TOWARD A SALMON CONJECTURE

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Abstract. By using a result from the numerical algebraic geometry package Bertini we show that (with extremely high probability) a set of degree 6 and degree 9 polynomials cut out the secant variety $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$. This, combined with an argument provided by Landsberg and Manivel, implies set-theoretic defining equations in degrees 5, 6 and 9 for a much larger set of secant varieties, including $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ which is of particular interest in light of the salmon prize offered by E. Allman for the ideal-theoretic defining equations.

1. Introduction

In 2007, E. Allman offered a prize of Alaskan salmon to anyone who finds the defining ideal of

$$\sigma_4\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right),$$

the Zariski closure of all points on secant $\mathbb{P}^3$'s to the Segre product $Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, [All10]. The ideal-theoretic question is still open. Our main result is Theorem 3.10 in which we give a geometric argument (relying on results of Landsberg and Manivel) combined with a calculation using numerical algebraic geometry to show that with extremely high probability, $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ is cut out set-theoretically by 1728 equations in degree 5, 1000 equations in degree 6 and 8000 equations in degree 9. Even though these dimensions are large, we show that in each degree, the large space of polynomials can be constructed from a small number of representatives via simple substitutions (see Remarks 3.2, 2.2 and 3.5). Theorem 3.10 gives evidence that Theorem 3.10 may also hold ideal-theoretically.

One practical interest of the secant variety $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ is in phylogenetics, where the secant variety is associated to the statistical model for evolution called the mixture model of independence models [AR08]. The main motivation to study this particular model is that Allman and Rhodes showed in [AR08, Theorem 11] that finding the polynomial invariants for this small evolutionary tree would provide all polynomial invariants for the statistical model for *any* binary evolutionary tree with *any* number of states.

Note that while Allman asks for the generators of the defining ideal of the secant variety, a collection of set-theoretic defining equations provides a necessary and sufficient test for membership on the model. Very recently Friedland [Fri10] has proved (without a computer) that a set of polynomials in degrees $5, 9$ and 16 define $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ set-theoretically. Indeed, Friedland's set of polynomials do (in theory) allow one to test whether a given set of data fits the model. Because it uses polynomials in smaller degree, Theorem 3.10 provides a more efficient practical membership test for the model. Please consult [CF09] for more practical issues regarding phylogenetic tree construction using algebraic methods.

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Our equations in degree 6 are not in the ideal of the equations in degree 5, thus they are non-trivial generators in the ideal, and Friedland's result cannot be a set of minimal generators of the ideal. We have not found any such obstructions to our result holding ideal-theoretically and this leads to a salmon conjecture that the ideal-theoretic version of Theorem 3.10 also holds.

This work was initiated in October 2008 when Bernd Sturmfels asked for a Macaulay2 readable file of the degree 6 polynomials in the ideal of $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$. Proposition 2.1 is a representation theoretic description of these polynomials and corrects minor errors in [LM04, Proposition 6.3], and [LM08, Remark 5.7]. In Section 2 we give a brief overview of how these polynomials were constructed from their representation theoretic description. These equations and other supplementary materials for this paper are available online at http://web.math.unifi.it/users/oeding/salmon_materials

At the December 2008 MSRI workshop on Algebraic Statistics, the first author presented Conjecture 3.8 which, when combined with an argument of Landsberg and Manivel, implies our main result. This argument is discussed in Section 3. The missing ingredient for the conjecture was to understand the zero-set of the degree 6 polynomials. Shortly after this workshop, the first author asked for help from the Bertini Team, including the second author.

The two authors worked together to get the correct mixture of initial input and computing strategies in order to find a computation that would finish in a reasonable amount of time. Finally on July 12, 2010, a computation that had taken approximately 2 weeks on 8 processors (two 2.66 GHz quad-core Xeon 5410s set up as one head processor and seven worker processors) finished, providing a numerical proof to Conjecture 3.8. Because our calculations use numerical approximations, we say that the proof holds with extremely high probability. In Section 4 we discuss our computational methods and the reliability of this result.

2. Symmetry and the equations in degree 6

In this section we recall well-known facts about the variety we are studying. The main purpose is to set up notation. The reader who is unfamiliar with these concepts may consult [FH91], or for a more detailed account related to secant varieties see [LM04, LM08, LW07] or the upcoming [Lan10].

Let A, B, C be vector spaces of dimensions a, b, c respectively. The symmetry group of $\sigma_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is the change of coordinates in each factor $GL(A) \times GL(B) \times GL(C)$ (or $GL(A) \times GL(B) \times GL(C) \rtimes \mathfrak{S}_3$ when $A \cong B \cong C$). When a large group acts we can use tools from representation theory to aid in our search for defining equations. Since much of this work has already been done, we only describe the equations relevant for our application.

The module $S^d(A^* \otimes B^* \otimes C^*)$ of degree d homogeneous polynomials on $A \otimes B \otimes C$ has an *isotypic decomposition* (see [LM04, Proposition 4.1])

$$S^d(A^* \otimes B^* \otimes C^*) = \bigoplus_{|\pi_1|=|\pi_2|=|\pi_3|=d} \left(S_{\pi_1}A^* \otimes S_{\pi_2}B^* \otimes S_{\pi_3}C^*\right)^{\oplus M_{\pi_1,\pi_2,\pi_3}},$$

where the π_i are partitions of d and the multiplicity M_{π_1,π_2,π_3} is the dimension of the highest weight space which can be computed via characters. The modules

$$\left(S_{\pi_1}A^* \otimes S_{\pi_2}B^* \otimes S_{\pi_3}C^*\right)^{M_{\pi_1,\pi_2,\pi_2}}$$

are called *isotypic components*, and the individual modules $S_{\pi_1}A^* \otimes S_{\pi_2}B^* \otimes S_{\pi_3}C^*$ are irreducible $GL(A) \times GL(B) \times GL(C)$-modules sometimes called *Schur modules*.

The ideal of any $GL(A)\times GL(B)\times GL(C)$-invariant variety in $\mathbb{P}(A\otimes B\otimes C)$ consists of a subset of the modules occurring in the isotypic decomposition.

If X is a projective variety, let $\mathcal{I}_s(X)$ denote the ideal of homogeneous degree s polynomials in the ideal of X. In general, if X is any variety, $\mathcal{I}_s(\sigma_k(X))=0$ for $s\leq k$, and in particular, $\mathcal{I}_s(\sigma_4(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))=0$ for $s\leq 4$. Also, one can calculate (by checking every irreducible module of degree 5 polynomials) that

$$\mathcal{I}_5\left(\sigma_4\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right)\right)=0.$$

Proposition 2.1. *Let $A\cong B\cong\mathbb{C}^3$ and $C\cong\mathbb{C}^4$. Then $\mathcal{I}_6\left(\sigma_4\left(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C\right)\right)$ is the $GL(A)\times GL(B)\times GL(C)$-module*

$$M_6:=S_{2,2,2}A^*\otimes S_{2,2,2}B^*\otimes S_{3,1,1,1}C^*.$$

Proof. This was found by following the ideal membership test described in [LM04]. We first decomposed $S^6(A^*\otimes B^*\otimes C^*)$ into its isotypic decomposition. Next we computed a basis of the highest weight space for each isotypic component. Then we checked to see if any linear subspace of the highest weight space of an isotypic component vanished on the variety. The only module which passed this test was M_6, which occurs with multiplicity one in $S^6(A^*\otimes B^*\otimes C^*)$. We used Maple to carry out this procedure, and a copy of our code with some examples can be found on the website for supplementary materials listed above. □

Note that $S_{2,2,2}\mathbb{C}^3$ is one-dimensional and as a vector space, the module $S_{3,1,1,1}\mathbb{C}^4$ is isomorphic to $S^2\mathbb{C}^4$, which is 10-dimensional. There is a correspondence between the 10 basis elements of the module and the 10 semi-standard fillings (strictly increasing in the columns and non-decreasing in the rows) of the tableau of shape $(3,1,1,1)$ with the numbers $1,2,3,4$. We list these fillings below. The basis of polynomials is available at the URL mentioned above, or may be obtained from the first author.

Here is a brief overview of an algorithm to construct the polynomials in $S_{\pi_1}A^*\otimes S_{\pi_2}B^*\otimes S_{\pi_3}C^*$. More details can be found in [Lan10, Oed09a, Oed09b] for example.

For concreteness, we fix the degree $d=6$. The input to the algorithm is the fillings of the tableau of shapes π_1,π_2,π_3. The first step is to construct a highest weight vector in $A^{\otimes 6}\otimes B^{\otimes 6}\otimes C^{\otimes 6}$. For this we work one vector space at a time. Suppose a_1,a_2,a_3 is a basis of A^*. Then $a_1\otimes a_1\otimes a_2\otimes a_2\otimes a_3\otimes a_3$ is a *pre-highest weight vector*. The Young symmetrizer

$$Y_{\pi_1}:A^*\otimes A^*\otimes A^*\otimes A^*\otimes A^*\otimes A^*\to A^*\otimes A^*\otimes A^*\otimes A^*\otimes A^*\otimes A^*$$

is the map that skew symmetrizes the vector spaces A^* in positions corresponding to the columns of the filling associated to π_1 and then symmetrizes the vector spaces corresponding to the rows of the filling associated to π_1. The result is a highest weight vector of $S_{\pi_1}A$ in $(A^*)^{\otimes 6}$. We perform the analogous construction in the B^* and C^* factors and take the tensor product of the resulting highest weight vectors.

The resulting vector we have constructed is in $S_{\pi_1}A^*\otimes S_{\pi_2}B^*\otimes S_{\pi_3}C^*$, however it is embedded in $(A^*)^{\otimes 6}\otimes(B^*)^{\otimes 6}\otimes(C^*)^{\otimes 6}$. The final step is to perform the re-ordering isomorphism $(A^*)^{\otimes 6}\otimes(B^*)^{\otimes 6}\otimes(C^*)^{\otimes 6}\to(A^*\otimes B^*\otimes C^*)^{\otimes 6}$, and then symmetrize the result to arrive at a polynomial in $S^6(A^*\otimes B^*\otimes C^*)$.

We computed the 10 polynomials in $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$ using the fixed fillings

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}$$

for π_1 and π_2 respectively with each of the following fillings for π_3

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,$$

$$\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&4&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ .$$

Notice that up to re-naming the numbers, the fillings for π_3 can be divided into two classes, depending on whether the last two numbers in the first row are equal. The four fillings of the first class (with the last two numbers in the first row equal) correspond to polynomials with 936 terms, whereas the six fillings of the second class correspond to polynomials with 576 terms. Moreover, a simple substitution $p_{i,j,k} \mapsto p_{i,j,k'}$ takes one polynomial to another in the same class. Here the indices satisfy $1 \leq i,j \leq 3$ and $1 \leq k,k' \leq 4$. This additional symmetry is useful for the Bertini computation below.

These fillings produce homogeneous polynomials that are, moreover, homogeneous in multi-degree. In general, the *multi-degree* of a monomial is a collection of vectors $[[l_1^A, l_2^A, l_3^A],$ $[l_1^B, l_2^B, l_3^B],$ $[l_1^C, l_2^C, l_3^C, l_4^C]]$, and is defined on a single variable $x_{i,j,k}$ by $l_{i'}^A$ is 0 (respectively 1) for $x_{i,j,k}$ if $i \neq i'$ (respectively $i = i'$) ($l_{j'}^B$ and $l_{k'}^C$ are defined similarly) and the multi-degree is defined for monomials by declaring it to be additive over products of variables. For example, the following is a sampling of terms in the highest weight polynomial corresponding to the filling $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$,

$$\cdots - x_{321}x_{113}x_{211}x_{221}x_{134}x_{332} - x_{321}x_{122}x_{231}^2x_{313}x_{114} + x_{211}x_{312}x_{131}x_{121}x_{334}x_{223}\dots$$

and one finds that this polynomial has multi-degree $[[2,2,2],[2,2,2],[3,1,1,1]]$.

Remark 2.2. Note that when $a = b = 3$ and $c = 4$, $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$ is 10-dimensional. When $a = b = c = 4$, the dimension of $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$ increases to 1000, however the basis of this larger space can still be constructed from the two polynomials that have 576 and 936 monomials via the substitutions $p_{i,j,k} \mapsto p_{i',j',k'}$ with $1 \leq i,i',j,j',k,k' \leq 4$.

3. Geometric techniques for secant varieties

Suppose $A' \subset A$, $B' \subset B$ and $C' \subset C$. Landsberg and Manivel have shown how to take equations on $\sigma_r(\mathbb{P}A' \times \mathbb{P}B' \times \mathbb{P}C')$ to equations on $\sigma_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ and call this procedure *inheritance* [LM04, Proposition 4.4].

Subspace varieties contain tensors that can be written using fewer variables. More specifically,

$$\begin{aligned}\mathrm{Sub}_{a',b',c'}(A\otimes B\otimes C) := \big\{&[T]\in\mathbb{P}(A\otimes B\otimes C) \mid \exists \mathbb{C}^{a'}\subseteq A,\\ &\mathbb{C}^{b'}\subseteq B, \mathbb{C}^{c'}\subseteq C, \text{ with } [T]\in\mathbb{P}(\mathbb{C}^{a'}\otimes\mathbb{C}^{b'}\otimes\mathbb{C}^{c'})\big\}.\end{aligned}$$

Landsberg and Weyman have shown that $\mathrm{Sub}_{a',b',c'}(A\otimes B\otimes C)$ is normal with rational singularities, and the ideal is generated by minors of flattenings [LW07, Theorem 3.1]. Recall that a flattening of a 3-tensor in $A\otimes B\otimes C$ is the choice to view it as a matrix in $A\otimes(B\otimes C)$, $B\otimes(A\otimes C)$ or $(A\otimes B)\otimes C)$.

The subspace varieties are important in light of equations because of the fact that $\mathrm{Sub}_{r,r,r}(A\otimes B\otimes C)\supseteq\sigma_r(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$, and therefore when non-trivial, the ideal of $\mathrm{Sub}_{r,r,r}$ gives equations of σ_r. There is an easy test for a module to be in the ideal of a subspace variety, namely $S_{\pi_1}A^*\otimes S_{\pi_2}B^*\otimes S_{\pi_3}C^*$ is in the ideal of $\mathrm{Sub}_{a',b',c'}(A\otimes B\otimes C)$ if and only if at least one of the following holds; $\#(\pi_1)>a'$, $\#(\pi_2)>b'$ or $\#(\pi_3)>c'$, where $\#(\cdot)$ is the number of parts of the partition.

Landsberg and Manivel made an important reduction for the salmon problem, which we record here. Let a,b,c respectively denote the dimensions of A,B,C.

Theorem 3.1 (Landsberg-Manivel '08 Corollary 5.6)**.** *As sets, for $a,b,c\geq 3$, $\sigma_4\left(\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1}\right)$ is the zero-set of the union of:*

(1) *Strassen's commutation conditions,*

$$\begin{aligned}M_5 := &S_{(3,1,1)}A^*\otimes S_{(2,1,1,1)}B^*\otimes S_{(2,1,1,1)}C^*\\ &\oplus S_{(2,1,1,1)}A^*\otimes S_{(3,1,1)}B^*\otimes S_{(2,1,1,1)}C^*\\ &\oplus S_{(2,1,1,1)}A^*\otimes S_{(2,1,1,1)}B^*\otimes S_{(3,1,1)}C^*,\end{aligned}$$

(2) *Equations inherited from $\sigma_4\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right)$, and*
(3) *Modules in $S^5(A^*\otimes B^*\otimes C^*)$ containing a $\bigwedge^5$,* i.e. *equations for* $\mathrm{Sub}_{4,4,4}$.

Note that when $a=b=c=4$, the third set of equations is trivial. The key point is that we will have a complete description of the set-theoretic defining equations of $\sigma_4(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3)$ as soon as we have the equations of $\sigma_4(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3)$.

Remark 3.2. The equations in degrees 5 and 9 were found by Strassen [Str83] and were described in terms of certain commutation conditions. Later, Landsberg and Manivel [LM08] reinterpreted these conditions from the geometric and representation theoretic point of view and provided generalizations in this language. In [Stu09] one finds a nice description of these equations requiring only basic linear algebra. Analogous to our description of the equations in degree 6, here we give the representation theoretic description of the polynomials of degree 5.

Note also that when $a=b=c=4$, M_5 is a 1728-dimensional irreducible G-module, for $G=GL(4)\times GL(4)\times GL(4)\times\mathfrak{S}_3$. A natural basis of M_5 can be constructed as in the previous section. For this we need to give the fillings for the triple of Young diagrams corresponding to the partitions $(2,1,1,1),(2,1,1,1),(3,1,1)$. We note that up to permutation, there is just one equivalence class for the fillings of the diagram for $(2,1,1,1)$ with representative

$\begin{ytableau} 1 & 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$. There are three equivalence classes for the fillings of the diagram for $(3,1,1)$ with representatives $\begin{ytableau} 1 & 1 & 1 \\ 2 \\ 3 \end{ytableau}$, $\begin{ytableau} 1 & 1 & 2 \\ 2 \\ 3 \end{ytableau}$ and $\begin{ytableau} 1 & 1 & 2 \\ 3 \\ 4 \end{ytableau}$.

Therefore to construct representatives for a basis of $S_{(2,1,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(3,1,1)}C^*$, we fix the representative filling for $(2,1,1,1)$ in both instances, and we let the filling for $(3,1,1)$ vary over the three representatives. Thus we construct three polynomials, one for each representative filling of the diagram for $(3,1,1)$ and respectively, these polynomials have 180, 360 and 540 monomials. After constructing these three polynomials, the rest of the polynomials in the basis of M_5 can be constructed by the simple substitutions $p_{i,j,k} \mapsto p_{i',j',k'}$.

Another important result for the salmon problem is from Strassen, which has been reinterpreted in representation theoretic language in [LM08].

Theorem 3.3 ([Str83])**.** *The ideal of the hypersurface $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^{26}$ is generated in degree 9 by a nonzero vector in the 1-dimensional module*

$$S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3.$$

Inheritance implies that $M_9 := S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^4 \in \mathcal{I}(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$.

Remark 3.4. Suppose $[T] \in \mathbb{P}(A \otimes B \otimes C)$, with $\dim(A) = 3$. Then write $T = a_1 \otimes T_1 + a_2 \otimes T_2 + a_3 \otimes T_3$, where the T_i are $b \times c$ matrices in $B \otimes C$ and the a_i are a basis of A.

Strassen described his equation in degree 9 as follows. On an open set one may assume that T_1 is invertible. Then consider the polynomial

$$\det(T_1)^2 \det(T_2 T_1^{-1} T_3 - T_3 T_1^{-1} T_2).$$

Strassen showed that this polynomial is irreducible, of degree 9, and vanishes on $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

A useful reformulation by Ottaviani of Strassen's equation is the following. As before write $T = a_1 \otimes T_1 + a_2 \otimes T_2 + a_3 \otimes T_3$. Here one does not require any of the slices T_1, T_2, T_3 to be invertible. Construct the block matrix

$$\psi_T = \begin{pmatrix} 0 & T_3 & -T_2 \\ -T_3 & 0 & T_1 \\ T_2 & -T_1 & 0 \end{pmatrix}. \tag{1}$$

One checks that ψ_T is linear in T, and that if $[T] \in Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ then $Rank(\psi_T) = 2$. Therefore if $[T]$ is a general point in $\sigma_k(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ it can be written as the sum of k points on $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ so $Rank(\psi_T) \leq 2k$. In particular, in the case $\dim(A) = \dim(B) = \dim(C) = 3$, the 9×9 determinant $\det(\psi_T)$ gives a non-trivial equation for $\sigma_4(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, which is also Strassen's equation. This polynomial has $9{,}216$ monomials.

Remark 3.5. In the case that $a = b = 3$ and $c = 4$, as a vector space, M_9 is isomorphic to $S^3\mathbb{C}^4$ so $\dim(M_9) = 20$. When the highest weight vector of a module has a determinantal representation (as in the case of M_9), it is typically much faster to compute a basis of the module from the highest weight vector using lowering operators. Using this method, we found that the natural basis of M_9 consists of polynomials with $9,216$ or $25,488$ or $43,668$ monomials. As in Remarks 3.2 and 2.2, these polynomials can be associated to representative polynomials, depending on fillings. In the A and B-factors, the diagram for $(3,3,3)$ can only have one semi-standard filling, namely $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&3\\\hline\end{array}$. In the C-factor, there are three classes of fillings, namely $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&3\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&4\\\hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline 3&4&4\\\hline\end{array}$. These fillings yield the representative polynomials consisting of $9,216$ or $25,488$ or $43,668$ monomials respectively. The rest of the polynomials in a basis of M_9 can be constructed by the substitution $p_{i,j,k} \mapsto p_{i,j,k'}$.

Alternately, a basis of M_9 can be constructed via Ottaviani's formulation. They are derived from the condition that the now 9×12 matrix appearing in (1) have rank 8 or less. However, the space of 9×9 minors of ψ_T is no longer irreducible when $a = b = 3$ and $c = 4$. Namely the space of 9×9 minors of the 9×12 matrix ψ_T is the following representation

$$\begin{gathered} S_{3,3,3}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,3}C^* \\ \oplus S_{4,3,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,2,1}C^* \\ \oplus S_{5,2,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,2,2,2}C^*. \end{gathered}$$

There are three equivalence classes of maximal minors of ψ_T depending only on the column index I of the maximal minor of $\Delta_I(\psi_T)$. Let $P = (P_1, P_2, P_3)$ be the partition of the set $\{1, \ldots, 12\}$ into three sets $P_1 = \{1, 2, 3, 4\}$, $P_2 = \{5, 6, 7, 8\}$, $P_3 = \{9, 10, 11, 12\}$. The representation $S_{3,3,3}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,3}C^*$ is associated to the minors $\Delta_I(\psi_T)$ such that $I \cap P_i = 3$ for $i = 1, 2, 3$. This condition precisely forces the minor of ψ_T to be constructed with 3×3 submatrices of T_1, T_2 and T_3. The representation $S_{4,3,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,2,1}C^*$ is associated to the minors $\Delta_I(\psi_T)$ such that $I \cap P_1 = 4$, $I \cap P_2 = 3$, $I \cap P_3 = 2$. The representation $S_{5,2,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,2,2,2}C^*$ is associated to the minors $\Delta_I(\psi_T)$ such that $I \cap P_1 = 4$, $I \cap P_2 = 4$, $I \cap P_3 = 1$.

Note that symmetry in the A and B factors implies that we may reverse the roles of A and B to find two more modules in the ideal, namely the two modules $S_{3,3,3}A^* \otimes S_{4,3,2}B^* \otimes S_{3,3,2,1}C^*$ and $S_{3,3,3}A^* \otimes S_{5,2,2}B^* \otimes S_{3,2,2,2}C^*$ must also vanish on $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

While we have described five modules of degree 9 equations which vanish on $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, we only use the module $M_9 = S_{3,3,3}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,3}C^*$ along with M_6 described above for our set-theoretic defining equations. We can conclude that $\langle M_9 \rangle \not\subset \langle M_6 \rangle$ by analyzing the shapes of the partitions involved. More specifically, in the C-factor the partition $(3,3,3)$ only has 3 parts, but if $S_{\pi_1}A^* \otimes S_{\pi_2}B^* \otimes S_{\pi_3}C^*$ is a module in the ideal generated by M_6 then π_3 must have at least 4 parts. However this argument fails for the other four degree 9 modules so it is possible that these equations are in the ideal generated by M_6. Moreover our set-theoretic result implies that it must be the case that the other degree 9 modules are in the ideal generated by M_6 (with very high probability).

Example 3.6 ([Fri10]**).** Friedland has shown that the equations in degree 9 are not sufficient to define $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ set-theoretically. We thank J.M. Landsberg for the following clarification of Friedland's example. Consider the point

$$P = (a_1\otimes b_1+a_2\otimes b_2)\otimes c_1+(a_1\otimes b_1+a_2\otimes b_3)\otimes c_2+(a_1\otimes b_1+a_3\otimes b_2)\otimes c_3+(a_1\otimes b_1+a_3\otimes b_3)\otimes c_4.$$

The span of $\{a_1, a_2, a_3\}$ and the span of $\{b_1, b_2, b_3\}$ are both no more than 3-dimensional, so P is a zero of M_5. One finds that $\psi_T(P)$ has rank 8 and therefore P is a zero of M_9. However P is not a point of $\sigma_4(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$. This geometric argument implies that more polynomials are needed than just the degree 5 and 9 equations. For this, Friedland produces equations of degree 16 which do not vanish on P.

On the other hand, P is not in the zero set of M_6, so M_6 is sufficient to rule out the possibility of points of the same form as P to have border rank 4.

Remark 3.7. To construct a basis of the 8000 dimensional space $S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4$, one can repeat the lowering operator procedure. Since these polynomials are very complicated, our experience is that, in practice, one should use the degree 9 equations in their determinantal form. To check if a point z vanishes on all of the polynomials in $S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4$, it is more efficient to first construct the matrix in (1) for the point z and check that the determinant vanishes. Then repeat this test for all allowable changes of coordinates, in other words, for every $g \in GL(4)\times GL(4)\times GL(4)$ construct the matrix in (1) for $g.z$ and check that the determinant still vanishes. If one only wants a quick check that z is in the zero-set with high probability, it suffices to check that $g.z$ is in the zero-set for a random g.

Since $(2, 2, 2)$ has 3 parts, and $(3, 1, 1, 1)$ has 4 parts, M_6 must vanish on the subspace varieties $\mathrm{Sub}_{2,3,4}\cup\mathrm{Sub}_{3,2,4}\cup\mathrm{Sub}_{3,3,3}$. Also, note that two of these subspace varieties are already contained in the secant variety, namely $\sigma_4\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right) \supset \mathrm{Sub}_{2,3,4}\cup\mathrm{Sub}_{3,2,4}$. Indeed, if $x \in \mathrm{Sub}_{2,3,4}$, there exists $A' \subset A$ such that $\dim(A') = 2$ and $x \in \mathbb{P}(A'\otimes B\otimes C)$. But in this case $\mathbb{P}(A'\otimes B\otimes C) = \sigma_4(\mathbb{P}A'\times\mathbb{P}B\times\mathbb{P}C) \subset \sigma_4(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$. The same argument is repeated for $\mathrm{Sub}_{3,2,4}$.

Based on this evidence, we make the conjecture

Conjecture 3.8. *As sets,*

$$\mathcal{V}(S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(3,1,1,1)}\mathbb{C}^4) = \sigma_4\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right)\cup\mathrm{Sub}_{3,3,3}\,.$$

Theorem 4.1 below implies that Conjecture 3.8 is true with extremely high probability.

Corollary 3.9 (Corollary to Theorem 4.1)**.** *Let $A \cong \mathbb{C}^3$, $B \cong \mathbb{C}^3$, $C \cong \mathbb{C}^4$. The secant variety $\sigma_4\left(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C\right)$ is defined set-theoretically by*

$$\begin{aligned} M_6 &= S_{(2,2,2)}A^*\otimes S_{(2,2,2)}B^*\otimes S_{(3,1,1,1)}C^* \\ M_9 &= S_{(3,3,3)}A^*\otimes S_{(3,3,3)}B^*\otimes S_{(3,3,3)}C^*. \end{aligned}$$

Proof. By Proposition 2.1 and by Strassen's Theorem 3.3 combined with inheritance we know that both M_6 and M_9 are in the ideal of $\sigma_4\left(\mathbb{P}A^*\times\mathbb{P}B^*\times\mathbb{P}C^*\right)$. So we know that $\sigma_4\left(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C\right) \subset \mathcal{V}(M_6\oplus M_9)$.

For the other inclusion, select a point z in the common zero locus of M_6 and M_9. Since $z \in \mathcal{V}(M_6)$, Conjecture 3.8 says that either z is on the secant variety, in which case we are done, or z is on the subspace variety. In the latter case, let $C' \subset C$ be a 3-dimensional vector space so that $z \in \mathbb{P}(A\otimes B\otimes C')$. Then z is a zero of $M_9 = S_{(3,3,3)}A^*\otimes S_{(3,3,3)}B^*\otimes S_{(3,3,3)}C^*$, and therefore is also a zero of the polynomials in the restriction $S_{(3,3,3)}A^* \otimes S_{(3,3,3)}B^* \otimes S_{(3,3,3)}C'^*$. So by Strassen's Theorem 3.3,

$$z \in \sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C').$$

We conclude because we have the obvious inclusion

$$\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C') \subset \sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C).$$

□

We used numerical algebraic geometry, specifically Bertini, to compute the decomposition of the zero set $\mathcal{V}(M_6)$ into irreducible varieties. In particular Theorem 4.1 verifies Conjecture 3.8 with extremely high probability. For completeness, we restate the combination of Landsberg and Manivel's result with our computations as follows:

Theorem 3.10. *As sets, for $a, b, c \geq 3$, with extremely high probability, $\sigma_4\left(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}\right)$, is the zero-set of:*

(1) *Strassen's commutation conditions,*

$$\begin{aligned} M_5 := {} & S_{(3,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(2,1,1,1)}C^* \\ & \oplus S_{(2,1,1,1)}A^* \otimes S_{(3,1,1)}B^* \otimes S_{(2,1,1,1)}C^* \\ & \oplus S_{(2,1,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(3,1,1)}C^*, \end{aligned}$$

(2) *equations inherited from $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$,*

$$\begin{aligned} M_6 &= S_{(2,2,2)}A^* \otimes S_{(2,2,2)}B^* \otimes S_{(3,1,1,1)}C^* \\ M_9 &= S_{(3,3,3)}A^* \otimes S_{(3,3,3)}B^* \otimes S_{(3,3,3)}C^*, \end{aligned}$$

(3) *and modules in $S^5(A^* \otimes B^* \otimes C^*)$ containing a $\bigwedge^5$,* i.e. *equations for* $\mathrm{Sub}_{4,4,4}$.

4. Results using numerical algebraic geometry

In this section, we provide a very brief overview of the basic methods of numerical algebraic geometry; references for further details are provided. We then describe the results of the run establishing the main result of this article and conclude with a short discussion regarding the reliability of numerical algebraic geometry methods and, more to the point, the reliability of this result.

4.1. **Brief overview of numerical algebraic geometry methods.** Given generators of an ideal of $\mathbb{C}[x_1, \dots, x_N]$, the methods of numerical algebraic geometry will produce a *numerical irreducible decomposition* for the associated variety $X \subset \mathbb{C}^N$. In particular, for each irreducible component Z of X, these methods will produce, with probability one, $\deg Z$ numerical approximations (to any number of digits) of generic points on Z. The end result is a catalog of all irreducible components of X, each indicated by a set of *witness points* on the component (together referred to as a *witness set* for the component), its dimension, and its degree.

The core method of numerical algebraic geometry is *homotopy continuation*, a method for approximating the complex zero-dimensional solution set of a polynomial system. The basic idea of homotopy continuation is to cast the given polynomial system F as a member of a parameterized family of polynomial systems, one of which, G, has known solutions or is otherwise easily solved. If done correctly, the solutions of G will vary continuously to those of F as the parameters are varied appropriately. By tracking these paths numerically (using predictor-corrector methods), one will arrive at numerical approximations of all complex zero-dimensional solutions of F. There have been many technical advances in this area that contribute heavily to the reliability of these methods. Please refer to [SW05, Li03] as general references and [BHSW08, BHSW09] regarding the use of adaptive precision methods for added reliability.

Pairing homotopy continuation with the use of hyperplane sections, monodromy, and a few other methods described fully in [SW05] yields the numerical irreducible decomposition. Briefly, a d-dimensional irreducible algebraic set in $\mathbb{C}^N$ will intersect a generic codimension d linear space in a set of points. This statement about genericity (along with similar assumptions of genericity throughout numerical algebraic geometry) is the reason for referring to these methods as *probability-one* methods, as described further below.

The computation of a numerical irreducible decomposition begins by searching for codimension one irreducible components (by adding $N-1$ linear polynomials to the set of generators and solving for zero-dimensional components via homotopy continuation), followed by codimension two components, etc. Once this sweep through all possible dimensions has been completed, we have a superset of the desired numerical irreducible decomposition, since a linear variety of codimension d will intersect any component of dimension *d or higher*. Sommese, Verschelde, and Wampler (and others) have developed methods for removing points in the "wrong dimension," i.e., those discovered while searching for components in dimension d which actually lie on higher-dimensional components, called *junk points*. They have also developed algorithms for performing pure-dimensional decompositions to yield witness sets on each irreducible component (instead of the initially-found witness sets for the union of all equidimensional irreducible components). Since this is intended as a very brief overview, please refer to [SW05] for further details.

There are three main software packages in this field: Bertini [BHSW10b], HOM4PS-2.0 [LLT10], and PHCpack [Ver10]. Each package has various benefits over the others [BHSW10a]. Since Bertini is typically the most efficient package for large, parallel, positive-dimensional problems as well as the package with the most reliability and precision features, we used Bertini in our computations for this article. In fairness, it should also be noted that the second author is a Bertini developer.

4.2. **Numerical results for the Salmon Problem.**

Theorem 4.1. *With extremely high probability, the zero-set of the* 10 *polynomials in a basis of M_6 (defined above) has precisely two irreducible components. One, in dimension 31, has degree 345. The other, in dimension 29, has degree 84.*

Indeed, it can be checked that $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$ is non-defective and has dimension 31 [BCS97, p. 5]. It is also straightforward to check that $\mathrm{Sub}_{3,3,3}$ has dimension 29, and by the pigeon-hole principle, these must be our components in the zero-set of M_6. Though these dimensions are sufficient information to identify our varieties, as additional information, we find that this secant variety has degree 345 and the subspace variety has degree 84.

Proof. The conclusion comes from the results of a calculation on Bertini [BHSW10b] using approximately 2 weeks of computing time on 8 processors, using tight controls including small tracking and final tolerances (10^{-10} or smaller), adaptive precision numerical methods, and a variety of checks and error controls built into Bertini (such as checking at $t = 0.1$ that no paths have crossed). $\square$

4.3. **Reliability of this result.** Can the result of the previous section be accepted as absolute proof? No, unfortunately, it may not. However, this numerical computation gives extremely strong evidence that Theorem 4.1 is indeed true even without the statement "with extremely high probability".

4.3.1. *Theoretical results with extremely high probability.* Many of the methods of numerical algebraic geometry are *probability-one* algorithms, meaning that the method will provide the desired result unless some random choices are degenerate. In particular, the success of these methods depends on the choice of random numbers from a Zariski open, dense set S of some parameter space rather than choosing some set of points in the complement of S. Since the complement of S is an algebraic set, we know that it must have positive codimension, making it a set of measure zero for any reasonable choice of measure. Thus, the methods will succeed with probability (measure) one.

This does not imply that these methods are untrustworthy, but it must be understood that these methods cannot provide complete certainty for this reason. In the case of the Salmon Problem, the parameter space from which random numbers are chosen is of very high dimension, so the likelihood of having chosen "bad" random numbers (particularly since Bertini chooses random numbers with some fixed level of precision, meaning any irrational number would be missed) is extremely low. While it is feasible to actually compute the algebraic set of "bad" choices, for example when choosing a linear space with which to slice some potential irreducible components, such computations are much more time-consuming than solving the original problem and, themselves, include the need to locate some further "bad" algebraic set, etc., *ad infinitum*. Thus, certainty cannot be attained by computing the set of bad choices. Methods for certifying path tracking are under development but are not yet available.

4.3.2. *Numerical concerns and thoughts about this particular computation.* The choice of "bad" random numbers (as in the previous discussion) or numbers very close to "bad" random numbers will typically lead to ill-conditioning during path-tracking. Bertini has been developed with reliability as the paramount concern, so paths experiencing significant numerical ill-conditioning (significant enough to overwhelm adaptive precision methods) will be recorded as "failed paths," *i.e.*, paths for which tracking from the initial parameter value to the final parameter value is not possible are said to have failed. It is not uncommon to complete a Bertini run with some path failures, meaning that tolerances must be made tighter (paths must be tracked more carefully, new random numbers must be chosen, etc.) to avoid this ill-conditioning.

The run for this article used a special equation-by-equation algorithm called regeneration [HSWar] and required the following of more than 200,000 paths. There were absolutely no path failures and no crossed paths detected among these 200,000 paths. The second author is a developer of Bertini and has worked in this field for several years. To have such a run end with no numerical warnings is very encouraging; this was a very "clean" run.

In the case of positive-dimensional tracking, there is an extra layer of reliability. In this case, suppose there is a component of dimension 28 that Bertini somehow missed while searching for components in dimension 28, in addition to the components of dimensions 29 and 31 found by Bertini (as stated in the theorem above). This would mean that (a) the linear variety of codimension 28 used when searching for 28-dimensional components was degenerate (in the "bad" algebraic set in the appropriate parameter space) and, far more importantly, (b) this component was also missed when searching for all lower-dimensional components. Indeed, while moving through the cascade of dimensions, junk points (see above) would, with extremely high probability, be found at each dimension below 28. Thus, given the large number of variables in this problem and the complete dearth of such extraneous junk points, it is very unlikely that Bertini missed a component.

Furthermore, when using monodromy (see [SW05]) to decompose the pure-dimensional components in dimensions 29 and 31, monodromy loops indicated directly that all witness points in each dimension fell on the same component. While this is not absolute proof that there is a unique component in dimension 29 and one in dimension 31, this again provides very strong numerical evidence. Indeed, if the breakup in dimensions 29 and 31 is incorrect, e.g., if there are actually two components in dimension 29 rather than just one, then the monodromy action used to carry out the equidimensional decomposition in these dimensions would not have sent points from one component into those of another, again, with extremely high probability.

Similarly, the trace test (see [SW05]) used to certify (with extremely high probability) the completeness of a witness set would fail if even a single witness point were missing. The trace test for each of the two components succeeded.

In summary, we cannot conclude with unquestionable certainty that Theorem 4.1 holds unconditionally, but we *can* state with an extremely high level of confidence that it is correct. Motivated by this result, we hope to find a direct argument to prove Conjecture 3.8.

4.3.3. *Numerical vs. symbolic computation.* Finally, one might wonder why we chose to use numerical methods to test this conjecture rather than symbolic methods that will provide certainty. The main reasons are simple: time and space. Regarding time we expect that without additional ideas to reduce the difficulty of computation, a related calculation using symbolic methods should take at least eight times as long as the calculation in Bertini because Gröbner basis algorithms are not parallelizable. In fact, based on the timings from an ongoing benchmarking project between the Bertini and Singular [DGPS10] development teams, we suspect that any symbolic computation will actually take far more than eight times as long. Regarding the issue of space we must consider data storage at intermediate stages. While the initial input and final result may be relatively small, Gröbner basis algorithms typically must store large intermediate results for subsequent calculations. On the other hand, homotopy continuation algorithms require a trivial amount of extra data in intermediate stages. Indeed, the amount of memory used grows linearly with the number of paths tracked (simply because the final point on each path must be stored). Bertini is thus much less likely to fail due to memory constraints.

References

[All10] E. Allman, *Open problem: Determine the ideal defining* $Sec_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, http://www.dms.uaf.edu/∼eallman/salmonPrize.pdf, 2010.

[AR08] E. Allman and J. Rhodes, *Phylogenetic ideals and varieties for the general Markov model*, Adv. in Appl. Math. **40** (2008), no. 2, 127–148. MR 2388607 (2008m:60145)

[BCS97] P. Bürgisser, M. Clausen, and M. Shokrollahi, *Algebraic complexity theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Berlin: Springer-Verlag, 1997. MR 1440179 (99c:68002)

[BHSW08] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, *Adaptive multiprecision path tracking*, SIAM J. Num. An. **46** (2008), 722–746.

[BHSW09] ———, *Stepsize control for adaptive precision path tracking*, Interactions of Classical and Numerical Algebraic Geometry, Contemporary Mathematics, vol. 496, American Mathematical Society, 2009, pp. 21–31.

[BHSW10a] ———, *Benchmarking bertini, software for numerical algebraic geometry*, 2010.

[BHSW10b] ———, *Bertini: Software for numerical algebraic geometry*, Available at http://www.nd.edu/∼sommese/bertini, 2010.

[CF09] M. Casanellas and J. Fernandez-Sanchez, *Relevant phylogenetic invariants of evolutionary models*, ArXiv e-prints (2009).

[DGPS10] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR *3-1-1 — A computer algebra system for polynomial computations*, 2010, http://www.singular.uni-kl.de.

[FH91] W. Fulton and J. Harris, *Representation theory: A first course*, Graduate Texts in Mathematics, vol. 129, New York: Springer-Verlag, 1991. MR 1153249 (93a:20069)

[Fri10] S. Friedland, *On tensors of border rank l in* $\mathbb{C}^{m\times n\times l}$, March 2010.

[HSWar] J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, *Regeneration homotopies for solving systems of polynomials*, Math. Comp. (to appear).

[Lan10] J. M. Landsberg, *The geometry of tensors: Applications to complexity, statistics and engineering*, in preparation, 2010.

[Li03] T.Y. Li, *Numerical solution of polynomial systems by homotopy continuation methods*, Handbook of Numerical Analysis, Special Volume: Foundations of Computational Mathematics, vol. XI, North-Holland, 2003, pp. 209–304.

[LLT10] T.-L. Lee, T.Y. Li, and C.-H. Tsai, *Hom4ps-2.0: A software package for solving polynomials systems by the polyhedral homotopy continuation method*, Available at http://www.mth.msu.edu/∼li/, 2010.

[LM04] J. M. Landsberg and L. Manivel, *On the ideals of secant varieties of Segre varieties*, Found. Comput. Math. **4** (2004), no. 4, 397–422. MR 2097214 (2005m:14101)

[LM08] J. M. Landsberg and Laurent Manivel, *Generalizations of Strassen's equations for secant varieties of Segre varieties*, Comm. Algebra **36** (2008), no. 2, 405–422. MR MR2387532 (2009f:14109)

[LW07] J. M. Landsberg and J. Weyman, *On the ideals and singularities of secant varieties of Segre varieties*, Bull. Lond. Math. Soc. **39** (2007), no. 4, 685–697. MR 2346950 (2008h:14055)

[Oed09a] L. Oeding, *G-varieties and the principal minors of symmetric matrices*, Ph.D. thesis, Texas A&M University, 2009.

[Oed09b] L. Oeding, *Set-theoretic defining equations of the tangential variety of the Segre variety*, ArXiv e-prints (2009).

[Str83] V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl. **52/53** (1983), 645–685. MR MR709378 (85b:15039)

[Stu09] Bernd Sturmfels, *Open problems in agrebraic statistics*, Emerging Applications of Algebraic Geometry (Seth Sullivant and Bernd Sturmfels, eds.), The IMA Volumes in Mathematics and its Applications, vol. 149, Springer New York, 2009, pp. 1–13.

[SW05] A.J. Sommese and C.W. Wampler, *Numerical solution of polynomial systems arising in engineering and science*, World Scientific, Singapore, 2005.

[Ver10] J. Verschelde, *PHCpack: a general-purpose solver for polynomial systems by homotopy continuation*, Available at http://www.math.uic.edu/∼jan/PHCpack/phcpack.html, 2010.

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