Minimizers of the Willmore functional under fixed conformal class

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Abstract: We prove the existence of a smooth minimizer of the Willmore energy in the class of conformal immersions of a given closed Riemann surface into $\mathbb{R}^n, n = 3, 4,$ if there is one conformal immersion with Willmore energy smaller than a certain bound $\mathcal{W}_{n,p}$ depending on codimension and genus p of the Riemann surface. For tori in codimension 1, we know $\mathcal{W}_{3,1} = 8\pi$.

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1 Introduction

For an immersed closed surface $f : \Sigma \to \mathbb{R}^n$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |\vec{\mathbf{H}}|^2 \, \mathrm{d}\mu_g,$$

where $\vec{\mathbf{H}}$ denotes the mean curvature vector of $f, g = f^* g_{euc}$ the pull-back metric and μ_g the induced area measure on Σ . The Gauß equations and the Gauß-Bonnet Theorem give rise to equivalent expressions

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |A|^2 \, \mathrm{d}\mu_g + 2\pi(1 - p(\Sigma)) = \frac{1}{2} \int_\Sigma |A^\circ|^2 \, \mathrm{d}\mu_g + 4\pi(1 - p(\Sigma)) \qquad (1.1)$$

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where A denotes the second fundamental form, $A^\circ = A - \frac{1}{2} g \otimes H$ its tracefree part and $p(\Sigma)$ is the genus of Σ.

We always have $\mathcal{W}(f) \geq 4\pi$ with equality only for round spheres, see [Wil82] in codimension one that is $n = 3$. On the other hand, if $\mathcal{W}(f) < 8\pi$ then f is an embedding by an inequality of Li and Yau in [LY82].

Critical points of $\mathcal{W}$ are called Willmore surfaces or more precisely Willmore immersions. They satisfy the Euler-Lagrange equation which is the fourth order, quasilinear geometric equation

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = 0$$

where the Laplacian of the normal bundle along f is used and $Q(A^0)$ acts linearly on normal vectors along f by

$$Q(A^0)\phi := g^{ik} g^{jl} A^0_{ij} \langle A^0_{kl}, \phi \rangle.$$

The Willmore functional is scale invariant and moreover invariant under the full Möbius group of $\mathbb{R}^n$. As the Möbius group is non-compact, minimizers of the Willmore energy cannot be found via the direct method.

In [KuSch06], we investigated the relation of the pull-back metric g to constant curvature metrics on Σ after dividing out the Möbius group. More precisely, we proved that for immersions $f : \Sigma \to \mathbb{R}^n, n = 3, 4,$ and genus $p = p(\Sigma) \geq 1$ satisfying $\mathcal{W}(f) \leq \mathcal{W}_{n,p} - \delta$ for some $\delta > 0$, where

$$\begin{aligned} \mathcal{W}_{3,p} &= \min(8\pi, 4\pi + \textstyle\sum_k (\beta^3_{p_k} - 4\pi) \mid \sum_k p_k = p, 0 \leq p_k < p), \\ \mathcal{W}_{4,p} &= \min(8\pi, \beta^4_p + 8\pi/3, 4\pi + \textstyle\sum_k (\beta^4_{p_k} - 4\pi) \mid \sum_k p_k = p, 0 \leq p_k < p), \end{aligned} \tag{1.2}$$

and

$$\beta^n_p := \inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \text{ immersion}, p(\Sigma) = p \}, \tag{1.3}$$

there exists a Möbius transformation Φ of the ambient space $\mathbb{R}^n$ such that the pull-back metric $\tilde{g} = (\Phi \circ f)^* g_{euc}$ differs from a constant curvature metric $e^{-2u}\tilde{g}$ by a bounded conformal factor, more precisely

$$\| u \|_{L^\infty(\Sigma)}, \| \nabla u \|_{L^2(\Sigma, \tilde{g})} \leq C(p, \delta).$$

In this paper, we consider conformal immersions $f : \Sigma \to \mathbb{R}^n$ of a fixed closed Riemann surface Σ and prove existence of minimizers in this conformal class under the above energy assumptions.

Corollary 7.3 *Let Σ be a closed Riemann surface of genus $p \geq 1$ with*

$$\inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \textit{ conformal immersion } \} < \mathcal{W}_{n,p},$$

where $\mathcal{W}_{n,p}$ is defined in (1.2) above and $n = 3, 4$.

Then there exists a smooth conformal immersion $f : \Sigma \to \mathbb{R}^n$ which minimizes the Willmore energy in the set of all conformal immersions. Moreover f satisfies the Euler-Lagrange equation

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = g^{ik} g^{jl} A^0_{ij} q_{kl} \quad \textit{on } \Sigma,$$

where q *is a smooth transverse traceless symmetric 2-covariant tensor with respect to* $g = f^* g_{euc}$. $\square$

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2 Direct method

Let Σ be a closed orientable surface of genus $p \geq 1$ with smooth metric g satisfying

$$\mathcal{W}(\Sigma, g, n) := \inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \text{ conformal immersion to } g\} < \mathcal{W}_{n,p},$$

as in the situation of Corollary 7.3 where $\mathcal{W}_{n,p}$ is defined in (1.2) above and $n = 3, 4$. To get a minimizer, we consider a minimizing sequence of immersions $f_m : \Sigma \to \mathbb{R}^n$ conformal to g in the sense

$$\mathcal{W}(f_m) \to \inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \text{ conformal immersion to } g\}.$$

After applying suitable Möbius transformations according to [KuSch06] Theorem 4.1 we will be able to estimate f_m in $W^{2,2}(\Sigma)$ and $W^{1,\infty}(\Sigma)$, see below, and after passing to a subsequence, we get a limit $f \in W^{2,2}(\Sigma) \cap W^{1,\infty}(\Sigma)$ which is an immersion in a weak sense, see (2.6) below. To prove that f is smooth, which implies that it is a minimizer, and that f satisfies the Euler-Lagrange equation in Corollary 7.3 we will consider variations, say of the form $f + V$. In general, these are not conformal to g anymore, and we want to correct it by $f + V + \lambda_r V_r$ for suitable selected variations V_r . Now even these are not conformal to g since the set of conformal metrics is quite small in the set of all metrics. To increase the set of admissible pull-back metrics, we observe that it suffices for $(f + V + \lambda_r V_r) \circ \phi$ being conformal to g for some diffeomorphism ϕ of Σ . In other words, the pullback metric $(f + V + \lambda_r V_r)^* g_{euc}$ need not be conformal to g , but has to coincide only in the modul space. Actually, we will consider the Teichmüller space, which is coarser than the modul space, but is instead a smooth open manifold, and the bundle projection $\pi : \mathcal{M} \to \mathcal{T}$ of the sets of metrics $\mathcal{M}$ into the Teichmüller space $\mathcal{T}$, see [FiTr84], [Tr]. Clearly $\mathcal{W}(\Sigma, g, n)$ depends only on the conformal structure defined by g , in particular descends to Teichmüller space and leads to the following definition.

Definition 2.1 *We define* $\mathcal{M}_{p,n} : \mathcal{T} \to [0, \infty]$ *for* $p \geq 1, n \geq 3,$ *by selecting a closed, orientable surface* Σ *of genus* p *and*

$$\mathcal{M}_{p,n}(\tau) := \inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \textit{ smooth immersion}, \pi(f^* g_{euc}) = \tau\}.$$

$\square$

We see

$$\mathcal{M}_{p,n}(\pi(g)) = \mathcal{W}(\Sigma, g, n).$$

Next, $\inf_\tau \mathcal{M}_{p,n}(\tau) = \beta_p^n$ for the infimum under fixed genus defined in (1.3), and, as the minimum is attained and $4\pi < \beta_p^n < 8\pi$, see [Sim93] and [BaKu03],

$$4\pi < \min_{\tau \in \mathcal{T}} \mathcal{M}_{p,n}(\tau) = \beta_p^n < 8\pi.$$

In the following proposition, we consider a slightly more general situation than above.

Proposition 2.2 *Let* $f_m : \Sigma \to \mathbb{R}^n, n = 3, 4,$ *be smooth immersions of a closed, orientable surface* Σ *of genus* $p \geq 1$ *satisfying*

$$\mathcal{W}(f_m) \leq \mathcal{W}_{n,p} - \delta \tag{2.1}$$

and

$$\pi(f_m^* g_{euc}) \to \tau_0 \text{ in } \mathcal{T}. \tag{2.2}$$

Then replacing f_m *by* $\Phi_m \circ f_m \circ \phi_m$ *for suitable Möbius transformations* Φ_m *and diffeomorphisms* ϕ_m *of* Σ *homotopic to the identity, we get*

$$\limsup_{m\to\infty} \| f_m \|_{W^{2,2}(\Sigma)} \leq C(p, \delta, \tau_0) \tag{2.3}$$

and $f_m^* g_{euc} = e^{2u_m} g_{poin,m}$ *for some unit volume constant curvature metrics* $g_{poin,m}$ *with*

$$\begin{gathered} \| u_m \|_{L^\infty(\Sigma)}, \| \nabla u_m \|_{L^2(\Sigma, g_{poin,m})} \leq C(p, \delta), \\ g_{poin,m} \to g_{poin} \quad \text{smoothly} \end{gathered} \tag{2.4}$$

with $\pi(g_{poin}) = \tau_0$ *. After passing to a subsequence*

$$\begin{gathered} f_m \to f \text{ weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma), \\ u_m \to u \text{ weakly in } W^{1,2}(\Sigma), \text{weakly}^* \text{ in } L^\infty(\Sigma), \end{gathered} \tag{2.5}$$

and

$$f^* g_{euc} = e^{2u} g_{poin} \tag{2.6}$$

where $(f^* g_{euc})(X, Y) := \langle \partial_X f, \partial_Y f \rangle$ *for* $X, Y \in T\Sigma$ *.*

Proof:

Clearly, replacing f_m by $\Phi_m \circ f_m \circ \phi_m$ as above does neither change the Willmore energy nor the projection into the Teichmüller space.

By [KuSch06] Theorem 4.1 after applying suitable Möbius transformations, the pull-back metric $g_m := f_m^* g_{euc}$ is conformal to a unique constant curvature metric $e^{-2u_m} g_m =: g_{poin,m}$ of unit volume with

$$osc_\Sigma u_m, \| \nabla u_m \|_{L^2(\Sigma, g_m)} \leq C(p, \delta).$$

Combining the Möbius transformations with suitable homotheties, we may further assume that f_m has unit volume. This yields

$$1 = \int_\Sigma \mathrm{d}\mu_{g_m} = \int_\Sigma e^{2u_m} \mathrm{d}\mu_{g_{poin,m}},$$

and, as $g_{poin,m}$ has unit volume as well, we conclude that u_m has a zero on Σ , hence

$$\| u_m \|_{L^\infty(\Sigma)}, \| \nabla u_m \|_{L^2(\Sigma, g_{poin,m})} \leq C(p, \delta)$$

and, as $f_m^* g_{euc} = g_m = e^{2u_m} g_{poin,m}$,

$$\| \nabla f_m \|_{L^\infty(\Sigma, g_{poin,m})} \leq C(p, \delta).$$

Next

$$\Delta_{g_{poin,m}} f_m = e^{2u_m} \Delta_{g_m} f_m = e^{2u_m} \vec{\mathbf{H}}_{f_m}$$

and

$$\| \Delta_{g_{poin,m}} f_m \|^2_{L^2(\Sigma, g_{poin,m})} \leq e^{2 \max u_m} \int_\Sigma |\vec{\mathbf{H}}_{f_m}|^2 e^{2u_m} \, \mathrm{d}\mu_{g_{poin,m}} =$$

$$= 4e^{2 \max u_m} \mathcal{W}(f_m) \leq C(p, \delta). \tag{2.7}$$

To get further estimates, we employ the convergence in Teichmüller space (2.2). We consider a slice $\mathcal{S}$ of unit volume constant curvature metrics for $\tau_0 \in \mathcal{T}$, see [FiTr84], [Tr]. There exist unique $\tilde{g}_{poin,m} \in \mathcal{S}$ with $\pi(\tilde{g}_{poin,m}) = \pi(g_m) \to \tau_0$ for m large enough, hence

$$\phi_m^* \tilde{g}_{poin,m} = g_{poin,m}$$

for suitable diffeomorphisms ϕ_m of Σ homotopic to the identity. Replacing f_m by $f_m \circ \phi_m$, we get $g_{poin,m} = \tilde{g}_{poin,m} \in \mathcal{S}$ and

$$g_{poin,m} \to g_{poin} \quad \text{smoothly}$$

with $g_{poin} \in \mathcal{S}, \pi(g_{poin}) = \tau_0$. Then translating f_m suitably, we obtain

$$\| f_m \|_{L^\infty(\Sigma)} \leq C(p, \delta).$$

Moreover standard elliptic theory, see [GT] Theorem 8.8, implies by (2.7),

$$\| f_m \|_{W^{2,2}(\Sigma)} \leq C(p, \delta, g_{poin})$$

for m large enough.

Selecting a subsequence, we get $f_m \to f$ weakly in $W^{2,2}(\Sigma), f \in W^{1,\infty}(\Sigma), Df_m \to Df$ pointwise almost everywhere, $u_m \to u$ weakly in $W^{1,2}(\Sigma)$, pointwise almost everywhere, and $u \in L^\infty(\Sigma)$. Putting $g := f^* g_{euc}$, that is $g(X,Y) := \langle \partial_X f, \partial_Y f \rangle$, we see

$$g(X,Y) \leftarrow \langle \partial_X f_m, \partial_Y f_m \rangle = g_m(X,Y) = e^{2u_m} g_{poin,m}(X,Y) \to e^{2u} g_{poin}(X,Y)$$

for $X, Y \in T\Sigma$ and pointwise almost everywhere on Σ, hence

$$f^* g_{euc} = g = e^{2u} g_{poin}.$$

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We call a mapping $f \in W^{1,\infty}(\Sigma, \mathbb{R}^n)$ with $g := f^* g_{euc} = e^{2u} g_{poin}, u \in L^\infty(\Sigma)$ as in (2.6) a weak local bilipschitz immersion. If further $f \in W^{2,2}(\Sigma, \mathbb{R}^n)$, we see $g, u \in W^{1,2}(\Sigma)$, hence we can define weak Christoffel symbols $\Gamma \in L^2$ in local charts, a weak second fundamental form A and a weak Riemann tensor R via the equations of Weingarten and Gauß

$$\partial_{ij} f = \Gamma_{ij}^k \partial_k f + A_{ij},$$

$$R_{ijkl} = \langle A_{ik}, A_{jl} \rangle - \langle A_{ij}, A_{kl} \rangle.$$

Of course this defines $\vec{\mathbf{H}}, A^0, K$ for f as well. Moreover we define the tangential and normal projections for $V \in W^{1,2}(\Sigma, \mathbb{R}^n) \cap L^\infty(\Sigma, \mathbb{R}^n)$

$$\pi_f.V := g^{ij} \langle \partial_i f, V \rangle \partial_j f, \quad \pi_f^\perp.V := V - \pi_\Sigma.V \in W^{1,2}(\Sigma, \mathbb{R}^n) \cap L^\infty(\Sigma, \mathbb{R}^n). \tag{2.8}$$

Mollifing f as in [SU83] §4 Proposition, we get smooth $f_m : \Sigma \to \mathbb{R}^n$ with

$$f_m \to f \quad \text{strongly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma) \tag{2.9}$$

and, as $Df \in W^{1,2}$, that locally uniformly $\sup_{|x-y| \leq C/m} d(Df_m(x), Df(y)) \to 0$. This implies for that the pull-backs are uniformly bounded from below and above

$$c_0 g \leq f_m^* g_{euc} \leq Cg \tag{2.10}$$

for some $0 < c_0 \leq C < \infty$ and m large, in particular f_m are smooth immersions.

3 The full rank case

For a smooth immersion $f : \Sigma \to \mathbb{R}^n$ of a closed, orientable surface Σ of genus $p \geq 1$ and $V \in C^\infty(\Sigma, \mathbb{R}^n)$, we see that the variations $f + tV$ are still immersions for small t . We put $g_t := (f + tV)^* g_{euc}, g = g_0 = f^* g_{euc}$ and define

$$\delta\pi_f.V := d\pi_g.\partial_t g_{t|t=0}. \tag{3.1}$$

The elements of

$$\mathcal{V}_f := \delta\pi_f.C^\infty(\Sigma, \mathbb{R}^n) \subseteq T_{\pi g}\mathcal{T}$$

can be considered as the infinitessimal variations of g in Teichmüller space obtained by ambient variations of f . We call f of full rank in Teichmüller space, if $\dim \mathcal{V}_f = \dim \mathcal{T}$. In this case, the necessary corrections in Teichmüller space mentioned in §2 can easily be achieved by the inverse function theorem, as we will see in this section.

Writing $g = e^{2u} g_{poin}$ for some unit volume constant curvature metric g_{poin} by Poincaré's Theorem, see [FiTr84], [Tr], we see $\pi(g_t) = \pi(e^{-2u} g_t)$, hence for an orthonormal basis $q^r(g_{poin}), r = 1, \ldots, \dim \mathcal{T},$ of transvere traceless tensors in $S_2^{TT}(g_{poin})$ with respect to g_{poin}

$$\begin{aligned} \delta\pi_f.V = d\pi_g.\partial_t g_{t|t=0} = d\pi_{g_{poin}}.e^{-2u} \partial_t g_{t|t=0} = \\ = \sum_{r=1}^{\dim \mathcal{T}} \langle e^{-2u} \partial_t g_{t|t=0}.q^r(g_{poin}) \rangle_{g_{poin}} d\pi_{g_{poin}}.q^r(g_{poin}). \end{aligned} \tag{3.2}$$

Calculating in local charts

$$g_{t,ij} = g_{ij} + t\langle \partial_i f, \partial_j V \rangle + t\langle \partial_j f, \partial_i V \rangle + t^2 \langle \partial_i V, \partial_j V \rangle, \tag{3.3}$$

we get

$$\langle e^{-2u} \partial_t g_{t|t=0}, q^r(g_{poin}) \rangle_{g_{poin}} = \int_\Sigma g_{poin}^{ik} g_{poin}^{jl} e^{-2u} \partial_t g_{t,ij\,|t=0} q_{kl}^r(g_{poin}) \ \mathrm{d}\mu_{g_{poin}} =$$

$$= 2 \int\limits_\Sigma g^{ik} g^{jl} \langle \partial_i f, \partial_j V \rangle q^r_{kl}(g_{poin}) \, \mathrm{d}\mu_g =$$

$$= -2 \int\limits_\Sigma g^{ik} g^{jl} \langle \nabla^g_j \nabla^g_i f, V \rangle q^r_{kl}(g_{poin}) \, \mathrm{d}\mu_g - 2 \int\limits_\Sigma g^{ik} g^{jl} \langle \partial_i f, V \rangle \nabla^g_j q^r_{kl}(g_{poin}) \, \mathrm{d}\mu_g =$$

$$= -2 \int\limits_\Sigma g^{ik} g^{jl} \langle A^0_{ij}, V \rangle q^r_{kl}(g_{poin}) \, \mathrm{d}\mu_g,$$

as $q \in S_2^{TT}(g_{poin}) = S_2^{TT}(g)$ is divergence- and tracefree with respect to g. Therefore

$$\delta\pi_f.V = \sum_{r=1}^{\dim \ \mathcal{T}} 2 \int\limits_\Sigma g^{ik} g^{jl} \langle \partial_i f, \partial_j V \rangle q^r_{kl}(g_{poin}) \, \mathrm{d}\mu_g \ d\pi_{g_{poin}}.q^r(g_{poin}) =$$

$$= \sum_{r=1}^{\dim \ \mathcal{T}} -2 \int\limits_\Sigma g^{ik} g^{jl} \langle A^0_{ij}, V \rangle q^r_{kl}(g_{poin}) \, \mathrm{d}\mu_g \ d\pi_{g_{poin}}.q^r(g_{poin}), \tag{3.4}$$

and $\delta\pi_f$ and $\mathcal{V}_f$ are well defined for weak local bilipschitz immersions f and $V \in W^{1,2}(\Sigma)$.

First, we select variations whoose image via $\delta\pi_f$ form a basis of $\mathcal{V}_f$.

Proposition 3.1 *For a weak local bilipschitz immersion f and finitely many points $x_0, \ldots, x_N \in \Sigma$, there exist $V_1, \ldots, V_{\dim \mathcal{V}_f} \in C_0^\infty(\Sigma - \{x_0, \ldots, x_N\}, \mathbb{R}^n)$ such that*

$$\mathcal{V}_f = span \ \{\delta\pi_f.V_s \mid s = 1, \ldots, \dim \ \mathcal{V}_f \ \}. \tag{3.5}$$

Proof:
Clearly $\delta\pi_f : C^\infty(\Sigma, \mathbb{R}^n) \to T_{\pi g_{poin}}\mathcal{T}$ is linear. For suitable $W \subseteq C^\infty(\Sigma, \mathbb{R}^n)$, we obtain a direct sum decomposition

$$C^\infty(\Sigma, \mathbb{R}^n) = ker \ \delta\pi_f \oplus W$$

and see that $\delta\pi_f | W \to im \ \delta\pi_f$ is bijective, hence $\dim \ W = \dim \ \mathcal{V}_f$ and a basis $V_1, \ldots, V_{\dim \mathcal{V}_f}$ of W satisfies (3.5).

Next we choose cutoff functions $\varphi_\varrho \in C_0^\infty(\cup_{k=0}^N B_\varrho(x_k))$ such that $0 \leq \varphi_\varrho \leq 1, \varphi_\varrho = 1$ on $\cup_{k=0}^N B_{\varrho/2}(x_k)$ and $|\nabla\varphi_\varrho|_g \leq C\varrho^{-1}$, hence $1 - \varphi_\varrho \in C_0^\infty(\Sigma - \{x_0, \ldots, x_N\}), 1 - \varphi_\varrho \to 1$ on $\Sigma - \{x_0, \ldots, x_N\}, \int_\Sigma |\nabla\varphi_\varrho|_g \, \mathrm{d}\mu_g \leq C\varrho \to 0$ for $\varrho \to 0$. Clearly $\varphi_m V_s \in C_0^\infty(\Sigma - \{x_0, \ldots, x_N\}, \mathbb{R}^n)$ and by (3.4)

$$\delta\pi_f.(\varphi_m V_s) \to \delta\pi_f.V_s,$$

hence $\varphi_m V_1, \ldots, \varphi_m V_{\dim \mathcal{V}_f}$ satisfies for large m all conclusions of the proposition.

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We continue with a convergence criterion for the first variation.

Proposition 3.2 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a weak local bilipschitz immersion approximated by smooth immersions* f_m *with pull-back metrics* $g = f^* g_{euc} = e^{2u} g_{poin}, g_m = f_m^* g_{euc} = e^{2u_m} g_{poin_m}$ *for some smooth unit volume constant curvature metrics* $g_{poin}, g_{poin,m}$ *and satisfying*

$$\begin{gathered} f_m \to f \quad \textit{weakly in } W^{2,2}(\Sigma), \textit{weakly}^* \textit{ in } W^{1,\infty}(\Sigma), \\ \Lambda^{-1} g_{poin} \le g_m \le \Lambda g_{poin}, \\ \| u_m \|_{L^\infty(\Sigma)} \le \Lambda \end{gathered} \tag{3.6}$$

for some $\Lambda < \infty$ *. Then for any* $W \in L^2(\Sigma, \mathbb{R}^n)$

$$\pi_{f_m}.W \to \pi_f.W$$

as $m \to \infty$ *.*

Proof:
By (3.6)

$$\begin{gathered} g_m = f_m^* g_{euc} \to f^* g_{euc} = g \quad \text{weakly in } W^{1,2}(\Sigma), \text{weakly}^* \text{ in } L^\infty(\Sigma), \\ \Gamma^k_{g_m,ij} \to \Gamma^k_{g,ij} \quad \text{weakly in } L^2, \\ A_{f_m,ij}, A^0_{f_m,ij} \to A_{f,ij}, A^0_{f,ij} \quad \text{weakly in } L^2, \\ \int_\Sigma |K_{g_m}| \, \mathrm{d}\mu_{g_m} \le \frac{1}{2} \int_\Sigma |A_{f_m}|^2 \, \mathrm{d}\mu_{g_m} \le C, \end{gathered} \tag{3.7}$$

hence by [FiTr84], [Tr],

$$\begin{gathered} q^r(g_{poin,m}) \to q^r(g_{poin}) \begin{cases} \text{weakly in } W^{1,2}(\Sigma), \\ \text{weakly}^* \text{ in } L^\infty(\Sigma), \end{cases} \\ d\pi_{g_{poin,m}}.q^r(g_{poin,m}) \to d\pi_{g_{poin}}.q^r(g_{poin}). \end{gathered} \tag{3.8}$$

Then by (3.4)

$$\delta\pi_{f_m}.W =$$

$$= \sum_{r=1}^{\dim \mathcal{T}} -2 \int_\Sigma g_m^{ik} g_m^{jl} \langle A^0_{f_m,ij}, W \rangle q^r_{kl}(g_{poin,m}) \, \mathrm{d}\mu_{g_m} \; d\pi_{g_{poin,m}}.q^r(g_{poin,m}) \to$$

$$\to \delta\pi_f.W.$$

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In the full rank case, the necessary corrections in Teichmüller space mentioned in §2 are achieved in the following lemma by the inverse function theorem.

Lemma 3.3 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a weak local bilipschitz immersion approximated by smooth immersions* f_m *satisfying (2.2) - (2.6).*

If f *is of full rank in Teichmüller space, then for arbitrary* $x_0 \in \Sigma$, *neighbourhood* $U_*(x_0) \subseteq \Sigma$ *of* x_0 *and* $\Lambda < \infty$ *, there exists a neighbourhood* $U(x_0) \subseteq$

$U_*(x_0)$ of x_0 , variations $V_1, \ldots, V_{\dim \mathcal{T}} \in C_0^\infty(\Sigma - \overline{U(x_0)}, \mathbb{R}^n)$, satisfying (3.5), and $\delta > 0, C < \infty, m_0 \in \mathbb{N}$ such that for any $V \in C_0^\infty(U(x_0), \mathbb{R}^n)$ with $f_m + V$ a smooth immersion for some $m \geq m_0$, and $V = 0$ or

$$\begin{gathered}\Lambda^{-1} g_{poin} \leq (f_m + V)^* g_{euc} \leq \Lambda g_{poin}, \\ \| V \|_{W^{2,2}(\Sigma)} \leq \Lambda, \\ \int_{U_*(x_0)} |A_{f_m+V}|^2 \, \mathrm{d}\mu_{f_m+V} \leq \varepsilon_0(n),\end{gathered} \tag{3.9}$$

where $\varepsilon_0(n)$ is as in Lemma A.1, and any $\tau \in \mathcal{T}$ with

$$d_{\mathcal{T}}(\tau, \tau_0) \leq \delta, \tag{3.10}$$

there exists $\lambda \in \mathbb{R}^{\dim \mathcal{T}}$ satisfying

$$\pi((f_m + V + \lambda_r V_r)^* g_{euc}) = \tau$$

and

$$|\lambda| \leq C d_{\mathcal{T}}\Big(\pi((f_m + V)^* g_{euc}), \tau\Big).$$

Proof:
By (2.3), (2.4) and Λ large enough, we may assume

$$\begin{gathered}\| u_m, Df_m \|_{W^{1,2}(\Sigma) \cap L^\infty(\Sigma)} \leq \Lambda, \\ \Lambda^{-1} g_{poin} \leq f_m^* g_{euc} \leq \Lambda g_{poin},\end{gathered} \tag{3.11}$$

in particular

$$\int_\Sigma |A_{f_m}|^2 \, \mathrm{d}\mu_{f_m} \leq C(\Sigma, g_{poin}, \Lambda). \tag{3.12}$$

Putting $\nu_m := |\nabla^2_{g_{poin}} f_m|^2_{g_{poin}} \mu_{g_{poin}}$, we see $\nu_m(\Sigma) \leq C(\Lambda, g_{poin})$ and for a subsequence $\nu_m \to \nu$ weakly* in $C_0^0(\Sigma)^*$. Clearly $\nu(\Sigma) < \infty$, and there are at most finitely many $y_1, \ldots, y_N \in \Sigma$ with $\nu(\{y_i\}) \geq \varepsilon_1$, where we choose $\varepsilon_1 > 0$ below.

As f is of full rank, we can select $V_1, \ldots, V_{\dim \mathcal{T}} \in C_0^\infty(\Sigma - \{x_0, y_1, \ldots, y_N\}, \mathbb{R}^n)$ with $span\{\delta \pi_f . V_r\} = T_{g_{poin}} \mathcal{T}$ by Proposition 3.1. We choose a neighbourhood $U_0(x_0) \subseteq U_*(x_0)$ of x_0 with a chart $\varphi_0 : U_0(x_0) \xrightarrow{\approx} B_2(0), \varphi_0(x_0) = 0$,

$$supp\ V_r \cap \overline{U_0(x_0)} = \emptyset \quad \text{for } r = 1, \ldots, \dim \mathcal{T},$$

put $x_0 \in U_\varrho(x_0) = \varphi_0^{-1}(B_\varrho(0))$ for $0 < \varrho \leq 2$ and choose $x_0 \in U(x_0) \subseteq U_{1/2}(x_0)$ small enough, as we will see below.

Next for any $x \in \cup_{r=1}^{\dim \mathcal{T}} supp\ V_r$, there exists a neighbourhood $U_0(x)$ of x with a chart $\varphi_x : U_0(x) \xrightarrow{\approx} B_2(0), \varphi_x(x) = 0, \overline{U_0(x)} \cap \overline{U_0(x_0)} = \emptyset, \nu(U_0(x)) < \varepsilon_1$ and in the coordinates of the chart φ_x

$$\int_{U_0(x)} g^{ik}_{poin} g^{jl}_{poin} g_{poin,rs} \Gamma^r_{g_{poin},ij} \Gamma^s_{g_{poin},kl} \, \mathrm{d}\mu_{g_{poin}} \leq \varepsilon_1. \tag{3.13}$$

Putting $x \in U_\varrho(x) = \varphi_x^{-1}(B_\varrho(0)) \subset\subset U_0(x)$ for $0 < \varrho \leq 2$, we see that there are finitely many $x_1, \ldots, x_M \in \cup_{r=1}^{\dim \mathcal{T}} supp\ V_r$ such that

$$\cup_{r=1}^{\dim \mathcal{T}} supp\ V_r \subseteq \cup_{k=1}^{M} U_{1/2}(x_k).$$

Then there exists $m_0 \in \mathbb{N}$ such that for $m \geq m_0$

$$\int_{U_1(x_k)} |\nabla^2_{g_{poin}} f_m|^2_{g_{poin}}\ d\mu_{g_{poin}} < \varepsilon_1 \quad \text{for } k = 1, \ldots, M.$$

For V and $m \geq m_0$ as above, we put $\tilde{f}_{m,\lambda} := f_m + V + \lambda_r V_r$. Clearly

$$supp(f_m - \tilde{f}_{m,\lambda}) \subseteq \cup_{k=0}^{M} U_{1/2}(x_k).$$

By (3.9), (3.11), (3.13), and $|\lambda| < \lambda_0 \leq 1/4$ small enough independent of m and V , $\tilde{f}_{m,\lambda}$ is a smooth immersion with

$$(2\Lambda)^{-1} g_{poin} \leq \tilde{g}_{m,\lambda} := \tilde{f}^*_{m,\lambda} g_{euc} \leq 2\Lambda g_{poin}, \tag{3.14}$$

and if $V \neq 0$ by (3.9) and the choice of $U_0(x)$ that

$$\int_{U_1(x_k)} |A_{\tilde{f}_{m,\lambda}}|^2\ d\mu_{\tilde{g}_{m,\lambda}} \leq \varepsilon_0(n) \quad \text{for } k = 0, \ldots, M$$

for $C(\Lambda, g_{poin})(\varepsilon_1 + \lambda_0) \leq \varepsilon_0(n)$. If $V = 0$, we see $supp(f_m - \tilde{f}_{m,\lambda}) \subseteq \cup_{k=1}^{M} U_{1/2}(x_k)$. Further by (3.12)

$$\int_\Sigma |K_{\tilde{g}_{m,\lambda}}|\ d\mu_{\tilde{g}_{m,\lambda}} \leq \frac{1}{2} \int_\Sigma |A_{\tilde{f}_{m,\lambda}}|^2\ d\mu_{\tilde{g}_{m,\lambda}} \leq$$

$$\leq \frac{1}{2} \int_\Sigma |A_{f_m}|^2\ d\mu_{f_m} + \frac{1}{2} \sum_{k=0}^{M} \int_{U_{1/2}(x_k)} |A_{\tilde{f}_{m,\lambda}}|^2\ d\mu_{\tilde{g}_{m,\lambda}} \leq C(\Sigma, g_{poin}, \Lambda) + (M+1)\varepsilon_0(n).$$

This verifies (A.3) and (A.4) for $f = f_m, \tilde{f} = \tilde{f}_{m,\lambda}, g_0 = g_{poin}$ and different, but appropriate Λ . (A.2) follows from (2.4) and (3.12). Then for the unit volume constant curvature metric $\tilde{g}_{poin,m,\lambda} = e^{-2\tilde{u}_{m,\lambda}} \tilde{g}_{m,\lambda}$ conformal to $\tilde{g}_{m,\lambda}$ by Poincaré's Theorem, see [FiTr84], [Tr], we get from Lemma A.1 that

$$\| \tilde{u}_{m,\lambda} \|_{L^\infty(\Sigma)}, \| \nabla \tilde{u}_{m,\lambda} \|_{L^2(\Sigma, g_{poin})} \leq C \tag{3.15}$$

with $C < \infty$ independent of m and V .

From (3.9), we have a $W^{2,2} \cap W^{1,\infty}-$bound on $\tilde{f}_0$, hence for $\tilde{f}_{m,\lambda}$. On $\Sigma - U(x_0)$, we get $\tilde{f}_{m,\lambda} = f_m + \lambda_r V_r \to f$ weakly in $W^{2,2}(\Sigma - U(x_0))$ and weakly* in $W^{1,\infty}(\Sigma - U(x_0))$ for $m_0 \to \infty, \lambda_0 \to 0$ by (2.5). If $V = 0$, then $\tilde{f}_{m,\lambda} = f_m + \lambda_r V_r \to f$ weakly in $W^{2,2}(\Sigma)$ and weakly* in $W^{1,\infty}(\Sigma)$ for $m_0 \to \infty, \lambda_0 \to 0$ by (2.5). Hence letting $m_0 \to \infty, \lambda_0 \to 0, U(x_0) \to \{x_0\}$, we conclude

$$\tilde{f}_{m,\lambda} \to f \quad \text{weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma), \tag{3.16}$$

in particular

$$\tilde{g}_{m,\lambda} \to f^* g_{euc} = e^{2u} g_{poin} =: g \begin{cases} \text{weakly in } W^{1,2}(\Sigma), \\ \text{weakly}^* \text{ in } L^\infty(\Sigma). \end{cases}$$

Together with (3.14), this implies by [FiTr84], [Tr],

$$\pi(\tilde{g}_{m,\lambda}) \to \tau_0. \tag{3.17}$$

We select a chart $\psi : U(\pi(g_{poin})) \subseteq \mathcal{T} \to \mathbb{R}^{\dim \mathcal{T}}$ and put $\hat{\pi} := \psi \circ \pi, \delta\hat{\pi} = d\psi \circ d\pi, \hat{\mathcal{V}}_f := d\psi_{\pi g_{poin}}.\mathcal{V}_f$. By (3.17) for m_0 large enough, λ_0 and $U(x_0)$ small enough independent of V , we get $\pi(\tilde{g}_{m,\lambda}) \in U(\tau_0)$ and define

$$\Phi_m(\lambda) := \hat{\pi}(\tilde{g}_{m,\lambda}).$$

This yields by (3.4), (3.14), (3.15), (3.16) and Proposition 3.2

$$D\Phi_m(\lambda) = (\delta\hat{\pi}_{\tilde{f}_{m,\lambda}}.V_r)_{r=1,\ldots,\dim \mathcal{T}} \to (\delta\hat{\pi}_f.V_r)_{r=1,\ldots,\dim \mathcal{T}} =: A \in \mathbb{R}^{\dim \mathcal{T} \times \dim \mathcal{T}}.$$

As $span\{\delta\pi_f.V_r\} = T_{g_{poin}}\mathcal{T} \cong \mathbb{R}^{\dim \mathcal{T}}$, the matrix A is invertible. Choosing m_0 large enough, λ_0 and $U(x_0)$ small enough, we obtain

$$\| D\Phi_m(\lambda) - A \| \leq 1/(2 \| A^{-1} \|),$$

hence by standard inverse function theorem for any $\xi \in \mathbb{R}^{\dim \mathcal{T}}$ with $|\xi - \Phi_m(0)| < \lambda_0/(2 \| A^{-1} \|)$ there exists $\lambda \in B_{\lambda_0}(0)$ with

$$\Phi_m(\lambda) = \xi,$$
$$|\lambda| \leq 2 \| A^{-1} \| \ |\xi - \Phi_m(0)|.$$

As

$$d_{\mathcal{T}}\Big(\pi((f_m + V)^* g_{euc}), \tau\Big) \leq d_{\mathcal{T}}(\pi(\tilde{g}_0), \tau_0) + d_{\mathcal{T}}(\tau_0, \tau) < d_{\mathcal{T}}(\pi(\tilde{g}_0), \tau_0) + \delta$$

by (3.10), we see for δ small enough, m_0 large enough and $U(x_0)$ small enough by (3.17) that there exists $\lambda \in \mathbb{R}^{\dim \mathcal{T}}$ satisfying

$$\pi((f_m + V + \lambda_r V_r)^* g_{euc}) = \pi(\tilde{g}_{m,\lambda}) = \tau,$$
$$|\lambda| \leq C d_{\mathcal{T}}\Big(\pi((f_m + V)^* g_{euc}), \tau\Big),$$

and the lemma is proved.

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4 The degenerate case

In this section, we consider the degenerate case when the immersion is not of full rank in Teichmüller space. First we see that the image in Teichmüller space looses at most one dimension.

Proposition 4.1 *For a weak local bilipschitz immersion* $f \in W^{2,2}(\Sigma, \mathbb{R}^n)$ *we always have*

$$\dim\ \mathcal{V}_f \geq \dim\ \mathcal{T} - 1. \tag{4.1}$$

If f *is not of full rank in Teichmüller space then* f *is isothermic locally around all but finitely many points of* Σ *, that is around all but finitely many points of* Σ *there exist local conformal principal curvature coordinates.*

Proof:
For $q \in S_2^{TT}(g_{poin})$, we put $\Lambda_q : C^\infty(\Sigma, \mathbb{R}^n) \to \mathbb{R}$

$$\Lambda_q.V := -2 \int_\Sigma g^{ik} g^{jl} \langle A^0_{ij}, V \rangle q_{kl}\ \mathrm{d}\mu_g$$

and define the annihilator

$$Ann := \{ q \in S_2^{TT}(g_{poin}) \mid \Lambda_q = 0\ \}.$$

As $d\pi_{g_{poin}} | S_2^{TT}(g_{poin}) \to T_{g_{poin}} \mathcal{T}$ is bijective, we see by (3.4) and elementary linear algebra

$$\dim\ \mathcal{T} = \dim\ \mathcal{V}_f + \dim\ Ann. \tag{4.2}$$

Clearly,

$$q \in Ann \Longleftrightarrow g^{ik} g^{jl} A^0_{ij} q_{kl} = 0. \tag{4.3}$$

Choosing a conformal chart with respect to the smooth metric g_{poin} , we see $g_{ij} = e^{2v}\delta_{ij}$ for some $v \in W^{1,2} \cap L^\infty$ and $A^0_{11} = -A^0_{22}, A^0_{12} = A^0_{21}, q_{11} = -q_{22}, q_{12} = q_{21}$, as both A^0 and $q \in S_2^{TT}(g_{poin})$ are symmetric and tracefree with respect to $g = e^{2u} g_{poin}$. This rewirtes (4.3) into

$$q \in Ann \Longleftrightarrow A^0_{11} q_{11} + A^0_{12} q_{12} = 0. \tag{4.4}$$

The correspondance between $S_2^{TT}(g_{poin})$ and the holomorphic quadratic differentials is exactly that in conformal coordinates

$$h := q_{11} - i q_{12} \text{ is holomorphic.} \tag{4.5}$$

Now if (4.1) were not true, there would be two linearly independent $q^1, q^2 \in Ann$ by (4.2). Likewise the holomorphic functions $h^k := q^k_{11} - i q^k_{12}$ are linearly independant over $\mathbb{R}$, in particular neither of them vanishes identically, hence these vanish at most at finitely many points, as Σ is closed. Then h^1/h^2 is meromorphic and moreover not a real constant. This implies that $Im(h^1/h^2)$ does not vanish identically, hence vanishes at most at finitely many points. Outside these finitely many points, we calculate

$$Im(h^1/h^2) = |h^2|^{-2} Im(h^1 \overline{h^2}) = |h^2|^{-2} \det \begin{pmatrix} q^1_{11} & q^1_{12} \\ q^2_{11} & q^2_{12} \end{pmatrix},$$

hence

$$\det \begin{pmatrix} q^1_{11} & q^1_{12} \\ q^2_{11} & q^2_{12} \end{pmatrix} \text{ vanishes at most at finitely many points}$$

and by (4.4)

$$A^0 = 0. \tag{4.6}$$

Approximating f by smmoth immersions as in (2.10), we get

$$\nabla_k \vec{\mathbf{H}} = 2g^{ij}\nabla_i A^0_{jk} = 0 \quad \text{weakly},$$

where $\nabla = D^\perp$ denotes the normal connection in the normal bundle along f. Therefore $|\vec{\mathbf{H}}|$ is constant and

$$\partial_k \vec{\mathbf{H}} = g^{ij}\langle \partial_k \vec{\mathbf{H}}, \partial_j f\rangle \partial_i f = -g^{ij}\langle \vec{\mathbf{H}}, A_{jk}\rangle \partial_i f \quad \text{weakly}.$$

Using $\Delta_{g_{poin}} f = e^{2u}\Delta_g f = e^{2u}\vec{\mathbf{H}}$ and $u \in W^{1,2} \cap L^\infty$, we conclude successively that $f \in C^\infty$. Then (4.6) implies that f parametrizes a round sphere or a plane, contradicting $p \geq 1$, and (4.1) is proved.

Next if f is not of full rank in Teichmüller space, there exits $q \in Ann - \{0\} \neq \emptyset$ by (4.2), and the holomorphic function h in (4.5) vanishes at most at finitely many points. In a neighbourhood of a point where h does not vanish, there is a holomorphic function w with $(w')^2 = ih$. Then w has a local inverse z and using w as new local conformal coordinates, h transforms into $(h \circ z)(z')^2 = -i$, hence $q_{11} = 0, q_{12} = 1$ in $w-$coordinates. By (4.4)

$$A_{12} = 0,$$

and w are local conformal principal curvature coordinates.

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Since we loose at most one dimension in the degenerate case, we will do the necessary corrections in Teichmüller space mentioned in §2 by investigating the second variation in Teichmüller space. To be more precise, for a smooth immersion $f : \Sigma \to \mathbb{R}^n$ with pull-back metric f^*g_{euc} conformal to a unit volume constant curvature metric $g_{poin} = e^{-2u}f^*g_{euc}$ by Poincaré's Theorem, see [FiTr84], [Tr], we select a chart $\psi : U(\pi(g_{poin})) \subseteq \mathcal{T} \to \mathbb{R}^{\dim \mathcal{T}}$, put $\hat{\pi} := \psi \circ \pi, \hat{\mathcal{V}}_f := d\psi_{\pi g_{poin}}.\mathcal{V}_f$, and define the second variation in Teichmüller space of f with respect to the chart ψ by

$$\delta^2 \hat{\pi}_f(V) := \left(\frac{d}{dt}\right)^2 \hat{\pi}((f + tV)^* g_{euc})_{|t=0}. \tag{4.7}$$

Putting $f_t := f + tV, g_t := f_t^* g_{euc}, g = g_0$, we see $\hat{\pi}(g_t) = \hat{\pi}(e^{-2u}g_t)$ and calculate

$$\delta^2 \hat{\pi}_f(V) = \left(\frac{d}{dt}\right)^2 \hat{\pi}(e^{-2u}g_t)_{|t=0} =$$

$$= d\hat{\pi}_{g_{poin}}.(e^{-2u}(\partial_{tt}g_t)_{|t=0}) + d^2\hat{\pi}_{g_{poin}}(e^{-2u}(\partial_t g_t)_{|t=0}, e^{-2u}(\partial_t g_t)_{|t=0}).$$

Decomposing

$$e^{-2u}(\partial_t g_t)_{|t=0} = \sigma g_{poin} + \mathcal{L}_X g_{poin} + q, \tag{4.8}$$

with $\sigma \in C^\infty(\Sigma), X \in \mathcal{X}(\Sigma), q \in S_2^{TT}(g_{poin})$, we continue recalling $\{\sigma g_{poin} + \mathcal{L}_X g_{poin}\} = ker\ d\hat{\pi}_{g_{poin}}$ by [FiTr84], [Tr], and

$$\delta^2 \hat{\pi}_f(V) = d\hat{\pi}_{g_{poin}}.(e^{-2u}(\partial_{tt}g_t)_{|t=0}) + d^2\hat{\pi}_{g_{poin}}(\sigma g_{poin} + \mathcal{L}_X g_{poin} + q, \sigma g_{poin} + \mathcal{L}_X g_{poin} + q) =$$

$$= d\hat{\pi}_{g_{poin}}.(e^{-2u}(\partial_{tt}g_t)_{|t=0})+d^2\hat{\pi}_{g_{poin}}(q,q)+d^2\hat{\pi}_{g_{poin}}(\sigma g_{poin}+\mathcal{L}_X g_{poin}, \sigma g_{poin}+\mathcal{L}_X g_{poin}+2q) =$$

$$= d\hat{\pi}_{g_{poin}}.\Big(e^{-2u}(\partial_{tt}g_t)_{|t=0} - \mathcal{L}_X\mathcal{L}_X g_{poin} - 2\sigma\mathcal{L}_X g_{poin} - 2\sigma q - 2\mathcal{L}_X q\Big) + d^2\hat{\pi}_{g_{poin}}(q,q).$$

For an orthonormal basis $q^r(g_{poin}), r = 1, \ldots, \dim\ \mathcal{T},$ of transverse traceless tensors in $S_2^{TT}(g_{poin})$ with respect to g_{poin} , we obtain

$$\delta^2\hat{\pi}_f(V) = \sum_{r=1}^{\dim\ \mathcal{T}} \alpha_r\ d\hat{\pi}_{g_{poin}}.q^r(g_{poin}) + \sum_{r,s=1}^{\dim\ \mathcal{T}} \beta_r\beta_s\ d^2\hat{\pi}_{g_{poin}}(q^r(g_{poin}), q^s(g_{poin})), \qquad (4.9)$$

where

$$\alpha_r :=$$

$$= \int_\Sigma g_{poin}^{ik} g_{poin}^{jl} \Big(e^{-2u}(\partial_{tt}g_t)_{|t=0} - \mathcal{L}_X\mathcal{L}_X g_{poin} - 2\sigma\mathcal{L}_X g_{poin} - 2\sigma q - 2\mathcal{L}_X q\Big)_{ij} q_{kl}^r(g_{poin})\ \mathrm{d}\mu_{g_{poin}},$$

$$\beta_r := \int_\Sigma g_{poin}^{ik} g_{poin}^{jl} q_{ij} q_{kl}^r(g_{poin})\ \mathrm{d}\mu_{g_{poin}}. \qquad (4.10)$$

Since

$$\mathcal{L}_X\mathcal{L}_X g_{poin,ij} = g_{poin}^{mo}\Big(X_m\nabla_o\mathcal{L}_X g_{poin,ij} + \nabla_i X_m\mathcal{L}_X g_{poin,jo} + \nabla_j X_m \mathcal{L}_X g_{poin,oi}\Big),$$

we get integrating by parts

$$\alpha_r = \int_\Sigma g_{poin}^{ik} g_{poin}^{jl} \Big(e^{-2u}(\partial_{tt}g_t)_{|t=0} - 2\sigma\mathcal{L}_X g_{poin} - 2\sigma q - 2\mathcal{L}_X q\Big)_{ij} q_{kl}^r(g_{poin})\ \mathrm{d}\mu_{g_{poin}} +$$

$$+ \int_\Sigma g_{poin}^{ik} g_{poin}^{jl} g_{poin}^{mo} \Big(\nabla_o X_m\mathcal{L}_X g_{poin,ij} - \nabla_i X_m \mathcal{L}_X g_{poin,jo} - \nabla_j X_m\mathcal{L}_X g_{poin,oi}\Big) q_{kl}^r(g_{poin})\,\mathrm{d}\mu_{g_{poin}} +$$

$$+ \int_\Sigma g_{poin}^{ik} g_{poin}^{jl} g_{poin}^{mo} X_m \mathcal{L}_X g_{poin,ij}\nabla_o q_{kl}^r(g_{poin})\ \mathrm{d}\mu_{g_{poin}}. \qquad (4.11)$$

For a weak local bilipschitz immersion $f \in W^{2,2}(\Sigma, \mathbb{R}^n)$ and $V \in W^{1,2}(\Sigma, \mathbb{R}^n)$, we see $g \in W^{1,2}(\Sigma), u \in W^{1,2}(\Sigma) \cap L^\infty(\Sigma), (\partial_t g_t)_{|t=0} \in L^2(\Sigma),$ and $(\partial_{tt}g_t)_{|t=0} \in L^1(\Sigma)$ by (3.3). Then we get a decomposition as in (4.8) with $\sigma \in L^2(\Sigma), X \in \mathcal{X}^1(\Sigma), q \in S_2^{TT}(\Sigma) \subseteq S_2(\Sigma)$. Observing that $q^r(g_{poin}) \in S_2^{TT}(\Sigma) \subseteq S_2(\Sigma)$, we conclude that $\delta^2\hat{\pi}_f$ is well defined for weak local bilipschitz immersions $f \in W^{2,2}(\Sigma, \mathbb{R}^n)$ and $V \in W^{1,2}(\Sigma, \mathbb{R}^n)$ by (4.9), (4.10) and (4.11).

Proposition 4.2 *For a weak local bilipschitz immersion* $f \in W^{2,2}(\Sigma, \mathbb{R}^n)$ *, which is not of full rank in Teichmüller space, and finitely many points* $x_1, \ldots, x_N \in \Sigma$ *, there exist* $V_1, \ldots, V_{\dim \mathcal{T} - 1}, V_\pm \in C_0^\infty(\Sigma - \{x_1, \ldots, x_N\}, \mathbb{R}^n)$ *such that*

$$\hat{\mathcal{V}}_f = span\ \{\delta\hat{\pi}_f.V_s \mid s = 1, \ldots, \dim\ \mathcal{T} - 1\ \} \qquad (4.12)$$

and for some $e \perp \hat{\mathcal{V}}_f, |e| = 1,$

$$\pm\langle\delta^2\hat{\pi}_f(V_\pm), e\rangle > 0, \quad \delta\hat{\pi}_f.V_\pm = 0. \qquad (4.13)$$

Proof:
By Proposition 4.1, there exists $x_0 \in \Sigma - \{x_1, \dots, x_N\}$ such that f is isothermic around x_0 . Moreover, since Σ is not a sphere, hence A^0 does not vanish almost everywhere with respect to μ_g , as we have seen in the argument after (4.6) in Proposition 4.1, we may assume that

$$x_0 \in supp\ |A^0|^2 \mu_g. \tag{4.14}$$

By Proposition 3.1, there exist $V_1, \dots, V_{\dim \mathcal{T}-1} \in C_0^\infty(\Sigma - \{x_0, \dots, x_N\}, \mathbb{R}^n)$ which satisfy (4.12). Next we select a chart $\varphi : U(x_0) \xrightarrow{\approx} B_1(0)$ of conformal principal curvature coordinates that is

$$g = e^{2v} g_{euc}, \quad A_{12} = 0, \tag{4.15}$$

in local coordinates of φ and where $v \in W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$. Moreover, we choose $U(x_0)$ so small that $\overline{U(x_0)} \cap supp\ V_s = \emptyset$ for $s = 1, \dots, \dim\ \mathcal{T} - 1,$ and $\overline{U(x_0)} \cap \{x_1, \dots, x_N\} = \emptyset$. For any $V_0 \in W^{1,2}(\Sigma, \mathbb{R}^n) \cap L^\infty(\Sigma)$ with $supp\ V_0 \subseteq U(x_0)$, there exists a unique $\gamma \in \mathbb{R}^{\dim \mathcal{T}-1}$ such that for $V := V_0 - \gamma_s V_s \in W^{1,2}(\Sigma, \mathbb{R}^n), supp\ V \subseteq \Sigma - \{x_1, \dots, x_N\}$

$$\delta \hat{\pi}_f . V = 0. \tag{4.16}$$

By (3.4)

$$|\gamma| \leq C |\delta \hat{\pi}_f . V_0| \leq C \parallel A^0 \parallel_{L^2(suppV_0, g)} \parallel V_0 \parallel_{L^\infty(U(x_0))}, \tag{4.17}$$

where C does not depend on V_0 . By (3.2), we get for the decomposition in (4.8) that $q = 0$. We select the orthonormal basis $q^r(g_{poin}), r = 1, \dots, \dim\ \mathcal{T},$ of $S_2^{TT}(g_{poin})$ with respect to g_{poin} , in such a way that $d\hat{\pi}_{g_{poin}} . q^r(g_{poin}) \in \hat{\mathcal{V}}_f$ for $r = 2, \dots, \dim\ \mathcal{T},$ and $\langle d\hat{\pi}_{g_{poin}} . q^1(g_{poin}), e \rangle > 0$. Putting

$$I(V_0) := \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} \Big(e^{-2u} (\partial_{tt} g_t)_{|t=0} - 2\sigma \mathcal{L}_X g_{poin} \Big)_{ij} q^1_{kl}(g_{poin})\ \mathrm{d}\mu_{g_{poin}} +$$

$$+ \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} g^{mo}_{poin} \Big(\nabla_o X_m \mathcal{L}_X g_{poin,ij} - \nabla_i X_m \mathcal{L}_X g_{poin,jo} - \nabla_j X_m \mathcal{L}_X g_{poin,oi} \Big) q^1_{kl}(g_{poin})\, \mathrm{d}\mu_{g_{poin}}$$

$$+ \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} g^{mo}_{poin} X_m \mathcal{L}_X g_{poin,ij} \nabla_o q^r_{kl}(g_{poin})\ \mathrm{d}\mu_{g_{poin}}. \tag{4.18}$$

we obtain from (4.9), (4.10) and (4.11)

$$\langle \delta^2 \hat{\pi}_f(V), e \rangle = \langle d\hat{\pi}_{g_{poin}} . q^1(g_{poin}), e \rangle\ I(V_0).$$

As $I(V_{0,m}) \to I(V_0)$ for $V_{0,m} \to V_0$ in $W^{1,2}(\Sigma, \mathbb{R}^n)$, it suffices to find V_0 respectively $V \in W^{1,2}(\Sigma)$ such that

$$\pm I(V_0) > 0. \tag{4.19}$$

Recalling $\delta \hat{\pi}_f . W \in \hat{\mathcal{V}}_f \perp e$ for any $W \in C^\infty(\Sigma, \mathbb{R}^n)$, we see in the same way by (3.4) that

$$0 = \langle \delta \hat{\pi}_f . W, e \rangle = -2 \int_\Sigma g^{ik} g^{jl} \langle A^0_{ij}, W \rangle q^1_{kl}(g_{poin})\ \mathrm{d}\mu_g\ \langle d\pi_{g_{poin}} . q^1(g_{poin}), e \rangle,$$

hence, as $\langle d\hat{\pi}_{g_{poin}}.q^1(g_{poin}), e\rangle > 0$ and $W \in C^\infty(\Sigma, \mathbb{R}^n)$ is arbitrary, $g^{ik}g^{jl}A^0_{ij}q^1_{kl}(g_{poin}) = 0$. We get by (4.15) that $A^0_{11}q^1_{11}(g_{poin}) = 0$ in $B_1(0) \cong U(x_0)$ and by (4.14), as $q^1_{11}(g_{poin}) - iq^1_{12}(g_{poin})$ is holomorphic,

$$\left.\begin{array}{c} q^1_{11}(g_{poin}) = -q^1_{22}(g_{poin}) = 0, \\ q^1_{12} \equiv: q^1 \in \mathbb{R} - \{0\} \end{array}\right\} \quad \text{in } B_1(0) \cong U(x_0) \tag{4.20}$$

in local coordinates of φ .

Next we consider V_0 to be normal at f and calculate by (3.3) that

$$(\partial_t g_{t,ij})_{|t=0} = \langle \partial_i f, \partial_j V\rangle + \langle \partial_j f, \partial_i V\rangle =$$

$$= -2\langle A_{ij}, V_0\rangle - \gamma_s\langle \partial_i f, \partial_j V_s\rangle - \gamma_s\langle \partial_j f, \partial_i V_s\rangle,$$

hence by (4.20)

$$\| (\partial_t g_t)_{|t=0} \|_{L^2(\Sigma)} \leq 2 \| A \|_{L^2(supp V_0, g)} \| V_0 \|_{L^\infty(\Sigma)} + C|\gamma| \leq$$

$$\leq C \| A \|_{L^2(supp V_0, g)} \| V_0 \|_{L^\infty(\Sigma)} .$$

As above we can select $\sigma \in L^2(\Sigma), X \in \mathcal{X}^1(\Sigma)$ in (4.8) such that

$$\| \sigma \|_{L^2(\Sigma)}, \| X \|_{W^{1,2}(\Sigma)} \leq C \| A \|_{L^2(supp V_0, g)} \| V_0 \|_{L^\infty(\Sigma)},$$

where C does not depend on V_0 . We get from (4.18)

$$\Big| I(V_0) - \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} e^{-2u} (\partial_{tt} g_{t,ij})_{|t=0}\ q^1_{kl}(g_{poin})\ \mathrm{d}\mu_{g_{poin}} \Big| \leq C \| A \|^2_{L^2(supp V_0, g)} \| V_0 \|^2_{L^\infty(\Sigma)}, \tag{4.21}$$

where C does not depend on V_0 .

We continue with (3.3) and get, as $supp\ V_0 \cap supp\ V_s = \emptyset$ for $s = 1, \ldots, \dim\ \mathcal{T} - 1$,

$$(\partial_{tt} g_{t,ij})_{|t=0} = 2\langle \partial_i V, \partial_j V\rangle = 2\langle \partial_i V_0, \partial_j V_0\rangle + \gamma_r\gamma_s\langle \partial_i V_r, \partial_j V_s\rangle.$$

As

$$\Big| \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} e^{-2u} 2\gamma_r\gamma_s \langle \partial_i V_r, \partial_j V_s\rangle q^1_{kl}(g_{poin})\ \mathrm{d}\mu_{g_{poin}} \Big| \leq$$

$$\leq C|\gamma|^2 \leq C \| A \|^2_{L^2(supp V_0, g)} \| V_0 \|^2_{L^\infty(\Sigma)}$$

and

$$\int_\Sigma g^{ik}_{poin} g^{jl}_{poin} e^{-2u} 2\langle \partial_i V_0, \partial_j V_0\rangle q^1_{kl}(g_{poin})\ \mathrm{d}\mu_{g_{poin}} = \int_{B_1(0)} 4q^1 e^{-2v} \langle \partial_1 V_0, \partial_2 V_0\rangle\ \mathrm{d}\mathcal{L}^2,$$

where we identify $U(x_0) \cong B_1(0)$, we get from (4.21)

$$\Big| I(V_0) - \int_{B_1(0)} 4q^1 e^{-2v} \langle \partial_1 V_0, \partial_2 V_0\rangle\ \mathrm{d}\mathcal{L}^2 \Big| \leq C \| A \|^2_{L^2(supp V_0, g)} \| V_0 \|^2_{L^\infty(\Sigma)}, \tag{4.22}$$

where C does not depend on V_0 .

Perturbing $x_0 \cong 0$ in $U(x_0) \cong B_1(0)$ slightly, we may assume that 0 is a Lebesgue point for ∇f and v . We select a vector $\mathcal{N}_0 \in \mathbb{R}^n$ normal in $f(0)$ at f and define via normal projection $\mathcal{N} := \pi_f^\perp \mathcal{N}_0 \in W^{1,2}(B_1(0), \mathbb{R}^n)$, see (2.8). Clearly, $\mathcal{N}(0) = \mathcal{N}_0$, and 0 is a Lebesgue point of $\mathcal{N}$. For $\eta \in C_0^\infty(B_1(0))$, we put $\eta_\varrho(y) := \eta(\varrho^{-1}y)$ and $V_0 := \eta_\varrho \mathcal{N}$ and calculate

$$\int\limits_{B_1(0)} \langle \partial_1 V_0, \partial_2 V_0 \rangle e^{-2v} \, \mathrm{d}\mathcal{L}^2 = \int\limits_{B_1(0)} \partial_1 \eta_\varrho \partial_2 \eta_\varrho |\mathcal{N}|^2 e^{-2v} \mathrm{d}\mathcal{L}^2 + \int\limits_{B_1(0)} (\partial_1 \eta_\varrho) \eta_\varrho \langle \mathcal{N}, \partial_2 \mathcal{N} \rangle e^{-2v} \mathrm{d}\mathcal{L}^2 +$$

$$+ \int\limits_{B_1(0)} \eta_\varrho \partial_2 \eta_\varrho \langle \partial_1 \mathcal{N}, \mathcal{N} \rangle e^{-2v} \, \mathrm{d}\mathcal{L}^2 + \int\limits_{B_1(0)} \eta_\varrho^2 \langle \partial_1 \mathcal{N}, \partial_2 \mathcal{N} \rangle e^{-2v} \, \mathrm{d}\mathcal{L}^2.$$

As $\| \nabla \eta_\varrho \|_{L^2(B_\varrho(0))} = \| \nabla \eta \|_{L^2(B_1(0))}$, we see

$$\lim_{\varrho \to 0} \int\limits_{B_1(0)} \langle \partial_1 V_0, \partial_2 V_0 \rangle e^{-2v} \, \mathrm{d}\mathcal{L}^2 =$$

$$= \lim_{\varrho \to 0} \int\limits_{B_1(0)} \partial_1 \eta(y) \partial_2 \eta(y) |\mathcal{N}(\varrho y)|^2 e^{-2v(\varrho y)} \, \mathrm{d}y = e^{-2v(0)} \int\limits_{B_1(0)} \partial_1 \eta \partial_2 \eta \, \mathrm{d}\mathcal{L}^2.$$

Since $\| V_0 \|_{L^\infty(\Sigma)} \leq \| \eta_\varrho \|_{L^\infty(B_\varrho(0))} = \| \eta \|_{L^\infty(B_1(0))}$ and $\| A \|_{L^2(supp V_0, g)} \to 0$ for $\varrho \to 0$, we get

$$\lim_{\varrho \to 0} I(\eta_\varrho \mathcal{N}) = 4 q^1 e^{-2v(0)} \int\limits_{B_1(0)} \partial_1 \eta \partial_2 \eta \, \mathrm{d}\mathcal{L}^2.$$

Introducing new coordinates $\tilde{y}_1 := (y_1 + y_2)/\sqrt{2}, \tilde{y}_2 := (-y_1 + y_2)/\sqrt{2}$, that is we rotate the coordiantes by 45 degrees, and putting $\eta(\tilde{y}_1, \tilde{y}_2) = \xi(\tilde{y}_1) \tau(\tilde{y}_2)$ with $\xi, \tau \in C_0^\infty(] - 1/2, 1/2[)$, we see

$$\int\limits_{B_1(0)} \partial_{y_1} \eta \partial_{y_2} \eta \, \mathrm{d}\mathcal{L}^2 = \frac{1}{2} \int |\xi'|^2 \int |\tau|^2 - \frac{1}{2} \int |\xi|^2 \int |\tau'|^2.$$

Choosing $\tau \in C_0^\infty(] - 1/2, 1/2[), \tau \not\equiv 0$ and $\xi(t) := \tau(2t)$, we get $\int |\xi'|^2 = 2 \int |\tau'|^2, 2 \int |\xi|^2 = \int |\tau|^2$ and

$$\int\limits_{B_1(0)} \partial_{y_1} \eta \partial_{y_2} \eta \, \mathrm{d}\mathcal{L}^2 = \frac{3}{4} \int |\tau|^2 \int |\tau'|^2 > 0.$$

Exchanging ξ and τ , we produce a negative sign. Choosing ϱ small enough and approximating $V_0 = \eta_\varrho \mathcal{N}$ smoothly, we obtain the desired $V_\pm \in C_0^\infty(\Sigma - \{x_1, \ldots, x_N\}, \mathbb{R}^n)$.

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Next we extend the convegence criterion in Proposition 3.2 to the second variation.

Proposition 4.3 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a weak local bilipschitz immersion approximated by smooth immersions* f_m *with pull-back metrics* $g = f^* g_{euc} = e^{2u} g_{poin}, g_m = f_m^* g_{euc} = e^{2u_m} g_{poin_m}$ *for some smooth unit volume constant curvature metrics* $g_{poin}, g_{poin,m}$ *and satisfying*

$$\begin{aligned} f_m \to f \quad \text{weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma), \\ \Lambda^{-1} g_{poin} \leq g_m \leq \Lambda g_{poin}, \\ \parallel u_m \parallel_{L^\infty(\Sigma)} \leq \Lambda \end{aligned} \tag{4.23}$$

for some $\Lambda < \infty$ *. Then for any chart* $\psi : U(\pi(g_{poin})) \subseteq \mathcal{T} \to \mathbb{R}^{\dim \mathcal{T}}$ *,* $\hat{\pi} := \psi \circ \pi, \delta\hat{\pi} = d\psi \circ \delta\pi, \delta^2\hat{\pi}$ *defined in (4.7) and any* $W \in W^{1,2}(\Sigma, \mathbb{R}^n)$

$$\delta\hat{\pi}_{f_m}.W \to \delta\hat{\pi}_f.W, \quad \delta^2\hat{\pi}_{f_m}(W) \to \delta^2\hat{\pi}_f(W)$$

as $m \to \infty$ *.*

Proof:
By Proposition 3.2, we know already $\delta\hat{\pi}_{f_m}.W \to \delta\hat{\pi}_f.W$ and get further (3.7) and (3.8).

Next we select a slice $\mathcal{S}(g_{poin})$ of unit volume constant curvature metrics for $\pi(g_{poin}) =: \tau_0 \in \mathcal{T}$ around g_{poin} with $\pi : \mathcal{S}(g_{poin}) \cong U(\tau_0)$, and $q^r(\tilde{g}_{poin}) \in S_2^{TT}(\tilde{g}_{poin})$ for $\pi(\tilde{g}_{poin}) \in U(\tau_0)$, see [FiTr84], [Tr].

As $\pi(g_{poin,m}) = \pi(g_m) \to \pi(g_{poin}) \in U(\tau_0) \cong \mathcal{S}(g_{poin})$, there exist for m large enough smooth diffeomorphisms ϕ_m of Σ homotopic to the identity with $\phi_m^* g_{poin,m} =: \tilde{g}_{poin,m} \in \mathcal{S}(g_{poin})$. As $\pi(\tilde{g}_{poin,m}) = \pi(g_{poin,m}) = \pi(g_m) \to \pi(g_{poin})$ and $\tilde{g}_{poin,m} \in \mathcal{S}(g_{poin})$, we get

$$\tilde{g}_{poin,m} \to g_{poin} \quad \text{smoothly.} \tag{4.24}$$

The Theorem of Ebin and Palais, see [FiTr84], [Tr], and the remarks following imply by (3.7), (4.23) and (4.24) that after appropriately modifying ϕ_m

$$\phi_m, \phi_m^{-1} \to id_\Sigma \quad \text{weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma), \tag{4.25}$$

in particular,

$$\parallel D\phi_m \parallel_{W^{1,2}(\Sigma) \cap L^\infty(\Sigma)}, \parallel D(\phi_m^{-1}) \parallel_{W^{1,2}(\Sigma) \cap L^\infty(\Sigma)} \leq C \tag{4.26}$$

with C independent of m and V .

For the second variations, we see by (3.3) and (4.23)

$$\partial_t((f_m + tW)^* g_{euc})_{ij\ |t=0} = \langle \partial_i f_m, \partial_j W \rangle + \langle \partial_j f_m, \partial_i W \rangle \to$$

$$\to \langle \partial_i f, \partial_j W \rangle + \langle \partial_j f, \partial_i W \rangle = \partial_t((f + tW)^* g_{euc})_{ij\ |t=0} \quad \text{weakly in } W^{1,2}(\Sigma).$$

Following (4.8), we decompose

$$\phi_m^* \partial_t((\tilde{f}_m + tW)^* g_{euc})_{|t=0} = \sigma_m \tilde{g}_{poin,m} + \mathcal{L}_{X_m} \tilde{g}_{poin,m} + q_m$$

with $q_m \in S_2^{TT}(\tilde{g}_{poin,m})$ and moreover recalling (4.26), we can achieve

$$\parallel \sigma_m \parallel_{W^{1,2}(\Sigma)}, \parallel X_m \parallel_{W^{2,2}(\Sigma)}, \parallel q_m \parallel_{C^2(\Sigma)} \leq C.$$

For a subsequence, we get $\sigma_m \to \sigma$ weakly in $W^{1,2}(\Sigma), X_m \to X$ weakly in $W^{2,2}(\Sigma)$, $q_m \to q$ strongly in $C^1(\Sigma)$ with $q \in S_2^{TT}(g_{poin})$ and by (4.25)

$$\partial_t((f+tW)^* g_{euc})_{|t=0} = \sigma g_{poin} + \mathcal{L}_X g_{poin} + q. \tag{4.27}$$

Observing from (3.3) that

$$(\partial_{tt} g_{t,ij})_{|t=0} = 2\langle \partial_i W, \partial_j W \rangle,$$

we get from (4.9), (4.10) and the equivariance of $\hat{\pi}$

$$\delta^2 \hat{\pi}_{f_m}(W) =$$

$$= \sum_{r=1}^{\dim \mathcal{T}} \alpha_{m,r} \; d\hat{\pi}_{\tilde{g}_{poin,m}}.q^r(\tilde{g}_{poin,m}) + \sum_{r,s=1}^{\dim \mathcal{T}} \beta_{m,r}\beta_{m,s} \; d^2\hat{\pi}_{\tilde{g}_{poin,m}}(q^r(\tilde{g}_{poin,m}), q^s(\tilde{g}_{poin,m})),$$

where

$$\alpha_{m,r} := \int_\Sigma g_m^{ik} g_m^{jl} 2\langle \partial_i W, \partial_j W \rangle q_{kl}^r(g_{poin,m}) \; \mathrm{d}\mu_{g_m} +$$
$$- \int_\Sigma \tilde{g}^{ik}_{poin,m} \tilde{g}^{jl}_{poin,m} \Big(\mathcal{L}_{X_m} \mathcal{L}_{X_m} \tilde{g}_{poin,m} + 2\sigma_m \mathcal{L}_{X_m} \tilde{g}_{poin,m} \Big)_{ij} q_{kl}^r(\tilde{g}_{poin,m}) \; \mathrm{d}\mu_{\tilde{g}_{poin,m}} +$$
$$- \int_\Sigma \tilde{g}^{ik}_{poin,m} \tilde{g}^{jl}_{poin,m} \Big(2\sigma_m q_m + 2\mathcal{L}_{X_m} q_m \Big)_{ij} q_{kl}^r(\tilde{g}_{poin,m}) \; \mathrm{d}\mu_{\tilde{g}_{poin,m}},$$

$$\beta_{m,r} := \int_\Sigma \tilde{g}^{ik}_{poin,m} \tilde{g}^{jl}_{poin,m} q_{m,ij} q_{kl}^r(\tilde{g}_{poin,m}) \; \mathrm{d}\mu_{\tilde{g}_{poin,m}}.$$

By the above convergences in particular by (3.7), (3.8) and (4.24), we obtain

$$\alpha_{m,r} \to \alpha_r := \int_\Sigma g^{ik} g^{jl} 2\langle \partial_i W, \partial_j W \rangle q_{kl}^r(g_{poin}) \; \mathrm{d}\mu_g +$$
$$- \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} \Big(\mathcal{L}_X \mathcal{L}_X g_{poin} + 2\sigma \mathcal{L}_X g_{poin} \Big)_{ij} q_{kl}^r(g_{poin}) \; \mathrm{d}\mu_{g_{poin}} +$$
$$- \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} \Big(2\sigma q + 2\mathcal{L}_X q \Big)_{ij} q_{kl}^r(g_{poin}) \; \mathrm{d}\mu_{g_{poin}},$$

$$\beta_{m,r} \to \beta_r := \int_\Sigma g^{ik}_{poin} g^{jl}_{poin} q_{ij} q_{kl}^r(g_{poin}) \; \mathrm{d}\mu_{g_{poin}}.$$

We see from (4.24)

$$d^2\hat{\pi}_{\tilde{g}_{poin,m}}(q^r(\tilde{g}_{poin,m}), q^s(\tilde{g}_{poin,m})) \to d^2\hat{\pi}_{g_{poin}}(q^r(g_{poin}), q^s(g_{poin})).$$

Observing (4.27), we get from (4.9), (4.10)

$$\delta^2 \hat{\pi}_{\tilde{f}_m}(W) \to \delta^2 \hat{\pi}_f(W) \quad \text{for any } W \in W^{1,2}(\Sigma, \mathbb{R}^n), \tag{4.28}$$

and the proposition is proved.

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Remark:
The bound on the conformal factor u_m in the above proposition is implied by Proposition A.2, when we replace the weak convergence of $f_m \to f$ in $W^{2,2}(\Sigma)$ by strong convergence. $\square$

Now we can extend the correction lemma 3.3 to the degenerate case.

Lemma 4.4 *Let $f : \Sigma \to \mathbb{R}^n$ be a weak local bilipschitz immersion approximated by smooth immersions f_m satisfying (2.2) - (2.6).*

If f is not of full rank in Teichmüller space, then for arbitrary $x_0 \in \Sigma$, neighbourhood $U_*(x_0) \subseteq \Sigma$ of x_0, and $\Lambda < \infty$, there exists a neighbourhood $U(x_0) \subseteq U_*(x_0)$ of x_0, variations $V_1, \ldots, V_{\dim \mathcal{T}-1}, V_\pm \in C_0^\infty(\Sigma - \overline{U(x_0)}, \mathbb{R}^n)$, satisfying (4.12) and (4.13), and $\delta > 0, C < \infty, m_0 \in \mathbb{N}$ such that for any $V \in C_0^\infty(U(x_0), \mathbb{R}^n)$ with $f_m + V$ a smooth immersion for some $m \geq m_0$, and $V = 0$ or

$$\begin{aligned} \Lambda^{-1} g_{poin} \leq (f_m + V)^* g_{euc} \leq \Lambda g_{poin}, \\ \| V \|_{W^{2,2}(\Sigma)} \leq \Lambda, \\ \int_{U_*(x_0)} |A_{f_m+V}|^2 \, d\mu_{f_m+V} \leq \varepsilon_0(n), \end{aligned} \tag{4.29}$$

where $\varepsilon_0(n)$ is as in Lemma A.1, and any $\tau \in \mathcal{T}$ with

$$d_{\mathcal{T}}(\tau, \tau_0) \leq \delta, \tag{4.30}$$

there exists $\lambda \in \mathbb{R}^{\dim \mathcal{T}-1}, \mu_\pm \in \mathbb{R}$, satisfying $\mu_+ \mu_- = 0$,

$$\pi((f_m + V + \lambda_r V_r + \mu_\pm V_\pm)^* g_{euc}) = \tau,$$

$$|\lambda|, |\mu_\pm| \leq C d_{\mathcal{T}}\Big(\pi((f_m + V)^* g_{euc}), \tau\Big)^{1/2}.$$

Further for any $\lambda_0 > 0$, one can choose m_0, δ in such a way that for $m \geq m_0$,

$$\| V \|_{W^{2,2}(\Sigma)} \leq \delta,$$

there exists $\tilde{\lambda} \in \mathbb{R}^{\dim \mathcal{T}-1}, \tilde{\mu}_\pm \in \mathbb{R}$, satisfying $\tilde{\mu}_+ \tilde{\mu}_- = 0$,

$$\begin{aligned} \pi((f_m + V + \tilde{\lambda}_r V_r + \tilde{\mu}_\pm V_\pm)^* g_{euc}) = \tau, \\ |\tilde{\lambda}|, |\tilde{\mu}_\pm| \leq \lambda_0, \end{aligned}$$

and

$$\begin{aligned} \mu_+ \tilde{\mu}_+ \leq 0, \tilde{\mu}_- = 0, \quad \text{if } \mu_+ \neq 0, \\ \mu_- \tilde{\mu}_- \leq 0, \tilde{\mu}_+ = 0, \quad \text{if } \mu_- \neq 0. \end{aligned}$$

Proof:
By (2.3), (2.4) and Λ large enough, we may assume

$$\begin{aligned} \| u_m, Df_m \|_{W^{1,2}(\Sigma) \cap L^\infty(\Sigma)} \leq \Lambda, \\ \Lambda^{-1} g_{poin} \leq f_m^* g_{euc} \leq \Lambda g_{poin}, \end{aligned} \tag{4.31}$$

in particular

$$\int\limits_\Sigma |A_{f_m}|^2 \, d\mu_{f_m} \leq C(\Sigma, g_{poin}, \Lambda). \tag{4.32}$$

Putting $\nu_m := |\nabla^2_{g_{poin}} f_m|^2_{g_{poin}} \mu_{g_{poin}}$, we see $\nu_m(\Sigma) \leq C(\Lambda, g_{poin})$ and for a subsequence $\nu_m \to \nu$ weakly* in $C^0_0(\Sigma)^*$. Clearly $\nu(\Sigma) < \infty$, and there are at most finitely many $y_1, \ldots, y_N \in \Sigma$ with $\nu(\{y_i\}) \geq \varepsilon_1$, where we choose $\varepsilon_1 > 0$ below.

By Proposition 4.2, we can select $V_1, \ldots, V_{\dim \mathcal{T}-1}, V_+ = V_{\dim \mathcal{T}}, V_- = V_{\dim \mathcal{T}+1} \in C^\infty_0(\Sigma - \{x_0, y_1, \ldots, y_N\}, \mathbb{R}^n)$ satisfying (4.12) and (4.13).

We choose a neighbourhood $U_0(x_0) \subseteq U_*(x_0)$ of x_0 with a chart $\varphi_0 : U_0(x_0) \xrightarrow{\approx} B_2(0), \varphi_0(x_0) = 0$,

$$supp\ V_r \cap \overline{U_0(x_0)} = \emptyset \quad \text{for } r = 1, \ldots, \dim \mathcal{T} + 1,$$

put $x_0 \in U_\varrho(x_0) = \varphi_0^{-1}(B_\varrho(0))$ for $0 < \varrho \leq 2$ and choose $x_0 \in U(x_0) \subseteq U_{1/2}(x_0)$ small enough, as we will see below.

Next for any $x \in \cup_{r=1}^{\dim \mathcal{T}+1} supp\ V_r$, there exists a neighbourhood $U_0(x)$ of x with a chart $\varphi_x : U_0(x) \xrightarrow{\approx} B_2(0), \varphi_x(x) = 0, \overline{U_0(x)} \cap \overline{U_0(x_0)} = \emptyset, \nu(U_0(x)) < \varepsilon_1$ and in the coordinates of the chart φ_x

$$\int\limits_{U_0(x)} g^{ik}_{poin} g^{jl}_{poin} g_{poin,rs} \Gamma^r_{g_{poin},ij} \Gamma^s_{g_{poin},kl} \, d\mu_{g_{poin}} \leq \varepsilon_1. \tag{4.33}$$

Putting $x \in U_\varrho(x) = \varphi_x^{-1}(B_\varrho(0)) \subset\subset U_0(x)$ for $0 < \varrho \leq 2$, we see that there are finitely many $x_1, \ldots, x_M \in \cup_{r=1}^{\dim \mathcal{T}+1} supp\ V_r$ such that

$$\cup_{r=1}^{\dim \mathcal{T}+1} supp\ V_r \subseteq \cup_{k=1}^M U_{1/2}(x_k).$$

Then there exists $m_0 \in \mathbb{N}$ such that for $m \geq m_0$

$$\int\limits_{U_1(x_k)} |\nabla^2_{g_{poin}} f_m|^2_{g_{poin}} \, d\mu_{g_{poin}} < \varepsilon_1 \quad \text{for } k = 1, \ldots, M.$$

For V and $m \geq m_0$ as above, we put $\tilde{f}_{m,\lambda,\mu} := f_m + V + \lambda_r V_r + \mu_\pm V_\pm$. Clearly

$$supp(f_m - \tilde{f}_{m,\lambda,\mu}) \subseteq \cup_{k=0}^M U_{1/2}(x_k).$$

By (4.29), (4.31), (4.33), and $|\lambda|, |\mu| < \lambda_0 \leq 1/4$ small enough independent of m and V , $\tilde{f}_{m,\lambda,\mu}$ is a smooth immersion with

$$(2\Lambda)^{-1} g_{poin} \leq \tilde{g}_{m,\lambda,\mu} := \tilde{f}^*_{m,\lambda,\mu} g_{euc} \leq 2\Lambda g_{poin}, \tag{4.34}$$

and if $V \neq 0$ by (4.29) and the choice of $U_0(x)$ that

$$\int\limits_{U_1(x_k)} |A_{\tilde{f}_{m,\lambda,\mu}}|^2 \, d\mu_{\tilde{g}_{m,\lambda,\mu}} \leq \varepsilon_0(n) \quad \text{for } k = 0, \ldots, M$$

for $C(\Lambda, g_{poin})(\varepsilon_1 + \lambda_0) \leq \varepsilon_0(n)$. If $V = 0$, we see $supp(f_m - \tilde{f}_{m,\lambda,\mu}) \subseteq \cup_{k=1}^M U_{1/2}(x_k)$. Further by (4.32)

$$\int_\Sigma |K_{\tilde{g}_{m,\lambda,\mu}}| \, d\mu_{\tilde{g}_{m,\lambda,\mu}} \leq \frac{1}{2} \int_\Sigma |A_{\tilde{f}_{m,\lambda,\mu}}|^2 \, d\mu_{\tilde{g}_{m,\lambda,\mu}} \leq$$

$$\leq \frac{1}{2} \int_\Sigma |A_{f_m}|^2 \, d\mu_{f_m} + \frac{1}{2} \sum_{k=0}^M \int_{U_{1/2}(x_k)} |A_{\tilde{f}_{m,\lambda,\mu}}|^2 \, d\mu_{\tilde{g}_{m,\lambda,\mu}} \leq C(\Sigma, g_{poin}, \Lambda) + (M+1)\varepsilon_0(n).$$

This verifies (A.3) and (A.4) for $f = f_m, \tilde{f} = \tilde{f}_{m,\lambda,\mu}, g_0 = g_{poin}$ and different, but appropriate Λ . (A.2) follows from (2.4) and (4.32). Then for the unit volume constant curvature metric $\tilde{g}_{poin,m,\lambda,\mu} = e^{-2\tilde{u}_{m,\lambda,\mu}} \tilde{g}_{m,\lambda,\mu}$ conformal to $\tilde{g}_{m,\lambda,\mu}$ by Poincaré's Theorem, see [FiTr84], [Tr], we get from Lemma A.1 that

$$\parallel \tilde{u}_{m,\lambda,\mu} \parallel_{L^\infty(\Sigma)}, \parallel \nabla \tilde{u}_{m,\lambda,\mu} \parallel_{L^2(\Sigma, g_{poin})} \leq C \tag{4.35}$$

with $C < \infty$ independent of m and V .

From (4.29), we have a $W^{2,2} \cap W^{1,\infty}$−bound on $\tilde{f}_{m,0,0}$, hence for $\tilde{f}_{m,\lambda,\mu}$. On $\Sigma - U(x_0)$, we get $\tilde{f}_{m,\lambda,\mu} = f_m + \lambda_r V_r + \mu_\pm V_\pm \to f$ weakly in $W^{2,2}(\Sigma - U(x_0))$ and weakly* in $W^{1,\infty}(\Sigma - U(x_0))$ for $m_0 \to \infty, \lambda_0 \to 0$ by (2.5). If $V = 0$, then $\tilde{f}_{m,\lambda,\mu} = f_m + \lambda_r V_r + \mu_\pm V_\pm \to f$ weakly in $W^{2,2}(\Sigma)$ and weakly* in $W^{1,\infty}(\Sigma)$ for $m_0 \to \infty, \lambda_0 \to 0$ by (2.5). Hence letting $m_0 \to \infty, \lambda_0 \to 0, U(x_0) \to \{x_0\}$, we conclude

$$\tilde{f}_{m,\lambda,\mu} \to f \quad \text{weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma), \tag{4.36}$$

Then by (4.34), (4.35), (4.36) and Proposition 4.3 for any $W \in W^{1,2}(\Sigma, \mathbb{R}^n)$

$$\delta\hat{\pi}_{\tilde{f}_{m,\lambda,\mu}}.W \to \delta\hat{\pi}_f.W, \quad \delta^2\hat{\pi}_{\tilde{f}_{m,\lambda,\mu}}(W) \to \delta^2\hat{\pi}_f(W) \tag{4.37}$$

for $m_0 \to \infty, \lambda_0 \to 0, U(x_0) \to \{x_0\}$.

Further from (4.36)

$$\tilde{g}_{m,\lambda,\mu} \to f^* g_{euc} = e^{2u} g_{poin} =: g \begin{cases} \text{weakly in } W^{1,2}(\Sigma), \\ \text{weakly}^* \text{ in } L^\infty(\Sigma), \end{cases}$$

and by (4.35), we see $\tilde{u}_{m,\lambda,\mu} \to \tilde{u}$ weakly in $W^{1,2}(\Sigma)$ and weakly* in $L^\infty(\Sigma)$, hence

$$\tilde{g}_{poin,m,\lambda,\mu} = e^{-2\tilde{u}_{m,\lambda,\mu}} \tilde{g}_{m,\lambda,\mu} \to e^{2(u-\tilde{u})} g_{poin} \begin{cases} \text{weakly in } W^{1,2}(\Sigma), \\ \text{weakly}^* \text{ in } L^\infty(\Sigma), \end{cases}$$

which together with (4.34) implies

$$\pi(\tilde{g}_{m,\lambda,\mu}) \to \tau_0. \tag{4.38}$$

We select a chart $\psi : U(\pi(g_{poin})) \subseteq \mathcal{T} \to \mathbb{R}^{\dim \mathcal{T}}$ and put $\hat{\pi} := \psi \circ \pi, \delta\hat{\pi} = d\psi \circ d\pi, \hat{\mathcal{V}}_f := d\psi_{\pi g_{poin}}.\mathcal{V}_f, \delta^2\hat{\pi}$ defined in (4.7). By (4.38) for m_0 large enough, λ_0 and $U(x_0)$ small enough independent of V , we get $\pi(\tilde{g}_{m,\lambda,\mu}) \in U(\tau_0)$ and define

$$\Phi_m(\lambda, \mu) := \hat{\pi}(\tilde{g}_{m,\lambda,\mu}).$$

We see by (4.37)

$$D\Phi_m(\lambda,\mu) = (\delta\hat{\pi}_{\tilde{f}_{m,\lambda,\mu}}.V_r)_{r=1,\ldots,\dim\mathcal{T}+1} \to (\delta\hat{\pi}_f.V_r)_{r=1,\ldots,\dim\mathcal{T}+1}.$$

After a change of coordinates, we may assume

$$\hat{\mathcal{V}}_f = \mathbb{R}^{\dim\mathcal{T}-1}\times\{0\}$$

and $e := e_{\dim\mathcal{T}} \perp \hat{\mathcal{V}}_f$. Writing $\Phi_m(\lambda,\mu) = (\Phi_{m,0}(\lambda,\mu),\varphi_m(\lambda,\mu)) \in \mathbb{R}^{\dim\mathcal{T}-1}\times\mathbb{R}\times\mathbb{R}$, we get

$$\partial_\lambda\Phi_{m,0}(\lambda,\mu) \to (\pi_{\hat{\mathcal{V}}_f}\delta\hat{\pi}_f.V_r)_{r=1,\ldots,\dim\mathcal{T}-1} =: A \in \mathbb{R}^{(\dim\mathcal{T}-1)\times(\dim\mathcal{T}-1)}.$$

From (4.12), we see that A is invertible, hence after a further change of coordiantes we may assume that $A = I_{(\dim\mathcal{T}-1)}$ and

$$\| \partial_\lambda\Phi_{m,0}(\lambda,\mu) - I_{(\dim\mathcal{T}-1)} \| \leq 1/2 \tag{4.39}$$

for m_0 large enough, λ_0 and $U(x_0)$ small enough independent of V . Next by (4.13), we obtain

$$\begin{aligned} \partial_{\mu_\pm}\Phi_m(\lambda,\mu) &\to \delta\hat{\pi}_f.V_\pm = 0,\\ \nabla\varphi_m(\lambda,\mu) &\to \langle(\delta\hat{\pi}_f.V_r)_{r=1,\ldots,\dim\mathcal{T}+1}, e\rangle = 0, \end{aligned} \tag{4.40}$$

as $\delta\hat{\pi}_f.V_r \in \hat{\mathcal{V}}_f \perp e$, hence

$$|\partial_{\mu_\pm}\Phi_m(\lambda,\mu)|, |\nabla\varphi_m(\lambda,\mu)| \leq \varepsilon_2 \tag{4.41}$$

for any $\varepsilon_2 > 0$ chosen below, if m_0 large enough, λ_0 and $U(x_0)$ small enough independent of V .

The second derivatives

$$\begin{aligned} \partial_{ss}\Phi_m(\lambda,\mu) &= \delta^2\hat{\pi}_{\tilde{f}_{m,\lambda,\mu}}(V_s),\\ 4\partial_{sr}\Phi_m(\lambda,\mu) &= \delta^2\hat{\pi}_{\tilde{f}_{m,\lambda,\mu}}(V_s+V_r) - \delta^2\hat{\pi}_{\tilde{f}_{m,\lambda,\mu}}(V_s-V_r) \end{aligned} \tag{4.42}$$

are given by the second variation in Teichmüller space. From (4.37) for $W = V_s, V_s \pm V_r \in C^\infty(\Sigma,\mathbb{R}^n)$, we conclude for m_0 large enough, λ_0 and $U(x_0)$ small enough independent of V that

$$|D^2\Phi_m(\lambda,\mu)| \leq \Lambda_1 \tag{4.43}$$

for some $1 \leq \Lambda_1 < \infty$ and

$$\begin{aligned} \partial_{\mu_+\mu_+}\Phi_m(\lambda,\mu) &\to \delta^2\hat{\pi}_f(V_+),\\ \partial_{\mu_-\mu_-}\Phi_m(\lambda,\mu) &\to \delta^2\hat{\pi}_f(V_-), \end{aligned} \tag{4.44}$$

hence by (4.13)

$$\pm\lim_{m,\lambda,\mu}\partial_{\mu_\pm\mu_\pm}\varphi_m(\lambda,\mu) = \pm\langle\delta^2\hat{\pi}_f(V_\pm), e\rangle > 0$$

and

$$\pm\partial_{\mu_\pm\mu_\pm}\varphi_m(\lambda,\mu) \geq \gamma \tag{4.45}$$

for some $0 < \gamma \leq 1/4$ and m_0 large enough, λ_0 and $U(x_0)$ small enough independent of V. Now, we choose $\varepsilon_2 > 0$ to satisfy $C\Lambda_1\varepsilon_2 \leq \gamma/4$. Choosing m_0 even larger and $U(x_0)$ even smaller we get by (4.38)

$$d_{\mathcal{T}}(\pi(\tilde{g}_0), \tau_0) \leq \delta,$$

where we choose δ now. As

$$d_{\mathcal{T}}\Big(\pi((f_m + V)^* g_{euc}), \tau\Big) \leq d_{\mathcal{T}}(\pi(\tilde{g}_0), \tau_0) + d_{\mathcal{T}}(\tau_0, \tau) < 2\delta$$

by (4.30), we choose $C_\psi \delta \leq \Lambda_1 \lambda_0^2, \lambda_0/8, \gamma\lambda_0^2/32$ and conclude from Proposition B.1 that there exists $\lambda \in \mathbb{R}^{\dim \mathcal{T} - 1}, \mu_\pm \in \mathbb{R}$ with $\mu_+\mu_- = 0$ and satisfying

$$\begin{gathered}\pi((f_m + V + \lambda_r V_r + \mu_\pm V_\pm)^* g_{euc}) = \pi(\tilde{g}_{m,\lambda,\mu}) = \tau, \\ |\lambda|, |\mu_\pm| \leq C d_{\mathcal{T}}\Big(\pi((f_m + V)^* g_{euc}), \tau\Big)^{1/2}.\end{gathered}$$

To obtain the second conclusion, we consider $\lambda_0 > 0$ such small that $C\Lambda_1\varepsilon_2 + C\Lambda_1\lambda_0 \leq \gamma/2$ and fix this λ_0. We assume $\parallel V \parallel_{W^{2,2}(\Sigma)} \leq \delta$ and see as in (4.36) that $\tilde{f}_{m,0,0} = f_m + V \to f$ weakly in $W^{2,2}(\Sigma)$ and weakly* in $W^{1,\infty}(\Sigma)$ for $m_0 \to \infty, \delta \to 0$ by (2.5). Again we get (4.37) and (4.40) for $\lambda, \mu = 0$, hence for m_0 large enough, δ small enough, but fixed $U(x_0)$,

$$|\nabla \varphi_m(0)| \leq \sigma$$

with $C\Lambda_1\varepsilon_2 + C\sigma\lambda_0^{-1} + C\Lambda_1\lambda_0 \leq \gamma$. Then by Proposition B.1, there exist further $\tilde{\lambda} \in \mathbb{R}^{\dim \mathcal{T} - 1}, \tilde{\mu}_\pm \in \mathbb{R}$ with $\tilde{\mu}_+\tilde{\mu}_- = 0$ and satisfying

$$\begin{gathered}\pi((f_m + V + \tilde{\lambda}_r V_r + \tilde{\mu}_\pm V_\pm)^* g_{euc}) = \pi(\tilde{g}_{\tilde{\lambda},\tilde{\mu}}) = \tau, \\ |\tilde{\lambda}|, |\tilde{\mu}_\pm| \leq \lambda_0, \\ \mu_+\tilde{\mu}_+ \leq 0, \tilde{\mu}_- = 0, \quad \text{if } \mu_- = 0, \\ \mu_-\tilde{\mu}_- \leq 0, \tilde{\mu}_+ = 0, \quad \text{if } \mu_+ = 0,\end{gathered}$$

and the lemma is proved.

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5 Elementary properties of $\mathcal{M}_{p,n}$

As a first application of our correction Lemmas 3.3 and 4.4 we establish upper semicontinuity of the minimal Willmore energy under fixed Teichmüller class $\mathcal{M}_{p,n}$.

Proposition 5.1 *$\mathcal{M}_{p,n} : \mathcal{T} \to [\beta_p^n, \infty]$ is upper semicontinuous. Moreover $\mathcal{M}_{p,n}$ is continuous at $\tau \in \mathcal{T}, n = 3, 4,$ with*

$$\mathcal{M}_{p,n}(\tau) \leq \mathcal{W}_{n,p}.$$

Proof:
For the upper semicontinuity, we have to prove

$$\limsup_{\tau \to \tau_0} \mathcal{M}_{p,n}(\tau) \le \mathcal{M}_{p,n}(\tau_0) \quad \text{for } \tau_0 \in \mathcal{T}. \tag{5.1}$$

It suffices to consider $\mathcal{M}_{p,n}(\tau_0) < \infty$. In this case, there exists for any $\varepsilon > 0$ a smooth immersion $f : \Sigma \to \mathbb{R}^n$ with $\pi(f^* g_{euc}) = \tau_0$ and $\mathcal{W}(f) < \mathcal{M}_{p,n}(\tau_0) + \varepsilon$.

By Lemmas 3.3 and 4.4 applied to the constant sequence $f_m = f$ and $V = 0$, there exist for any τ close enough to τ_0 in Teichmüller space $\lambda_\tau \in \mathbb{R}^{\dim \mathcal{T}+1}$ with

$$\pi((f + \lambda_{\tau,r} V_r)^* g_{euc}) = \tau,$$
$$\lambda_\tau \to 0 \text{ for } \tau \to \tau_0.$$

Clearly $f + \lambda_{\tau,r} V_r \to f$ smoothly, hence

$$\limsup_{\tau \to \tau_0} \mathcal{M}_{p,n}(\tau) \le \lim_{\tau \to \tau_0} \mathcal{W}(f + \lambda_{\tau,r} V_r) = \mathcal{W}(f) \le \mathcal{M}_{p,n}(\tau_0) + \varepsilon$$

and (5.1) follows.

If $\mathcal{M}_{p,n}$ were not continuous at $\tau_0 \in \mathcal{T}$ with $\mathcal{M}_{p,n}(\tau_0) \le \mathcal{W}_{n,p}$, by upper semicontinuity of $\mathcal{M}_{p,n}$ proved above, there exists $\delta > 0$ and a sequence $\tau_m \to \tau_0$ in $\mathcal{T}$ with

$$\mathcal{M}_{p,n}(\tau_m) \le \mathcal{M}_{p,n}(\tau_0) - 2\delta.$$

We select smooth immersions $f_m : \Sigma \to \mathbb{R}^n$ with $\pi(f_m^* g_{euc}) = \tau_m \to \tau_0$ and

$$\mathcal{W}(f_m) \le \mathcal{M}_{p,n}(\tau_m) + 1/m, \tag{5.2}$$

hence for m large enough $\mathcal{W}(f_m) \le \mathcal{M}_{p,n}(\tau_0) - \delta \le \mathcal{W}_{n,p} - \delta$. Replacing f_m by $\Phi_m \circ f_m \circ \phi_m$ as in Proposition 2.2 does neither change the Willmore energy nor its projections in Teichmüller space, and we may assume that f_m, f satisfy (2.2) - (2.6). By Lemmas 3.3 and 4.4 applied to $f_m, V = 0$ and $\tau = \tau_0$, there exist $\lambda_m \in \mathbb{R}^{\dim \mathcal{T}+1}$ for m large enough with

$$\pi((f_m + \lambda_{m,r} V_r)^* g_{euc}) = \tau_0,$$
$$\lambda_m \to 0 \text{ for } m \to \infty.$$

This yields

$$\mathcal{M}_{p,n}(\tau_0) \le \liminf_{m \to \infty} \mathcal{W}(f_m + \lambda_{m,r} V_r) \le \liminf_{m \to \infty} (\mathcal{W}(f_m) + C|\lambda_m|) \le \mathcal{M}_{p,n}(\tau_0) - \delta,$$

which is a contradiction, and the proposition is proved.

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Secondly, we prove that the infimum taken in Definition 2.1 over smooth immersions is not improved by weak local bilipschitz immersion in $W^{2,2}$.

Proposition 5.2 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a weak local bilipschitz* $W^{2,2}-$*immersion with pull-back metric* $f^* g_{euc} = e^{2u} g_{poin}$ *conformal to a smooth unit volume constant curvature metric* g_{poin} . *Then*

$$\mathcal{M}_{p,n}(\pi(g_{poin})) \le \mathcal{W}(f) = \frac{1}{4} \int_\Sigma |A_f|^2 \, \mathrm{d}\mu_f + 2\pi(1-p).$$

Proof:
We approximate f by smmoth immersions f_m as in (2.9) and (2.10). Putting $g := f^* g_{euc}, g_m := f_m^* g_{euc}$ and writing $g_m = e^{2u_m} g_{poin,m}$ for some unit volume constant curvature metric $g_{poin,m}$ by Poincaré's Theorem, see [FiTr84], [Tr], we get by Proposition A.2

$$\| u_m \|_{L^\infty(\Sigma)}, \| \nabla u_m \|_{L^2(\Sigma)} \leq C \tag{5.3}$$

for some $C < \infty$ independent of m. In local charts, we see

$$g_m \to g \quad \text{strongly in } W^{1,2}, \text{weakly}^* \text{ in } L^\infty,$$
$$\Gamma^k_{g_m,ij} \to \Gamma^k_{g,ij} \quad \text{strongly in } L^2,$$

and

$$A_{f_m,ij} = \nabla_i^{g_m} \nabla_j^{g_m} f_m \to \partial_{ij} f - \Gamma^k_{g,ij} \partial_k f =$$
$$= \nabla_i^g \nabla_j^g f = A_{f,ij} \quad \text{strongly in } L^2.$$

Therefore by (1.1)

$$\mathcal{W}(f_m) \to \mathcal{W}(f) = \frac{1}{4} \int_\Sigma |A_f|_g^2 \, \mathrm{d}\mu_g + 2\pi(1-p) \tag{5.4}$$

and

$$\pi(g_m) \to \pi(g_{poin}). \tag{5.5}$$

Next for a slice $\mathcal{S}(g_{poin})$ of unit volume constant curvature metrics for $\pi(g_{poin}) \in \mathcal{T}$ around g_{poin}, see [FiTr84], [Tr], there exist unique $\tilde{g}_{poin,m} \in \mathcal{S}(g_{poin})$ with $\pi(\tilde{g}_{poin,m}) = \pi(g_m) \to \pi(g_{poin})$ for m large enough, hence

$$\phi_m^* g_{poin,m} = \tilde{g}_{poin,m} \to g_{poin} \quad \text{smoothly}$$

for suitable diffeomorphisms ϕ_m of Σ homotopic to the identity.

Next by (5.3)

$$\| D\phi_m \|_{L^\infty(\Sigma)}, \| D\phi_m^{-1} \|_{L^\infty(\Sigma)} \leq C \tag{5.6}$$

and

$$\| g_{poin,m} \|_{W^{1,2}(\Sigma)} \leq \| e^{-2u_m} \|_{W^{1,2}(\Sigma) \cap L^\infty(\Sigma)} \| g_m \|_{W^{1,2}(\Sigma) \cap L^\infty(\Sigma)} \leq C$$

and for a subsequence $g_{poin,m} \to \tilde{g}, u_m \to \tilde{u}$ weakly in $W^{1,2}(\Sigma)$, in particular $\tilde{g} \leftarrow g_{poin,m} = e^{-2u_m} g_m \to e^{-2\tilde{u}} g$, and

$$\tilde{g} = e^{-2\tilde{u}} g = e^{2(u-\tilde{u})} g_{poin} \tag{5.7}$$

is conformal to the smooth metric g_{poin}.

The Theorem of Ebin and Palais, see [FiTr84], [Tr], and the remarks following imply after appropriately modifying ϕ_m

$$\phi_m \to id_\Sigma \quad \text{weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma) \tag{5.8}$$

and $g_{poin} = id_\Sigma^* \tilde{g} = e^{2(u-\tilde{u})} g_{poin}$ by (5.7), hence

$$u_m \to \tilde{u} = u \quad \text{weakly in } W^{1,2}(\Sigma), \text{weakly}^* \text{ in } L^\infty(\Sigma). \tag{5.9}$$

Putting $\tilde{f}_m := f_m \circ \phi_m, \tilde{u}_m := u_m \circ \phi_m$, we see $\tilde{f}_m^* g_{euc} = \phi_m^*(e^{2u_m} g_{poin,m}) = e^{2\tilde{u}_m} \tilde{g}_{poin,m}$ and by (5.6), (5.8) and (5.9)

$$\tilde{f}_m \to f \text{ weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma),$$
$$\tilde{u}_m \to u \text{ weakly in } W^{1,2}(\Sigma), \text{weakly}^* \text{ in } L^\infty(\Sigma).$$

Therefore $\tilde{f}_m, f$ satisfy (2.2) - (2.6). By Lemmas 3.3 and 4.4 applied to $V = 0$, there exist $\lambda_m \in \mathbb{R}^{\dim \mathcal{T}+1}$ for m large enough with

$$\pi((\tilde{f}_m + \lambda_{m,r} V_r)^* g_{euc}) = \pi(g_{poin}),$$
$$\lambda_m \to 0 \text{ for } m \to \infty,$$

as $\pi(\tilde{f}_m^* g_{euc}) = \pi(g_m) \to \pi(g_{poin})$ by (5.5). Then

$$\pi((f_m + \lambda_{m,r}(V_r \circ \phi_m^{-1}))^* g_{euc}) = \pi(g_{poin}),$$
$$f_m + \lambda_{m,r}(V_r \circ \phi_m^{-1}) \to f \quad \text{strongly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma),$$
$$c_0 \le \Big(f_m + \lambda_{m,r}(V_r \circ \phi_m^{-1})\Big)^* g_{euc} \le C$$

for some $0 < c_0 \le C < \infty$ and m large, when observing that ϕ_m^{-1} is bounded in $W^{2,2}(\Sigma) \cap W^{1,\infty}(\Sigma)$ by (5.6), (5.8) and $|D^2(\phi_m^{-1})| \le C|D^2(\phi_m) \circ \phi_m^{-1}|$. As in (5.4) we get

$$\mathcal{M}_{p,n}(\pi(g_{poin})) \le \lim_{m \to \infty} \mathcal{W}(f_m + \lambda_{m,r}(V_r \circ \phi_m^{-1})) = \mathcal{W}(f),$$

and recalling (5.4) again, the proposition follows.

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This proposition implies strong convergence of minimizing sequences.

Proposition 5.3 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a weak local bilipschitz immersion approximated by smooth immersions* f_m *satisfying (2.2) - (2.6) and*

$$\mathcal{W}(f_m) \le \mathcal{M}_{p,n}(\pi(f_m^* g_{euc})) + \varepsilon_m \tag{5.10}$$

with $\varepsilon_m \to 0$ *. Then*

$$f_m \to f \quad \textit{strongly in } W^{2,2}(\Sigma) \tag{5.11}$$

and

$$\mathcal{W}(f) = \mathcal{M}_{p,n}(\tau_0). \tag{5.12}$$

Proof:
By (2.3), (2.4), (2.5) and Λ_0 large enough, we may assume after relabeling the sequence f_m

$$\begin{gathered} \| Df_m \|_{W^{1,2}(\Sigma)\cap L^\infty(\Sigma)} \le \Lambda_0, \\ \tfrac{1}{2} g_{poin} \le g_{poin,m} \le 2 g_{poin}, \\ \Lambda_0^{-1} g_{poin} \le g_m = f_m^* g_{euc} \le \Lambda_0 g_{poin}, \\ \int_\Sigma |A_{f_m}|^2 \, d\mu_{f_m} \le \Lambda_0. \end{gathered} \tag{5.13}$$

In local charts, we see

$$g_m \to g \quad \text{weakly in } W^{1,2}, \text{weakly}^* \text{ in } L^\infty,$$
$$\Gamma^k_{g_m,ij} \to \Gamma^k_{g,ij} \quad \text{weakly in } L^2,$$

and

$$A_{f_m,ij} = \nabla_i^{g_m} \nabla_j^{g_m} f_m \to \partial_{ij} f - \Gamma^k_{g,ij} \partial_k f =$$
$$= \nabla_i^g \nabla_j^g f = A_{f,ij} \quad \text{weakly in } L^2.$$

We conclude by Propositions 5.1, 5.2 and (5.10), as $\pi(f_m^* g_{euc}) \to \tau_0$ by (2.2),

$$\mathcal{W}(f) \leq \liminf_{m\to\infty} \mathcal{W}(f_m) \leq \liminf_{m\to\infty} \mathcal{M}_{p,n}(f_m^* g_{euc}) \leq \mathcal{M}_{p,n}(\tau_0) \leq \mathcal{W}(f),$$

hence $\vec{\mathbf{H}}_{f_m} \to \vec{\mathbf{H}}_f$ strongly in L^2 . This yields using (2.4)

$$\Delta_{g_{poin,m}} f_m = e^{2u_m} \vec{\mathbf{H}}_{f_m} \to e^{2u} \vec{\mathbf{H}}_f = \Delta_{g_{poin}} f \quad \text{strongly in } L^2$$

and

$$\partial_i \Big(g^{ij}_{poin} \sqrt{g_{poin}} \partial_j (f_m - f) \Big) = \partial_i \Big((g^{ij}_{poin} \sqrt{g_{poin}} - g^{ij}_{poin,m} \sqrt{g_{poin,m}}) \partial_j f_m \Big) +$$
$$+ \sqrt{g_{poin,m}} \Delta_{g_{poin,m}} f_m - \sqrt{g_{poin}} \Delta_{g_{poin}} f \to 0 \quad \text{strongly in } L^2$$

recalling that $\partial_j f_m$ is bounded in L^∞ and $g_{poin,m} \to g_{poin}$ smoothly by (2.4), hence

$$f_m \to f \quad \text{strongly in } W^{2,2}(\Sigma),$$

and the proposition is proved.

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6 Decay of the second derivative

In this section, we add to our assumptions on f, f_m , as considered in §2 - §4, that f_m is approximately minimizing in its Teichmüller class, see (6.1). The aim is to prove in the following proposition a decay for the second derivatives which implies that the limits in $C^{1,\alpha}$.

Proposition 6.1 *Let $f : \Sigma \to \mathbb{R}^n$ be a weak local bilipschitz immersion approximated by smooth immersions f_m satisfying (2.2) - (2.6) and*

$$\mathcal{W}(f_m) \leq \mathcal{M}_{p,n}(\pi(f_m^* g_{euc})) + \varepsilon_m \tag{6.1}$$

with $\varepsilon_m \to 0$. Then there exists $\alpha > 0, C < \infty$ such that

$$\int_{B_\varrho^{g_{poin}}(x)} |\nabla^2_{g_{poin}} f|^2_{g_{poin}} \, \mathrm{d}\mu_{g_{poin}} \leq C \varrho^{2\alpha} \quad \text{for any } x \in \Sigma, \varrho > 0, \tag{6.2}$$

in particular $f \in C^{1,\alpha}(\Sigma)$.

Proof:
By (2.3), (2.4) and Λ_0 large enough, we may assume after relabeling the sequence f_m

$$\begin{gathered}\| u_m, Df_m \|_{W^{1,2}(\Sigma)\cap L^\infty(\Sigma)} \leq \Lambda_0,\\ \tfrac{1}{2} g_{poin} \leq g_{poin,m} \leq 2 g_{poin},\\ \Lambda_0^{-1} g_{poin} \leq g_m = f_m^* g_{euc} \leq \Lambda_0 g_{poin},\\ \int\limits_\Sigma |A_{f_m}|^2 \, \mathrm{d}\mu_{f_m} \leq \Lambda_0.\end{gathered} \tag{6.3}$$

Putting $\nu_m := |\nabla^2_{g_{poin}} f_m|^2_{g_{poin}} \mu_{g_{poin}}$, we see $\nu_m(\Sigma) \leq C(\Lambda_0, g_{poin})$ and for a subsequence $\nu_m \to \nu$ weakly* in $C_0^0(\Sigma)^*$ with $\nu(\Sigma) < \infty$.

We consider $x_0 \in \Sigma$ with a neighbourhood $U_0(x_0)$ satisfying

$$\nu(\overline{U_0(x_0)} - \{x_0\}) < \varepsilon_1, \tag{6.4}$$

where we choose $\varepsilon_1 = \varepsilon_1(\Lambda_0, n) > 0$ below, together with a chart $\varphi : U_0(x_0) \xrightarrow{\approx} B_{2\varrho_0}(0), \varphi(x_0) = 0, U_\varrho(x_0) := \varphi^{-1}(B_\varrho(0))$ for $0 < \varrho \leq 2\varrho_0 \leq 2$, such that

$$\frac{1}{2} g_{euc} \leq (\varphi^{-1})^* g_{poin} \leq 2 g_{euc} \tag{6.5}$$

and in the coordinates of the chart φ

$$\int\limits_{B_{\varrho_0}(0)} g^{ik}_{poin} g^{jl}_{poin} g_{poin,rs} \Gamma^r_{g_{poin},ij} \Gamma^s_{g_{poin},kl} \, \mathrm{d}\mu_{g_{poin}} < \varepsilon_1. \tag{6.6}$$

Moreover we select

$$U_{\varrho_1}(x_0) \subseteq U(x_0) \subseteq U_{\varrho_0}(x_0), \tag{6.7}$$

variations $V_1, \ldots, V_{\dim \mathcal{T} - 1}, V_\pm \in C_0^\infty(\Sigma - \overline{U(x_0)}, \mathbb{R}^n)$ and $\delta > 0, C = C_{x_0,\varphi} < \infty, m_0 \in \mathbb{N}$, as in Lemmas 3.3 and 4.4 for $x_0, U_{\varrho_0}(x_0)$ and $\Lambda := C(\Lambda_0)$ defined below. As $\pi(f_m^*) \to \pi(g_{poin})$ by (2.2), we get for m_0 large enough

$$d_{\mathcal{T}}(\pi(f_m^* g_{euc}), \pi(g_{poin})) < \varepsilon \quad \text{for } m \geq m_0. \tag{6.8}$$

Clearly for each $x_0 \in \Sigma$, there exist $U_0(x_0), \varrho_0$ as above, since $\nu(B_\varrho^{g_{poin}}(x_0) - \{x_0\}) \to \nu(\emptyset) = 0$ for $\varrho \to 0$.

For $x_0, U_0(x_0)$ as above and $0 < \varrho \leq \varrho_0$, there exists $m_1 \geq m_0$ such that $\nu_m(U_{\varrho_0}(x_0) - U_{\varrho/2}(x_0)) < \varepsilon_1$ for $m \geq m_1$, hence in the coordinates of the chart φ

$$\int\limits_{B_{\varrho_0}(0) - B_{\varrho/2}(0)} |D^2 f_m|^2 \, \mathrm{d}\mathcal{L}^2 \leq C \int\limits_{B_{\varrho_0}(0) - B_{\varrho/2}(0)} g^{ik}_{poin} g^{jl}_{poin} \langle \partial_{ij} f_m, \partial_{kl} f_m \rangle \, \mathrm{d}\mu_{g_{poin}} \leq$$

$$\leq 2 \int\limits_{U_{\varrho_0}(0) - U_{\varrho/2}(x_0)} |\nabla_{g_{poin}} f_m|^2_{g_{poin}} \, \mathrm{d}\mu_{g_{poin}} + 2 \int\limits_{B_{\varrho_0}(x)} g^{ik}_{poin} g^{jl}_{poin} \Gamma^r_{g_{poin},ij} \Gamma^s_{g_{poin},kl} \langle \partial_r f_m, \partial_s f_m \rangle \, \mathrm{d}\mu_{g_{poin}} \leq$$

$$\leq 2\varepsilon_1 + C(\Lambda_0) \int\limits_{B_{\varrho_0}(0)} g_{poin}^{ik} g_{poin}^{jl} g_{poin,rs} \Gamma^r_{g_{poin},ij} \Gamma^s_{g_{poin},kl} \, \mathrm{d}\mu_{g_{poin}} \leq C(\Lambda_0)\varepsilon_1. \tag{6.9}$$

There exists $\sigma \in]3/\varrho/4, 7\varrho/8[$ satisfying by Co-Area formula, see [Sim] §12,

$$\int\limits_{\partial B_\sigma(0)} |D^2 f_m|^2 \, \mathrm{d}\mathcal{H}^1 \leq 8\varrho^{-1} \int\limits_{3\varrho/4}^{7\varrho/8} \int\limits_{\partial B_r(0)} |D^2 f_m|^2 \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}r =$$

$$= 8\varrho^{-1} \int\limits_{B_{7\varrho/8}(0) - B_{3\varrho/4}(0)} |D^2 f_m|^2 \, \mathrm{d}\mathcal{L}^2 \leq C(\Lambda_0)\varepsilon_1 \varrho^{-1}. \tag{6.10}$$

First we conclude that

$$osc_{\partial B_\sigma(0)} |Df_m| \leq C \int\limits_{\partial B_\sigma(0)} |D^2 f_m| \, \mathrm{d}\mathcal{H}^1 \leq C(\Lambda_0)\varepsilon_1^{1/2},$$

hence for any $x \in \partial B_\sigma(0)$ and the affine function $l(y) := f_m(x) + Df_m(x)(y-x)$

$$\sigma^{-1} \parallel f_m - l \parallel_{L^\infty(\partial B_\sigma(0))} + \parallel D(f_m - l) \parallel_{L^\infty(\partial B_\sigma(0))}) \leq C(\Lambda_0)\varepsilon_1^{1/2}.$$

Moreover by (6.3) and (6.5)

$$c_0(\Lambda_0)(\delta_{ij})_{ij} \leq (\langle \partial_i l, \partial_j l \rangle)_{ij} = g_m(x) \leq C(\Lambda_0)(\delta_{ij})_{ij}$$

hence $Dl \in \mathbb{R}^{2\times 2}$ is invertible and

$$\parallel Dl \parallel, \parallel (Dl)^{-1} \parallel \leq C(\Lambda_0).$$

Next by standard trace extension lemma, there exists $\tilde{f}_m \in C^2(\overline{B_\sigma(0)})$ such that

$$\begin{gathered} \tilde{f}_m = f_m, D\tilde{f}_m = Df_m \quad \text{on } \partial B_\sigma(0), \\ \sigma^{-1}|\tilde{f}_m - l| + |D(\tilde{f}_m - l)| \leq \\ \leq C(\sigma^{-1} \parallel f_m - l \parallel_{L^\infty(\partial B_\sigma(0))} + \parallel D(f_m - l) \parallel_{L^\infty(\partial B_\sigma(0))}) \leq C(\Lambda_0)\varepsilon_1^{1/2}, \\ \int\limits_{B_\sigma(0)} |D^2 \tilde{f}_m|^2 \, \mathrm{d}\mathcal{L}^2 \leq C\sigma \int\limits_{\partial B_\sigma(0)} |D^2 f_m|^2 \, \mathrm{d}\mathcal{H}^1 \leq C(\Lambda_0)\varepsilon_1. \end{gathered} \tag{6.11}$$

We see

$$\parallel D\tilde{f}_m - Dl \parallel \leq C(\Lambda_0)\varepsilon_1^{1/2} \leq \parallel (Dl)^{-1} \parallel^{-1} /2$$

for $\varepsilon_1 = \varepsilon_1(\Lambda_0) \leq 1$ small enough, and $D\tilde{f}_m$ is of full rank everywhere with

$$\parallel D\tilde{f}_m \parallel, \parallel (D\tilde{f}_m)^{-1} \parallel \leq C(\Lambda_0).$$

Extending $\tilde{f}_m = f_m$ on $\Sigma - U_\sigma(x_0)$, we see that $\tilde{f}_m : \Sigma \to \mathbb{R}^n$ is a $C^{1,1}-$immersion with pullback metric satisfying by (6.5)

$$c_0(\Lambda_0)g_{poin} \leq c_0(\Lambda_0)g_{euc} \leq \tilde{g}_m := \tilde{f}_m^* g_{euc} \leq C(\Lambda_0)g_{euc} \leq C(\Lambda_0)g_{poin}. \tag{6.12}$$

and by (6.6), (6.9) and (6.11)

$$\int\limits_{B_{\varrho_0}(0)} |D^2\tilde{f}_m|^2 \,\mathrm{d}\mathcal{L}^2, \int\limits_{B_{\varrho_0}(0)} |\nabla_{g_{poin}}\tilde{f}_m|^2_{g_{poin}} \,\mathrm{d}\mu_{g_{poin}} \leq C(\Lambda_0)\varepsilon_1, \tag{6.13}$$

in particular, as $|A_{\tilde{f}_m,ij}| \leq |\partial_{ij}\tilde{f}_m|$ in local coordinates,

$$\int\limits_{U_{\varrho_0}(x_0)} |A_{\tilde{f}_m}|^2 \,\mathrm{d}\mu_{\tilde{g}_m} \leq C(\Lambda_0)\varepsilon_1 \leq \varepsilon_0(n) \tag{6.14}$$

for $\varepsilon_1 = \varepsilon_1(\Lambda_0, n)$ small enough. Putting $V := \tilde{f}_m - f_m$, we see *supp* $V \subset\subset U_\varrho(x_0) \subseteq U(x_0)$ for $0 < \varrho \leq \varrho_1$ from (6.7) and

$$\parallel V \parallel_{W^{2,2}(\Sigma)} \leq C \parallel V \parallel_{W^{2,2}(B_\sigma(0))} \leq C(\Lambda_0).$$

Together with (6.12) and (6.14), this verifies (3.9) and (4.29) for $\Lambda = C(\Lambda_0)$ in Lemmas 3.3 and 4.4, respectively. After slightly smoothing V , there exists $\lambda_m \in \mathbb{R}^{\dim \mathcal{T}+1}$ by Lemmas 3.3 and 4.4 and (6.8) with

$$\begin{gathered} \pi((\tilde{f}_m + \lambda_{m,r}V_r)^* g_{euc}) = \pi(f_m^* g_{euc}), \\ |\lambda_m| \leq C_{x_0,\varphi} d_{\mathcal{T}}\Big(\pi(\tilde{f}_m^* g_{euc}), \pi(f_m^* g_{euc})\Big)^{1/2}. \end{gathered} \tag{6.15}$$

By the minimizing property (6.1), and the Gauß-Bonnet Theorem in (1.1), we get

$$\frac{1}{4}\int\limits_{\Sigma} |A_{f_m}|^2 \,\mathrm{d}\mu_{f_m} - \varepsilon_m \leq \mathcal{M}_{p,n}(\pi(f_m^* g_{euc})) + 2\pi(p-1) \leq \frac{1}{4}\int\limits_{\Sigma} |A_{\tilde{f}_m+\lambda_{m,r}V_r}|^2 \,\mathrm{d}\mu_{\tilde{f}_m+\lambda_{m,r}V_r},$$

hence, as $\tilde{f}_m = f_m$ in $\Sigma - U_\sigma(x_0)$ and *supp* $V_r \cap \overline{U_\sigma(x_0)} = \emptyset$,

$$\int\limits_{U_\sigma(x_0)} |A_{f_m}|^2 \,\mathrm{d}\mu_{f_m} \leq \int\limits_{U_\sigma(x_0)} |A_{\tilde{f}_m}|^2 \,\mathrm{d}\mu_{\tilde{f}_m} + C_{x_0,\varphi}(\Lambda_0, g_{poin})|\lambda_m| + 4\varepsilon_m. \tag{6.16}$$

We continue, using $|A_{\tilde{f}_m,ij}| \leq |\partial_{ij}\tilde{f}_m|$ in local coordinates, (6.10) and (6.11),

$$\begin{aligned} &\int\limits_{U_\sigma(x_0)} |A_{\tilde{f}_m}|^2 \,\mathrm{d}\mu_{\tilde{f}_m} \leq \\ &\leq C(\Lambda_0) \int\limits_{B_\sigma(0)} |D^2\tilde{f}_m|^2 \,\mathrm{d}\mathcal{L}^2 \leq C(\Lambda_0) \int\limits_{B_{7\varrho/8}(0)-B_{3\varrho/4}(0)} |D^2 f_m|^2 \,\mathrm{d}\mathcal{L}^2 \leq \\ &\leq C(\Lambda_0) \int\limits_{U_{7\varrho/8}(x_0)-U_{3\varrho/4}(x_0)} |\nabla^2_{g_{poin}} f_m|^2 \,\mathrm{d}\mu_{g_{poin}} + C(\Lambda_0) \parallel \Gamma_{g_{poin}} \parallel^2_{L^\infty(B_{\varrho_0}(0))} \varrho^2 \leq \\ &\leq C(\Lambda_0)\nu_m\Big(U_{7\varrho/8}(x_0) - U_{3\varrho/4}(x_0)\Big) + C_{x_0,\varphi}(\Lambda_0, g_{poin})\varrho^2. \end{aligned} \tag{6.17}$$

We calculate in local coordinates in $B_\sigma(0)$ that

$$\Delta_{g_{poin,m}} f_m = e^{2u_m} \Delta_{g_m} f_m = e^{2u_m} \vec{\mathbf{H}}_{f_m}$$

and

$$|\Delta_{g_{poin,m}}(f_m - \tilde{f}_m)| \leq C(\Lambda_0)(|\vec{\mathbf{H}}_{f_m}| + |D^2 \tilde{f}_m| + |\Gamma_{g_{poin,m}}|\ |D\tilde{f}_m|).$$

By standard elliptic theory, see [GT] Theorem 8.8, from (2.4) for $m \geq m_1$ large enough, as $f_m = \tilde{f}_m$ on $\partial B_\sigma(0)$, we get

$$\int\limits_{B_\sigma(0)} |D^2 f_m|^2 \ \mathrm{d}\mathcal{L}^2 \leq$$

$$\leq C_{x_0,\varphi}(\Lambda_0, g_{poin})\Big(\int\limits_{U_\sigma(x_0)} |\vec{\mathbf{H}}_{f_m}|^2 \ \mathrm{d}\mu_{f_m} + \int\limits_{B_\sigma(0)} |D^2 \tilde{f}_m|^2 \ \mathrm{d}\mathcal{L}^2 + \int\limits_{B_\sigma(0)} |D\tilde{f}_m|^2 \ \mathrm{d}\mathcal{L}^2 \Big) \leq$$

$$\leq C_{x_0,\varphi}(\Lambda_0, g_{poin})\Big(\int\limits_{U_\sigma(x_0)} |\vec{\mathbf{H}}_{f_m}|^2 \ \mathrm{d}\mu_{f_m} + \nu_m\Big(U_{7\varrho/8}(x_0) - U_{3\varrho/4}(x_0)\Big) + \varrho^2 \Big), \qquad (6.18)$$

where we have used (6.17). Putting (6.16), (6.17) and (6.18) together yields

$$\nu_m(U_{\varrho/2}(x_0)) = \int\limits_{U_{\varrho/2}(x_0)} |\nabla^2_{g_{poin}} f_m|^2 \ \mathrm{d}\mu_{g_{poin}} \leq$$

$$\leq C_{x_0,\varphi}(\Lambda_0, g_{poin})\Big(\nu_m\Big(U_{7\varrho/8}(x_0) - U_{3\varrho/4}(x_0)\Big) + \varrho^2 + |\lambda_m| + \varepsilon_m \Big). \qquad (6.19)$$

To estimate λ_m, we continue observing that $\tilde{f}_m^* g_{euc} = \tilde{g}_m$ and $f_m^* g_{euc} = g_m$ coincide on $\Sigma - U_\varrho(x_0)$,

$$d_{\mathcal{T}}(\pi(\tilde{f}_m^* g_{euc}), \pi(f_m^* g_{euc}))^2 \leq$$

$$\leq 2 d_{\mathcal{T}}(\pi(\tilde{f}_m^* g_{euc}), \pi(g_{poin}))^2 + 2 d_{\mathcal{T}}(\pi(f_m^* g_{euc}), \pi(g_{poin}))^2 \leq$$

$$\leq C_{\tau_0} \int\limits_\Sigma \Big(\frac{1}{2} g_{poin,ij} \tilde{g}_m^{ij} \sqrt{\tilde{g}_m} - \sqrt{g_{poin}} \Big) \ \mathrm{d}x + C_{\tau_0} \int\limits_\Sigma \Big(\frac{1}{2} g_{poin,ij} g_m^{ij} \sqrt{g_m} - \sqrt{g_{poin}} \Big) \ \mathrm{d}x =$$

$$= 2C_{\tau_0} \int\limits_{\Sigma - U_\varrho(x_0)} \Big(\frac{1}{2} g_{poin,ij} g_m^{ij} \sqrt{g_m} - \sqrt{g_{poin}} \Big) \ \mathrm{d}x +$$

$$+ C_{\tau_0} \int\limits_{U_\varrho(x_0)} \Big(\frac{1}{2} g_{poin,ij} \tilde{g}_m^{ij} \sqrt{\tilde{g}_m} - \sqrt{g_{poin}} \Big) \ \mathrm{d}x + C_{\tau_0} \int\limits_{U_\varrho(x_0)} \Big(\frac{1}{2} g_{poin,ij} g_m^{ij} \sqrt{g_m} - \sqrt{g_{poin}} \Big) \ \mathrm{d}x \leq$$

$$\leq 2C_{\tau_0} \int\limits_{\Sigma - U_\varrho(x_0)} \Big| \frac{1}{2} g_{poin,ij} g_m^{ij} \sqrt{g_m} - \sqrt{g_{poin}} \Big| \ \mathrm{d}x + C_{\tau_0}(\Lambda_0) \varrho^2.$$

where we have used (6.3) and (6.12). As $g_m \to g = e^{2u} g_{poin}$ pointwise and bounded on Σ, we get from Lebesgue's convergence theorem

$$\limsup_{m \to \infty} d_{\mathcal{T}}(\pi(\tilde{f}_m^* g_{euc}), \pi(f_m^* g_{euc})) \leq C_{\tau_0}(\Lambda_0) \varrho$$

and from (6.15)

$$\limsup_{m\to\infty} |\lambda_m| \le C_{x_0,\varrho_0 g_{poin}}(\Lambda_0)\varrho^{1/4}.$$

Plugging into (6.19) and passing to the limit $m \to \infty$, we obtain

$$\nu(U_{\varrho/2}(x_0)) \le C_{x_0,\varphi}(\Lambda_0, g_{poin})\nu\Big(U_\varrho(x_0) - U_{\varrho/2}(x_0)\Big) + C_{x_0,\varphi}(\Lambda_0, g_{poin})\varrho^{1/4}$$

and by hole-fllling

$$\nu(U_{\varrho/2}(x_0)) \le \gamma\nu(U_\varrho(x_0)) + C_{x_0,\varphi}(\Lambda_0, g_{poin})\varrho^{1/4}$$

with $\gamma = C/(C+1) < 1$. Iterating with [GT] Lemma 8.23, we arrive at

$$\nu(B_\varrho^{g_{poin}}(x_0)) \le C_{x_0,\varphi}(\Lambda_0, g_{poin})\varrho^{2\alpha}\varrho_1^{-2\alpha} \quad \text{for all } \varrho > 0 \tag{6.20}$$

and some $0 < \alpha = \alpha_{x_0,\varphi}(\Lambda_0, g_{poin}) < 1$. Since $\nu(B_\varrho^{g_{poin}}(x_0)) \to \nu(\{x_0\})$ for $\varrho \to 0$, we first conclude

$$\nu(\{x_0\}) = 0. \tag{6.21}$$

Then we can improve the choice of $U_0(x_0)$ in (6.4) to

$$\nu(\overline{U_0(x_0)}) < \varepsilon_1,$$

and we can repeat the above iteration for any $x \in U_{\varrho_1/2}(x_0), 0 < \varrho \le \varrho_1/2$ to obtain

$$\nu(B_\varrho^{g_{poin}}(x)) \le C_{x_0,\varphi}(\Lambda_0, g_{poin})\varrho^{2\alpha}\varrho_1^{-2\alpha} \quad \text{for all } x \in U_{\varrho_1/2}(x_0)\varrho > 0.$$

By a finite covering, this yields (6.2). Since in the coordinates of the chart φ

$$\int\limits_{B_\varrho(x)} |D^2 f_m|^2 \,\mathrm{d}\mathcal{L}^2 \le C \int\limits_{B_\varrho(x)} g^{ik}_{poin} g^{jl}_{poin} \langle \partial_{ij} f_m, \partial_{kl} f_m\rangle \,\mathrm{d}\mu_{g_{poin}} \le$$

$$\le 2 \int\limits_{B_\varrho(x)} |\nabla_{g_{poin}} f_m|^2_{g_{poin}} \,\mathrm{d}\mu_{g_{poin}} + 2 \int\limits_{B_\varrho(x)} g^{ik}_{poin} g^{jl}_{poin} \Gamma^r_{g_{poin},ij} \Gamma^s_{g_{poin},kl} \langle \partial_r f_m, \partial_s f_m\rangle \,\mathrm{d}\mu_{g_{poin}} \le$$

$$\le C_{x_0,\varphi}(\Lambda_0, g_{poin})\varrho^{2\alpha}\varrho_1^{-2\alpha} + C(\Lambda_0, g_{poin})\varrho^2,$$

we conclude by Morrey's lemma, see [GT] Theorem 7.19, that $f \in C^{1,\alpha}(\Sigma)$, and the proposition is proved.

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7 The Euler-Lagrange equation

The aim of this section is to prove the Euler-Lagrange equation for the limit of immersions approximately minimizing under fixed Teichmüller class. From this we will conclude full regularity of the limit.

Theorem 7.1 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a weak local bilipschitz immersion approximated by smooth immersions* f_m *satisfying (2.2) - (2.6) and*

$$\mathcal{W}(f_m) \leq \mathcal{M}_{p,n}(\pi(f_m^* g_{euc})) + \varepsilon_m \tag{7.1}$$

with $\varepsilon_m \to 0$ *. Then* f *is a smooth minimizer of the Willmore energy under fixed Teichmüller class*

$$\mathcal{W}(f) = \mathcal{M}_{p,n}(\tau_0) \tag{7.2}$$

and satisfies the Euler-Lagrange equation

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = g^{ik} g^{jl} A^0_{ij} q_{kl} \quad \text{on } \Sigma, \tag{7.3}$$

where q *is a smooth transverse traceless symmetric 2-covariant tensor with respect to* $g = f^* g_{euc}$ *.*

Proof:
By Propositions 3.1 and 4.2, we select variations variations $V_1, \ldots, V_{\dim \mathcal{T}} \in C_0^\infty(\Sigma, \mathbb{R}^n)$, satisfying (3.5) or variations $V_1, \ldots, V_{\dim \mathcal{T}-1}, V_\pm \in C_0^\infty(\Sigma, \mathbb{R}^n)$, satisfying (4.12) and (4.13), depending on whether f has full rank in Teichmüller space or not.

For $V \in C^\infty(\Sigma, \mathbb{R}^n)$ and putting $f_{m,t,\lambda,\mu} := f_m + tV + \lambda_r V_r + \mu_\pm V_\pm$, we see for $|t| \leq t_0$ for some $t_0 = t_0(V, \Lambda_0, n) > 0$ small enough and $|\lambda|, |\mu| \leq \lambda_0$ for some $\lambda_0 = \lambda_0(V_r, V_\pm, \Lambda_0) > 0$ small enough that

$$\begin{aligned} \| Df_{m,t,\lambda,\mu} \|_{W^{1,2}(\Sigma) \cap L^\infty(\Sigma)} &\leq 2\Lambda_0, \\ (2\Lambda_0)^{-1} g_{poin} \leq f^*_{m,t,\lambda,\mu} g_{euc} &\leq 2\Lambda_0 g_{poin} \end{aligned}$$

and all $m \in \mathbb{N}$. As $f_{m,t,\lambda,\mu} \to f$ strongly in $W^{2,2}(\Sigma)$ and weakly* in $W^{1,\infty}(\Sigma)$ for $m \to \infty, t, \lambda, \mu \to 0$ by Proposition 5.3, we get from Proposition 4.3 and the remark following for any chart $\psi : U(\pi(g_{poin})) \subseteq \mathcal{T} \to \mathbb{R}^{\dim \mathcal{T}}$, and put $\hat{\pi} := \psi \circ \pi, \delta\hat{\pi} = d\psi \circ \delta\pi, \hat{\mathcal{V}}_f := d\psi_{\pi g_{poin}}.\mathcal{V}_f, \delta^2\hat{\pi}$ defined in (4.7), and any $W \in C^\infty(\Sigma, \mathbb{R}^n)$

$$\begin{gathered} \pi(f^*_{m,t,\lambda,\mu} g_{euc}) \to \tau_0, \\ \delta\hat{\pi}_{f_{m,\lambda,\mu}}.W \to \delta\hat{\pi}_f.W, \quad \delta^2\hat{\pi}_{f_{m,t,\lambda,\mu}}(W) \to \delta^2\hat{\pi}_f(W) \end{gathered} \tag{7.4}$$

as $m \to \infty, t, \lambda, \mu \to 0$. If f is of full rank in Teichmüller space, then $\hat{\mathcal{V}}_f = \mathbb{R}^{\dim \mathcal{T}}$, and we put $d = \dim \mathcal{T}$. If f is not of full rank in Teichmüller space, we may assume fter a change of coordinates,

$$\hat{\mathcal{V}}_f = \mathbb{R}^{\dim \mathcal{T} - 1} \times \{0\}$$

and put $d := \dim \mathcal{T} - 1$ and $e := e_{\dim \mathcal{T}} \perp \hat{\mathcal{V}}_f$. By (3.5) or (4.12), we see for the orthogonal projection $\pi_{\hat{\mathcal{V}}_f} : \mathbb{R}^{\dim \mathcal{T}} \to \hat{\mathcal{V}}_f$ that

$$(\pi_{\hat{\mathcal{V}}_f} \delta\hat{\pi}_f.V_r)_{r=1,\ldots,d} =: A \in \mathbb{R}^{d \times d} \tag{7.5}$$

is invertible, hence after a further change of variable, we may assume that $A = I_d$. In the degenerate case, we further know

$$\langle \delta\hat{\pi}_f.V_r, e \rangle = 0. \tag{7.6}$$

By (4.13)

$$\pm\langle\delta^2\hat{\pi}_f(V_\pm), e\rangle \geq 2\gamma, \quad \delta\hat{\pi}_f.V_\pm = 0, \tag{7.7}$$

for some $\gamma > 0$.

Next we put for m large enough and t_0, λ_0 small enough

$$\Phi_m(t,\lambda,\mu) := \hat{\pi}(f^*_{m,t,\lambda,\mu}g_{euc}).$$

Clearly, Φ_m is smooth. We get from (7.4), (7.5) (7.6) and (7.7) for some $\Lambda_1 < \infty$ and any $0 < \varepsilon \leq 1$ that

$$\begin{gathered}\| D^2\Phi_m(t,\lambda) \| \leq \Lambda_1,\\ osc\ D^2\Phi_m \leq \varepsilon,\\ \| \partial_\lambda\Phi_m(t,\lambda) - I_d \| \leq \varepsilon \leq 1/2,\\ \partial_\lambda\Phi_m(t,\lambda) \to (\delta\hat{\pi}_f.V_r)_{r=1,\ldots,d},\end{gathered} \tag{7.8}$$

in the full rank case, and writing $\Phi_m(t,\lambda,\mu) = (\Phi_{m,0}(t,\lambda,\mu), \varphi_m(t,\lambda,\mu))$ in the degenerate case that

$$\begin{gathered}\| D^2\Phi_m(t,\lambda,\mu) \| \leq \Lambda_1,\\ osc\ D^2\Phi_m \leq \varepsilon,\\ \| \partial_\lambda\Phi_m(\lambda) - I_d \| \leq \varepsilon \leq 1/2,\\ \pm\partial_{\mu_\pm\mu_\pm}\varphi_m(t,\lambda,\mu) \geq \gamma,\\ |\partial_\mu\Phi_m(t,\lambda,\mu)|, |D\varphi_m(t,\lambda,\mu)| \leq \varepsilon,\\ \partial_\lambda\Phi_m(t,\lambda,\mu) \to (\delta\hat{\pi}_f.V_r)_{r=1,\ldots,d},\\ \partial_{\mu_\pm\mu_\pm}\varphi_m(t,\lambda,\mu) \to \langle\delta^2\hat{\pi}_f(V_\pm), e\rangle,\\ \partial_\mu\Phi_m(t,\lambda,\mu), D\varphi_m(t,\lambda,\mu) \to 0,\end{gathered} \tag{7.9}$$

all for $m \geq m_0$ large enough and $|t| \leq t_0, |\lambda|, |\mu| \leq \lambda_0$ small enough or respectively $t, \lambda, \mu \to 0$. We choose ε, λ_0 smaller to satisfy $C\Lambda_1\varepsilon + C\Lambda_1\lambda_0 \leq \gamma/2$. Moreover choosing m_0 large enough and t_0 small enough, we can further achieve

$$|D\varphi_m(t,0,0)| \leq \sigma$$

with $C\Lambda_1\varepsilon + C\sigma\lambda_0^{-1} + C\Lambda_1\lambda_0 \leq \gamma$, and

$$|\Phi_m(t,0,0) - \Phi_m(0,0,0)| \leq Ct_0 \leq \Lambda_1\lambda_0^2, \lambda_0/8, \gamma\lambda_0^2/32.$$

By Proposition B.1 there exist $|\lambda_{m,r}(t)|, |\mu_{m,\pm}(t)|, |\tilde{\lambda}_{m,r}(t)|, |\tilde{\mu}_{m,\pm}(t)| \leq \lambda_0$ with $\mu_{m,+}(t)\mu_{m,-}(t) = 0, \tilde{\mu}_{m,+}(t)\tilde{\mu}_{m,-}(t) = 0$ and

$$\Phi_m(t,\lambda_m(t),\mu_m(t)) = \Phi_m(0,0,0) = \Phi_m(t,\tilde{\lambda}_m(t),\tilde{\mu}_m(t)), \tag{7.10}$$

which means

$$\begin{gathered}\pi\Big((f_m + tV + \lambda_{m,r}(t)V_r + \mu_{m,\pm}(t)V_\pm)^*g_{euc}\Big) = \pi(f_m^*g_{euc}) =\\ = \pi\Big((f_m + tV + \tilde{\lambda}_{m,r}(t)V_r + \tilde{\mu}_{m,\pm}(t)V_\pm)^*g_{euc}\Big),\end{gathered} \tag{7.11}$$

and

$$\mu_{m,+}(t)\tilde{\mu}_{m,+}(t) \le 0, \tilde{\mu}_{m,-}(t) = 0, \quad \text{if } \mu_{m,+}(t) \neq 0,$$
$$\mu_{m,-}(t)\tilde{\mu}_{m,-}(t) \le 0, \tilde{\mu}_{m,+}(t) = 0, \quad \text{if } \mu_{m,-}(t) \neq 0. \tag{7.12}$$

By (7.8), (7.9) and (7.10), we get from a Taylor expansion of the smooth function Φ_m at 0 that

$$|t\partial_t\Phi_m(0) + \lambda_m(t)\partial_\lambda\Phi_m(0) + \mu_m(t)\partial_\mu\Phi_m(0) +$$
$$+\frac{1}{2}(t, \lambda_m(t), \mu_m(t))^T D^2\Phi_m(0)(t, \lambda_m(t), \mu_m(t))| \le$$
$$\le C\varepsilon(|t|^2 + |\lambda_m(t)|^2 + |\mu_m(t)|^2),$$

hence, as $\mu_{m,+}(t)\mu_{m,-}(t) = 0$,

$$|t\partial_t\Phi_m(0) + \lambda_m(t)\partial_\lambda\Phi_m(0) + \mu_{m,\pm}(t)\partial_{\mu_\pm}\Phi_m(0) + \frac{1}{2}\mu_{m,\pm}(t)^2 \; \partial_{\mu_\pm\mu_\pm}\Phi_m(0)| \le$$
$$\le C\Lambda_1\varepsilon^{-1}(|t|^2 + |\lambda_m(t)|^2) + C\varepsilon|\mu_{m,\pm}(t)|^2).$$

Passing to the limit $m \to \infty$, we get for subsequences $\lambda_m(t) \to \lambda(t), \mu_{m,\pm}(t) \to \mu(t)$ with $|\lambda(t)|, |\mu(t)| \le \lambda_0$ and by (7.8) and (7.9)

$$|t\delta\hat{\pi}_f.V + \lambda_r(t)\delta\hat{\pi}_f.V_r + \frac{1}{2}\mu_\pm(t)^2 \; \delta^2\hat{\pi}_f(V_\pm)| \le$$
$$\le C\Lambda_1\varepsilon^{-1}(|t|^2 + |\lambda(t)|^2) + C\varepsilon|\mu_\pm(t)|^2). \tag{7.13}$$

In the degenerate case, we recall $\delta\hat{\pi}_f.V_{(r)} \in \hat{\mathcal{V}}_f \perp e$, hence by (7.7) and (7.13)

$$\gamma\mu_\pm(t)^2 \le \frac{1}{2}\mu_\pm(t)^2\langle\delta^2\hat{\pi}_f(V_\pm), e\rangle \le$$
$$\le |t\delta\hat{\pi}_f.V + \lambda_r(t)\delta\hat{\pi}_f.V_r + \frac{1}{2}\mu_{m,\pm}(t)^2 \; \delta^2\hat{\pi}_f(V_\pm)| \le$$
$$\le C\Lambda_1\varepsilon^{-1}(|t|^2 + |\lambda(t)|^2) + C\varepsilon|\mu_\pm(t)|^2)$$

and for $C\varepsilon \le \gamma/2$ small enough

$$|\mu_\pm(t)| \le C(|t| + |\lambda(t)|). \tag{7.14}$$

Then again by (7.13)

$$|t\delta\hat{\pi}_f.V + \lambda_r(t)\delta\hat{\pi}_f.V_r| \le C(|t|^2 + |\lambda(t)|^2). \tag{7.15}$$

In the full rank case, we have $\mu_\pm = 0$, and (7.15) directly follows from (7.13). As $(\delta\hat{\pi}_f.V_r)_{r=1,\ldots,d}$ are linearly independent by (7.5), we continue

$$|\lambda(t)| \le C(|t| + |\lambda(t)|^2) \le C|t| + C\lambda_0|\lambda(t)|,$$

hence for λ_0 small enough $|\lambda(t)| \le C|t|$. We get from (7.14)

$$|\mu_\pm(t)| \le C|t| \tag{7.16}$$

and from (7.15)

$$|t\delta\hat{\pi}_f.V + \lambda_r(t)\delta\hat{\pi}_f.V_r| \leq C|t|^2.$$

Therefore λ is differentiable at $t = 0$ with

$$\lambda_r'(0)\delta\hat{\pi}_f.V_r = -\delta\hat{\pi}_f.V. \tag{7.17}$$

Writing for the inverse $A^{-1} = (b_{rs})_{r,s=1,\dots,d}$ in (7.5), we continue with (3.4)

$$\lambda_r'(0) = -b_{rs}\langle\delta\hat{\pi}_f.V, e_s\rangle =$$

$$= \sum_{\sigma=1}^{\dim\ \mathcal{T}} 2\int_\Sigma g^{ik}g^{jl}\langle A_{ij}^0, V\rangle q_{kl}^\sigma(g_{poin})\ \mathrm{d}\mu_g\ \langle d\hat{\pi}_{g_{poin}}.q^\sigma(g_{poin}), b_{rs}e_s\rangle,$$

hence putting

$$q_{kl}^r := \sum_{\sigma=1}^{\dim\ \mathcal{T}} 2q_{kl}^\sigma(g_{poin})\langle d\hat{\pi}_{g_{poin}}.q^\sigma(g_{poin}), b_{rs}e_s\rangle \in S_2^{TT}(g_{poin}),$$

we get

$$\lambda_r'(0) = \int_\Sigma g^{ik}g^{jl}\langle A_{ij}^0, V\rangle q_{kl}^r\ \mathrm{d}\mu_g. \tag{7.18}$$

Moreover

$$f_m + tV + \lambda_{m,r}(t)V_r + \mu_{m,\pm}(t)V_\pm \to f + tV + \lambda_r(t)V_r + \mu_\pm(t)V_\pm$$

strongly in $W^{2,2}(\Sigma)$ and weakly* in $W^{1,\infty}(\Sigma)$, hence recalling (7.11)

$$\mathcal{W}(f + tV + \lambda_r(t)V_r + \mu_\pm(t)V_\pm) \leftarrow \mathcal{W}(f_m + tV + \lambda_{m,r}(t)V_r + \mu_{m,\pm}(t)V_\pm) \geq$$

$$\geq \mathcal{M}_{p,n}(\pi(f_m^* g_{euc})) \geq \mathcal{W}(f_m) - \varepsilon_m \to \mathcal{W}(f).$$

Since $(t,\lambda,\mu) \mapsto \mathcal{W}(f + tV + \lambda_r V_r + \mu_\pm V_\pm)$ is smooth, we get again by a Taylor expansion, (7.16) and (7.17)

$$0 \leq \mathcal{W}(f + tV + \lambda_r(t)V_r + \mu_\pm(t)V_\pm) - \mathcal{W}(f) =$$

$$= t\delta\mathcal{W}_f.V + \lambda_r(t)\delta\mathcal{W}_f.V_r + \mu_\pm(t)\delta\mathcal{W}_f.V_\pm + O(|t|^2).$$

As $\mu_+(t)\mu_-(t) = 0$ and by (7.12), we can adjust the sign of $\mu_\pm(t)$ according to the sign of $\delta\mathcal{W}_f.V_\pm$ and improve to

$$0 \leq t\delta\mathcal{W}_f.V + \lambda_r(t)\delta\mathcal{W}_f.V_r + O(|t|^2).$$

Differentiating by t at $t = 0$, we conclude from (7.17) and (7.18)

$$\delta\mathcal{W}_f.V = -\lambda_r'(0)\delta\mathcal{W}_f.V_r = -\int_\Sigma g^{ik}g^{jl}\langle A_{ij}^0, V\rangle q_{kl}^r\ \delta\mathcal{W}_f.V_r\ \mathrm{d}\mu_g,$$

hence putting $q_{kl} := q^r_{kl}\ \delta\mathcal{W}_f.V_r \in S_2^{TT}(g_{poin})$, we get

$$\delta\mathcal{W}_f.V = \int\limits_\Sigma g^{ik}g^{jl}\langle A^0_{ij}, V\rangle q_{kl}\ \mathrm{d}\mu_g \quad \text{for all } V \in C^\infty(\Sigma, \mathbb{R}^n). \tag{7.19}$$

As $f \in W^{2,2} \cap C^{1,\alpha}$ by Proposition 6.1, we can write f as a graph, more precisely for any $x_0 \in \Sigma$ there exists a neighbourhood $U(x_0)$ of x_0 such that after a translation, rotation and a homothetie, which leaves $\mathcal{W}$ as conformal transformation invariant, there is a $(W^{2,2} \cap C^{1,\alpha})-$ inverse chart $\phi : B_1(0) \xrightarrow{\approx} U(x_0), \phi(0) = x_0$, with $\tilde{f}(y) := (f \circ \phi)(y) = (y, u(y))$ for some $u \in (W^{2,2} \cap C^{1,\alpha})(B_1(0), \mathbb{R}^{n-2})$. Moreover, we may assume $|u|, |Du| \leq 1$ and from (6.2) that

$$\int\limits_{B_\varrho} |D^2u|^2\ \mathrm{d}\mathcal{L}^2 \leq C\varrho^{2\alpha} \quad \text{for any Ball } B_\varrho. \tag{7.20}$$

We calculate the square integral of the second fundamental form for a graph as

$$\mathcal{A}(u) := \int\limits_{B_1(0)} |A_{\tilde{f}}|^2\ \mathrm{d}\mu_{\tilde{f}} = \int\limits_{B_1(0)} (\delta_{rs} - d_{rs})g^{ij}g^{kl}\partial_{ik}u^r\partial_{jl}u^s\sqrt{g}\ \mathrm{d}\mathcal{L}^2,$$

where $g_{ij} := \delta_{ij} + \partial_i u\partial_j u, (g^{ij}) = (g_{ij})^{-1}, d_{rs} := g^{kl}\partial_k u^r\partial_l u^s$, see [Sim93] p. 310.

By the Gauß-Bonnet Theorem in (1.1), we see for any $v \in C_0^\infty(B_1(0), \mathbb{R}^{n-2})$ that $4\mathcal{W}(graph(u + v)) = \mathcal{A}(u + v) + 8\pi(1 - p)$, hence from (7.19)

$$\delta\mathcal{A}_u.v = \int\limits_{B_1(0)} (g^{ik}g^{jl} - \frac{1}{2}g^{ij}g^{kl})\tilde{q}_{kl}\langle\partial_{ij}u - \Gamma^m_{ij}\partial_m u, v\rangle\ \mathrm{d}\mu_g,$$

where $\tilde{q} = \phi^*q, \Gamma^m_{ij} = g^{mk}\langle\partial_{ij}u, \partial_k u\rangle$. From this we conclude that

$$\partial_{jl}(2a^{ijkl}_{rs}\partial_{ik}u^s) - \partial_j\Big((\partial_{\partial_j u^r}a^{imkl}_{ts})\partial_{ik}u^s\partial_{ml}u^t\Big) = b^{ijkl}_{rs}(\partial u)\tilde{q}_{kl}\partial_{ij}u^s \tag{7.21}$$

weakly for testfunctions $v \in C_0^\infty(B_1(0), \mathbb{R}^{n-2})$, where

$$a^{ijkl}_{rs}(Du) := (\delta_{rs} - d_{rs})g^{ij}g^{kl}\sqrt{g},$$
$$b^{ijkl}_{rs}(Du) = (g^{ik}g^{jl} - \tfrac{1}{2}g^{ij}g^{kl})(\delta_{rs} - d_{rs})\sqrt{g}.$$

Then we conclude from [Sim] Lemma 3.2 and (7.20) that $u \in (W^{3,2}_{loc} \cap C^{2,\alpha}_{loc})(B_1(0))$.

Full Regularity is now obtained by [ADN59], [ADN64]. First we conclude by finite differences that $u \in W^{4,2}_{loc}(B_1(0))$ and

$$2a^{ijkl}_{rs}(Du)\partial_{ijkl}u^s + \tilde{b}_r(Du, D^2u) * (1 + D^3u) + b_r(Du) * \tilde{q}(.) * D^2u = 0 \quad \text{strongly in } B_1(0),$$

with $a^{ijkl}_{rs}, \tilde{b}_r, b_r$ are smooth in Du and D^2u , whereas $\tilde{q} = \phi^*q \in (W^{1,2} \cap C^{0,\alpha})(B_1(0))$. As $W^{4,2} \hookrightarrow W^{3,p}$ for all $1 \leq p < \infty$, we see $u \in W^{4,p}_{loc}(B_1(0)) \hookrightarrow C^{3,\alpha}_{loc}(B_1(0))$ and then $u \in C^{4,\alpha}_{loc}(B_1(0))$.

Now we proceed by induction assuming $u, \tilde{f} \in C^{k,\alpha}_{loc}(B_1(0))$ for some $k \geq 4$. We see $\tilde{g} := \tilde{f}^* g_{euc} \in C^{k-1,\alpha}_{loc}$, hence we get locally conformal $C^{k,\alpha}$ − charts $\varphi : U \subseteq B_1(0) \xrightarrow{\approx} \Omega \subseteq \mathbb{R}^2$ with $\varphi^{-1,*}\tilde{g} = e^{2v} g_{euc}$. On the other hand, as g_{poin} is smooth, there exists a smooth conformal chart $\psi : U(x_0) \xrightarrow{\approx} \Omega_0 \subseteq \mathbb{R}^2$ with $\psi^{-1,*} g_{poin} = e^{2u_0} g_{euc}$, when choosing $U(x_0)$ small enough. As $g = f^* g_{euc} = e^{2u} g_{poin}$, we see that $\psi \circ \phi \circ \varphi^{-1} : \Omega \to \Omega_0$ is a regular conformal mapping with respect to standard euclidian metric, in particular holomorphic or anti-holomorphic, hence smooth. We conclude that $\phi \in C^{k,\alpha}_{loc}(B_1(0))$ and $\tilde{q} = \phi^* q \in C^{k-1,\alpha}_{loc}(B_1(0))$, as $q \in S_2^{TT}(g_{poin})$ is smooth. Then we conlude $u \in C^{k+1,\alpha}_{loc}(B_1(0))$ and by induction $u, \tilde{f}, \phi$ and $f = \tilde{f} \circ \phi^{-1}$ are smooth.

In [KuSch02] §2, the first variation of the Willmore functional with a different factor was calculated for variations V to be

$$\delta \mathcal{W}(f).V := \frac{d}{dt} \mathcal{W}(f + tV) = \int_\Sigma \frac{1}{2} \langle \Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}}, V \rangle \; d\mu_g,$$

and we obtain from (7.19)

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = 2 g^{ik} g^{jl} A^0_{ij} q_{kl} \quad \text{on } \Sigma,$$

which is (7.3) up to a factor for q . (7.2) was already obtained in (5.12). This concludes the proof of the theorem.

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As a corollary we get minimizers under fixed Teichmüller or conformal class, when the infimum is smaller than the bound $\mathcal{W}_{n,p}$ in (1.2).

Corollary 7.2 *Let Σ be a closed orientable surface of genus $p \geq 1$ and $\tau_0 \in \mathcal{T}$ satisfying*

$$\mathcal{M}_{p,n}(\tau_0) < \mathcal{W}_{n,p}$$

where $\mathcal{W}_{n,p}$ is defined in (1.2) and $n = 3, 4$.

Then there exists a smooth immersion $f : \Sigma \to \mathbb{R}^n$ which minimizes the Willmore energy in the fixed Teichmüller class $\tau_0 = \pi(f^* g_{euc})$

$$\mathcal{W}(f) = \mathcal{M}_{p,n}(\tau_0).$$

Moreover f satisfies the Euler-Lagrange equation

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = g^{ik} g^{jl} A^0_{ij} q_{kl} \quad \text{on } \Sigma,$$

where q is a smooth transverse traceless symmetric 2-covariant tensor with respect to $g = f^* g_{euc}$.

Proof:
We select a minimizing sequence of smooth immersions $f_m : \Sigma \to \mathbb{R}^n$ with $\pi(f_m^* g_{euc}) = \tau_0$

$$\mathcal{W}(f_m) \to \mathcal{M}_{p,n}(\tau_0). \tag{7.22}$$

We may assume that $\mathcal{W}(f_m) \leq \mathcal{W}_{n,p} - \delta$ for some $\delta > 0$. Replacing f_m by $\Phi_m \circ f_m \circ \phi_m$ for suitable Möbius transformations Φ_m and diffeomorphisms ϕ_m of Σ homotopic to the identity, which does neither change the Willmore energy nor the projection into the Teichmüller space, we may further assume by Proposition 2.2 that $f_m \to f$ weakly in $W^{2,2}(\Sigma)$ and satisfies (2.2) - (2.6). Since (7.1) is implied by (7.22), Theorem 7.1) yields that f is a smooth immersion which minimizes the Willmore energy in the fixed Teichmüller class $\tau_0 = \pi(f^* g_{euc})$ and satisfies the above Euler-Lagrange equation.

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Corollary 7.3 *Let* Σ *be a closed Riemann surface of genus* $p \geq 1$ *with*

$$\inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \textit{ conformal immersion } \} < \mathcal{W}_{n,p},$$

where $\mathcal{W}_{n,p}$ *is defined in (1.2) and* $n = 3, 4$ *.*

Then there exists a smooth conformal immersion $f : \Sigma \to \mathbb{R}^n$ *which minimizes the Willmore energy in the set of all conformal immersions. Moreover* f *satisfies the Euler-Lagrange equation*

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = g^{ik} g^{jl} A^0_{ij} q_{kl} \quad \textit{on } \Sigma, \tag{7.23}$$

where q *is a smooth transverse traceless symmetric 2-covariant tensor with respect to the Riemann surface* Σ *, that is with respect to* $g = f^* g_{euc}$ *.*

Proof:
We select select a smooth conformal metric g_0 for the Riemann surface Σ and put $\tau_0 := \pi(g_0)$. By invariance, we see

$$\mathcal{M}_{p,n}(\tau_0) = \inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \text{ conformal immersion } \} < \mathcal{W}_{n,p}.$$

Therefore by Corollary 7.2, there exists a smooth immersion $f : \Sigma \to \mathbb{R}^n$ which minimizes the Willmore energy in the fixed Teichmüller class $\tau_0 = \pi(f^* g_{euc})$ and satisfies the above Euler-Lagrange equation. Moreover there exists a diffeomorphism ϕ of Σ homotopic to the identity such that $(f \circ \phi)^* g_{euc}$ is conformal to g_0 . Then $\tilde{f} := f \circ \phi$ is a smooth conformal immersion of the Riemann surface Σ which minimizes the Willmore energy in the set of all conformal immersions and moreover satisfies the Euler-Lagrange equation.

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Appendix

A Conformal factor

Lemma A.1 *Let* Σ *be a closed, orientable surface of genus* $p \geq 1$ *,* g_0 *a given smooth metric on* Σ *,* $x_1, \ldots, x_M \in \Sigma$ *with charts* $\varphi_k : U(x_k) \xrightarrow{\approx} B_1(0), \varphi_k(x_k) = 0, U_\varrho(x_k) := \varphi_k^{-1}(B_\varrho(0))$ *for* $0 < \varrho \leq 1$,

$$\Lambda^{-1} g_{euc} \leq (\varphi_k^{-1})^* g_0 \leq \Lambda g_{euc} \quad \textit{for } k = 1, \ldots, M. \tag{A.1}$$

Let $f : \Sigma \to \mathbb{R}^n$ *be a smooth immersion with* $g := f^* g_{euc} = e^{2u} g_{poin}$ *for some unit volume constant curvature metric* g_{poin} *and*

$$\begin{gathered} \Lambda^{-1} g_0 \le g \le \Lambda g_0, \\ \parallel u \parallel_{L^\infty(\Sigma)}, \int_\Sigma |K_g| \ \mathrm{d}\mu_g \le \Lambda, \end{gathered} \tag{A.2}$$

and $\tilde{f} : \Sigma \to \mathbb{R}^n$ *a smooth immersion with* $\tilde{g} := \tilde{f}^* g_{euc} = e^{2\tilde{u}} \tilde{g}_{poin}$ *for some unit volume constant curvature metric* $\tilde{g}_{poin}$ *,*

$$\begin{gathered} \Lambda^{-1} g_0 \le \tilde{g} \le \Lambda g_0, \\ \int_\Sigma |K_{\tilde{g}}| \ \mathrm{d}\mu_{\tilde{g}} \le \Lambda, \end{gathered} \tag{A.3}$$

supp$(f - \tilde{f}) \subseteq \cup_{k=1}^M U_{1/2}(x_k)$ *and*

$$\int\limits_{U_1(x_k)} |\tilde{A}|^2 \ \mathrm{d}\mu_{\tilde{g}} \le \varepsilon_0(n) \quad \textit{for } k = 1, \ldots, M \tag{A.4}$$

for some universal $0 < \varepsilon_0(n) < 1$ *. Then*

$$\parallel \tilde{u} \parallel_{L^\infty(\Sigma)}, \parallel \nabla \tilde{u} \parallel_{L^2(\Sigma, \tilde{g})} \le C(\Sigma, g_0, \Lambda, p). \tag{A.5}$$

Proof:
We know

$$-\Delta_g u + K_p e^{-2u} = K_g, \quad -\Delta_{\tilde{g}} \tilde{u} + K_p e^{-2\tilde{u}} = K_{\tilde{g}} \quad \text{on } \Sigma. \tag{A.6}$$

Observing $\int_\Sigma | - K_p e^{-2\tilde{u}}| \ \mathrm{d}\mu_{\tilde{g}} = -K_p = 4\pi(p-1)$ and multiplying (A.6) by $\tilde{u} - \tilde{\lambda}$ for any $\tilde{\lambda} \in \mathbb{R}$, we get recalling (A.3)

$$c_0(\Lambda) \int\limits_\Sigma |\nabla \tilde{u}|^2_{g_0} \ \mathrm{d}\mu_{g_0} \le \int\limits_\Sigma |\nabla \tilde{u}|^2_{\tilde{g}} \ \mathrm{d}\mu_{\tilde{g}} =$$

$$= \int\limits_\Sigma (-K_p e^{-2\tilde{u}} + K_{\tilde{g}})(\tilde{u} - \tilde{\lambda}) \ \mathrm{d}\mu_{\tilde{g}} \le C(\Lambda, p) \parallel \tilde{u} - \tilde{\lambda} \parallel_{L^\infty(\Sigma)},$$

hence for $\tilde{\lambda} = -\!\!\!\!\!\!\int_\Sigma \tilde{u} \ \mathrm{d}\mu_{g_0}$ by Poincaré inequality

$$\parallel \tilde{u} - \tilde{\lambda} \parallel_{L^2(\Sigma, g_0)} \le C(\Sigma, g_0) \parallel \nabla \tilde{u} \parallel_{L^2(\Sigma, g_0)} \le C(\Sigma, g_0, \Lambda, p) \sqrt{osc_\Sigma \tilde{u}}. \tag{A.7}$$

Next by the uniformization theorem, see [FaKr] Theorem IV.4.1, we can parametrize $\tilde{f} \circ \varphi_k^{-1} : B_1(0) \to \mathbb{R}^n$ conformally with respect to the euclidean metric on $B_1(0)$, possibly after replacing $B_1(0)$ by a slightly smaller ball. Then by [MuSv95] Theorem 4.2.1 for $\varepsilon_0(n)$ small enough, there exist $v_k \in C^\infty(U_1(x_k))$ with

$$\begin{gathered} -\Delta_{\tilde{g}} v_k = K_{\tilde{g}} \quad \text{on } U_1(x_k), \\ \parallel v_k \parallel_{L^\infty(U_1(x_k))} \le C_n \int_{U_1(x_k)} |\tilde{A}|^2 \ \mathrm{d}\mu_{\tilde{g}} \le C_n \varepsilon_0(n) \le 1 \end{gathered} \tag{A.8}$$

for $\varepsilon_0(n)$ small enough. We get

$$\left.\begin{aligned} -\Delta_{\tilde g}(\tilde u - v_k) = -K_p e^{-2\tilde u} \geq 0, \\ -\Delta_{\tilde g}(\tilde u - v_k) + K_p(e^{-2\tilde u} - e^{-2v_k}) = -K_p e^{-2v_k} \geq 0, \end{aligned}\right\} \quad \text{on } U_1(x_k) \tag{A.9}$$

for $k = 1, \dots, M$, and, as $g = \tilde g$ on $\Sigma - \cup_{k=1}^M \overline{U_{1/2}(x_k)}$,

$$\left.\begin{aligned} -\Delta_g(\tilde u - u) + K_p(e^{-2\tilde u} - e^{-2u}) = 0, \\ -\Delta_g(\tilde u - u)_+ \leq 0, \end{aligned}\right\} \quad \text{on } \Sigma - \cup_{k=1}^M \overline{U_{1/2}(x_k)}. \tag{A.10}$$

Therefore $\tilde u - u$ cannot have positive interior maxima nor negative interior minima in $\Sigma - \cup_{k=1}^M \overline{U_{1/2}(x_k)}$ as $K_p \leq 0$ by standard maximum principle, hence putting $\Gamma := \cup_{k=1}^M \partial U_{3/4}(x_k)$ we get

$$\sup_{\Sigma - \cup_{k=1}^M \overline{U_{3/4}(x_k)}} (\tilde u - u)_\pm = \max_\Gamma (\tilde u - u)_\pm .$$

From above, we see from (A.1), (A.3), (A.8), (A.9),

$$0 \leq -\partial_i \Big(\tilde g^{ij} \sqrt{\tilde g}\ \partial_j(\tilde u - v_k) \Big) + K_p \sqrt{\tilde g}(e^{-2\tilde u} - e^{-2v_k}) = -K_p e^{-2v_k} \sqrt{\tilde g} \leq C(\Lambda, p), \tag{A.11}$$

hence using [GT] Theorem 8.16

$$\sup_{U_{3/4}(x_k)} (\tilde u - v_k)_\pm \leq \max_\Gamma (\tilde u - v_k)_\pm + C(\Lambda, p).$$

Together, we get from (A.2) and (A.8)

$$\max_\Sigma \tilde u_\pm \leq \max_\Gamma \tilde u_\pm + C(\Lambda, p). \tag{A.12}$$

From (A.1), (A.2), (A.3), and g_{poin} having unit volume, we get

$$c_0(\Lambda) \leq \mu_g(\Sigma), \mu_{g_0}(\Sigma), \mu_{\tilde g}(\Sigma) \leq C(\Lambda). \tag{A.13}$$

Now if $\min_\Sigma \tilde u \leq -C(\Lambda, p)$, there exists $x \in \Gamma$ with $\tilde u(x) \leq \min_\Sigma \tilde u + C(\Lambda, p) \leq 0$. As $\tilde u - v_k \geq \min_\Sigma \tilde u - 1 =: \lambda$, we get identifying $\varphi_k : U_1(x_k) \cong B_1(0)$ by the weak Harnack inequality, see [GT] Theorem 8.18, from (A.11)

$$\| \tilde u - v_k - \lambda \|_{L^2(B_{1/8}(x))} \leq C(\Lambda) \inf_{B_{1/8}(x)} (\tilde u - v_k - \lambda) \leq$$

$$\leq C(\Lambda)(\tilde u(x) - \min_\Sigma \tilde u + 2) \leq C(\Lambda, p),$$

and

$$\| \tilde u - \min_\Sigma \tilde u \|_{L^2(B_{1/8}(x))} \leq C(\Lambda, p).$$

We see from (A.1) and (A.7)

$$c_0 |\tilde\lambda - \min_\Sigma \tilde u| \leq \| \tilde\lambda - \min_\Sigma \tilde u \|_{L^2(B_{1/8}(x))} \leq$$

$$\leq C(\Lambda) \parallel \tilde{u} - \tilde{\lambda} \parallel_{L^2(\Sigma,g_0)} + \parallel \tilde{u} - \min_\Sigma \tilde{u} \parallel_{L^2(B_{1/8}(x))} \leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}}).$$

Using (A.13), we have proved

$$\min_\Sigma \tilde{u} \leq -C(\Lambda, p) \Longrightarrow$$

$$\parallel \tilde{u} - \min_\Sigma \tilde{u} \parallel_{L^2(\Sigma,g_0)} \leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}}). \tag{A.14}$$

If further $\min_\Sigma \tilde{u} \ll -C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}})$, we put $A := [\min_\Sigma \tilde{u} \leq \tilde{u} \leq \min_\Sigma \tilde{u}]$ and see $\tilde{u} - \min_\Sigma \tilde{u} \geq |\min_\Sigma \tilde{u}/2|$ on $\Sigma - A$ and using (A.3)

$$\frac{1}{2} |\min_\Sigma \tilde{u}| \; \mu_{\tilde{g}}(\Sigma - A) \leq \int\limits_{\Sigma - A} |\tilde{u} - \min_\Sigma \tilde{u}| \; \mathrm{d}\mu_{\tilde{g}} \leq$$

$$\leq C(\Sigma, g_0, \Lambda, p) \int\limits_\Sigma |\tilde{u} - \min_\Sigma \tilde{u}| \; \mathrm{d}\mu_{g_0} \leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}}),$$

hence

$$\mu_{\tilde{g}}(\Sigma - A) \leq \frac{C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}})}{|\min_\Sigma \tilde{u}|} \leq c_0(\Lambda)/2,$$

if $|\min_\Sigma \tilde{u}| \gg C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}})$ is large enough. This yields using (A.13) and $\mu_{\tilde{g}_{poin}}(\Sigma) = 1$

$$c_0(\Lambda)/2 \leq \mu_{\tilde{g}}(A) = \int\limits_A e^{2\tilde{u}} \; \mathrm{d}\mu_{\tilde{g}_{poin}} \leq \mu_{\tilde{g}_{poin}}(\Sigma) \exp(\min_\Sigma \tilde{u}) = \exp(\min_\Sigma \tilde{u}),$$

and we conclude

$$\min_\Sigma \tilde{u} \geq -C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}}). \tag{A.15}$$

In the same way as above, if $\max_\Sigma \tilde{u} \geq -C(\Lambda, p)$, we get from (A.12) that there exists $x \in \Gamma$ with $\tilde{u}(x) \geq \max_\Sigma \tilde{u} - C(\Lambda, p) \geq 0$. As $\tilde{u} - v_k \leq \max_\Sigma \tilde{u} + 1 =: \lambda$, we get by the weak Harnack inequality, see [GT] Theorem 8.18, from (A.11)

$$\parallel \tilde{u} - v_k - \lambda \parallel_{L^2(B_{1/8}(x))} \leq C(\Lambda) \inf_{B_{1/8}(x)} (\lambda - \tilde{u} + v_k) \leq$$

$$\leq C(\Lambda)(\max_\Sigma \tilde{u} - \tilde{u}(x) + 2) \leq C(\Lambda, p),$$

and

$$\parallel \max_\Sigma \tilde{u} - \tilde{u} \parallel_{L^2(B_{1/8}(x))} \leq C(\Lambda, p).$$

We see from (A.1) and (A.7)

$$c_0 |\max_\Sigma \tilde{u} - \tilde{\lambda}| \leq \parallel \max_\Sigma \tilde{u} - \tilde{\lambda} \parallel_{L^2(B_{1/8}(x))} \leq$$

$$\leq C(\Lambda) \parallel \tilde{u} - \tilde{\lambda} \parallel_{L^2(\Sigma,g_0)} + \parallel \max_\Sigma \tilde{u} - \tilde{u} \parallel_{L^2(B_{1/8}(x))} \leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_\Sigma \tilde{u}}).$$

Using (A.13), we have proved

$$\max_\Sigma \tilde{u} \geq C(\Lambda, p) \Longrightarrow$$

$$\| \max_{\Sigma} \tilde{u} - \tilde{u} \|_{L^2(\Sigma, g_0)} \leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_{\Sigma}\tilde{u}}). \tag{A.16}$$

Now if $\max_{\Sigma} \tilde{u} \gg C(\Sigma, g_0, \Lambda, p)(1+\sqrt{osc_{\Sigma}\tilde{u}})$, we need a lower bound on $\tilde{u}$. If $\min_{\Sigma} \tilde{u} \leq -C(\Lambda, p)$, we see from (A.14) and (A.16) that

$$\min_{\Sigma} \tilde{u} \geq \max_{\Sigma} \tilde{u} - C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_{\Sigma}\tilde{u}}) \geq 0,$$

therefore $\min_{\Sigma} \tilde{u} \geq -C(\Lambda, p)$. We put $A := [\max_{\Sigma} \tilde{u}/2 \leq \tilde{u} \leq \max_{\Sigma} \tilde{u}]$ and see $\max_{\Sigma} \tilde{u} - \tilde{u} \geq \max_{\Sigma} \tilde{u}/2$ on $\Sigma - A$, hence using (A.7)

$$\frac{1}{2} \max_{\Sigma} \tilde{u} \ \mu_{\tilde{g}_{poin}}(\Sigma - A) \leq \int\limits_{\Sigma - A} |\max_{\Sigma} \tilde{u} - \tilde{u}| e^{-2\tilde{u}} \ \mathrm{d}\mu_{\tilde{g}} \leq$$

$$\leq C(\Sigma, g_0, \Lambda, p) \int\limits_{\Sigma} |\max_{\Sigma} \tilde{u} - \tilde{u}| \ \mathrm{d}\mu_{g_0} \leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_{\Sigma}\tilde{u}})$$

and

$$\mu_{\tilde{g}_{poin}}(\Sigma - A) \leq \frac{C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_{\Sigma}\tilde{u}})}{\max_{\Sigma} \tilde{u}} \leq \frac{1}{2},$$

if $\max_{\Sigma} \tilde{u} \gg C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_{\Sigma}\tilde{u}})$ is large enough. This yields $\mu_{\tilde{g}_{poin}}(A) \geq 1/2$ and by (A.13)

$$C(\Lambda) \geq \mu_{\tilde{g}}(\Sigma) \geq \int\limits_{A} e^{2\tilde{u}} \ \mathrm{d}\mu_{\tilde{g}_{poin}} \geq \mu_{\tilde{g}_{poin}}(A) \exp(\max_{\Sigma} \tilde{u}) \geq \exp(\max_{\Sigma} \tilde{u})/2,$$

and we conclude

$$\max_{\Sigma} \tilde{u} \leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_{\Sigma}\tilde{u}}). \tag{A.17}$$

(A.14) and (A.16) yield

$$osc_{\Sigma}\tilde{u} = \max_{\Sigma} \tilde{u} - \min_{\Sigma} \tilde{u} \leq$$

$$\leq C(\Sigma, g_0, \Lambda, p)(1 + \sqrt{osc_{\Sigma}\tilde{u}}) \leq C(\Sigma, g_0, \Lambda, p) + \frac{1}{2} osc_{\Sigma}\tilde{u},$$

hence $osc_{\Sigma}\tilde{u} \leq C(\Sigma, g_0, \Lambda, p)$. Then (A.5) follows from (A.14), (A.16) and (A.7), and the lemma is proved.

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Here, we use this lemma to get a bound on the conformal factor for sequences strongly converging in $W^{2,2}$.

Proposition A.2 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a weak local bilipschitz immersion approximated by smooth immersions* f_m *with pull-back metrics* $g = f^* g_{euc} = e^{2u} g_{poin}, g_m = f_m^* g_{euc} = e^{2u_m} g_{poin_m}$ *for some smooth unit volume constant curvature metrics* $g_{poin}, g_{poin,m}$ *and satisfying*

$$f_m \to f \quad \textit{strongly in } W^{2,2}(\Sigma), \textit{weakly}^* \textit{ in } W^{1,\infty}(\Sigma),$$

$$\Lambda^{-1} g_{poin} \leq g_m \leq \Lambda g_{poin}.$$

Then

$$\sup_{m \in \mathbb{N}} \Big(\| u_m \|_{L^\infty(\Sigma)}, \| \nabla u_m \|_{L^2(\Sigma, g_m)} \Big) < \infty.$$

Proof:
We want to apply Lemma A.1 to $f = f_1, \tilde{f} = f_m, g_0 = g_{poin}$. (A.2) and (A.3) are immediate by the above assumptions for appropriate possibly larger $\Lambda < \infty$.

In local charts, we have

$$\begin{gathered} g_m \to g \quad \text{strongly in } W^{1,2}, \text{weakly}^* \text{ in } L^\infty(\Sigma), \\ \Gamma^k_{g_m,ij} \to \Gamma^k_{g,ij} \quad \text{strongly in } L^2, \end{gathered}$$

and

$$\begin{gathered} A_{f_m,ij} = \nabla^{g_m}_i \nabla^{g_m}_j f_m \to \partial_{ij} f - \Gamma^k_{g,ij} \partial_k f = \\ = \nabla^g_i \nabla^g_j f = A_{f,ij} \quad \text{strongly in } L^2. \end{gathered}$$

Therefore $|A_{f_m}|^2_{g_m} \sqrt{g_m} \to |A_f|^2_g \sqrt{g}$ strongly in L^1 , hence for each $x \in \Sigma$ there exists a neighbourhood $U(x)$ of x with

$$\int\limits_{U(x)} |A_{f_m}|^2_{g_m} \, \mathrm{d}\mu_{g_m} \le \varepsilon_0(n) \quad \text{for all } m \in \mathbb{N},$$

where $\varepsilon_0(n)$ is as in Lemma A.1. Choosing $U(x)$ even smaller, we may assume that there are charts $\varphi_x : U(x) \xrightarrow{\approx} B_1(0)$ with $\varphi_x(x) = 0$ and $c_{0,x} g_{euc} \le (\varphi^{-1})^* g_{poin} \le C_x g_{euc}$. Selecting a finite cover $\Sigma = \cup_{k=1}^M \varphi_{x_k}^{-1}(B_{1/2}(0))$, we obtain (A.1) and (A.4) for appropriate $\Lambda < \infty$. As clearly $supp(f_1 - f_m) \subseteq \Sigma$, the assertion follows from Lemma A.1.

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B Analysis

Proposition B.1 *Let* $\Phi = (\Phi_0, \varphi) : B^{M+2}_{\lambda_0}(0) \to \mathbb{R}^{M+1} = \mathbb{R}^M \times \mathbb{R}, M \in \mathbb{N},$ *be twice differentiable satisfying for* $\xi = (\lambda, \mu, \nu) \in B^{M+2}_{\lambda_0}(0) \subseteq \mathbb{R}^M \times \mathbb{R} \times \mathbb{R}$

$$\begin{gathered} \| \partial_\lambda \Phi_0 - I_M \| \le 1/2, \\ |\partial_{(\mu,\nu)} \Phi_0|, |\partial_\lambda \varphi| \le \varepsilon, \\ \| D^2 \Phi \| \le \Lambda, \\ \partial_{\mu\mu} \varphi, -\partial_{\nu\nu} \varphi \ge \gamma \end{gathered}$$

with $0 < \varepsilon, \gamma, \lambda_0 \le 1/4, 1 \le \Lambda < \infty,$

$$C \Lambda \varepsilon \le \gamma. \tag{B.1}$$

Then for $\eta = (\eta_0, \bar{\eta}) \in \mathbb{R}^M \times \mathbb{R}$ *with*

$$\begin{gathered} |\Phi_0(0) - \eta_0| \le \min(\Lambda \lambda_0^2, \lambda_0/8), \\ |\varphi(0) - \bar{\eta}| \le \gamma \lambda_0^2/32, \end{gathered}$$

there exists $\xi \in B^{M+2}_{\lambda_0}(0)$ *with* $\mu\nu = 0$ *and satisfying*

$$\Phi(\xi) = \eta$$

with

$$|\xi| \le C\gamma^{-1/2}|\Phi(0) - \eta|^{1/2}. \tag{B.2}$$

If further

$$\begin{gathered} |D\varphi(0)| \le \sigma, \\ C\Lambda\varepsilon + C\sigma\lambda_0^{-1} + C\Lambda\lambda_0 \le \gamma, \end{gathered} \tag{B.3}$$

there exists a solution $\tilde{\xi} \in B^{M+2}_{\lambda_0}(0)$ *of* $\Phi(\tilde{\xi}) = \eta, \tilde{\mu}\tilde{\nu} = 0$ *with*

$$\begin{aligned} \mu\tilde{\mu} \le 0, \tilde{\nu} = 0, \quad & \textit{if } \nu = 0, \\ \nu\tilde{\nu} \le 0, \tilde{\mu} = 0, \quad & \textit{if } \mu = 0. \end{aligned}$$

Proof:

After the choice in (B.10) below, we will need only one variable μ or ν . Therefore to simplify the notation, we put $\nu = 0$ and omit ν .

First there exists a twice differentiable function $\lambda :]-\lambda_0/2, \lambda_0/2[\to B^M_{\lambda_0/2}(0)$

$$\Phi_0(\lambda(\mu), \mu) = \eta_0. \tag{B.4}$$

Indeed putting

$$T_\mu(\lambda) := \lambda - \Phi_0(\lambda, \mu) + \eta_0 \quad \text{ for } |\lambda| \le \lambda_0/2,$$

we see

$$\| T'_\mu \| \le \| I_M - \partial_\lambda \Phi_0 \| \le 1/2,$$

hence

$$|T_\mu(\lambda_1) - T_\mu(\lambda_2)| \le \frac{1}{2}|\lambda_1 - \lambda_2|.$$

As

$$\begin{gathered} |T_\mu(0)| = |\Phi_0(0, \mu) - \eta_0| \le |\Phi_0(0, \mu) - \Phi_0(0, 0)| + |\Phi_0(0) - \eta_0| \le \\ \le \| \partial_\mu \Phi_0 \| \, |\mu| + \lambda_0/8 < (\varepsilon/2 + 1/8)\lambda_0 \le \lambda_0/4, \end{gathered}$$

we get

$$|T_\mu(\lambda)| \le |\lambda|/2 + |T_\mu(0)| < \lambda_0/2$$

and by Banach's fixed point theorem, (B.4) has a unique solution $\lambda = \lambda(\mu) \in B^M_{\lambda_0/2}(0)$. In particular

$$|\lambda(0)| \le 2|T_0(0)| \le 2|\Phi_0(0) - \eta_0| \le 2\Lambda\lambda_0^2. \tag{B.5}$$

By implict function theorem, λ is twice differentiable and

$$\begin{gathered} \partial_\lambda \Phi_0 \partial_\mu \lambda + \partial_\mu \Phi_0 = 0, \\ \partial_\mu \lambda^T \partial_{\lambda\lambda} \Phi_0 \partial_\mu \lambda + 2\partial_{\lambda\mu} \Phi_0 \partial_\mu \lambda + \partial_\lambda \Phi_0 \partial_{\mu\mu} \lambda + \partial_{\mu\mu} \Phi_0 = 0. \end{gathered}$$

This yields

$$|\partial_\mu \lambda| \le 2|\partial_\mu \Phi_0| \le 2\varepsilon \le 1/2, \qquad \text{(B.6)}$$
$$|\partial_{\mu\mu}\lambda| \le 2|\partial_\mu \lambda^T \partial_{\lambda\lambda}\Phi_0 \partial_\mu \lambda + 2\partial_{\lambda\mu}\Phi_0\partial_\mu\lambda + \partial_{\mu\mu}\Phi_0| \le C\Lambda.$$

Next we put

$$\psi(\mu) := \varphi(\lambda(\mu),\mu) - \bar{\eta} \quad \text{for } |\mu| < \lambda_0/2$$

and see by (B.5)

$$|\psi(0)| \le |\varphi(\lambda(0),0) - \varphi(0)| + |\varphi(0) - \bar{\eta}| \le$$
$$\le \| \partial_\lambda \varphi \| \, |\lambda(0)| + \gamma\lambda_0^2/32 \le C\Lambda\varepsilon\lambda_0^2 + \gamma\lambda_0^2/32 \le \gamma\lambda_0^2/16. \qquad \text{(B.7)}$$

Clearly ψ is twice differentiable and

$$\partial_\mu \psi = \partial_\lambda \varphi \partial_\mu \lambda + \partial_\mu \varphi, \qquad \text{(B.8)}$$
$$\partial_{\mu\mu}\psi = \partial_\mu \lambda^T \partial_{\lambda\lambda}\varphi \partial_\mu \lambda + 2\partial_{\lambda\mu}\varphi\partial_\mu\lambda + \partial_\lambda\varphi\partial_{\mu\mu}\lambda + \partial_{\mu\mu}\varphi.$$

This yields using (B.6)

$$|\partial_{\mu\mu}\psi - \partial_{\mu\mu}\varphi| \le |\partial_\mu \lambda^T \partial_{\lambda\lambda}\varphi\partial_\mu\lambda + 2\partial_{\lambda\mu}\varphi\partial_\mu\lambda + \partial_\lambda\varphi\partial_{\mu\mu}\lambda| \le \qquad \text{(B.9)}$$
$$\le C \| D^2\varphi \| \, |\partial_\mu\lambda| + \| \partial_\lambda \varphi \| \, |\partial_{\mu\mu}\lambda| \le C\Lambda\varepsilon.$$

Since the assumptions and conclusions are the equivalent for $-\Phi$ and $-\eta$, we assume after possibly exchanging μ by ν that

$$\psi(0) \le 0. \qquad \text{(B.10)}$$

Replacing μ by $-\mu$, we may further assume $\partial_\mu\psi(0) \ge 0$. On the other hand by (B.7) and (B.9)

$$\liminf_{\mu \nearrow \lambda_0/2} \psi(\mu) \ge \psi(0) + \partial_\mu\psi(0)\lambda_0/2 + (\inf \partial_{\mu\mu}\psi/2)(\lambda_0/2)^2 \ge$$
$$\ge -\gamma\lambda_0^2/16 + (\gamma - C\Lambda\varepsilon)\lambda_0^2/8 > 0,$$

when using (B.1). Therefore there exists $0 \le \mu < \lambda_0/2$ with $\psi(\mu) = 0$, hence putting $\xi := (\lambda(\mu),\mu,0) \in B_{\lambda_0}^{M+2}(0)$

$$\Phi(\xi) = \Big(\Phi_0(\lambda(\mu),\mu), \varphi(\lambda(\mu),\mu)\Big) = \Big(\eta_0, \psi(\mu) + \bar{\eta}\Big) = \eta.$$

Choosing λ_0 small enough, we further obtain (B.2).

If further (B.3) is satisfied for the original λ_0, we see from (B.5), (B.6) and (B.8) that

$$|\partial_\mu\psi(0)| \le C|D\varphi(\lambda(0),0)| \le C|D\varphi(0)| + C \| D^2\varphi \| \, |\lambda(0)| \le C\sigma + C\Lambda\lambda_0^2.$$

Proceeding as above with (B.7) and (B.9), we calculate

$$\liminf_{\mu \searrow -\lambda_0/2} \psi(\mu) \ge \psi(0) + \partial_\mu\psi(0)\lambda_0/2 + (\inf \partial_{\mu\mu}\psi/2)(\lambda_0/2)^2 \ge$$
$$\ge -\gamma\lambda_0^2/16 - (C\sigma + C\Lambda\lambda_0^2)\lambda_0 + (\gamma - C\Lambda\varepsilon)\lambda_0^2/8 =$$
$$\ge (\gamma/16 - C\Lambda\varepsilon - C\sigma\lambda_0^{-1} - C\Lambda\lambda_0)\lambda_0^2 > 0,$$

when using (B.3). Therefore there exists $-\lambda_0/2 < \tilde{\mu} \le 0$ with $\psi(\tilde{\mu}) = 0$, hence putting $\tilde{x}i := (\lambda(\tilde{\mu}),\tilde{\mu},0) \in B_{\lambda_0}^{M+2}(0)$, we get $\Phi(\tilde{\xi}) = \eta$ and $\mu\tilde{\mu} \le 0$.

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