TWO CLASSES OF SLANT SURFACES IN NEARLY KÄHLER SIX SPHERE

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ABSTRACT. In this paper we find examples of slant surfaces in the nearly Kähler six sphere. First, we characterize two-dimensional small and great spheres which are slant. Their description is given in terms of the associative 3-form in $\mathrm{Im}\,\mathbb{O}$. Later on, we classify the slant surfaces of S^6 which are orbits of maximal torus in G_2. We show that these orbits are flat tori which are linearly full in $S^5 \subset S^6$ and that their slant angle is between $\arccos\frac{1}{3}$ and $\frac{\pi}{2}$. Among them we find one parameter family of minimal orbits.

1. Introduction

It is known that S^2 and S^6 are the only spheres that admit an almost complex structure. The best known Hermitian almost complex structure J on S^6 is defined using octonionic multiplication. It not integrable, but satisfies condition $(\nabla_X J)X = 0$, for the Levi-Civita connection ∇ and every vector field X on S^6. Therefore, sphere S^6 with this structure J is usually referred as nearly Kähler six sphere.

Submanifolds of nearly Kähler sphere S^6 are subject of intensive research. A. Gray [7] proved that almost complex submanifolds of nearly Kähler S^6 are necessarily two-dimensional and minimal. In paper [3] Bryant showed that any Riemannian surface can be embedded in the six sphere as an almost complex submanifold. Almost complex surfaces were further investigated in paper [1] and classified into four types.

Totally real submanifolds of S^6 can be of dimension two or three. Three-dimensional totally real submanifolds are investigated in [8] where N. Ejiri proved that they have to be minimal and orientable. In paper [2] authors classify totally real, minimal surfaces with constant curvature. Totaly real and minimal surfaces with Gauss curvature $K \in [0, 1]$ (and compact) or those with K constant must have either $K = 0$ or $K = 1$ (see [6]).

Slant submanifold N of an almost Hermitian manifold (M, g, J) is a generalization of totaly real and almost complex submanifold. The latter have slant angle $\frac{\pi}{2}$ and 0, respectively. Slant submanifolds with slant angle $\theta \in (0, \frac{\pi}{2})$ are called proper slant submanifolds. Note that the notion of surface with constant Kähler angle coincides with the notion of slant surface and in this paper we use the latter one. General theory regarding slant submanifolds and some classification theorems of slant surfaces in $\mathbb{C}^2$ can be found in [4]. Because of dimensional reasons (see [4]) a proper slant submanifold of six sphere is two-dimensional. Very few examples of

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slant surfaces of almost Kähler six sphere are known and all of them are minimal. In paper [1] it is shown that rotation of an almost complex curve of type (III) results in minimal slant surface that is linearly full in some $S^5 \subset S^6$. According to [12], a minimal slant surface of S^6 that has non-negative Gauss curvature $K \geq 0$ must have either $K \equiv 0$ or $K \equiv 1$. Classification of such surfaces is given in paper [13]. In the present paper we find two classes of slant surfaces of S^6 with $K \equiv 0$ and $K \equiv 1$. Some of them are minimal and therefore known, but most of them are not minimal.

In the Section 2 we recall some basic facts about octonions and define the notion of a slant submanifold (Definition 2.1). Lemma 2.2 is simple, but it is not found in [4]. In Section 3 we consider slant two-dimensional spheres which are intersection of an affine 3-plane and the six sphere. We give their characterization in terms of associative 3-form in Theorem 3.1. Finally, in the Section 4 we investigate two-dimensional orbits of a Cartan subgroup of group G_2 on the sphere S^6. We found that these orbits are flat, two-dimensional tori which are always slant, but it is interesting that their slant angle is in the range $[\arccos\frac{1}{3}, \frac{\pi}{2}]$ (see Theorem 4.1). In the Theorem 4.2 we find one-parameter family of minimal orbits. Note that similar method was used in paper [10] for obtaining three-dimensional orbits which are CR submanifolds of sphere S^6.

2. Preliminaries

Let $\mathbb{H}$ be a field of quaternions. The Cayley algebra, or algebra of octonions, is vector space $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^8$ with multiplication defined in terms of quaternionic multiplication:

$$(q, r) \cdot (s, t) := (qs - \bar{t}r, tq + r\bar{s}), \quad q, s, r, t \in \mathbb{H}.$$

In the sequel we omit the multiplication sign. Conjugation of Cayley numbers is defined by

$$\overline{(q, r)} := (\bar{q}, -r), \quad q, r \in \mathbb{H},$$

and inner product by:

$$\langle x, y \rangle := \frac{1}{2}(x\bar{y} + y\bar{x}), \quad x, y \in \mathbb{O}. \tag{1}$$

If we denote by $1, i, j, k$ the standard orthonormal basis of $\mathbb{H}$, then $e_0 = (1, 0), e_1 = (i, 0), e_2 = (j, 0), e_3 = (k, 0), e_4 = (0, 1), e_5 = (0, i), e_6 = (0, j), e_7 = (0, k)$ is orthonormal basis of $\mathbb{O}$. It is easy to check that the multiplication in that basis is given by the following table.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The Cayley numbers are not associative, so we define *associator*

$$[x, y, z] := (xy)z - x(yz), \quad x, y, z \in \mathbb{O}.$$

Denote by

$$\operatorname{Im}\mathbb{O} := \{x \in \mathbb{O} \mid x + \overline{x} = 0\},$$

the subspace of imaginary Cayley numbers. Then we have the orthogonal decomposition

$$\mathbb{O} = \mathbb{R} \oplus \operatorname{Im}\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7. \tag{2}$$

On the subspace of imaginary Cayley numbers $\operatorname{Im}\mathbb{O}$ we define the vector product

$$x \times y := \frac{1}{2}(xy - yx),$$

that shares many properties with the vector product in $\mathbb{R}^3$.
We state some well known properties of Cayley numbers without a proof.

Lemma 2.1. 1) *If $x, y \in Im\mathbb{O}$ then*

$$xy = -\langle x, y\rangle + x \times y.$$

2) *For all $x, y, z \in \mathbb{O}$ we have*

$$\overline{x}(xy) = (\overline{x}x)y,$$

$$\langle xy, xz\rangle = \langle x, x\rangle\langle y, z\rangle = \langle yx, zx\rangle.$$

3) *If $x, y, z \in \mathbb{O}$ are mutually orthogonal unit vectors then*

$$x(yz) = y(zx) = z(xy).$$

Exceptional group G_2 is usually defined as a group of automorphisms of Cayley numbers. Since it preserves the multiplication it also preserves the inner product (1) and the decomposition (2) and therefore it is subgroup of group $O(7)$. Actually, it is subgroup of group $SO(7)$.
For any point $p \in S^6 \subset \operatorname{Im}\mathbb{O}$ and a tangent vector $X \in T_pS^6$ we define automorphism $J_p : T_pS^6 \to T_pS^6$ by

$$J_p(X) := p \cdot X = p \times X.$$

One can easily show that six-dimensional sphere $(S^6, \langle\rangle, J)$ is almost Hermitian manifold, i.e. J_p satisfies

$$J_p^2 = -\mathrm{Id}\,, \qquad \langle J_pX, J_pY\rangle = \langle X, Y\rangle,$$

for all $X, Y \in T_pS^6$. The unit six-dimensional sphere $S^6 \subset \operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ posses almost complex structure J defined by

$$J_p(X) = pX = p \times X, \quad p \in S^6, X \in T_pS^6.$$

Obviously, the group G_2 preserves the structure J.

Definition 2.1. *Let (M, g, J) be an almost Hermitian manifold and $N \subset M$ be a submanifold of M. For each $p \in N$ and $X \in T_pN$ we define Wirtinger angle*

$$\theta_p(X) := \angle(JX, T_pN).$$

We say that N is slant submanifold *if its Wirtinger angle θ is constant, i.e. it doesn't depend on the point $p \in N$ and tangent vector $X \in T_pN$.*

Slant submanifold with angle $\theta \equiv 0$ is usually called *almost complex submanifold*, and slant submanifold with angle $\theta \equiv \frac{\pi}{2}$ is called *totally real submanifold.* Slant submanifold that is neither almost complex nor totally real is called *proper slant submanifold.*
It is known (see [4]) that proper slant submanifold $N \subset M$ has to be of even dimension and its dimension is less that half of the dimension of M. The next lemma shows that in the case $\dim N = 2$, the Wirtinger angle of N is always independent on vector $X \in T_pM$.

Lemma 2.2. *Let (M, g, J) be an almost-Hermitian manifold and $N \subset M$ surface. The Wirtinger angle $\theta_p(X)$ doesn't depend on the vector $X \in T_pN$ and*

$$\theta_p(Z) = |g(X, JY)|$$

for all $Z \in T_pN$, where (X, Y) is any orthonormal basis of T_pN.

Proof: For each $Z \in T_pN$ there is an orthogonal decomposition

$$JZ = PZ + FZ,$$

to the tangent component $PZ \in T_pN$ and normal component $FZ \in T_p^{\perp}N$.
Therefore,

$$\cos\theta_p(Z) = \frac{g(JZ, PZ)}{\|JZ\|\|PZ\|} = \frac{\|PZ\|^2}{\|Z\|\|PZ\|} = \frac{\|PZ\|}{\|Z\|}.$$

Since J is an isometry we have

$$g(JX, Y) = -g(X, JY),$$

for all $X, Y \in T_pM$ and particularly for $X = Y$

$$g(JX, X) = 0.$$

Let (X, Y) be an orthonormal basis of T_pN. For any $Z = aX + bY \in T_pN$

$$\begin{aligned} \|PZ\|^2 &= g(PZ, X)^2 + g(PZ, Y)^2 = g(JZ, X)^2 + g(JZ, Y)^2 = \\ &= g(aJX + bJY, X)^2 + g(aJX + bJY, Y)^2 = \\ &= (a^2 + b^2)g(X, JY)^2, \end{aligned}$$

so we have

$$\cos\theta_p(Z) = \frac{\sqrt{a^2 + b^2}\,|g(X, JY)|}{\sqrt{a^2 + b^2}} = |g(X, JY)|.$$

□

3. Slant two-dimensional spheres in S^6

Definition 3.1. *If $\pi \in G_{\mathbb{R}}(3, Im\,\mathbb{O})$ is imaginary part or quaternionic subalgebra of Cayley numbers $\mathbb{O}$ we call π associative 3-plane. Denote by $ASSOC \subset G_{\mathbb{R}}(3, Im\,\mathbb{O})$ set of all associative planes.*

Since the quaternions are associative, the associator of any three vectors of an associative plane is equal zero. Vice versa, if associator of some three vectors from $\mathrm{Im}\,\mathbb{O}$ vanishes, then these three elements span an associative plane.
On the vector space $\mathrm{Im}\,\mathbb{O} = \mathbb{R}^7$ we define 3-form ϕ by the formula

$$\phi(x, y, z) := \langle x, yz\rangle.$$

In ([9]) this form is called *associative* 3*-form* and it is shown that the form ϕ is calibration with contact set $ASSOC$.

We use the following notation for the associative 3-form ϕ and the associator of a 3-dimensional plane $\pi \in \mathrm{Im}\,\mathbb{O}$

$$\phi(\pi) := |\phi(f_1, f_2, f_3)|, \quad \text{and} \quad [\pi] := [f_1, f_2, f_3],$$

where f_1, f_2, f_3 is orthonormal basis of π. One can show that these definitions do not depend on the choice of orthonormal basis f_1, f_2, f_3 of the plane π. Furthermore, both the form ϕ and the associator are G_2 invariant.
From the above definition it follows that $\phi(\pi) \in [0, 1]$ for any 3-dimensional plane ϕ and that associative planes are characterized by the condition $\phi(\pi) = 1$.
The associator and the associative 3-form ϕ are related by the formula

$$\phi^2(\pi) + \frac{1}{4}\|[\pi]\|^2 = 1, \tag{-1}$$

which we prove in the Lemma 3.1.
In the remainder of this section we characterize two-dimensional spheres that are intersection of 3-dimensional affine plane and the sphere S^6. For the proof of Theorem 3.1 we need the following lemmas.

Lemma 3.1. *Let f_1, f_2, f_3 be an orthonormal basis of the plane π. The Gram matrix of the set of vectors*

$$f = (f_1, f_2, f_3, f_2f_3, f_3f_1, f_1f_2, [f_1, f_2, f_3])$$

is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \phi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \phi & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \phi & 0 \\ \phi & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \phi & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4(1-\phi^2) \end{pmatrix},$$

where we abbreviate $\phi = \phi(f_1, f_2, f_3)$. Particularly, the set f spans $Im\mathbb{O}$ if and only if $\phi(\pi) \neq 1$, i.e. the plane π is not associative.

Proof: Most of the scalar product are simple to calculate using properties of Cayley numbers from Lemma 2.1. For example:

$$\langle f_2f_3, f_3f_1 \rangle = -\langle f_2f_3, f_1f_3 \rangle = \langle f_2, f_1 \rangle |f_3|^2 = 0.$$

Now we prove the most complicate inner product $\langle [f_1, f_2, f_3], [f_1, f_2, f_3] \rangle$. First, note that all f_2f_3, f_3f_1, f_1f_2 are imaginary. Using simple transformations one can show

$$f_3(f_1f_2) = -2\phi(f_1, f_2, f_3) - (f_1f_2)f_3. \tag{-1}$$

Using the previous formula we get

$$\begin{aligned} \langle [f_1, f_2, f_3], [f_1, f_2, f_3] \rangle &= 4(\langle (f_1f_2)f_3, (f_1f_2)f_3 \rangle + 2\langle (f_1f_2)f_3, \phi \rangle + \langle \phi, \phi \rangle) = \\ &= 4(1 - 2\phi^2 + \phi^2) = 4(1 - \phi^2). \end{aligned}$$

$\square$

Theorem 3.1. *Let $S_r^2 = \pi' \cap S^6$, $r \in (0, 1]$ be a two-dimensional sphere of radius r. Denote by $\pi \in G_{\mathbb{R}}(3, Im\mathbb{O})$ 3-dimensional plane parallel to the affine plane π' containing S_r^2.*

a) *If S_1^2 is great sphere, that is $\pi = \pi'$, then it is slant with the slant angle $\theta = \arccos \phi(\pi)$. Therefore, S_1^2 is proper slant sphere if π is not associative.*

b) *If the plane π is associative then the small sphere S^2_r is slant with the slant angle $\theta = \arccos r$.*
c) *If the plane π is not associative then the small sphere S^2_r is slant with the slant angle $\theta = \arccos(r\,\phi(\pi))$ if and only if its center is point $C = \pm\frac{\sqrt{1-r^2}}{\|[\pi]\|}[\pi]$.*

Proof: Let X, Y be an orthonormal basis of the tangent space $T_pS^2_r$, $p \in S^2_r$. According to the Lemma 2.2 we have

$$\cos\theta_p = |\langle X, JY\rangle| = |\langle X, pY\rangle| = |\phi(p, X, Y)|.$$

Particularly, if $p \in S^2_1$ then p, X, Y is an orthonormal basis of the plane $\pi = \pi'$ and we have

$$\cos\theta_p = |\phi(p, X, Y)| = \phi(\pi)$$

which proves the statement a).
Now we suppose that $\pi \neq \pi'$. Let f_1, f_2, f_3 be an orthonormal basis of π. Since π' is parallel to π we can write $\pi' = \pi + \sqrt{1-r^2}\,\xi$ for a unit vector $\xi \in \pi^\perp$. Therefore $p \in S^2_r$ is of the form

$$p = rp_0 + \sqrt{1-r^2}\,\xi$$

for some point $p_0 = p_1f_1 + p_2f_2 + p_3f_3 \in S^2_1$. As before, let X, Y be an orthonormal basis of the tangent space $T_pS^2_r$. Then vectors p_0, X, Y form an orthonormal basis of π.

$$\begin{aligned}\cos\theta_p &= |\phi(p, X, Y)| = |\phi(r\,p_0 + \sqrt{1-r^2}\,\xi, X, Y)| = \\ &= |r\,\phi(p_0, X, Y) + \sqrt{1-r^2}\,\phi(\xi, X, Y)|.\end{aligned}$$

Since $|\phi(p_0, X, Y)| = \phi(\pi)$, it remains to calculate $\phi(\xi, X, Y)$. Note that if the plane π is associative then $XY \in \pi$ and we have

$$\phi(\xi, X, Y) = \langle \xi, XY\rangle = 0.$$

Therefore, in the case of the associative plane π we have

$$\cos\theta_p = r\phi(\pi) = r,$$

what proves the statement b).
Let $\pi \neq \pi'$ be a non-associative plane. Since $X, Y \in \pi$ we can write them in the form

$$X = x_1f_1 + x_2f_2 + x_3f_3,$$
$$Y = y_1f_1 + y_2f_2 + y_3f_3.$$

The vectors X and Y are orthogonal to the point $p_0 \in \pi$ its coordinates are

$$p_0 = (p_1, p_2, p_3) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Now, easy calculation yields

$$\begin{aligned}\phi(\xi, X, Y) &= \langle \xi, XY\rangle = \langle \xi, (x_1f_1 + x_2f_2 + x_3f_3)(y_1f_1 + y_2f_2 + y_3f_3)\rangle = \\ &= \langle \xi, f_2f_3p_1 - f_1f_3p_2 + f_1f_2p_3\rangle = \\ &= \phi(\xi, f_2, f_3)p_1 - \phi(\xi, f_1, f_3)p_2 + \phi(\xi, f_1, f_2)p_3.\end{aligned}$$

The sphere S^2_r is slant if this expression doesn't depend on point $p \in S^2_r$, that is, if and only if

$$\phi(\xi, f_2, f_3) = 0, \quad \phi(\xi, f_1, f_3) = 0, \quad \phi(\xi, f_1, f_2) = 0.$$

This condition means that the vector ξ is orthogonal to vectors f_2f_3, f_3f_1, f_1f_2, and since $\xi \in \pi^\perp$, according to the Lemma 3.1, the only possibility for the unit vector ξ is

$$\xi = \pm\frac{[\pi]}{||[\pi]||}$$

and therefore the statement c) holds. □

Lemma 3.2. *Two 3-dimensional planes $\pi_1, \pi_2 \in G_{\mathbb{R}}(3, Im\,\mathbb{O})$ are G_2 equivalent if and only if $\phi(\pi_1) = \phi(\pi_2)$.*

Proof: If the planes π_1 and π_2 are equivalent by a G_2 transformation then we have $\phi(\pi_1) = \phi(\pi_2)$ because the form ϕ is G_2 invariant. Lets prove the converse.
If $\phi(\pi_1) = 1 = \phi(\pi_2)$, i.e. the planes are associative, they are G_2 equivalent. Namely, if the planes π_1 and π_2 are spanned by orthonormal bases f_1, f_2, f_1f_2 and g_1, g_2, g_1g_2 respectively, than any G_2 transformation that maps f_1, f_2 to g_1, g_2 also maps π_1 onto π_2.
Suppose that $\phi(\pi) = \phi \neq 1$ and f_1, f_2, f_3 is orthonormal basis of π. Denote by $F_1 = f_1, F_2 = f_2, F_3 = f_1f_2$. From Lemma 3.1 it follows that $F_4 = \frac{1}{2\sqrt{1-\phi^2}}[f_1, f_2, f_3]$ is unit vector orthogonal to F_1, F_2 and F_3. Following the Cayley-Dixon process the set of vectors F_1, F_2, F_3, F_4, $F_5 = F_1F_4$, $F_6 = F_2F_4$, $F_7 = F_3F_4$ is G_2 basis of $\mathrm{Im}\,\mathbb{O}$, i.e. satisfies the same multiplication properties as standard basis $e_1, \dots, e_7$ from Section 2. One can easily check the following relations

$$\begin{aligned}
F_5 &= \frac{1}{\sqrt{1-\phi^2}}(\phi f_1 + f_2f_3),\\
F_6 &= \frac{1}{\sqrt{1-\phi^2}}(\phi f_2 + f_3f_1),\\
F_7 &= \frac{1}{\sqrt{1-\phi^2}}(-f_3 + \phi f_1f_2).
\end{aligned} \tag{-9}$$

There is a G_2 transformation that maps vectors F_1, F_2, F_4 to vectors e_1, e_2, e_4, respectively. According to the relation (-9) the image of vector f_3 is $\phi\, e_3 - \sqrt{1-\phi^2}\, e_7$. Therefore, in the case $\phi(\pi) = \phi \neq 1$, plane π is G_2 equivalent to the plane

$$\pi_0^\phi = \mathbb{R}\langle e_1, e_2, \phi\, e_3 - \sqrt{1-\phi^2}\, e_7\rangle.$$

We conclude that any two planes π_1, π_2 with $\phi(\pi_1) = \phi(\pi_2)$ are G_2 equivalent, as claimed. □

Corrolary 3.1.1. *Let $\pi' \cap S^6$ and $\tau' \cap S^6$ be two-dimensional slant spheres from the Theorem 3.1. They are G_2 equivalent if and only if they are of the same radius and $\phi(\pi) = \phi(\tau)$.*

Remark 3.1.1. *Up to G_2 equivalence there is only one almost complex two-dimensional sphere. It is great sphere belonging to associative plane π, i.e $\phi(\pi) = 1$.*

Remark 3.1.2. *Totally real two-dimensional spheres are $S_r^2 = \pi' \cap S^6$ where π' is affine 3-plane parallel to the plane π with $\phi(\pi) = 0$. Up to a G_2 equivalence there is unique such sphere for each radius $r \in (0, 1]$. The one with radius $r = 1$ is minimal. It is found in classification of totally real minimal surfaces of constant curvature (see* [2]*, Theorem 6.5 (a)).*

Remark 3.1.3. *Great spheres S_1^2 (Theorem 3.1 a)) are exactly those from the classification of minimal slant surfaces with $K \equiv 1$ in S^6 (see* [13]*, Example 3.1).*

4. Two-dimensional slant orbits in S^6

In this section we consider orbits of two-dimensional subgroup $H \subset G_2 \subset SO(7)$ under the natural action on S^6. Since such action preserves both metric on S^6 and its almost complex structure all points on a fixed orbit have the same Wirtinger angle. Therefore, all such two-dimensional orbits are slant surfaces of S^6.
One can show that two-dimensional subgroup of G_2 is its Cartan subgroup, i.e. maximal tori of G_2. Since any two Cartan subgroups of G_2 are conjugate by some element of G_2, they have the same set of orbits. Therefore it is not a loss of generality if we pick any particular Cartan subgroup $H \subset G_2$ to work with.
Denote by $E_{[i,j]} = \frac{E_{ij}-E_{ji}}{2}$, $i,j = 1,2,...,7,\ i<j$, the standard basis of Lie algebra $\mathfrak{so}(7)$ of $SO(7)$. Then a basis of Lie algebra $\mathfrak{g}_2$ of Lie group G_2 is

$$\begin{array}{ll}
P_0 = E_{[3,2]} + E_{[6,7]}, & Q_0 = E_{[4,5]} + E_{[6,7]},\\
P_1 = E_{[1,3]} + E_{[5,7]}, & Q_1 = E_{[6,4]} + E_{[5,7]},\\
P_2 = E_{[2,1]} + E_{[7,4]}, & Q_2 = E_{[6,5]} + E_{[7,4]},\\
P_3 = E_{[1,4]} + E_{[7,2]}, & Q_3 = E_{[3,6]} + E_{[7,2]},\\
P_4 = E_{[5,1]} + E_{[3,7]}, & Q_4 = E_{[2,6]} + E_{[3,7]},\\
P_5 = E_{[1,7]} + E_{[3,5]}, & Q_5 = E_{[4,2]} + E_{[3,5]},\\
P_6 = E_{[6,1]} + E_{[1,3]}, & Q_6 = E_{[5,2]} + E_{[1,3]},
\end{array}$$

where a Cartan subalgebra $\mathfrak{h}$ is spanned by P_0 and Q_0. The elements of the corresponding group $\mathbb{H} = S^1 \times S^1$ are of the form

$$g_{t,s} = \exp(tP_0 + sQ_0) = (\exp tP_0)(\exp sQ_0), \quad t,s \in \mathbb{R}.$$

It is easy to show that the action of the element $g_{t,s} \in H$ on a point $p = (x_1, x_2, x_3, y_0, y_1, y_2, y_3) \in S^6 \subset \mathbb{R}^7$ is given by

$$\begin{aligned}
(-8)\quad g_{t,s}p \;=\; & (x_1, x_2\cos t - x_3 \sin t, x_3 \cos t + x_2 \sin t,\\
& y_0 \cos s + y_1 \sin s, y_1 \cos s + y_0 \sin s,\\
& y_2 \cos(s-t) - y_3 \sin(s-t), y_3 \cos(s-t) + y_2 \sin(s-t)).
\end{aligned}$$

Note that the action of H preserves x_1 coordinate, so the orbit $\mathcal{O}_p$ of any point p belongs to the hyperplane $x_1 = const$ and therefore to the totally geodesic sphere $S^5 \subset S^6$. One can easily check that the orbit $\mathcal{O}_p$ doesn't belong to any other hyperplane, i.e. the orbit $\mathcal{O}_p$ is not contained in a sphere of smaller dimension.
The tangent space of the orbit $\mathcal{O}_p$ in the point $p = (x_1, x_2, x_3, y_0, y_1, y_2, y_3) \in S^6$ is spanned by the vectors

$$\begin{aligned}
\overline{X} &= \frac{d}{dt}(g_{t,s}p)|_{(t,s)=(0,0)} = (0, -x_3, x_2, 0, 0, y_3, -y_2),\\
(-10)\qquad \overline{Y} &= \frac{d}{ds}(g_{t,s}p)|_{(t,s)=(0,0)} = (0,0,0,-y_1, y_0, -y_3, y_2).
\end{aligned}$$

These two vectors are linearly independent, i.e. the orbit $\mathcal{O}_p$ is two-dimensional if the following conditions are satisfied

$$(-10)\qquad \begin{aligned}
\alpha &= x_2^2 + x_3^2 + y_0^2 + y_1^2 \neq 0,\\
\beta &= x_2^2 + x_3^2 + y_2^2 + y_3^2 \neq 0,\\
\gamma &= y_0^2 + y_1^2 + y_2^2 + y_3^2 \neq 0.
\end{aligned}$$

Note that the corresponding equations represent three two-dimensional spheres on S^6. In the sequel we consider the orbit of a point $p \in S^6$ satisfying relations (-10).

An orthonormal basis of tangent space $T_p\mathcal{O}_p$ reads

$$\begin{aligned} X &= \frac{1}{\sqrt{\beta}}(0,-x_3,x_2,0,0,y_3,-y_2), \\ Y &= \frac{(0,-x_3(y_2^2+y_3^2),x_2(y_2^2+y_3^2),-\beta y_1,\beta y_0,y_3(x_2^2+x_3^2),y_2(x_2^2+x_3^2))}{\sqrt{\beta}\sqrt{\alpha\beta-(x_2^2+x_3^2)^2}}. \end{aligned}$$

Using Lemma 2.2 we get the slant angle of the orbit $\mathcal{O}_p$ in the point p which, as we know, is constant along the orbit

$$cos\theta_p = |\langle X, pY\rangle| = \frac{|x_3y_1y_2 - x_2y_0y_2 - x_2y_1y_3 - x_3y_0y_3|}{\sqrt{\alpha\beta-(x_2^2+x_3^2)^2}}. \tag{-13}$$

Now, we would like to know if we can find an orbit with slant angle θ for any given angle $\theta \in [0, \frac{\pi}{2}]$. From the formulas of the action (-8) it follows that it is sufficient to consider orbits of points satisfying $x_2 = 0 = y_0$, i.e. $p = (x_1, 0, x_3, 0, y_1, y_2, y_3)$. Having in mind the slant angle (4) we parameterize such points p in the following way

$$\begin{aligned} x3 &= \sqrt{1-x_1^2}\sin\theta\cos\varphi, \\ y1 &= \sqrt{1-x_1^2}\sin\varphi, \\ y2 &= \sqrt{1-x_1^2}\cos\theta\sin\phi\cos\varphi, \\ y3 &= \sqrt{1-x_1^2}\cos\theta\cos\phi\cos\varphi, \end{aligned}$$

where $x_1 \in [-1, 1]$, $\theta, \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\phi \in [0, 2\pi)$.
In the new coordinates the slant angle (4) becomes

$$\cos\theta_p = \frac{\sin 2\varphi\, \sin 2\theta}{2\sqrt{4\sin^2\varphi + \cos^2\varphi\,\sin^2 2\theta}}\sqrt{1-x_1^2}\,\sin\phi. \tag{-17}$$

Careful analysis of the above expression shows that the orbits $\mathcal{O}_p$ with minimal slant angle $\theta = \arccos\frac{1}{3}$ correspond tho points

$$p = \pm\frac{1}{\sqrt{3}}(0, 0, 1, 0, 1, 1, 0).$$

It is clear that there also exist points p with totaly real orbits $\mathcal{O}_p$, i.e. with $\cos\theta = 0$. Note that there are no almost complex orbits. By the previous consideration we prove the following theorem.

Theorem 4.1. *Let $p \in S^6$ be a point satisfying relations (-10). The orbit $\mathcal{O}_p$ under the action of Cartan subgroup $H \subset G_2$ on the sphere S^6 is slant torus fully contained in totally geodesic $S^5 \subset S^6$. Its slant angle is given by the formula (4) and takes all values from the interval* $[\arccos\frac{1}{3}, \frac{\pi}{2}]$.

Now, we analyze the geometry of the orbit $\mathcal{O}_p$ as a submanifold of sphere S^6. Starting from the basis (-10) of $T_p\mathcal{O}_p$ one can calculate the induced connection and second fundamental form in the sphere S^6. Then, one can check that the Gausian curvature of the orbit $\mathcal{O}_p$ vanishes. This also trivially follows from Gauss-Bonnet formula and the fact that Gausian curvature is constant along the orbit that is topologically a tori.

One can check that the mean curvature vector of the orbit $\mathcal{O}_p$ in the point $p = (x_1, 0, x_3, 0, y_1, y_2, y_3)$ is given by the formula

$$H(p) = (2x_1, 0, \frac{N(x_3, y_1)}{D}, 0, \frac{N(y_1, x_3)}{D}, y_2(2 - \frac{x_3^2 + y_1^2}{D}), y_3(2 - \frac{x3^2 + y1^2}{D})),$$

where

$$\begin{aligned} D &= (y_1^2 + y_2^2 + y_3^2)x_3^2 + y_1^2(y_2^2 + y_3^2), \\ N(y_1, x_3) &= y_1((2y_1^2 + 2y_2^2 + 2y_3^2 - 1)x_3^2 + (2y_1^2 - 1)(y_2^2 + y_3^2)). \end{aligned}$$

One can easily check that $H \equiv 0$ for all points of the orbit $\mathcal{O}_p$, corresponding to the point

$$p(\phi) = \frac{1}{\sqrt{3}}(0, 0, 1, 0, 1, \cos\phi, \sin\phi), \quad \phi \in [0, 2\pi), \tag{-19}$$

i.e. exactly those orbits are minimal surfaces of S^6.
From the formula (4) we get the slant angle for all minimal orbits

$$\theta = \arccos\frac{\cos\phi}{3}.$$

Therefore, we have the following theorem.

Theorem 4.2. *Let point $p \in S^6$ satisfies conditions (-10). Its orbit $\mathcal{O}_p$ under the action of Cartan subgroup $H \subset G_2$ is flat tori of S^6. There exist one-parameter family of minimal orbits $\mathcal{O}_p$ corresponding to the points (4). For each $\theta \in [\arccos\frac{1}{3}, \frac{\pi}{2}]$ there exist exactly two minimal orbits with slant angle θ.*

Remark 4.2.1. *Totally real minimal orbits are obtained for $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$. They are found in the classification of totally real minimal surfaces of constant curvature (see* [2]*, Theorem 6.5 (c)). Minimal orbits with arbitrary slant angle are those from the classification of minimal, flat, slant surfaces of S^6 (see* [13]*, Example 3.2).*

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