ω_∞ is a semipositive $(1,1)$-form on $\mathbb{P}E^*$. Applying Lemma 2.6 implies that

$$\Big| dvol_{\widehat{H_k}} - \frac{\omega_k^r}{r!}\Big| \leq C\epsilon\frac{\omega_k^r}{r!}. \tag{4.4}$$

Thus,

$$
\begin{aligned}
\Big|\int_{\mathbb{P}E^*}\langle \widehat{s}_i,\widehat{s}_j\rangle_{\widehat{H_k}} dvol_{\widehat{H_k}} - \int_{\mathbb{P}E^*}\langle \widehat{s}_i,\widehat{s}_j\rangle_{\widehat{H_k}}\frac{\omega_k^r}{r!}\Big| &\leq \int_{\mathbb{P}E^*}|\langle \widehat{s}_i,\widehat{s}_j\rangle_{\widehat{H_k}}||dvol_{\widehat{H_k}}-\frac{\omega_k^r}{r!}|\\
&\leq C\epsilon\int_{\mathbb{P}E^*}|s_i|_{H_k}|s_j|_{H_k}\frac{\omega_k^r}{r!}\\
&\leq C(d\mu+k)\epsilon\int_{\mathbb{P}E^*}|s_i|_{H_k}|s_j|_{H_k}\frac{\omega_0^{r-1}}{(r-1)!}\wedge\omega_\infty\\
&= C(d\mu+k)\epsilon\int_X|s_i|_{H_k}|s_j|_{H_k}\omega_\infty\\
&\leq C\epsilon\Big(\int_X|s_i|^2_{H_k}\omega_\infty\Big)^{\frac12}\Big(\int_X|s_j|^2_{H_k}\omega_\infty\Big)^{\frac12}\leq C\epsilon.
\end{aligned}
$$

Here we used (4.3), (4.4) and the fact $\sup_{\mathbb{P}E_x^*}|\widehat{s}_i|_{\widehat{H_k}} = |s_i(x)|_{H_k}$. We have

$$
\begin{aligned}
&\Big|\int_{\mathbb{P}E^*}\langle \widehat{s}_i,\widehat{s}_j\rangle_{\widehat{H_k}} dvol_{\widehat{H_k}} - C_{r,d}(d\mu+k)\delta_{ij}\Big|\\
&\leq \int_{\mathbb{P}E^*}|\langle \widehat{s}_i,\widehat{s}_j\rangle_{\widehat{H_k}}||dvol_{\widehat{H_k}}-\frac{\omega_k^r}{r!}| + \Big|\int_{\mathbb{P}E^*}\langle \widehat{s}_i,\widehat{s}_j\rangle_{\widehat{H_k}}\frac{\omega_k^r}{r!} - C_{r,d}(d\mu+k)\delta_{ij}\Big|\\
&\leq C(d\mu+k)\epsilon.
\end{aligned}
$$

□

Theorem 4.5. *Let X be a compact Riemann surface and $L\to X$ be an ample line bundle equipped with a Hermitian metric g. Suppose that $i\bar\partial\partial\log g=\omega_\infty$ is a Kähler form on X. Let E be a stable holomorphic vector bundle of rank r on X and h_∞ is a Hermitian-Einstein metric on E. Let R be the rank of Sym^dE. Suppose that $\{s_i^{(k)}\}_{i=1}^{N_k}$ is a sequence of bases for $H^0(X,Sym^dE\otimes L^{\otimes k})$ and $H^{(k)}$ is a sequence of Hermitian metrics on $Sym^dE\otimes L^{\otimes k}$ such that*

$$
\begin{aligned}
&\sum_{i=1}^{N_k}s_i^{(k)}\otimes(s_i^{(k)})^{*_{H^{(k)}}} = I_{Sym^dE\otimes L^k},\\
&\int_X\langle s_i^{(k)},s_j^{(k)}\rangle_{H^{(k)}}\omega_\infty = \frac{RVol(X,\omega_\infty)}{N_k}\delta_{ij},\\
&||H^{(k)}\otimes g^{\otimes(-k)} - Sym^dh_\infty||_{C^2(Sym^dh_\infty)} = O(k^{-\infty}).
\end{aligned}
$$

Then the sequence of Hermitian metrics $H^{(k)}$ on $\mathcal{O}_{\mathbb{P}E^*}(d)\otimes L^k$ and ordered bases $\underline{s}^{(k)}=(\widehat{s_1^{(k)}},\dots,\widehat{s_{N_k}^{(k)}})$ of $H^0(\mathbb{P}E^*,\mathcal{O}_{\mathbb{P}E^*}(d)\otimes L^k)$ is almost balanced of order q for any positive integer q.

Proof. Let p be a positive integer. There exists a constant C independent of k such that

$$||H^{(k)}\otimes g^{\otimes(-k)} - \mathrm{Sym}^dh_\infty||_{C^2(\mathrm{Sym}^dh_\infty)}\leq Ck^{-p}.$$

Fix $k\gg0$. The basis $\{\sqrt{R^{-1}N_k}s_1^{(k)},\dots,\sqrt{R^{-1}N_k}s_{N_k}^{(k)}\}$ is an orthonormal basis for $H^0(X,\mathrm{Sym}^dE\otimes L^k)$ with respect to $L^2(H^{(k)},\omega_\infty)$. Applying Proposition 4.4 to $H=H^{(k)}\otimes g^{\otimes -k}$ implies that

$$\Big|\int_{\mathbb{P}E^*}\langle\widehat{s_i^{(k)}},\widehat{s_j^{(k)}}\rangle_{\widehat{H^{(k)}}}dvol_{\widehat{H^{(k)}}} - C_{r,d}(d\mu+k)\delta_{ij}\Big|\leq Ck^{-p}(d\mu+k). \tag{4.5}$$

Define

$$M^{(k)} = \int_{\mathbb{P}E^*} \langle \widehat{s_i^{(k)}}, \widehat{s_j^{(k)}} \rangle_{\widehat{H^{(k)}}} dvol_{\widehat{H^{(k)}}} - C_{r,d}(d\mu + k)\delta_{ij},$$

$$D^{(k)} = C_{r,d}(d\mu + k).$$

We have

$$\int_{\mathbb{P}E^*} \langle \widehat{s_i^{(k)}}, \widehat{s_j^{(k)}} \rangle_{\widehat{H^{(k)}}} dvol_{\widehat{H^{(k)}}} = D^{(k)} I + M^{(k)}.$$

A simple calculation shows that

$$D^{(k)} \to C_{r,d} \quad \text{as} \quad k \to \infty.$$

On the other hand, (4.5) implies

$$||M^{(k)}||_{op} \leq \sum_{ij} |(M^{(k)})_{ij}| \leq Ck^{-p}(d\mu + k)N_k^2 \leq C' k^{3-p}.$$

Note that $N_k = O(k)$ by Riemann-Roch theorem. Therefore for any positive integer q, $||M^{(k)}|| = O(k^{-q-1})$ which means that the sequence of Hermitian metrics $H^{(k)}$ on $\mathcal{O}_{\mathbb{E}^*}(d) \otimes L^k$ and ordered bases $\underline{s}^{(k)} = (\widehat{s_1^{(k)}}, \dots, \widehat{s_{N_k}^{(k)}})$ of $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(d) \otimes L^k)$ is almost balanced of order q for any positive integer q.

□

Proof of Theorem 1.3. Fix a positive integer $a \geq 4$. Let ω_∞ be the kähler form on X with constant curvature. Since E is a stable bundle, there exists a Hermitian metric h_∞ on E satisfies the Hermitian-Einstein equation (4.1). Therefore Theorem 3.4 and Theorem 3.5 imply that there exists a sequence of balanced metrics $H^{(k)}$ on $\mathrm{Sym}^d E \otimes L^k$ such that

$$||H^{(k)} \otimes g^{\otimes(-k)} - \mathrm{Sym}^d h_\infty||_{C^a(\mathrm{Sym}^d h_\infty)} = O(k^{-\infty}). \tag{4.6}$$

By definition of balanced metrics on vector bundles (Definition 3.1), there exists a sequence of bases $\{s_i^{(k)}\}_{i=1}^{N_k}$ for $H^0(X, \mathrm{Sym}^d E \otimes L^{\otimes k})$ such that

$$\sum_{i=1}^{N_k} s_i^{(k)} \otimes (s_i^{(k)})^{*_{H^{(k)}}} = I_{\mathrm{Sym}^d E \otimes L^k},$$

$$\int_X \langle s_i^{(k)}, s_j^{(k)} \rangle_{H^{(k)}} \omega_\infty = \frac{RVol(X, \omega_\infty)}{N_k} \delta_{ij},$$

where R is the rank of $\mathrm{Sym}^d E$. Hence

$$\sum_{i=1}^{N_k} |\widehat{s_i^{(k)}}|^2_{\widehat{H^{(k)}}} = 1. \tag{4.7}$$

Define $\omega_0 = i\bar{\partial}\partial \log \widehat{H_\infty}$ and $\widetilde{\omega_k} = i\bar{\partial}\partial \log \widehat{H^{(k)}}$. Thus (4.6) implies

$$||\widetilde{\omega_k} - \omega_k||_{C^a(\omega_k)} \leq ||(\widetilde{\omega_k} - k\omega_\infty) - \omega_0||_{C^a(\omega_0)} = O(k^{-\infty}),$$

$$|\log \widehat{H^{(k)}} - \log(\widehat{H_\infty} \otimes g^{\otimes k})|_{C^{a+2}} = |\log(\widehat{H^{(k)}} \otimes g^{\otimes(-k)}) - \log \widehat{H_\infty}|_{C^{a+2}} = O(k^{-\infty}).$$

On the other hand, Theorem 4.5 and (4.7) imply that the sequence of Hermitian metrics $\widehat{H^{(k)}}$ on $\mathcal{O}_{\mathbb{P}E^*}(d) \otimes L^k$ and ordered bases $\underline{s}^{(k)} = (\widehat{s_1^{(k)}}, \dots \widehat{s_{N_k}^{(k)}})$ for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(d) \otimes L^k)$ is almost balanced of order q for any positive integer q. Since $\mathbb{P}E^*$ has no nontrivial holomorphic vector fields, we can perturb these almost balanced metrics to get balanced metrics on $\mathcal{O}_{\mathbb{P}E^*}(d) \otimes \pi^* L^k$ for $k \gg 0$ (see [S, Theorem 4.6]).

□

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