ACTION OF NON ABELIAN GROUP GENERATED BY AFFINE HOMOTHETIES ON $\mathbb{R}^n$

ADLENE AYADI AND YAHYA N'DAO

ABSTRACT. In this paper, we study the action of non abelian group G generated by affine homotheties on $\mathbb{R}^n$. We prove that G satisfies one of the following properties: (i) there exist a subgroup Λ_G of $\mathbb{R}^*$ containing 0 in its closure (i.e. $0 \in \overline{\Lambda_G}$), a G-invariant affine subspace E_G of $\mathbb{R}^n$ and $a \in E_G$ such that $\overline{G(x)} = \overline{\Lambda_G}(x-a) + E_G$ for every $x \in \mathbb{R}^n$. In particular, $\overline{G(x)} = E_G$ for every $x \in E_G$ and every orbit in $U = \mathbb{R}^n \backslash E_G$ is minimal in U. (ii) there exists a closed subgroup H_G of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ we have $\overline{G(x)} = (x + H_G) \cup (-x + a + H_G)$.

1. Introduction

A map $f \ : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called an affine homothety if there exists $\lambda \in \mathbb{R}\backslash\{-1, 0, 1\}$ and $a \in \mathbb{R}^n$ such that $f(x) = \lambda(x-a)+a$ for every $x \in \mathbb{R}^n$. (i.e. $f = T_a \circ (\lambda.id_{\mathbb{R}^n}) \circ T_{-a}$, $T_a : x \longmapsto x + a$). Write $f = (a, \lambda)$ and we call a *the center* of f and λ the ratio of f. Denote by

$$\mathcal{H}(n,\mathbb{R}) := \{\ f : x \ \longmapsto \ \lambda x + a;\ a \in \mathbb{R}^n,\ \lambda \in \mathbb{R}^*\}$$

the affine group generated by all affine homotheties of $\mathbb{R}^n$. We let $\mathcal{T}_n(\mathbb{R})$ the group of translation of $\mathbb{R}^n$. We say a *group of affine homotheties* any subgroup of $\mathcal{H}(n,\mathbb{R})$.

Denote by $\mathcal{S}_n$ the subgroup of $\mathcal{H}(n,\mathbb{R})$ of affine symmetries, i.e.

$$\mathcal{S}_n := \{\ f : x \ \longmapsto \ \varepsilon x + a;\ a \in \mathbb{R}^n,\ \varepsilon \in \{-1, 1\}\}$$

Write $f = (a, \varepsilon)$, for every $f \in \mathcal{S}_n$ defined by $f(x) = \varepsilon x + a$. Under above notation, a map $f = (a, \lambda) \in \mathcal{H}(n,\mathbb{R})$ is either an affine homothety if $\lambda \notin \{-1, 0, 1\}$, and here $a = f(a)$, or an affine symmetry (resp. translation) if $\lambda = -1$ (resp. $\lambda = 1$), and in this case $a = f(0)$.

Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$. There is a natural action $\mathcal{H}(n,\mathbb{R}) \times \mathbb{R}^n : \longrightarrow \ \mathbb{R}^n$. $(f, v) \ \longmapsto \ f(v)$. For a vector $v \in \mathbb{R}^n$, denote by $G(v) := \{f(v) :\ f \in G\} \subset \mathbb{R}^n$ the *orbit* of G through v. A subset $A \subset \mathbb{R}^n$ is called G-*invariant* if $f(A) \subset A$ for any $f \in G$; that is A is a union of orbits

2000 *Mathematics Subject Classification.* 37C85.
Key words and phrases. Homothety, orbit, density, minimal, non abelian, action, dynamic,...
This work is supported by the research unit: systèmes dynamiques et combinatoire: 99UR15-15.

and denote by $\overline{A}$ (resp. $\overset{\circ}{A}$) the closure (resp. interior) of A.
If U is an open G-invariant set, the orbit $G(v) \subset U$ is called *minimal in U* if $\overline{G(v)} \cap U = \overline{G(w)} \cap U$ for every $w \in \overline{G(v)} \cap U$.

We say that F is an *affine subspace* of $\mathbb{R}^n$ with dimension p if $F = E + a$, for some $a \in \mathbb{R}^n$ and some vector subspace E of $\mathbb{R}^n$ with dimension p. For every subset A of $\mathbb{R}^n$, denote by $vect(A)$ the vector subspace of $\mathbb{R}^n$ generated by all elements of A.

Denote by:
- $id_{\mathbb{R}^n}$ the identity map of $\mathbb{R}^n$.
- $\Lambda_G := \{\lambda : \ f = (a, \lambda) \in G\}$. It is obvious that Λ_G is a subgroup of $\mathbb{R}^*$ (see Lemma 2.1).
- $\mathrm{Fix}(f) := \{x \in \mathbb{R}^n : \ f(x) = x\}$, for every $f \in \mathcal{H}(n, \mathbb{R})$.
- $\Gamma_G := \begin{cases} \bigcup\limits_{f \in G\backslash \mathcal{S}_n} \mathrm{Fix}(f), & \text{if } \ G\backslash \mathcal{S}_n \neq \emptyset. \\ \emptyset, & \text{if } \ G \subset \mathcal{S}_n \end{cases}$
- $G_1 := G \cap \mathcal{T}_n$, we have G_1 is a subgroup of $\mathcal{T}_n$.
- $H_G = G_1(0)$, we have H_G is an additif subgroup of $\mathbb{R}^n$.
- $\gamma_G := \{f(0), \ \ f \in G \cap \mathcal{S}_n\}$.
- $\Omega_G := \Gamma_G \cup \gamma_G$.
- $\delta_G := \{f(0), \ \ f \in G \cap (\mathcal{S}_n \backslash \mathcal{T}_n)\}$.
- $E_G = Aff(\Omega_G)$ the smaller affine subspace of $\mathbb{R}^n$ containing Ω_G.

Remark that $\Omega_G \neq \emptyset$, since $\Gamma_G \neq \emptyset$ or $\gamma_G \neq \emptyset$, and so $E_G \neq \emptyset$.

We describe here, closure of all orbit defined by action of non abelian subgroups of $\mathcal{H}(n, \mathbb{R})$. We distinct two considerable states. When $G\backslash \mathcal{S}_n \neq \emptyset$, this means that G contains an affine homothety different to a symmetry (i.e. its homothety ratio λ has a modulus $|\lambda| \neq 1$). In this case we prove that closure of any orbit is an affine subspace of $\mathbb{R}^n$ or it is union of countable affine subspaces of $\mathbb{R}^n$. As consequence, we deduce that G has a minimal set in $\mathbb{R}^n$, which is contained in closure of all orbit.

In the other state, G is a non abelian subgroup of $\mathcal{S}_n$, then it contains necessarily an affine symmetry $f \in \mathcal{S}_n \backslash \mathcal{T}_n$. In this case we prove that closure of any orbit of G is union of at most two closed subgroups of $\mathbb{R}^n$. As consequence, we deduce that every orbit is minimal in $\mathbb{R}^n$.

I learned that Zhukova have, independently proved in [1] similar results to Lemma 3.4, Proposition 3.6 and Corollary 1.2.(ii). The methods of proof in [1] and in this paper are quite different and have different consequences.

For $n = 1$, we prove that action for every non abelian affine group is minimal (i.e. all orbits of G are dense in $\mathbb{R}$). In [2] and [3], the authors are interested to the semigroup case, in [3], Mohamed Javaheri has proved a strong density results for the orbits of real numbers under the action of the semigroup generated by the affine transformations $T_0(x) = \frac{x}{a}$ and $T_1(x) = bx+1$, where $a, b > 1$. These density results are formulated as generalizations of the Dirichlet approximation theorem and improve the results of Bergelson, Misiurewicz, and Senti in [2].

Our principal results can be stated as follows:

Theorem 1.1. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$. One has:*
(1) *If $G\backslash\mathcal{S}_n \neq \emptyset$, then:*

- (i) $0 \in \overline{\Lambda_G}$ *and E_G is a G-invariant affine subspace of $\mathbb{R}^n$ with dimension $p \geq 1$.*
- (ii) $\overline{G(x)} = E_G$, *for every $x \in E_G$.*
- (iii) *there exists $a \in E_G$ such that $\overline{G(x)} = \overline{\Lambda_G}(x-a) + E_G$, for every $x \in U = \mathbb{R}^n\backslash E_G$.*

(2) *If $G \subset S_n$, then H_G is a closed subgroup of $\mathbb{R}^n$ and there exists $a \in \mathbb{R}^n$ such that $\overline{G(x)} = (x + H_G) \cup (-x + a + H_G)$, for every $x \in \mathbb{R}^n$.*

Corollary 1.2. *Under notations of Theorem 1.1. If $G\backslash\mathcal{S}_n \neq \emptyset$, then:*

- (i) *Every orbit in U is minimal in U.*
- (ii) *E_G is a minimal set of G in $\mathbb{R}^n$ contained in the closure of every orbit of G.*
- (iii) *All orbit in U are homeomorphic.*

Corollary 1.3. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$. Then:*

- (i) *If $G\backslash\mathcal{S}_n \neq \emptyset$, then G has no periodic orbit. Moreover, if G is countable then it has no closed orbit.*
- (ii) *If $G \subset \mathcal{S}_n$, then every orbit of G is minimal in $\mathbb{R}^n$.*

Corollary 1.4. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$. Then the following assertions are equivalents:*

- (1) *G has a dense orbit in in $\mathbb{R}^n$.*
- (2) *Every orbit of U is dense in $\mathbb{R}^n$.*
- (3) *G satisfies one of the following:*
  - (i) $E_G = \mathbb{R}^n$
  - (ii) $dim(E_G) = n-1$ *and* $\overline{\Lambda_G} = \mathbb{R}$.

Remark 1.5. Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$.

- (i) Suppose that $dim(E_G) = n-1$ and there exist $\lambda, \mu \in \Lambda_G$ such that $\lambda\mu < 0$ and $\frac{log|\lambda|}{log|\mu|} \notin \mathbb{Q}$, then G has a dense orbit. (Indeed, in Lemma 3.12 we will prove that $\overline{\Lambda_G} = \mathbb{R}$ and we apply Corollary 1.4,(3).(ii)).

(ii) If $\overline{\Lambda_G} = \mathbb{R}$ and $dim(E_G) < n-1$, then by Theorem 1.1.(ii), G has no dense orbit and every orbit of U is dense in an affine subspace of $\mathbb{R}^n$ with dimension $dim(E_G)+1$.

Corollary 1.6. *Let G be a non abelian subgroup of $\mathcal{S}_n$. Then the following assertions are equivalent:*

(i) *G has a dense orbit.*
(ii) *Every orbit of G has a dense orbit.*
(iii) *The orbit $G(0)$ is dense in $\mathbb{R}^n$.*
(iv) *H_G is dense in $\mathbb{R}^n$.*

Remark 1.7. Let $\mathcal{H}_n = \{\ f : x \ \longmapsto\ \alpha(x-a)+a;\ a \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^*\}$ be the set of all affine homotheties of $\mathbb{R}^n$. Then:

(i) $\mathcal{H}(n,\mathbb{R}) = \mathcal{H}_n \cup \mathcal{T}_n(\mathbb{R})$.
(ii) $\mathcal{H}_n$ is not a group. (Indeed; $\mathcal{H}_n \cap \mathcal{T}_n(\mathbb{R}) = \{id_{\mathbb{R}^n}\}$. For $f = (a,2)$ and $g = \left(2a, \frac{1}{2}\right)$ one has $f \circ g = T_a \in \mathcal{T}_n(\mathbb{R})$, with $T_a : x \longmapsto x+a$.)
(iii) There exists a subgroup of $\mathcal{S}_n$, having two orbits non homeomorphic. (See example 6.4).

For $n=1$, remark that any subgroup of $\mathcal{H}(1,\mathbb{R})$ is a group of affine maps of $\mathbb{R}$. As consequence for Theorem 1.1, we establish the following strong result:

Corollary 1.8. *Let G be a non abelian group of affine maps of $\mathbb{R}$.*

(i) *If $G\backslash\mathcal{S}_1 \neq \emptyset$ then every orbit of G is dense in $\mathbb{R}$.*
(ii) *If $G \subset \mathcal{S}_1$ then all orbits of G are dense in $\mathbb{R}$ or all orbits are closed and discrete.*

This paper is organized as follows: In Section 2, we introduce some preliminaries Lemmas. Section 3 is devoted to given some results in the case $G\backslash\mathcal{S}_n \neq \emptyset$. Results in the case when G is a subgroup of $\mathcal{S}_n$ are given in Section 4. In Section 5, we prove Theorem 1.1, Corollaries 1.2, 1.3, 1.4, 1.6 and 1.8. In Section 7, we give four examples.

2. **Preliminaries Lemmas**

Recall that $\text{Fix}(f) := \{x \in \mathbb{R}^n :\ f(x) = x\}$, for every $f \in \mathcal{H}(n,\mathbb{R})$.
So

$$\text{Fix}(f) := \begin{cases} \emptyset, & \text{if } \ f \in \mathcal{T}_n \\ \left\{\frac{a}{2}\right\}, & \text{if } \ f = (a,\varepsilon) \in \mathcal{S}_n\backslash\mathcal{T}_n \\ \{a\}, & \text{if } \ f = (a,\lambda) \in \mathcal{H}(n,\mathbb{R})\backslash\mathcal{S}_n \end{cases}$$

Lemma 2.1. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$. The set Λ_G is a subgroup of $\mathbb{R}^*$. Moreover, if $G\backslash\mathcal{S}_n \neq \emptyset$, then $0 \in \overline{\Lambda_G}$.*

Proof. Since $id_{\mathbb{R}^n} \in G$, so $1 \in \Lambda_G$. Let $\lambda, \mu \in \Lambda_G$ and $f, g \in G$ defined by $f : x \longmapsto \lambda x + a$, and $g : x \longmapsto \mu x + b$, $x \in \mathbb{R}^n$, so $f \circ g^{-1}(x) = f\left(\frac{x}{\mu} - \frac{b}{\mu}\right) = \frac{\lambda}{\mu}x - \frac{\lambda b}{\mu} + a$. Hence $\frac{\lambda}{\mu} \in \Lambda_G$. Moreover, if $G\backslash\mathcal{S}_n \neq \emptyset$, $\Gamma_G\backslash\{-1, 1\} \neq \emptyset$. So $\underset{m\to\pm\infty}{lim} \lambda^m = 0$, for any $\lambda \in \Gamma_G$. It follows that $0 \in \overline{\Lambda_G}$. This proves the Lemma. □

Lemma 2.2.

(i) *Let $f = (a, \alpha)$, $g = (b, \beta) \in \mathcal{H}(n, \mathbb{R})\backslash\mathcal{S}_n$ then $f \circ g = g \circ f$ if and only if $a = b$ or $\alpha = 1$ or $\beta = 1$.*

(ii) *If $Fix(f) = Fix(g)$ then $f \circ g = g \circ f$.*

(iii) *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$, then there exist $f = (a, \alpha)$, $g = (b, \beta) \in G$ such that $a \neq b$. Moreover, there exist $a, b \in \Gamma_G$ such that $a \neq b$.*

Proof. (i) If $f \circ g(x) = g \circ f(x)$, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \text{then} \qquad \lambda(\mu(x-b)+b-a)+a &= \mu(\lambda(x-a)+a-b)+b, \\ \text{so} \qquad -\lambda\mu b+\lambda(b-a)+a &= -\mu\lambda a+\mu(a-b)+b, \end{aligned}$$

thus $(a - b)(\lambda\mu - \lambda - \mu + 1) = 0$. Hence $(a - b)(\lambda - 1)(\mu - 1) = 0$. This proves the lemma.

(ii) There are two cases:
• If $Fix(f) = Fix(g) = \emptyset$ then $f = T_a$ and $g = T_b$ for some $a, b \in \mathbb{R}^n$, so $f \circ g = g \circ f$.
• If $Fix(f) = Fix(g) = a \neq \emptyset$, so $f, g \in \mathcal{H}(n, \mathbb{R})\backslash\mathcal{T}_n$, there are four cases:
- $f = (a, \lambda)$, $g = (a, \mu) \in \mathcal{H}(n, \mathbb{R})\backslash\mathcal{S}_n$, then $f \circ g(x) = \lambda(\mu(x-a)+a-a)+a = \lambda\mu(x - a) + a = g \circ f(x)$, for every $x \in \mathbb{R}^n$.
-$f = g = (2a, -1) \in \mathcal{S}_n\backslash\mathcal{T}_n$, so $f \circ g = g \circ f$.
-$f(2a, -1) \in \mathcal{S}_n\backslash\mathcal{T}_n$ and $g = (a, \mu) \in \mathcal{H}(n, \mathbb{R})\backslash\mathcal{S}_n$, so for every $x \in \mathbb{R}^n$,

$$\begin{aligned} & f \circ g(x) = -(\mu(x-a)+a)+2a \\ \text{and} \qquad & g \circ f(x) = \mu(-x+2a-a)+a \\ \text{so} \qquad & f \circ g(x) = g \circ f(x) = -\mu x+(1+\mu)a \end{aligned}$$

hence $f \circ g = g \circ f$.
- $f = (a, \lambda) \in \mathcal{H}(n, \mathbb{R})\backslash\mathcal{S}_n$ and $g(2a, -1) \in \mathcal{S}_n\backslash\mathcal{T}_n$, so as above $f \circ g = g \circ f$.
This completes the proof.

(iii) Since G is non abelian then the proof of (iii) results from (ii).
Moreover, we have $\Gamma_G \neq \emptyset$ and let $h = (c, \lambda) \in G\backslash\mathcal{S}_n$. Since G is non abelian then there exists $h' \in G$ such that $h \circ h' \neq h' \circ h$. By (ii), $Fix(h) \neq Fix(h')$,

so $h'(c) \neq c$. Then $h' \circ h \circ h'^{-1} = (h'(c), \lambda) \in G\backslash\mathcal{S}_n$, hence $h'(c), c \in \Gamma_G$. □

Lemma 2.3. *Let $\mathcal{B} = (a_1, \dots, a_n)$ be a basis of $\mathbb{R}^n$. Then the smaller affine subspace $\mathcal{A}ff(\mathcal{B})$ of $\mathbb{R}^n$ containing $\{a_1, \dots, a_n\}$ is defined by*

$$\mathcal{A}ff(\mathcal{B}) := \left\{ x = \sum_{k=1}^{n} \alpha_k a_k : \ \alpha_k \in \mathbb{R}, \ \sum_{k=1}^{n} \alpha_k = 1 \right\}.$$

Proof. Let $E := T_{-a_1}(\mathcal{A}ff(\mathcal{B}))$. Then E is a vector subspace of $\mathbb{R}^n$ generated by $\{a_2 - a_1, \dots, a_n - a_1\}$. Since E is the smaller vector space containing $0, a_2 - a_1, \dots, a_n - a_1$, so $T_{a_1}(E)$ is the smaller affine subspace of $\mathbb{R}^n$ containing $\{a_1, \dots, a_n\}$ □

Remark 2.4. As consequence of Lemma 2.3, if E_G contains $a, a_1, \dots, a_n$ such that $(a_1, \dots, a_n)$ is a basis of $\mathbb{R}^n$, and $a = \sum\limits_{k=1}^{n} \alpha_k a_k$ with $\sum\limits_{k=1}^{n} \alpha_k \neq 1$. Then $E_G = \mathbb{R}^n$.

Lemma 2.5. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$. Then for every $x \in \mathbb{R}^n$ we have $\Gamma_G \subset \overline{G(x)}$.*

Proof. Let $x \in \mathbb{R}^n$ and $a \in \Gamma_G$. Since $G\backslash\mathcal{S}_n \neq \emptyset$ then there exists $f = (a, \lambda) \in G\backslash\mathcal{S}_n$, so $|\lambda| \neq 1$. Suppose that $|\lambda| > 1$ and so

$$\lim_{k \longrightarrow -\infty} f^k(x) = \lim_{k \longrightarrow -\infty} \lambda^k (x - a) + a = a.$$

Hence $a \in \overline{G(x)}$. It follows that $\Gamma_G \subset \overline{G(x)}$. □

Lemma 2.6. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$. Then:*

(i) *if E_G is a vector space, there exist $a_1, \dots, a_p \in \Gamma_G$ such that $(a_1, \dots, a_p)$ is a basis of E_G.*

(ii) *if $G' = T_{-a} \circ G \circ T_a$ for some $a \in \Gamma_G$, then $E_{G'} = T_{-a}(E_G)$ and $\Lambda_{G'} = \Lambda_G$.*

Proof. (i) Since E_G is a vector subspace of $\mathbb{R}^n$ with dimension p, so $E_G = vect(\Omega_G)$ where $\Omega_G = \Gamma_G \cup \gamma_G$. As $G\backslash\mathcal{S}_n \neq \emptyset$ then $\Gamma_G \neq \emptyset$. Let $a_1, \dots, a_k \in \Gamma_G$ and $b_{k+1}, \dots, b_p \in \gamma_G$ such that $\mathcal{B}_1 = (a_1, \dots, a_k, b_{k+1}, \dots, b_p)$ is a basis of E_G and $(a_1, \dots, a_k)$ is a basis of $vect(\Gamma_G)$. For every $k+1 \leq i \leq p$ there exists $g_i \in \mathcal{S}_n \cap G$ such that $g_i(0) = b_i$. Since $a_1 \in \Gamma_G$ then there exists $f \in G\backslash\mathcal{S}_n$ with $f = (a_1, \lambda)$. Write $f_i = g_i \circ f \circ g_i^{-1}$, for every $k+1 \leq i \leq p$. We have $f_i = (g_i(a_1), \lambda) \in G\backslash\mathcal{S}_n$. See that $g_i(a_1) = \varepsilon_i a_1 + b_i$, with $\varepsilon_i \in \{-1, 1\}$,

$k+1 \leq i \leq p$, so $g_i(a_1) \in \Gamma_G$, for every $k+1 \leq i \leq p$.

Let's show that $\mathcal{B}_2 = (a_1, \dots, a_k, g_{k+1}(a_1), \dots, g_p(a_1))$ is a basis of E_G:
Let

$$M = \begin{bmatrix} I_k & A \\ 0 & I_{p-k} \end{bmatrix}, \quad \text{with} \quad A = \begin{bmatrix} \varepsilon_{k+1} & \dots & \varepsilon_p \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix},$$

and I_k, I_{p-k} are respectively the identity matrix of $M_k(\mathbb{R})$ and $M_{p-k}(\mathbb{R})$. So M is invertible and $M(\mathcal{B}_1) = \mathcal{B}_2$. So $\mathcal{B}_2$ is a basis of E_G contained in Γ_G, a contradiction. We conclude that $k = p$.

(ii) Second, suppose that E_G is an affine subspace of $\mathbb{R}^n$ with dimension p. Let $a \in \Gamma_G$ and $G' = T_{-a} \circ G \circ T_a$. Set $f = (a, \lambda) \in G \backslash \mathcal{S}_n$, then $T_{-a} \circ f \circ T_a = (0, \lambda) \in G' \backslash \mathcal{S}_n$, so $0 \in \Gamma_{G'} \subset E_{G'}$, hence $E_{G'}$ is a vector space. By (i) there exists a basis $(a'_1, \dots, a'_p)$ of $E_{G'}$ contained in $\Gamma_{G'}$. Since $\Gamma_{G'} = T_{-a}(\Gamma_G)$, we let $a_k = T_a(a'_k)$, $1 \leq k \leq p$, so $a_1, \dots, a_p \in \Gamma_G$. We have $\Gamma_{G'} = T_{-a}(\Gamma_G) \subset T_{-a}(E_G)$ and $T_{-a}(E_G)$ is a vector subspace of $\mathbb{R}^n$ with dimension p, containing $a'_1, \dots, a'_p$. So $E_{G'} = T_{-a}(E_G)$.
See that for every $f = (b, \lambda) \in G \backslash \mathcal{S}_n$, $T_{-a} \circ f \circ T_a = (b - a, \lambda)$, so $\Lambda_{G'} = \Lambda_G$. □

Lemma 2.7. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G \backslash \mathcal{S}_n \neq \emptyset$ and E_G is a vector space. If $a_1, ..., a_p \in \Gamma_G$ such that $\mathcal{B}_1 = (a_1, ..., a_p)$ is a bases of E_G then there exists $a \in \Gamma_G$ such that $\mathcal{B}_2 = (a_1 - a, ..., a_p - a)$ is also a basis of E_G.*

Proof. Since $a_{p-1} \in \Gamma_G$, then there exists $\lambda \in \Lambda_G \backslash \{-1, 1\}$ such that $f_{p-1} = (a_{p-1}, \lambda) \in G \backslash \mathcal{S}_n$. Let $a = f_{p-1}(a_p) = \lambda(a_p - a_{p-1}) + a_{p-1}$. Then $a = \lambda a_p + (1 - \lambda) a_{p-1}$. By Lemma 3.1.(ii), Γ_G is G-invariant, so $a \in \Gamma_G$.
We have $P(\mathcal{B}_1) = \mathcal{B}_2$ where

$$P = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ \lambda - 1 & \dots & \dots & \lambda - 1 & \lambda & \lambda - 1 \\ -\lambda & \dots & \dots & -\lambda & \lambda & 1 - \lambda \end{bmatrix}.$$

Since $\det(P) = 2\lambda(1 - \lambda) \neq 0$ then P is invertible and so $\mathcal{B}_2$ is a basis of $\mathbb{R}^n$. □

Lemma 2.8. *Let G be the subgroup of $\mathcal{H}(n,\mathbb{R})$ generated by $f_1=(a_1,\lambda_1),\dots,f_p=(a_p,\lambda_p)\in\mathcal{H}(n,\mathbb{R})\backslash\mathcal{S}_n$. Then $E_G=\mathcal{A}ff(\{a_1,\dots,a_p\})$.*

Proof. Since $E_G=Aff(\Omega_G)$, it suffices to show that $\Omega_G\subset Aff(\{a_1,\dots,a_p\})$.

(i) First, suppose that E_G is a vector space. We will prove that $\Omega_G\subset\mathrm{vect}(\{a_1,\dots,a_p\})$:
Let $f\in G$, so $f=f_{i_1}^{n_1}\circ\dots\circ f_{i_q}^{n_q}$ for some $q\in\mathbb{N}^*$, $n_1,\dots,n_q\in\mathbb{Z}$ and $i_1,\dots,i_q\in\{1,\dots,p\}$. So for every $x\in\mathbb{R}^n$,

$$\begin{aligned}
f(x) &= f_{i_1}^{n_1}\circ\dots\circ f_{i_q}^{n_q}(x)\\
&= \lambda_{i_1}^{n_1}(\lambda_{i_2}^{n_2}(\dots(\lambda_{i_{q-1}}^{n_{q-1}}(\lambda_{i_q}^{n_q}(x-a_{i_q})+a_{i_q}-a_{i_{q-1}})+a_{i_{q-1}}\dots)\dots-a_{i_1})+a_{i_1}\\
&= \lambda_{i_1}^{n_1}\dots\lambda_{i_q}^{n_q}(x-a_{i_q})+\sum_{k=1}^{q-1}\lambda_{i_1}^{n_1}\dots\lambda_{i_k}^{n_k}(a_{i_{k+1}}-a_{i_k})+a_{i_1}\\
&= \lambda_{i_1}^{n_1}\dots\lambda_{i_q}^{n_q}x+\left(\sum_{k=1}^{q-1}\lambda_{i_1}^{n_1}\dots\lambda_{i_k}^{n_k}a_{i_{k+1}}-\sum_{k=1}^{q}\lambda_{i_1}^{n_1}\dots\lambda_{i_k}^{n_k}a_{i_k}+a_{i_1}\right)\\
&= \lambda x+a
\end{aligned}$$

where

$$(1)\qquad \begin{cases}\lambda=\lambda_{i_1}^{n_1}\dots\lambda_{i_q}^{n_q}\\ a=\sum\limits_{k=1}^{q-1}\lambda_{i_1}^{n_1}\dots\lambda_{i_k}^{n_k}a_{i_{k+1}}-\sum\limits_{k=1}^{q}\lambda_{i_1}^{n_1}\dots\lambda_{i_k}^{n_k}a_{i_k}+a_{i_1}\end{cases}$$

- If $|\lambda|\neq 1$, then $f(x)=\lambda x+a=\lambda\left(x-\frac{a}{1-\lambda}\right)+\frac{a}{1-\lambda}$, $x\in\mathbb{R}^n$, so $f=\left(\frac{a}{1-\lambda},\ \lambda\right)$, with $\frac{a}{1-\lambda}\in\mathrm{vect}(\{a_1,\dots,a_p\})$, so $\Gamma_G\subset\mathrm{vect}(a_1,\dots,a_p)$.
- If $|\lambda|=1$, then $f=\left(a,\ \frac{\lambda}{|\lambda|}\right)$, with $a=f(0)\in\mathrm{vect}(\{a_1,\dots,a_p\})$, so $\gamma_G\subset\mathrm{vect}(a_1,\dots,a_p)$.
It follows that $\Omega_G\subset\mathrm{vect}(\{a_1,\dots,a_p\})$.

(ii) Second, suppose that E_G is an affine space. We will prove that $\Omega_G\subset\mathcal{A}ff(\{a_1,\dots,a_p\})$: Let $G'=T_{-a_1}\circ G\circ T_{a_1}$, so G' is generated by $f'_k=T_{-a_1}\circ f_k\circ T_{a_1}=(a_k-a_1,\lambda_k)$, $k=1,\dots,p$. By Lemma 3.12.(ii), $E_{G'}=T_{-a_1}(E_G)$ is a vector space and by (i) we have $E_{G'}\subset vect(\{a_2-a_1,\dots,a_p-a_1\})$, so $E_G=T_{a_1}(E_{G'})\subset T_{a_1}(\mathrm{vect}(\{a_2-a_1,\dots,a_p-a_1\}))=\mathcal{A}ff(\{a_1,a_2,\dots,a_p\})$, so the proof is complete. □

3. Some results in the case $G\backslash\mathcal{S}_n\neq\emptyset$

In this case, G contains an affine homothety having a ratio with module different to 1. In the following, we give some Lemmas and propositions, will be used to prove Theorem 1.1.

Lemma 3.1. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$, then:*

(i) *If E_G is a vector subspace of $\mathbb{R}^n$, then $\{f(0), f\in G\}\subset E_G$.*
(ii) *If $\Gamma_G\neq\emptyset$, then Γ_G and E_G are G-invariant.*

Proof. (i) Let $f\in G$, there are two cases:
- If $f\in G\backslash\mathcal{S}_n$, then $f=(a,\lambda)$, for some $\lambda\in\Lambda_G$ and $a\in\Gamma_G\subset E_G$. Therefore $f(x)=\lambda(x-a)+a$, $x\in\mathbb{R}^n$ and $f(0)=(1-\lambda)a$, so $f(0)\in E_G$ since E_G is a vector space.
- If $f\in\mathcal{S}_n$, then $f=(a,\varepsilon)$, with $|\varepsilon|=1$ and so $f(0)=a\in E_G$.

(ii) Suppose that $\Gamma_G\neq\emptyset$:
Γ_G is G-invariant: Let $a\in\Gamma_G$ and $g\in G$ then there exists $\lambda\in\mathbb{R}\backslash\{-1,1\}$ such that $f=(a,\lambda)\in G\backslash S_n$. We let $h=g\circ f\circ g^{-1}\in G$. We obtain $h=(a',\lambda)\in G\backslash S_n$ with $a'=g(a)$. It follows that $g(a)\in\Gamma_G$ and so Γ_G is G-invariant.
E_G is G-invariant: Let $a\in\Gamma_G$ and $G'=T_{-a}\circ G\circ T_a$. We have G' is a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ and $E_{G'}=T_{-a}(E_G)$ is a vector subspace of $\mathbb{R}^n$. Let $f\in G'$ having the form $f(x)=\lambda x+b$, $x\in\mathbb{R}^n$. By (i), $b=f(0)\in\Gamma_{G'}\subset E_{G'}$. So for every $x\in E_{G'}$, $f(x)\in E_{G'}$, hence $E_{G'}$ is G'-invariant. By Lemma 2.6.(ii) one has $E_G=T_{-a}(E_{G'})$, so it is G-invariant. □

Lemma 3.2. *Let $\lambda>1$ and $H^{\lambda}:=\{q\lambda^p(1-\lambda^p),\ \ p,q\in\mathbb{Z}\}$. Then H^{λ} is dense in $\mathbb{R}$.*

Proof. Let $x,y\in\mathbb{R}^*_+$, such that $x<y$. Since $\lim\limits_{p\longrightarrow-\infty}\frac{y-x}{\lambda^p(1-\lambda^p)}=+\infty$ then there exists $p\in\mathbb{Z}^*_-$ such that $\frac{y-x}{\lambda^p(1-\lambda^p)}>1$. Therefore there exists $q\in\mathbb{Z}$ such that $\frac{x}{\lambda^p(1-\lambda^p)}<q<\frac{y}{\lambda^p(1-\lambda^p)}$. Since $\lambda>1$ and $p\neq0$ then $1-\lambda^p>0$ and so $x<q\lambda^p(1-\lambda^p)<y$ and $-y<-q\lambda^p(1-\lambda^p)<-x$. Hence $\mathbb{R}^*_+\subset\overline{H^{\lambda}}$ and $\mathbb{R}^*_-\subset\overline{H^{\lambda}}$. It follows that H^{λ} is dense in $\mathbb{R}$. □

Lemma 3.3. *Let $\lambda>1$, $a\in\mathbb{R}^n\backslash\{0\}$ and $H^{\lambda}_a:=\{q\lambda^p(1-\lambda^p)a+a,\ \ p,q\in\mathbb{Z}\}$. If G is the group generated by $f=(a,\lambda)\in\mathcal{H}(n,\mathbb{R})\backslash\mathcal{S}_n$ and $h=\lambda.id_{\mathbb{R}^n}$, then $H^{\lambda}_a\subset G(a)\subset\mathbb{R}a$. Moreover, $T_{(1-\lambda^m)a}\in G$, for every $m\in\mathbb{Z}$.*

Proof. Let $p, q \in \mathbb{Z}$. For every $z \in \mathbb{R}^n$ we have

$$h^{-p} \circ f^p(z) = \lambda^{-p}(\lambda^p(z-a)+a) = z + (\lambda^{-p}-1)a, \qquad (1)$$

$$\text{so} \quad \left(h^{-p} \circ f^p\right)^{q-1}(z) = z + (q-1)(\lambda^{-p}-1)a.$$

For $z = h^{-p}(a)$ we have

$$\left(h^{-p} \circ f^p\right)^{q-1}(h^{-p}(a)) = \lambda^{-p}a + (q-1)(\lambda^{-p}-1)a = q\lambda^{-p}a - (q-1)a.$$

Then

$$f^{2p} \circ \left(h^{-p} \circ f^p\right)^{q-1} \circ h^{-p}(a) = \lambda^{2p}\left(q\lambda^{-p}a - (q-1)a - a\right) + a.$$

$$= q\lambda^p(1-\lambda^p)a + a$$

It follows that $q\lambda^p(1-\lambda^p)a + a \in G(a)$ and so $H_a^\lambda \subset G(a)$. Since $h(\mathbb{R}a) = \mathbb{R}a$ and $f(\alpha a) = \lambda(\alpha a - a) + a = (\lambda\alpha - \lambda + 1)a$ then $\mathbb{R}a$ is G-invariant, so $G(a) \subset \mathbb{R}a$.
Moreover, by taking $p = -m$ in (1), for some $m \in \mathbb{Z}$, we obtain

$$h^m \circ f^m(z) = z + (\lambda^m - 1)a, \quad z \in \mathbb{R}^n.$$

Then $T_{(\lambda^m-1)a} \in G$ and so $T_{(1-\lambda^m)a} = T^{-1}_{(\lambda^m-1)a} \in G$. The proof is complete. □

Lemma 3.4. *Let $\lambda > 1$, $a, b \in \mathbb{R}^n$ with $a \neq b$. If G is the group generated by $f = (a, \lambda)$ and $g = (b, \lambda)$ then $\overline{G(a)} = \mathbb{R}(b-a) + a$.*

Proof. Let $\lambda > 1$, $a, b \in \mathbb{R}^n$ with $a \neq b$ and G be the group generated by $f = (a, \lambda)$ and $g = (b, \lambda)$. Denote by $G' = T_{-b} \circ G \circ T_b$, then G' is a subgroup of $\mathcal{H}(n, \mathbb{R})$ and it is generated by $h = T_{-b} \circ f \circ T_b$ and $g' = T_{-b} \circ g \circ T_b$. We obtain $h = \lambda.id_{\mathbb{R}^n}$ and $g' = (a-b, \lambda)$.

Since $\lambda > 1$, $g' \in G' \backslash \mathcal{S}_n$, so by Lemma 3.3, we have $H^\lambda_{a-b} \subset G'(a-b)$, where $H^\lambda_{a-b} := \{q\lambda^p(1-\lambda^p)(a-b) + a - b,\ p, q \in \mathbb{Z}\}$. Since $a - b \neq 0$ then H^λ_{a-b} and $H^\lambda := \{q\lambda^p(1-\lambda^p),\ p, q \in \mathbb{Z}\}$ are homeomorphic. By Lemma 3.2, we have H^λ is dense in $\mathbb{R}$ so H^λ_{a-b} is dense in $\mathbb{R}(a-b)$. Since $H^\lambda_{a-b} \subset G'(a-b)$ and by Lemma 3.3, $G'(a-b) \subset \mathbb{R}(a-b)$, so $\overline{G'(a-b)} = \mathbb{R}(a-b)$. We conclude that $\overline{G(a)} = T_b(\mathbb{R}(a-b)) = \mathbb{R}(a-b) + b$. As $\mathbb{R}(a-b) + b = \mathbb{R}(a-b) + (a-b) + b = \mathbb{R}(b-a) + a$. It follows that $\overline{G(a)} = \mathbb{R}(b-a) + a$. □

Lemma 3.5. *Let $\lambda > 1$, $\mu \in \mathbb{R}\backslash\{0, 1\}$ $a, b \in \mathbb{R}^n$ with $a \neq b$. If G is the group generated by $f = (a, \lambda)$ and $g = (b, \mu)$ then $\overline{G(a)} = \mathbb{R}(b-a) + a$.*

Proof. Let $\lambda > 1$, $\mu \in \mathbb{R}\backslash\{0,1\}$ $a, b \in \mathbb{R}^n$ with $a \neq b$ and G is the group generated by $f = (a, \lambda)$ and $g = (b, \mu)$, then by Lemma 2.2.(i), G is non abelian.
(i) First, we will show that $\mathbb{R}(b-a)+a$ is G-invariant:
Let $\alpha \in \mathbb{R}$, and $x = \alpha(b-a)+a$ we have

$$\begin{aligned} f(x) &= \lambda(\alpha(b-a)+a-a)+a \\ &= \lambda\alpha(b-a)+a \end{aligned}$$

and

$$\begin{aligned} f(x) &= \mu(\alpha(b-a)+a-b)+b \\ &= \mu(\alpha-1)(b-a)+b-a+a. \\ &= (1+\mu(\alpha-1))(b-a)+a \end{aligned}$$

So $f(x),\ g(x) \in \mathbb{R}(b-a)+a$.

(ii) Second, we let $g' = g \circ f \circ g^{-1}$, we have $g' = (g(a), \lambda) \in G\backslash\mathcal{S}_n$. Since $a \neq b$ and $\mu \neq 1$ then

$$g(a) - a = \mu(a-b) + b - a = (1-\mu)(b-a) \neq 0.$$

If G' is the subgroup of G generated by f and g' then by Lemma 3.4 we have $\overline{G'(a)} = \mathbb{R}(g(a)-a)+a$. Since $g(a) = \mu(a-b)+b$ then

$$\begin{aligned} \mathbb{R}(g(a)-a)+a &= \mathbb{R}(\mu(a-b)+b-a)+a \\ &= \mathbb{R}(1-\mu)(b-a)+a \\ &= \mathbb{R}(b-a)+a. \end{aligned}$$

By (i), we have $\mathbb{R}(b-a)+a$ is G-invariant so $\overline{G'(a)} \subset \overline{G(a)} \subset \mathbb{R}(b-a)+a$, hence $\overline{G(a)} = \mathbb{R}(b-a)+a$. □

Proposition 3.6. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$. Then for every $x \in E_G$, we have $\overline{G(x)} = E_G$.*

To prove the above Proposition, we need the following Lemmas:

Lemma 3.7. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$. Let $f \in G$, $u, v \in \mathbb{R}^n$ then $f(\mathbb{R}u+v) = \mathbb{R}u + f(v)$.*

Proof. Every $f \in G$ has the form $f(x) = \lambda x + a$, $x \in \mathbb{R}^n$. Let $\alpha \in \mathbb{R}$ then $f(\alpha u + v) = \lambda(\alpha u + v) + a = \lambda\alpha u + (\lambda u + v) = \lambda\alpha u + f(v)$. So $f(\mathbb{R}u+v) = \mathbb{R}u + f(v)$. □

Lemma 3.8. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ such that E_G is a vector subspace of $\mathbb{R}^n$ and $\Gamma_G \neq \emptyset$. Let $a, a_1, \dots, a_p \in \Gamma_G$ such that $(a_1, \dots, a_p)$ and $(a_1 - a, \dots, a_p - a)$ are two basis of E_G and let $D_k = \mathbb{R}(a_k - a) + a$, $1 \leq k \leq p$. If $D_k \subset \overline{G(a)}$ for every $1 \leq k \leq p$, then $\overline{G(a)} = E_G$.*

Proof. The proof is done by induction on $\dim(E_G) = p \geq 1$.
For $p = 1$, by Lemma 2.2.(iii) there exist $a, b \in \Gamma_G$ with $a \neq b$, since G is non abelian and $\Gamma_G \neq \emptyset$. In this case $D_1 = \mathbb{R}(b-a) + a = \mathbb{R} = E_G$, then if $D_1 \subset \overline{G(a)}$ so $\overline{G(a)} = E_G$. $\overline{\Gamma_G} = E_G$.
Suppose that Lemma 3.8 is true until dimension $p-1$. Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ with $\Gamma_G \neq \emptyset$ and let $a, a_1, \dots, a_p \in \Gamma_G$ such that $(a_1, \dots, a_p)$ is a basis of E_G. Suppose that $D_k \subset \overline{G(a)}$ for every $1 \leq k \leq p$.
Denote by H the vector subspace of E_G generated by $(a_1 - a), \dots, (a_{p-1} - a)$ and $\Delta_{p-1} = T_a(H)$. We have Δ_{p-1} is an affine subspace of E_G and it contains $a, a_1, \dots, a_{p-1}$.
Set $\lambda, \lambda_k \in \Gamma_G$, $1 \leq k \leq p-1$ such that $f = (a, \lambda), f_k = (a_k, \lambda_k) \in G \backslash \mathcal{S}_n$. Suppose that $\lambda > 1$ and $\lambda_k > 1$, for every $1 \leq k \leq p-1$ (leaving to replace f and f_k respectively by f^2 or f^{-2} and by f_k^2 or f_k^{-2}). Let G_k be the group generated by f and f_k, $1 \leq k \leq p-1$. By Lemma 3.5 we have $\overline{G_k(a)} = D_k$. Let G' be the subgroup of G generated by f, $f_1, \dots, f_{p-1}$, then $D_k \subset \overline{G'(a)}$ for every $1 \leq k \leq p-1$.
By Lemma 2.8 we have $E_{G'} = \Delta_{p-1}$. Let $G'' = T_{-a} \circ G' \circ T_a$, by Lemma 2.6.(ii) we have $E_{G''} = T_{-a}(\Delta_{p-1}) = H$ and $D'_k = T_{-a}(D_k) \subset \overline{G''(0)}$ for every $1 \leq k \leq p-1$. By induction hypothesis applied to G'' we have $\overline{G''(0)} = H$ so $\overline{G'(a)} = \Delta_{p-1}$. Since $G'(a) \subset G(a)$, then

$$\Delta_{p-1} \subset \overline{G(a)} \qquad (1).$$

Let $x \in E_G \backslash \Delta_{p-1}$ and $D = \mathbb{R}(a_p - a) + x$. Since $(a_1 - a, \dots, a_p - a)$ is a basis of E_G, so $H \oplus \mathbb{R}(a_p - a) = E_G$ with $x, a \in E_G$, then $x - a = z + \alpha(a_p - a)$ with $z \in H$ and $\alpha \in \mathbb{R}$. Let $y = z + a$, as $H + a = \Delta_{p-1}$ we have $y \in \Delta_{p-1}$, and

$$\begin{aligned} y &= -\alpha(a_p - a) + x \in D \\ &= z + a \\ &= x - a - \alpha(a_p - a) + a \\ &= -\alpha(a_p - a) + x \in D \end{aligned}$$

Hence $y \in \Delta_{p-1} \cap D$.

By (1) we have $y \in \overline{G(a)}$. Then there exists a sequence $(f_m)_{m \in \mathbb{N}}$ in G such that $\underset{m \longrightarrow +\infty}{lim} f_m(a) = y$. For every $m \in \mathbb{N}$ denote by $f_m = (b_m, \lambda_m)$.

By Lemma 3.7 we have $f_m(D_p) = f_m(\mathbb{R}(a_p - a) + a) = \mathbb{R}(a_p - a) + f_m(a)$. Since $\underset{m\longrightarrow+\infty}{lim} f_m(a) = y$ then

$$\underset{m\longrightarrow+\infty}{lim} f_m(D_p) = \mathbb{R}(a_p - a) + y$$

As $y \in D$ then $y - x \in \mathbb{R}(a_p - a)$, thus $\mathbb{R}(a_p - a) + y = \mathbb{R}(a_p - a) + x = D$ and so $\underset{m\longrightarrow+\infty}{lim} f_m(D_p) = D$.
Since $D_p \subset \overline{G(a)}$ then $D \subset \overline{G(a)}$, so $x \in \overline{G(a)}$, hence $E_G\backslash\Delta_{p-1} \subset \overline{G(a)}$. By (1) we obtain $E_G \subset \overline{G(a)}$. Since $\Gamma_G \neq \emptyset$ then by Lemma 3.1.(ii), we have E_G is G-invariant, so $G(a) \subset E_G$ since $a \in E_G$. It follows that $\overline{G(a)} = E_G$. □

Proof of Proposition 3.6. Let G be a non abelian subgroup of $\mathcal{H}(n, \mathcal{R})$. Since $G\backslash\mathcal{S}_n \neq \emptyset$ then $\Gamma_G \neq \emptyset$ and suppose that E_G is a vector subspace of $\mathbb{R}^n$, (one can replace G by $G' = T_{-a} \circ G \circ T_a$, for some $a \in \Gamma_G$).

First, we will prove that there exists $a \in \Gamma_G$ such that $\overline{G(a)} = E_G$. By Lemmas 2.6,(i) and 2.7, there exists $a, a_1, \dots, a_p \in \Gamma_G$ such that $(a_1, \dots, a_p)$ and $(a_1 - a, \dots, a_p - a)$ are two basis of E_G. Denote by $D_k = \mathbb{R}(a_k - a) + a$, $1 \leq k \leq p$. Since $a \in \Gamma_G$, then there exists $f \in G$ such that $f = (a, \lambda)$. Suppose that $\lambda > 1$ (one can replace f by f^2 or f^{-2}). By Lemma 3.5, $D_k \subset \overline{G(a)}$, for every $1 \leq k \leq p$. By Lemma 3.8, we have $\overline{G(a)} = E_G$.

Second, let $x \in E_G$ and by Lemma 2.5 we have $\Gamma_G \subset \overline{G(x)}$ and by Lemma 3.1.(ii), Γ_G is G-invariant. Since $a \in \Gamma_G$ then

$$E_G = \overline{G(a)} \subset \overline{\Gamma_G} \subset \overline{G(x)}.$$

It follows that $\overline{G(x)} = E_G$ since E_G is G-invariant (Lemma 3.1,(ii)). The proof is complete. □

Proposition 3.9. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$. Suppose that $G\backslash\mathcal{S}_n \neq \emptyset$ and E_G is a vector space. Then for every $x \in \mathbb{R}^n\backslash E_G$, we have $\overline{G(x)} = \overline{\Lambda_G}.x + E_G$.*

To prove the above Proposition, we need the following Lemma:

Lemma 3.10. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$. For every $\lambda \in \Lambda_G\backslash\{-1, 1\}$ and for every $b \in E_G$, there exists a sequence $(f_m)_{m\in\mathbb{N}}$ in G such that $\underset{m\longrightarrow+\infty}{lim} f_m = f$, with $f = (b, \lambda) \in \mathcal{H}(n, \mathbb{R})\backslash\mathcal{S}_n$.*

Proof. Let $\lambda \in \Lambda_G\backslash\{-1, 1\}$ and $b \in E_G$. Given $g = (a, \lambda) \in G\backslash\mathcal{S}_n$, so $a \in \Gamma_G \subset E_G$. By Proposition 3.6, we have $\overline{G(a)} = E_G$. Then there exists a sequence $(g_m)_{m\in\mathbb{N}}$ in G such that $\underset{m\longrightarrow+\infty}{lim} g_m(a) = b$. For every $m \in \mathbb{N}$, denote by $f_m = g_m \circ g \circ g_m^{-1}$, so $f_m = (g_m(a), \lambda)$. Hence $\underset{m\longrightarrow+\infty}{lim} f_m = f$, with $f = (b, \lambda)$. □

Lemma 3.11. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ such that E_G is a vector space and $G\backslash\mathcal{S}_n \neq \emptyset$. Then:*
(i) For every $b \in E_G$ there exists a sequence $(T_{b_m})_{m\in\mathbb{Z}}$ in $G\cap\mathcal{T}_n$ such that $\underset{m\to-\infty}{lim} T_{b_m} = T_b$.
(ii) If $-1 \in \Lambda_G$, then for every $b \in E_G$ there exists a sequence $(S_m)_{m\in\mathbb{N}}$ in $G\cap(\mathcal{S}_n\backslash\mathcal{T}_n)$ such that $\underset{m\longrightarrow+\infty}{lim} S_m = S = (b, -1)$.

Proof. (i) • First, suppose that $b \in \Gamma_G$, then there exists $f \in G\backslash\mathcal{S}_n$ with $f = (b, \lambda)$. Since G is non abelian set $g \in G$ such that $f\circ g \neq g\circ f$. Set $h = g\circ f\circ g^{-1}$, so $h = (g(b), \lambda)$. Let $G' = T_{-g(b)}\circ G\circ T_{g(b)}$, $f' = T_{-g(b)}\circ f\circ T_{g(b)}$ and $h' = T_{-g(b)}\circ h\circ T_{g(b)}$, so $f' = (b - g(b), \lambda)$ and $h' = (0, \lambda) = \lambda id_{\mathbb{R}^n}$. By Lemma 3.4, for every $m \in \mathbb{Z}$, $T'_m = T_{(1-\lambda^m)(b-g(b))} \in G'$. Write $T_{b_m} = T_{g(b)}\circ T'_m\circ T_{-g(b)}$, so

$$b_m = (1-\lambda^m)(b - g(b)) + g(b) = (1-\lambda^m)b + \lambda^m g(b), \quad m \in \mathbb{Z}.$$

Since $|\lambda| \neq 1$, suppose that $\lambda > 1$, so $\underset{m\to-\infty}{lim}(1-\lambda^m)a + \lambda^m g(b) = b$. It follows that the sequence $(T_{b_m})_m \in G\cap\mathcal{T}_n$ and $\underset{m\to-\infty}{lim} T_{b_m} = T_b$.

• Now, suppose that $b \in E_G$ and let $a \in \Gamma_G$. By Proposition 3.6, $\overline{G(a)} = E_G$, so there exists a sequence $(g_k)_k$ in G such that $\underset{k\longrightarrow+\infty}{lim} g_k(a) = b$. By above state, there exists a sequence $(T_{a_m})_m \in G\cap\mathcal{T}_n$ such that $\underset{m\to-\infty}{lim} T_{a_m} = T_a$. Set $T_{b_{m,k}} = g_k\circ T_{a_m}\circ g_k^{-1}$, one has $T_{b_{m,k}} = (g_k(a_m), 1) \in G\cap\mathcal{T}_n$. We have $\underset{m\to-\infty}{lim} b_{m,k} = \underset{m\to-\infty}{lim} g_k(a_m) = g_k(a)$, so

$$\underset{\|(k,-m)\|\to+\infty}{lim} b_{m,k} = \underset{k\to+\infty}{lim} g_k(a) = b.$$

So $\underset{\|(k,-m)\|\to+\infty}{lim} T_{b_{m,k}} = T_b$. This complete the proof of (i).

(ii) Suppose that $-1 \in \Lambda_G$ and let $b \in E_G$. Then there exists $f = (a, -1) \in G\cap\mathcal{S}_n$. By Lemma 3.1.(i), $a = f(0) \in E_G$, so $b - a \in E_G$, since E_G is a vector space. By (i), there exists a sequence $(T_{a_m})_{m\in\mathbb{Z}}$ in $G\cap\mathcal{T}_n$ such that $\underset{m\to-\infty}{lim} T_{a_m} = T_{b-a}$. Set $S_m = T_{a_m}\circ f$. We have $S_m = (a + a_m, -1) \in G\cap\mathcal{S}_n$. Since $\underset{m\to+\infty}{lim} a_m = b - a$, then $\underset{m\to+\infty}{lim} a + a_m = b$, so $\underset{m\to+\infty}{lim} S_m = S = (b, -1)$. The proof is complete. □

Proof of Proposition 3.9. Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$ and E_G is a vector space. Let $x \in U = \mathbb{R}^n\backslash E_G$.
Lest' prove that $\overline{\Lambda_G}.x + E_G \subset \overline{G(x)}$: Let $\alpha \in \Lambda_G$ and $a \in E_G$.
• Suppose that $\alpha \in \Lambda_G\backslash\{-1,1\}$. Since E_G is a vector space, $a' = \frac{a}{1-\alpha} \in E_G$. By Lemma 3.10 there exists a sequence $(f_m)_m$ in G such that $\underset{m\longrightarrow+\infty}{lim} f_m = f = (a',\alpha) \in G\backslash\mathcal{S}_n$. Then

$$\begin{aligned} f(x) &= \alpha(x-a') + a' \\ &= \alpha x + (1-\alpha)a' \\ &= \alpha x + a \in \overline{G(x)}, \end{aligned}$$

so

$$(\Lambda_G\backslash\{1,1\})\,.x + E_G \subset \overline{G(x)}.$$

• Suppose that $\alpha \in \Lambda_G \cap \{-1,1\}$.
- If $\alpha = 1$, by Lemma 3.11.(i), there exists a sequence $(T_{a_m})_m$ in G such that $\underset{m\longrightarrow+\infty}{lim} T_{a_m} = T_a$. So $T_a(x) = x + a \in \overline{G(x)}$.
- If $\alpha = 1$, by Lemma 3.11.(i), there exists a sequence $(S_m)_m$ in $G \cap (\mathcal{S}_n\backslash\mathcal{T}_n)$ such that $\underset{m\longrightarrow+\infty}{lim} S_m = S = (a,-1)$. So $S(x) = -x + a \in \overline{G(x)}$.
It follows that $\alpha x + a \in \overline{G(x)}$ and so

$$(\Lambda_G \cap \{-1,1\})\, x + E_G \subset \overline{G(x)}.$$

This proves that $\overline{\Lambda_G}.x + E_G \subset \overline{G(x)}$.

Conversely, let's prove that $G(x) \subset \Lambda_G.x + E_G$. Let $f \in G$.
• Suppose that $f = (a,\lambda) \in G\backslash\mathcal{S}_n$. By Lemma 3.1.(i), $f(0) = (1-\lambda)a \in E_G$ since E_G is a vector space.
Then $f(x) = \lambda(x-a) + a = \lambda x + (1-\lambda)a \in \Lambda_G.x + E_G$.
• Suppose that $f = (a,\varepsilon) \in G \cap \mathcal{S}_n$, so $f(x) = \varepsilon x + a \in \Lambda_G.x + E_G$, since by Lemma 3.1.(i), $f(0) = a \in E_G$.
It follows that $G(x) \subset \Lambda_G.x + E_G$. Therefore $\overline{G(x)} \subset \overline{\Lambda_G}.x + E_G$. Hence $\overline{G(x)} = \overline{\Lambda_G}.x + E_G$. □

Lemma 3.12. *If there exists $\lambda,\mu \in \Lambda_G$ such that $\lambda\mu < 0$ and $\frac{log|\lambda|}{log|\mu|} \notin \mathbb{Q}$, then $\overline{\Lambda_G} = \mathbb{R}$.*

Proof. Suppose that $\lambda < 0 < \mu$. Let $H_+ := \{\lambda^{2p}\mu^{2q}, p,q \in \mathbb{Z}\}$ and $H_- := \lambda.H_+$. See that $H_- \subset \Lambda_G$ and so $H_+ \cup H_- \subset \Lambda_G$. Set $f :]0,+\infty[\longrightarrow \mathbb{R}$, the homeomorphism defined by $f(x) = logx$, so $f(H_+) := \mathbb{Z} + \frac{log|\lambda|}{log|\mu|}\mathbb{Z}$. As $\frac{log|\lambda|}{log|\mu|} \notin \mathbb{Q}$ then $f(H_+)$ is dense in $\mathbb{R}$, so H_+ and H_- are dense respectively in $]0,+\infty[$ and in $]-\infty,0[$. We deduce that $\overline{\Lambda_G} = \mathbb{R}$. □

4. Some results for non abelian subgroup of $\mathcal{S}_n$

In this case, G is a non abelian subgroup of $\mathcal{S}_n$, then it contains necessarily an affine symmetry. In the following, recall that $G_1 = G\cap\mathcal{T}_n$ and every $f\in\mathcal{S}_n$ is denoted by $f=(a,\varepsilon)$, where $f : x\longmapsto \varepsilon x + a$. Denote by $\delta_G := \{f(0),\quad f\in G\cap(\mathcal{S}_n\backslash\mathcal{T}_n)\}$.
We use the following lemmas and propositions to prove Theorem 1.1 and above Corollaries:

Lemma 4.1. *Let G be a non abelian subgroup of $\mathcal{S}_n$. Then:*
- (i) *$G_1(0)$ is an additif subgroup of $\mathbb{R}^n$.*
- (ii) *$\delta_G\neq\emptyset$.*
- (iii) *For every $f\in G\backslash\mathcal{T}_n$, we have $f(G_1(0))=\delta_G$ and $f(\delta_G)=G_1(0)$.*

Proof. The proof of (i) is obvious.
(ii) If $\delta_G=\emptyset$ then $G\cap(\mathcal{S}_n\backslash\mathcal{T}_n)=\emptyset$, so G is a subgroup of $\mathcal{T}_n(\mathbb{R})$, hence G is abelian, a contradiction.

(iii) Let $f\in G\backslash\mathcal{T}_n$, $b\in G_1(0)$ and $g=(b,1)\in G_1$. Then for every $x\in\mathbb{R}^n$ we have $f\circ g(x)=f(x+b)=-x-b+a$, so $f\circ g=(-b+a,-1)$. Hence $f(b)=f\circ g(0)=-b+a\in\delta_G$.
Conversely, let $b\in\delta_G$ and $g=(b,-1)\in G\backslash\mathcal{T}_n$ such that $g(0)=b$. We have $f\circ g(x)=f(-x+b)=x-b+a$, so $f\circ g=(-b+a,1)\in G_1$, thus $c=-b+a\in G_1(0)$. Hence $b=-c+a=f(c)$ and so $b\in f(G_1(0))$. It follows that $f(G_1(0))=\delta_G$. As $f^{-1}=f$ so $f(\delta_G)=G_1(0)$. □

Proposition 4.2. *Let G be a non abelian subgroup of $\mathcal{S}_n$, $a\in\delta_G$ and $x\in\mathbb{R}^n$. Then:*
$$\overline{G(x)}=(x+\overline{G_1(0)})\cup(-x+a+\overline{G_1(0)}).$$

Proof. Let G be a non abelian subgroup of $\mathcal{S}_n$ and $x\in\mathbb{R}^n$. We have
$$G(x)=\{g(x)=\varepsilon x+b,\ g=(b,\varepsilon)\in G\}=(x+G_1(0))\cup(-x+\delta_G).\ \text{So}$$
$$\overline{G(x)}=\left(x+\overline{G_1(0)}\right)\cup\left(-x+\overline{\delta_G}\right).\qquad (1)$$

Since G is non abelian then $G\backslash\mathcal{T}_n\neq\emptyset$, so let $f=(a,-1)\in G\backslash\mathcal{T}_n$ with $a\in\delta_G$. By Lemma 4.1.(iii) we have $\delta_G=f(G_1(0))$. By Lemma 4.1.(i), $G_1(0)$ is an additif subgroup of $\mathbb{R}^n$ then $f(G_1(0))=G_1(0)+a$, so $\delta_G=G_1(0)+a$. Hence $-x+\overline{\delta_G}=-x+\overline{G_1(0)}+a$. By (1) we conclude that
$$\overline{G(x)}=\left(x+\overline{G_1(0)}\right)\cup\left(-x+a+\overline{G_1(0)}\right).$$
□

5. **Proof of main results**

Proof of Theorem 1.1. Let $a \in E_G$ and $G' = T_{-a} \circ G \circ T_a$. By Lemma 2.6.(ii), $E_{G'} = T_{-a}(E_G)$, so $E_{G'}$ is a vector subspace of $\mathbb{R}^n$. Then :

- *Proof of* (1).(*i*)*:* One has $\Gamma_G \neq \emptyset$ since $G \backslash \mathcal{S}_n \neq \emptyset$. Then by Lemma 2.1, $0 \in \overline{\Lambda_G}$ and by Lemma 3.1.(ii), E_G is G-invariant. As $G \backslash \mathcal{S}_n \neq \emptyset$, there exist $b, c \in \Gamma_G \subset E_G$, with $b \neq c$ (Lemma 2.2.(iii)), so $\dim(E_G) \geq 1$.
- *Proof of* (1).(*ii*)*:* By Proposition 3.6, $\overline{G'(x-a)} = E_{G'}$, for every $x \in E_G$. So $T_{-a}(\overline{G(x)}) = E_{G'}$, it follows that $\overline{G(x)} = T_a(E_{G'}) = E_G$. So the proof of (1)(i) is complete.
- *Proof of* (1).(*iii*)*:* By Proposition 3.9, $\overline{G'(x-a)} = \overline{\Lambda_{G'}}.(x-a) + E'_G$, for every $x \in U$. So by Lemma 2.6,(ii), $T_{-a}(\overline{G(x)}) = \overline{\Lambda_G}.(x-a) + E_G - a$, it follows that $\overline{G(x)} = \overline{\Lambda_G}.(x-a) + E_G$. So the proof of (1)(ii) is complete.
- *Proof of* (2)*:* The proof of (2) results from Lemma 4.1.(i) and Proposition 4.2, since $H_G = G_1(0)$. □

We will use the following Lemmas to prove Corollary 1.2.

Lemma 5.1. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ with $G \backslash \mathcal{S}_n \neq \emptyset$, then for every $x \in U$ we have $\overline{G(x)} = \overline{G(y)}$.*

Proof. Suppose that E_G is a vector space (leaving to replace G by $G' = T_{-a} \circ G \circ T_a$ for some $a \in E_G$, and by Lemma 2.6.(ii), $E_{G'} = T_{-a}(E_G)$ is a vector space). Let $x \in U$ and $y \in \overline{G(x)} \cap U$. By Theorem 1.1.(1).(iii), there exists $a \in E_G$ such that $\overline{G(x)} = \overline{\Lambda_G}(x-a) + E_G$. Since E_G is a vector space and $a \in E_G$ then $\overline{G(x)} = \overline{\Lambda_G}x + E_G$. In the same way,

$$\overline{G(y)} = \overline{\Lambda_G}y + E_G, \qquad (1).$$

See that $\overline{G(x)} \cap U = (\overline{\Lambda_G} \backslash \{0\})x + E_G$. Write $y = \alpha x + b$, where $\alpha \in \overline{\Lambda_G} \backslash \{0\}$ and $b \in E_G$. So by (1),

$$\overline{G(y)} = \overline{\Lambda_G}y + E_G = \overline{\Lambda_G}(\alpha x + b) + E_G = \alpha \overline{\Lambda_G}x + E_G.$$

Since $\alpha \in \Lambda_G$ and by Lemma 2.1, $\overline{\Lambda_G} \backslash \{0\}$ is a subgroup of $\mathbb{R}^*$, then $\alpha \overline{\Lambda_G} = \overline{\Lambda_G}$. Therefore $\overline{G(y)} = \overline{\Lambda_G}x + E_G = \overline{G(x)}$. □

Lemma 5.2. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that E_G is a vector subspace of $\mathbb{R}^n$. Let $x \in U$ then the vector subspace $H_x = \mathbb{R}x \oplus E_G$ of $\mathbb{R}^n$ is G-invariant.*

Proof. Let $x \in \mathbb{R}^n \backslash E_G$ and $H_x = \mathbb{R}x + E_G$. Let $f \in G$ having the form $f(z) = \lambda z + a$, $z \in \mathbb{R}^n$, then by Lemma 1.2.(i), $a = f(0) \in E_G$. For every $\alpha \in \mathbb{R}$, $b \in E_G$, we have $f(\alpha x + b) = \lambda(\alpha x + b) + a = \lambda \alpha x + \lambda b + a$. Since E_G is a vector space, then $\lambda b + a \in E_G$ and so $f(\alpha x + b) \in H_x$. □

Proof of Corollary 1.2.
• *The proof of* (i)*:* The proof results from Lemma 5.1.
• *The proof of* (ii)*:* As $G\backslash\mathcal{S}_n \neq \emptyset$, then by Lemma 2.1, $0 \in \overline{\Lambda_G}$. So the proof of (ii) results from Theorem 1,1.(1).(ii).
• *The proof of* (iii)*:* Suppose that E_G is a vector subspace of $\mathbb{R}^n$ (leaving, by Lemma 1.2.(ii), to replace G by $G' = T_{-a} \circ G \circ T_a$, for some $a \in E_G$).

Recall that $U = \mathbb{R}^n \backslash E_G$ and let $x, y \in U$ with $x \neq y$. Denote by $H_x = \mathbb{R}.x \oplus E_G$ and by $H_y = \mathbb{R}.y \oplus E_G$. By lemma 5.2 we have H_x and H_y are G-invariant. Let $\varphi: H_x \longrightarrow H_y$ be the homeomorphism defined by $\varphi(\alpha x + v) = \alpha y + v$ for every $\alpha \in \mathbb{R}$ and $v \in E_G$. For every $f \in G$, with the form $f(z) = \lambda z + a$, $z \in \mathbb{R}^n$, then by Lemma 3.1.(i), $a = f(0) \in E_G$ and so $\varphi(f(x)) = \varphi(\lambda x + a) = \lambda y + a = f(y)$. It follows that $\varphi(G(x)) = G(y)$. □

Proof of Corollary 1.3.
• *The proof of* (i)*:* From Corollary 1.2.(ii), the closure of every orbit of G contains E_G and by Theorem 1.1.(1), we have $\dim(E_G) \geq 1$, so G has no periodic orbit. Moreover, if G is countable then every orbit O is also countable, hence O can not be closed.
• *The proof of* (ii)*:* Let $x \in \mathbb{R}^n$ and $y \in \overline{G(x)}$. By Proposition 4.2 we have $\overline{G(x)} = \left(x + \overline{G_1(0)}\right) \cup \left(-x + a + \overline{G_1(0)}\right)$. Suppose that $y \in (x + \overline{G_1(0)})$ then $y = x + b$ for some $b \in \overline{G_1(0)}$. By Lemma 4.1.(i), $G_1(0)$ is an additif group, so $b + G_1(0) = G_1(0)$. Therefore, by Proposition 4.2 we have

$$\begin{aligned}\overline{G(y)} &= \left(x + b + \overline{G_1(0)}\right) \cup \left(-x - b + a + \overline{G_1(0)}\right) \\ &= \left(x + \overline{G_1(0)}\right) \cup \left(-x + a + \overline{G_1(0)}\right) = \overline{G(x)}.\end{aligned}$$

The same proof is used if $y \in (x + a + \overline{G_1(0)})$. □

Proof of Corollary 1.4. Let G is a non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$ such that $G\backslash\mathcal{S}_n \neq \emptyset$. Suppose that E_G is a vector subspace of $\mathbb{R}^n$ (leaving, by Lemma 2.6.(ii), to replace G by $T_{-a} \circ G \circ T_a$, for some $a \in E_G$.)

• Let's prove that (1) and (2) are equivalent: if $\overline{G(x)} = \mathbb{R}^n$, for some $x \in \mathbb{R}^n$, so $x \in U$. Let $y \in U$, then by Corollary 1.2.(i), $\overline{G(y)} \cap U = \overline{G(x)} \cap U = U$. Since U is dense in $\mathbb{R}^n$, $\overline{G(y)} = \mathbb{R}^n$. Conversely, the proof is obvious.

• (3).$(i) \Longrightarrow$ (1): If $E_G = \mathbb{R}^n$ then by Theorem 1.1.(1).(ii) we have $\overline{G(x)} = E_G$, for every $x \in E_G$. So G has a dense orbit.

• (3).$(ii) \Longrightarrow$ (1): If Λ_G is dense in $\mathbb{R}$ then by Theorem 1.1.(1).(ii) we have $\overline{G(x)} = \overline{\Lambda_G}x + E_G$, for every $x \in U$. So G has a dense orbit.

• (1) $\Longrightarrow$ (3): Suppose that G has a dense orbit $G(x)$, for some $x \in \mathbb{R}^n$. There are tow cases:
- If $E_G = \mathbb{R}^n$, then we obtain (3).(i).
- If $E_G \neq \mathbb{R}^n$ then $\dim(E_G) \leq n-1$, so and $U \neq \emptyset$, hence $x \in U$. By Theorem 1.1.(1).(iii) we have $\overline{G(x)} = \mathbb{R}x + E_G$, so $dim(E_G) = n-1$ and Λ_G is dense in $\mathbb{R}$. Then (3).(ii) follows. □

We use the following Lemma to prove Corollary 1.6:

Lemma 5.3. *Let H be an additif subgroup of $\mathbb{R}^n$. Then*

$$\overset{\circ}{\overline{H}} \neq \emptyset \quad if\ and\ only\ if \quad \overline{H} = \mathbb{R}^n$$

Proof. Suppose that $\overset{\circ}{\overline{H}} \neq \emptyset$ and let $a \in \overset{\circ}{\overline{H}} \neq \emptyset$. Then there exists $\varepsilon > 0$ such that $B_{(a,\varepsilon)} \subset \overline{H} \neq \emptyset$, where $B_{(a,\varepsilon)} = \{x \in \mathbb{R}^n : \ \|x - a\| < \varepsilon\}$ and $\|.\|$ is the euclidian norm. Since $\overline{H} \neq \emptyset$ is an additif group, it follows that $B_{(0,\varepsilon)} = T_{-a}\left(B_{(a,\varepsilon)}\right) \subset \overline{H}$. Moreover, we also have $B_{(0,m\varepsilon)} = mB_{(0,\varepsilon)} \subset \overline{H}$, for every $m \in \mathbb{N}^*$. As $\mathbb{R}^n = \bigcup\limits_{m\in\mathbb{N}^*} B_{(0,m\varepsilon)} \subset \overline{H}$, it follows that $\overline{H} = \mathbb{R}^n$. Conversely, the proof is obvious. □

Proof of Corollary 1.6. Let G be a non abelian subgroup of $\mathcal{S}_n$. By Lemma 3.1.(i), one has $\delta_G \neq \emptyset$. Let $a \in \delta_G$ and $f = (a, -1) \in G$.
• First, by Corollary 1.3.(ii) we prove that (i) , (ii) and (iii) are equivalent.
• Second, let's prove that (iii) and (iv) are equivalent: Suppose that $\overline{G(0)} = \mathbb{R}^n$. By Proposition 4.2 we have $\overline{G(0)} = \overline{G_1(0)} \cup (a + \overline{G_1(0)})$. Since $\overset{\circ}{\overline{G(0)}} \neq \emptyset$ then $\overset{\circ}{\overline{G_1(0)}} \neq \emptyset$. By Lemma 4.1.(i), $H_G = G_1(0)$ is an additive subgroup of $\mathbb{R}^n$ then by Lemma 5.3, $\overline{H_G} = \mathbb{R}^n$. Conversely, if $\overline{H_G} = \mathbb{R}^n$ then by By Proposition 4.2 we have $\overline{G(0)} = \overline{G_1(0)} \cup (a + \overline{G_1(0)}) = \mathbb{R}^n$. □

Proof of Corollary 1.8. For $n = 1$, G is a non abelian group of affine maps of $\mathbb{R}$.
• *The proof of* (i)*:* If $G\backslash\mathcal{S}_1 \neq \emptyset$, then by Theorem 1.1.(1), E_G is a G-invariant affine subspace of $\mathbb{R}$ with dimension $p = 1$ such that every orbit of E_G is dense in it. In this case $E_G = \mathbb{R}$.
• *The proof of* (ii)*:* If $G \subset \mathcal{S}_1$, then by Theorem 1.1.(2), H_G is a G-invariant closed subgroup of $\mathbb{R}$ and there exists $a \in E_G$ such that for every $x \in \mathbb{R}$, we have $\overline{G(x)} = (x + H_G) \cup (-x + a + H_G)$. Then there are two cases:
◇ If H_G is dense in $\mathbb{R}$, so every orbit of G is dense in $\mathbb{R}$.
◇ If H_G is discrete then every orbit is closed and discrete.

6. Examples

Example 6.1. Let G be a subgroup of $\mathcal{H}(2,\mathbb{R})$ generated by $f_1 = (a_1, \alpha_1)$ and $f_2 = (a_2, \alpha_2)$ and $f_3 = (a_3, \alpha_3)$, where $\alpha_k \neq 1$, for every $1 \leq k \leq 3$ and $a_1 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $a_3 = \begin{bmatrix} -\sqrt{3} \\ -\sqrt{2} \end{bmatrix}$. Then every orbit of G is dense in $\mathbb{R}$.

Indeed, by Lemma 2.2.(i), G is non abelian. By Proposition 3.6, for every $x \in E_G$, we have $\overline{G(x)} = E_G$. In this case, by Remark 2.4, $E_G = \mathbb{R}^2$ so every orbit of G is dense in $\mathbb{R}^2$.

Example 6.2. Let $(a_1, \dots, a_n)$ be a basis of $\mathbb{R}^n$ and $a = \sum_{k=1}^{n} \alpha_k a_k$, with $\sum_{k=1}^{n} \alpha_k a_k \neq 1$, then for every $t > 1$, the subgroup G of $\mathcal{H}(n,\mathbb{R})$ generated by $\{f = (a,t),\ T_{a_k}, 2 \leq k \leq n\}$ is minimal. (i.e. every orbit of G is dense in $\mathbb{R}^n$).

Indeed; By Remark 2.4, we have $E_G = \mathbb{R}^n$ and by Proposition 3.6, every orbit of G is dense in $\mathbb{R}^n$.

Example 6.3. Let $(a_1, \dots, a_n)$ be a basis of $\mathbb{R}^n$ and $\lambda \in \mathbb{R}\backslash\{0,1\}$. Then every orbit of the group generated by $T_{a_1}, \dots, T_{a_n}$, λId is dense in $\mathbb{R}^n$.

Indeed, By Remark 2.4 we have $E_G = \mathbb{R}^n$ and by Proposition 3.6 every orbit of G is dense in $\mathbb{R}^n$.

Example 6.4. Let $a \in \mathbb{R}^n$ and G be the group generated by $f = T_a$, $g = (a,-1)$ and $h = T_{\sqrt{2}a}$. Then for every $x \in \mathbb{R}^n\backslash\mathbb{R}a$, we have $G(0)$ and $G(x)$ are not homeomorphic.

Proof. Remark that for every $\varphi \in G_1$, there exist $n_1, m_1, p_1, \dots, n_r, m_r, p_r \in \mathbb{Z}$ such that $\varphi = (f^{n_1} \circ g^{m_1} \circ h^{p_1}) \circ \dots \circ (f^{n_r} \circ g^{m_r} \circ h^{p_r})$, for some $r \in \mathbb{N}^*$.

• First, let's show by induction on $r \geq 1$ that

$$\varphi(0) \in (\mathbb{Z} + \sqrt{2}\mathbb{Z})a \quad (i).$$

For $r = 1$, we have

$$\begin{aligned}\varphi(0) &= f^{n_1} \circ g^{m_1} \circ h^{p_1}(0) \\ &= -p_1\sqrt{2}a + m_1 a + n_1 a \\ &= (m_1 + n_1 + \sqrt{2}p_1)a.\end{aligned}$$

So $\varphi(0) \in\in (\mathbb{Z}+\sqrt{2}\mathbb{Z})a$.
Suppose property (i) is true up to order $r-1$. If

$$\varphi = (f^{n_1}\circ g^{m_1}\circ h^{p_1})\circ(f^{n_2}\circ g^{m_2}\circ h^{p_2}\circ\cdots\circ f^{n_r}\circ g^{m_r}\circ h^{p_r}),$$

then by induction property there exists $p, q \in \mathbb{Z}$ such that

$$f^{n_2}\circ g^{m_2}\circ h^{p_2}\circ\cdots\circ f^{n_r}\circ g^{m_r}\circ h^{p_r}(0) = (p+\sqrt{2}q)a.$$

So $\varphi(0) = f^{n_1}\circ g^{m_1}\circ h^{p_1}((p+\sqrt{2}q)a)$, thus

$$\varphi(0) = \begin{cases} -\big((p+\sqrt{2}q)a + p_1 a\big) + a + n_1 a, & \text{if } \ m_1 \text{ is odd,} \\ \big((p+\sqrt{2}q)a + p_1 a\big) + n_1 a, & \text{if } \ m_1 \text{ is even.} \end{cases}$$

Hence, $\varphi(0) \in (\mathbb{Z}+\sqrt{2}\mathbb{Z})a$.

It follows that

$$G_1(0) \subset (\mathbb{Z}+\sqrt{2}\mathbb{Z})a \qquad (1)$$

• Second, we will proof that $G_1(0) = (\mathbb{Z}+\sqrt{2}\mathbb{Z})a$. let $p, q \in \mathbb{Z}$, we have $f^p \circ h^q = T_{(p+\sqrt{2}q)a}$, thus $f^p\circ h^q(0) = (p+\sqrt{2}q)a \in G_1(0)$. It follows by (1) that $G_1(0) = (\mathbb{Z}+\sqrt{2}\mathbb{Z})a$. With the same proof we can show that $\delta_G = G_1(0)$.

• Thirdly, by Proposition 4.2 for every $x \in \mathbb{R}^n$, we have $\overline{G(x)} = (x + \overline{G_1(0)}) \cup (-x + \overline{G_1(0)})$. Therefore $\overline{G(0)} = \overline{G_1(0)} = \overline{(\mathbb{Z}+\sqrt{2}\mathbb{Z})a} = \mathbb{R}a$ and it is connected. But $\overline{G(x)} = (x+\mathbb{R}a)\cup(-x+\mathbb{R}a)$, is not connected for every $x \in \mathbb{R}^n \backslash \mathbb{R}a$. Hence $G(0)$ and $G(x)$ can not be homeomorphic. $\square$

Remark 6.5. Remark that the form of φ used in the proof of Example 6.4 is general of every $\varphi \in G$ and the order $(f^{n_k}\circ g^{m_k}\circ h^{p_k})$ is not particular of φ. For example, if $\varphi = g^m \circ f^n \circ h^p$, we write

$$\varphi = (f^{n_1}\circ g^{m_1}\circ h^{p_1})\circ(f^{n_2}\circ g^{m_2}\circ h^{p_2})\circ(f^{n_3}\circ g^{m_3}\circ h^{p_3})$$

with $n_1 = p_1 = m_2 = p_2 = n_3 = m_3 = 0$, $m_1 = m$, $n_2 = n$ and $p_3 = p$.

References

1. N.I.Zhukova, *Minimal sets of Cartan Foliations*, Trudy Mathematicheskogo Institua imeni V.A. Steklova (2007), vol 256, 115147.
2. Vitaly.B, Michal.M and Samuel.S, *Affine actions of a free semigroup on the real line*, Ergod. Th. and Dynam. Sys. (2006), 26, 12851305.
3. Mohamed.J,*A generalization of Dirichlet approximation theorem for the affine actions on real line*, Journal of Number Theory 128 (2008) 11461156.

Adlene Ayadi, Department of Mathematics, Faculty of Sciences of Gafsa, Gafsa, Tunisia.

Yahya N'dao, Department of Mathematics, Faculty of Sciences of Gafsa, Gafsa, Tunisia
E-mail address: adleneso@yahoo.fr, yahiandao@yahoo.fr