STRUCTURE OF CHINESE ALGEBRAS

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Abstract. The structure of the algebra $K[M]$ of the Chinese monoid M over a field K is studied. The minimal prime ideals are described. They are determined by certain homogeneous congruences on M and they are in a one to one correspondence with diagrams of certain special type. There are finitely many such ideals. It is also shown that the prime radical $B(K[M])$ of $K[M]$ coincides with the Jacobson radical and the monoid M embeds into the algebra $K[M]/B(K[M])$. A new representation of M as a submonoid of the product $B^d \times \mathbb{Z}^e$ for some natural numbers d, e, where B stands for the bicyclic monoid, is derived. Consequently, M satisfies a nontrivial identity.

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Introduction

For a positive integer n we consider the monoid $M = \langle a_1, \ldots, a_n \rangle$ defined by the relations

$$a_j a_i a_k = a_j a_k a_i = a_k a_j a_i \text{ for } i \leq k \leq j. \tag{1}$$

It is called the Chinese monoid of rank n. It is known that every element of M has a unique presentation of the form

$$x = b_1 b_2 \cdots b_n, \tag{2}$$

where

$$\begin{aligned}
b_1 &= a_1^{k_{11}} \\
b_2 &= (a_2 a_1)^{k_{21}} a_2^{k_{22}} \\
b_3 &= (a_3 a_1)^{k_{31}} (a_3 a_2)^{k_{32}} a_3^{k_{33}} \\
&\cdots \\
b_n &= (a_n a_1)^{k_{n1}} (a_n a_2)^{k_{n2}} \cdots (a_n a_{n-1})^{k_{n(n-1)}} a_n^{k_{nn}},
\end{aligned}$$

with all exponents nonnegative [2]. We call it the canonical form of the element $x \in M$. In particular, M has polynomial growth of degree $n(n-1)/2$, [13]. The Chinese monoid is related to the so called plactic monoid, introduced and studied in [16, 17]. Both constructions are strongly related to Young tableaux, and therefore to representation theory and algebraic combinatorics. The latter construction has already been established as a classical and powerful tool of the respective theories [6]. The Chinese monoid appeared in the classification of classes of monoids with the growth function coinciding with that of the plactic monoid [5]. Combinatorial properties of M were studied in detail in [2]. In case $n = 2$, the Chinese and the plactic monoids coincide. The monoid algebra $K[M]$ over a field K is the unital algebra defined by the algebra presentation determined by relations (1). It is called the Chinese algebra of rank n. If $n = 2$, the structure of $K[M]$ is described in [3]. In particular, this algebra is prime and semiprimitive, it is not noetherian and it does not satisfy any polynomial identity. For $n = 3$, some information on $K[M]$ was obtained in [8]. In particular the Jacobson radical of $K[M]$ is nonzero but it is nilpotent and the prime spectrum of $K[M]$ is pretty well understood. One of the motivations for a study of the Chinese monoid is based on an expectation that it can play a similar role as the plactic monoid in several aspects of representation theory, quantum algebras, and in algebraic combinatorics. Another motivation stems from difficult open problems concerning the radical of finitely presented algebras.

The results of this paper contribute to the general program of studying finitely presented algebras defined by homogeneous semigroup presentations. We say that an algebra A with unity is defined by homogeneous semigroup relations if it is given by a presentation $A = \langle X \colon R \rangle$, where X is a set of free generators of a free algebra over K and R is a set of relations of the form $u = w$, where u, w are words of equal lengths in the generators from X. In this case A may be identified with the semigroup algebra $K[S]$, where S is the monoid defined by the same presentation, [18]. Notice that there is a natural length function on the underlying monoid S. Certain important classes of such algebras, and of the underlying monoids, have been recently considered, in particular see [3, 7, 11]. Clearly, the Chinese algebra $K[M]$ is of this type. Also, the plactic algebra is defined by semigroup relations of degree 3. Algebras corresponding to the set theoretic solutions of the Yang-Baxter equation are defined by quadratic semigroup relations [11].

For certain important constructions of algebras defined by homogeneous semigroup relations it was shown that the minimal prime ideals have a very special form, which proved to have far reaching consequences for the properties of the algebra, [10, 11]. One might expect that this is a more general phenomenon occurring in this class of algebras. Our aim is to consider problems of this type for the class of Chinese algebras. We establish a remarkable form of minimal prime ideals of the algebra $K[M]$ and derive several consequences.

By $J(K[S]), B(K[S])$ we denote the Jacobson and the prime radical of $K[S]$, respectively. If η is a congruence on a semigroup S then $\mathcal{I}_\eta$ stands for the ideal of $K[S]$ spanned as a vector space over K by the set $\{s-t\colon s,t\in S, (s,t)\in\eta\}$. So $K[S]/\mathcal{I}_\eta = K[S/\eta]$. If $\phi : S\to T$ is a semigroup homomorphism, then by $ker(\phi)$ we mean the congruence on S determined by ϕ.

The paper is organized as follows. First, certain basic equalities of the form $\alpha K[M]\beta = 0$ for $\alpha,\beta\in K[M]$ are established. They are used to introduce two finite families of ideals of $K[M]$ such that every prime ideal P of $K[M]$ contains one of these ideals. Each of these ideals is of the form $\mathcal{I}_\rho$ for a congruence ρ on M. It is also shown that ρ is a homogeneous congruence on M, which means that $(s,t)\in\rho$ implies that s,t have equal length. A more involved construction allows us to continue this process by showing that every prime P contains an ideal of the form $\mathcal{I}_{\rho_2}$ for some homogeneous congruence ρ_2 containing ρ. Proceeding this way, we construct a finite tree D whose vertices correspond to certain homogeneous congruences on M and such that $\rho\subseteq\rho'$ if the vertex in D corresponding to ρ is above that corresponding to ρ'. Moreover, ideals corresponding to vertices lying in different branches of the tree D are incomparable under inclusion. It turns out that the leaves of this tree correspond to prime ideals of $K[M]$. This is used to prove the main result of the paper, asserting that the leaves of D are in a one to one correspondence with the minimal prime ideals of $K[M]$. The proof provides us with a procedure to construct every such prime P. In particular, every minimal prime P has a remarkable form $P=\mathcal{I}_{\rho_P}$, where ρ_P is the congruence on M defined by $\rho_P=\{(s,t)\in M\times M\colon s-t\in P\}$. Consequently, $K[M]/P\simeq K[M/\rho_P]$, so $K[M]/P$ inherits the natural $\mathbb{Z}$-gradation and therefore this algebra is again defined by a homogeneous semigroup presentation.

Moreover, our construction implies that every M/ρ_P is contained in a product $B^i\times\mathbb{Z}^j$ for some i,j, where $B=\langle p,q\colon qp=1\rangle$ is the bicyclic monoid. The latter plays an important role in ring theory and in semigroup theory, [4, 15]. We then show that M embeds into the product $\prod_P K[M]/P$, where P runs over the set of all minimal primes in $K[M]$. Hence M embeds into some $B^d\times\mathbb{Z}^e$. However, the algebra $K[M]$ is not semiprime if $n\geq 3$. This entirely new representation of the Chinese monoid M implies in particular that M satisfies certain explicitly given semigroup identity. Since the leaves of D correspond to diagrams of certain special type, one can enumerate the minimal primes of $K[M]$. It turns out that their number is equal to the so called n-th Tribonacci number. Moreover, the description of minimal primes P of $K[M]$ allows us to prove that every $K[M]/P$ is semiprimitive. In particular, the prime radical of $K[M]$ coincides with the Jacobson radical.

1. Special types of ideals and congruences

In this Section, two families of ideals of $K[M]$ are defined in Part 1.1. They will play a crucial role in the approach developed in the paper. Basic properties of these ideals are then presented in Part 1.2. Throughout, M stands for a Chinese monoid of rank $n \geq 3$.

1.1. Ideals of $\heartsuit$ and $\diamondsuit$ type.

We start with describing certain relations that hold in $K[M]$.

1.1.1 Theorem. The following equalities hold in $K[M]$:

$$(a_ia_j - a_ja_i)K[M](a_ka_l - a_la_k) = 0 \quad \text{for} \quad i > j \geq k > l \tag{3}$$

$$(a_ia_j - a_ja_i)K[M](a_{j+1}a_l - a_la_{j+1})a_m = 0 \quad \text{for} \quad i \geq j+1 > j \geq m > l \tag{4}$$

$$a_m(a_ia_j - a_ja_i)K[M](a_{j+1}a_l - a_la_{j+1}) = 0 \quad \text{for} \quad i > m \geq j+1 > j \geq l \tag{5}$$

Proof. We use the canonical form (2) of elements of M. To shorten the notation, we write only i instead of a_i. Also, we write each exponent as $*$ if it may be equal 0 and as $+$ if it is positive. Thus, the canonical form of an element $w \in M$ is

$$w = (1)^* \ (21)^*(2)^* \ (31)^*(32)^*(3)^* \dots (n1)^*(n2)^* \dots (n)^*$$

and the desired equalities may be written as

$$(ij - ji)\ w\ (kl - lk) = 0 \quad \text{for} \quad i > j \geq k > l \tag{3}$$

$$(ij - ji)\ w\ (kl - lk)m = 0 \quad \text{for} \quad i \geq k = j+1 > j \geq m > l \tag{4}$$

$$m(ij - ji)\ w\ (kl - lk) = 0 \quad \text{for} \quad i > m \geq k = j+1 > j \geq l \tag{5}$$

Notice that all three equalities are of the form $\alpha w \beta = 0$. We proceed by induction on the length of w. If w has length 0, so it is the unity of M, by using the defining relations and bringing the involved elements of M to the canonical form we get

$$\begin{aligned}(ij - ji)(kl - lk) &= (ijk)l - (ijl)k - j(ikl) + j(ilk) = \\ &= j(ikl) - j(il)k - j(il)k + (jk)(il) = (jk)(il) - (jk)(il) - (jk)(il) + (jk)(il) = 0,\end{aligned}$$

and similarly

$$\begin{aligned}(ij - ji)(kl - lk)m &= (ijk)lm - (ijl)(km) - j(ikl)m + j(il)(km) = \\ &= j(ikl)m - (il)j(km) - j(km)(il) + j(km)(il) = j(km)(il) - j(km)(il) = 0,\end{aligned}$$

$$\begin{aligned}m(ij - ji)(kl - lk) &= m(ijk)l - (mj)(ikl) - m(ijl)k + (mj)(ilk) = \\ &= mk(ijl) - k(mj)(il) - (mj)k(il) + k(mj)(il) = k(mj)(il) - k(mj)(il) = 0.\end{aligned}$$

So, assume the length of w is positive and assume the equalities hold for all $w' \in M$ shorter than w. The following regularities hold in all three cases being considered.

If y is the last letter of w and $y \geq k$, then $y(kl - lk) = k(yl) - k(yl) = 0$, which completes the proof. So assume that $y < k$. Two possibilities can occur.

1) If

$$w = (1)^* \ (21)^*(2)^* \ (31)^*(32)^*(3)^* \dots (y1)^*(y2)^* \dots (y)^+$$

then all letters of the word w are smaller than k, so not greater then j, in particular this applies to the first letter — we denote it by x. Therefore, $(ij - ji)x = j(ix) - j(ix) = 0$, which proves the assertion.

2) We now assume that for some $x > y$

$$w = (1)^* \ (21)^*(2)^* \ (31)^*(32)^*(3)^* \dots (x1)^*(x2)^* \dots (xy)^+.$$

If $x < k$ then again the first letter of w is smaller than k, so not greater then j. As above, we obtain $(ij - ji)x = j(ix) - j(ix) = 0$, as desired. Hence, assume that $x \geq k$. We know that $k > l$ and $k > y$. Two possibilities arise.

2a) $l \geq y$, so $x \geq k > l \geq y$. Let w' be the initial subword of the word w such that $w = w'(xy)$. Then w' is shorter than w, so by the induction hypothesis $\alpha w' \beta = 0$. Moreover, in all three equalities $\alpha w \beta = 0$ considered above, xy commutes with β, because xy commutes with all letters of z such that $x > z > y$. Thus we get

$$\alpha w \beta = \alpha(w'(xy))\beta = (\alpha w' \beta)(xy) = 0 \cdot (xy) = 0,$$

which completes the proof in this case.

2b) $l < y$, so $x \geq k > y > l$. Then

$$\begin{aligned}(xy)(kl - lk) &= (xyk)l - (xyl)k = (kxy)l - (yxl)k = k(xyl) - y(xlk) = \\ &= k(yxl) - y(kxl) = ky(xl) - yk(xl) = (ky - yk)(xl),\end{aligned}$$

and xl also commutes with all m such that $x > m > l$. Therefore, $(xy)\beta = \beta'(xl)$, where β' is of the same form as β, but has an y instead of the l.

Thus the following equality holds:

$$\alpha w \beta = \alpha w'(xy)\beta = \alpha w' \beta'(xl).$$

If $y < m$, the indices i, j, k, y, m in α, β' satisfy the inequalities mentioned in hypotheses of the theorem. Then $\alpha w \beta = \alpha w' \beta'(xl) = 0$ holds by the induction hypothesis, because w' is shorter than w. This completes the proof in this case.

If $y \geq m$ (this can occur in the case of equality (4)), we have $\beta' = (ky - yk)m = y(km) - y(km) = 0$. Therefore in this case the desired equality holds as well. This completes the proof. ∎

1.1.2 Notation. Pairs of elements $\alpha, \beta \in K[M]$ satisfying $\alpha K[M] \beta = 0$ and of a form as in Theorem 1.1.1 will be called *pairs of type* $\boxplus$. We denote by $\boxplus = \{(\alpha_i, \beta_i) \colon i \in I\}$ the set of all such pairs; I is a finite set of indices.

If P is a prime ideal of $K[M]$, then for each of the equalities $\alpha K[M] \beta = 0$ in Theorem 1.1.1 one of the elements α or β must belong to P. So P must contain a set of the form $\{\gamma_i \colon \forall_{i \in I} \, (\gamma_i = \alpha_i \text{ or } \gamma_i = \beta_i)\}$. In this manner we obtain a number of different sets $X_1, X_2, \ldots$. We shall use indices $\gamma_{i,j}$ for the elements of the set X_j. By (X_j) we denote the ideal generated by the set X_j in $K[M]$. So, let

$$P_j = (X_j) = \sum_{\gamma_{i,j} \in X_j} K[M]\gamma_{i,j}K[M] \triangleleft K[M].$$

Since every element γ_i is of the form $l_i - p_i$ for some $l_i, p_i \in M$, it follows that $P_j = \mathcal{I}_{\rho_j}$, where ρ_j is a congruence generated by the pairs (l_i, p_i) for $i \in I$.

1.1.3 Definition. An **ideal of $\heartsuit$ type**, for $s = 2, 3, \ldots, n-1$, is the ideal of $K[M]$ generated by the elements:

$$(\heartsuit) \qquad \begin{array}{lll} a_m a_i - a_i a_m & \text{for} & s \leq m, i, \\ a_l a_m - a_m a_l & \text{for} & l, m \leq s. \end{array}$$

Notice that, modulo such an ideal, the corresponding element a_s is central.

An **ideal of $\diamondsuit$ type**, for $s = 2, 3, \ldots, n$, is the ideal generated by the elements:

$$(\diamondsuit) \qquad \begin{array}{llll} a_m a_i - a_i a_m, & a_i a_{s-1} a_m - a_m a_{s-1} a_i & \text{for} & s \leq m, i, \\ a_l a_m - a_m a_l, & a_l a_s a_m - a_m a_s a_l & \text{for} & l, m \leq s-1. \end{array}$$

Notice that, modulo such an ideal, the corresponding element $a_s a_{s-1}$ is central.

We say a congruence ρ on M is of $\heartsuit$ or $\diamondsuit$ *type* if the ideal $\mathcal{I}_\rho$ o $K[M]$ generated by ρ is of $\heartsuit$ or $\diamondsuit$ type, respectively. We write $M_\rho = M/\rho$ in this case.

If $\mathcal{I}$ is an ideal of a ring R then $\overline{w}$ denotes the image of the element $w \in R$ in $R/\mathcal{I}$. Sometimes, to simplify notation, we shall write w instead of $\overline{w}$ if from the context it is clear that we mean the image in $R/\mathcal{I}$.

1.1.4 Theorem. Every prime ideal P in $K[M]$ contains one of the above mentioned $2n-3$ ideals $\mathcal{I}_\rho$ of $\heartsuit$ or $\diamondsuit$ type.

Proof. If all elements a_i in $K[M]$ commute modulo P, then $K[M]/P$ satisfies all equalities (3)-(5) of Theorem 1.1.1. Hence P contains all ideals of $\heartsuit$ and $\diamondsuit$ type.

Hence, we will assume that for some $u > v$ the element $a_u a_v - a_v a_u$ does not belong to the ideal P. Since P is prime, for each equality of the type $\alpha K[M]\beta = 0$ either $\alpha \in P$ or $\beta \in P$ holds. In particular, for an equality of type (3), all elements $a_k a_l - a_l a_k$ for $v \geq k > l$ and all $a_i a_j - a_j a_i$ for $i > j \geq u$ must belong to P.

Therefore, in $K[M]/P$, the elements $a_1, a_2, \ldots, a_v$ commute and $a_u, a_{u+1}, \ldots, a_n$ also commute. Let s be the smallest index greater than v and such that in $K[M]/P$ the element a_s does not commute with an element a_i for some $i \in \{1, 2, \ldots, s-1\}$. Such an s exists and $s \leq u$, because for $s = u$ the elements $a_s = a_u$ and a_v by assumption do not commute in $K[M]/P$. Since s is minimal, the elements $a_1, a_2, \ldots, a_{s-1}$ commute in $K[M]/P$.

Since a_s and a_i do not commute in $K[M]/P$, in the equalities of (3) type, in which $\beta = a_i a_s - a_s a_i$, we obtain $\alpha \in P$. Therefore, the elements $a_s, a_{s+1}, \ldots, a_n$ must commute in $K[M]/P$. Thus, we have found such an $s > 1$ that the elements $a_1, a_2, \ldots, a_{s-1}$ commute in $K[M]/P$ and the elements $a_s, a_{s+1}, \ldots, a_n$ also commute in $K[M]/P$, so P contains α or β for each equality of (3) type.

The prime ideal P must also contain α or β for each equality of (4) and (5) type. We know that P does not contain the element $a_u a_v - a_v a_u$, so elements a_u, a_v do not commute in $K[M]/P$.

Assume that in $K[M]/P$ the element a_{s-1} commutes with each of $a_s, a_{s+1}, \ldots, a_n$ (so, in view of the earlier assumptions, a_{s-1} is central in $K[M]/P$) or a_s commutes in $K[M]/P$ with all elements $a_1, a_2, \ldots, a_{s-1}$ (so, similarly a_s is central in $K[M]/P$). If in some equality (3)-(5) in one of the parentheses there is a noncommuting pair (so their difference is not in P), then the pairs in the other parentheses must commute in $K[M]/P$. Therefore, if elements $a_1, a_2, \ldots, a_{s-1}$ commute in $K[M]/P$, the elements $a_s, a_{s+1}, \ldots, a_n$ commute in $K[M]/P$ and one of the elements a_{s-1}, a_s is central in $K[M]/P$, then P contains an element α or β from each equality of (3)-(5) type. The properties described above lead to the conclusion that P contains some **ideal of $\heartsuit$ type**. There are $n-2$ such ideals (because $1 < s < n$).

Assume now that in $K[M]/P$ an element a_{s-1} does not commute with an element a_i for some $i \in \{s, s+1, \ldots, n\}$ and a_s does not commute with an element a_l for some $l \in \{1, 2, \ldots, s-1\}$.

Consider an equality of (4) type. The ideal P is prime, so it must contain α or β from that equality. This condition is of course fulfilled, if P contains the expression from one of the parentheses. However, if $j = s-1$ (so $j+1 = s$), by our assumption, there exists an i such that a_i, a_j do not commute in $K[M]/P$ and there exists an l such that a_{j+1}, a_l do not commute in $K[M]/P$. Therefore, both expressions in the parentheses in our equality may not belong to P. Then, if P is to satisfy the above condition for such $i \geq j+1 = s > s-1 = j \geq m > l$, it must contain $(a_{j+1}a_l - a_l a_{j+1})a_m$. This means that in $K[M]/P$ the following equality holds:

$$a_s a_l a_m = a_l a_s a_m \tag{6}$$

for every a_l not commuting with $a_{j+1} = a_s$ and every m such that $l < m \leq s-1$. Notice that if a_l, a_s commute in $K[M]/P$, this equality is of course also satisfied. So, we may rewrite condition (6) in a more general form (using the relations in M):

$$a_m a_s a_l = a_l a_s a_m \quad \text{for } l, m \text{ such that } \ l, m < s. \tag{7}$$

Similarly, for equalities of (5) type we obtain

$$a_m a_{s-1} a_i = a_i a_{s-1} a_m \quad \text{for } i, m \text{ such that } \ s-1 < m, i. \tag{8}$$

Therefore, in this case P must contain all elements $a_m a_s a_l - a_l a_s a_m$ for l, m such that $l, m < s$ and $a_m a_{s-1} a_i - a_i a_{s-1} a_m$ for i, m such that $s-1 < m, i$, as well as the previously mentioned elements $a_m a_i - a_i a_m$ for $s \leq m, i$ and $a_l a_m - a_m a_l$ for $l, m \leq s-1$. This means that P contains an **ideal of $\diamondsuit$ type**. Notice that there are $n-1$ such ideals (because $1 < s \leq n$). Moreover, the element $a_s a_{s-1}$ is central modulo such an ideal (so also modulo P). Namely, by (7) for $m = s-1$, it commutes in $K[M]/P$ with a_l for $l < s-1$, similarly by (8) for $m = s$, it commutes in $K[M]/P$ with a_i for $i > s$, an finally, by the equalities in M, it commutes with a_{s-1}, a_s.

We have considered all the possible cases. The result follows. ∎

1.2. **The form of M_ρ for ρ of $\heartsuit$ type and $M_\rho\langle(\overline{a_s a_{s-1}})^{-1}\rangle$ for ρ of $\diamondsuit$ type.**

1.2.1 Notation. For a congruence ρ of type $\heartsuit$ or $\diamondsuit$, let ψ be the natural homomorphism $M \to M_\rho = M/\rho$. We also denote by ψ its natural extension to a map $K[M] \to K[M_\rho]$. The image of an element $x \in K[M]$ under ψ will be also denoted by $\overline{x}$. Since ρ is a homogeneous congruence, in M_ρ we still have a natural $\mathbb{Z}$-gradation given by the lengths of words, so we may consider the degrees of elements of M_ρ.

For ρ of $\heartsuit$ or $\diamondsuit$ type, we denote the homomorphism ψ by $\psi_\heartsuit$ or $\psi_\diamondsuit$, respectively.

If ρ is of $\heartsuit$ type with the distinguished generator a_s then by M^s_{n-1} we denote the Chinese monoid with generators $a_1, \dots, a_{s-1}, a_{s+1}, \dots, a_n$ and by $\overline{M^s_{n-1}}$ its image under $\psi|_{M^s_{n-1}}$. Then it is easy to see that

$$\overline{M^s_{n-1}} = \psi|_{M^s_{n-1}}(M^s_{n-1}) = M^s_{n-1}/\left(\rho|_{M^s_{n-1}}\right) = M^s_{n-1}\Big/\left(\begin{smallmatrix} a_1,\dots,a_{s-1} \text{ commute} \\ a_{s+1},\dots,a_n \text{ commute}\end{smallmatrix}\right).$$

If ρ is of $\diamondsuit$ type with the distinguished generators a_s, a_{s-1}, by $M^{s-1,s}_{n-2}$ we denote the Chinese monoid with generators $a_1, \dots, a_{s-2}, a_{s+1}, \dots, a_n$ and by $\overline{M^{s-1,s}_{n-2}}$ its image under $\psi|_{M^{s-1,s}_{n-2}}$. Then, using Definition 1.1.3, it is easy to see that

$$\overline{M^{s-1,s}_{n-2}} = \psi|_{M^{s-1,s}_{n-2}}(M^{s-1,s}_{n-2}) = M^{s-1,s}_{n-2}/\left(\rho|_{M^{s-1,s}_{n-2}}\right) = M^{s-1,s}_{n-2}\Big/\left(\begin{smallmatrix} a_1,\dots,a_{s-2} \text{ commute} \\ a_{s+1},\dots,a_n \text{ commute}\end{smallmatrix}\right).$$

1.2.2 Remark. By Definition 1.1.3, we know that $\overline{a_s}$ is central in $K[M_\rho]$ for the congruence ρ of $\heartsuit$ type with the distinguished generator a_s.

1.2.3 Lemma. If the congruence ρ is of $\heartsuit$ type with the distinguished generator a_s, then the element $\overline{a_s}$ is regular in $K[M_\rho]$. Moreover, $M_\rho \simeq \overline{M^s_{n-1}} \times \langle\overline{a_s}\rangle \simeq \overline{M^s_{n-1}} \times \mathbb{N}$.

Proof. An easy degree argument shows that the element $\overline{a_s}$ is non-zero in $K[M_\rho]$.

By Remark 1.2.3, $\overline{a_s}$ is central in M_ρ and every element $w \in M_\rho$ is of the form $w = w_0 \cdot \overline{a_s}^k$, where $w_0 \in \overline{M^s_{n-1}}$ and $k \in \mathbb{N}$. Therefore $M_\rho = \overline{M^s_{n-1}} \cdot \langle\overline{a_s}\rangle$.

We now introduce in M_ρ a new relation $\overline{a_s} = 1$. Then the corresponding image of the whole M_ρ coincides with $\overline{M^s_{n-1}}$ and w_0 is the image of w. If $w = w_0 \cdot \overline{a_s}^k$ is equal to some $w' = w'_0 \cdot \overline{a_s}^{k'}$ for $w'_0 \in \overline{M^s_{n-1}}$, then their images after introducing the relation $\overline{a_s} = 1$ are also equal. Therefore $w_0 = w'_0$.

Moreover, if the elements w, w' are equal in M_ρ, then the exponents, with which $\overline{a_s}$ appears in them, must also be equal (by a degree argument). Therefore the equality $k = k'$ also holds. So the product $M_\rho = \overline{M^s_{n-1}} \cdot \langle\overline{a_s}\rangle$ is direct:

$$M_\rho \simeq \overline{M^s_{n-1}} \times \langle\overline{a_s}\rangle \simeq \overline{M^s_{n-1}} \times \mathbb{N}.$$

In particular $\overline{a_s}$ is a regular element in $K[M_\rho]$. ∎

For the future convenience, we reformulate the above lemma, introducing an additional notation.

1.2.4 Corollary. Let $\widehat{\psi}_\heartsuit \colon M \to \overline{M^s_{n-1}} \times \langle \overline{a_s} \rangle$ be the homomorphism defined by:

$$\begin{cases} \widehat{\psi}_\heartsuit(a_s) = (1, \overline{a_s}) \\ \widehat{\psi}_\heartsuit(a_i) = (\overline{a_i}, 1) \quad \text{for } i \neq s. \end{cases}$$

Let $\lambda \colon M_\rho \to \overline{M^s_{n-1}} \times \langle \overline{a_s} \rangle$ be the isomorphism $w \mapsto (w_0, \overline{a_s}^k)$ introduced in the proof of Lemma 1.2.3. Then $\widehat{\psi}_\heartsuit$ is an epimorphism, $ker(\widehat{\psi}_\heartsuit) = ker(\psi_\heartsuit)$ and the following diagram commutes:

$$\begin{array}{lll} M & \xrightarrow{\quad\widehat{\psi}_\heartsuit\quad} & \\ \downarrow{\scriptstyle \psi_\heartsuit} & & \\ M_\rho & \underset{\lambda}{\simeq} & \overline{M^s_{n-1}} \times \langle \overline{a_s} \rangle \simeq \overline{M^s_{n-1}} \times \mathbb{N} \end{array}$$

1.2.5 Remark. By Definition 1.1.3, we know that the element $\overline{a_s a_{s-1}}$ is central in $K[M_\rho]$ for the congruence ρ of $\diamondsuit$ type with distinguished generators a_{s-1}, a_s.

1.2.6 Lemma. The element $\overline{a_s a_{s-1}}$ is regular in $K[M_\rho]$, where ρ is of $\diamondsuit$ type with distinguished generators a_{s-1}, a_s.

Proof. A degree argument easily implies that $\overline{a_s a_{s-1}} \neq 0$. In view of Remark 1.2.5, it suffices to prove that for any elements $x, y \in M_\rho$, from the equality $\overline{a_s} x \overline{a_{s-1}} y = 0$ it follows that $x = y$.

As in the proof of Theorem 1.1.1, to simplify notation we shall write i instead of $\overline{a_i}$ and we shall write $*$ instead of the exponents (they may be equal 0). Then, by (2), we know that elements $w \in M$ have the canonical form

$$\begin{aligned} w = &(1)^* \\ &(21)^*(2)^* \\ &(31)^*(32)^*(3)^* \\ &\dots \\ &(n1)^*(n2)^* \dots (n)^*. \end{aligned}$$

Relations in M_ρ in particular imply that the generators $1, 2, \dots, s-1$ commute. Therefore, the element $w \in M_\rho$ may be written in the form

$$\begin{aligned} w = &(1)^*(2)^* \dots (s-1)^* \\ &(s1)^*(s2)^* \dots (s)^* \\ &(s+1\ 1)^*(s+1\ 2)^* \dots (s+1)^* \\ &\dots \\ &(n1)^*(n2)^* \dots (n)^*. \end{aligned}$$

A presentation in this form is not unique, because for example $i(sj) = j(si)$ for $i, j < s$, where the element j commutes with all j' for $j' \leq s-1$ and the element si commutes with all si' for $i' < s$. We can therefore perform all such possible changes for $j < i$, coming to the following form of w:

$$\begin{aligned} w = &(1)^*(2)^* \dots (i)^* \\ &(sj)^*(s\ j+1)^* \dots (s)^* \\ &(s+1\ 1)^*(s+1\ 2)^* \dots (s+1)^* \\ &\dots \\ &(n1)^*(n2)^* \dots (n)^*, \end{aligned}$$

where the first or the second row (or both) may disappear or in the second row only $(s)^*$ may be left. If in both of these rows some elements with non-zero exponents are left (other than $(s)^*$), then w may be written in the above form with i, j satisfying the condition $i \leq j \leq s-1$ and the exponents of (i) and of (sj) are positive.

Using the commutativity of elements $s, s+1, \dots, n$, each segment $(t1)^*(t2)^* \dots (t)^*$, for $t > s$, can be replaced by a segment of the form $(t1)^*(t2)^* \dots (t\ s-1)^*(s)^*(s+1)^* \dots (t)^*$. Moreover, the whole product $(s)^*(s+1)^* \dots (t)^*$ commutes with every product $(t'1)^*(t'2)^* \dots (t'\ s-1)^*$ for any $t' > t$, and $(s)^*$ commutes with the product $(s+1\ 1)^*(s+1\ 2)^* \dots (s+1\ s-1)^*$. Hence, w can be rewritten as

$$\begin{aligned} w = &(1)^*(2)^* \dots (i)^* \\ &(sj)^*(s\ j+1)^* \dots (s\ s-1)^* \\ &(s+1\ 1)^*(s+1\ 2)^* \dots (s+1\ s-1)^* \\ &\dots \\ &(n1)^*(n2)^* \dots (n\ s-1)^* \\ &(s)^*(s+1)^* \dots (n)^*, \end{aligned}$$

where the first or second row (or both) may disappear. If in both of these rows some elements with non-zero exponents are left, then w may be written in the above form with i, j satisfying the condition $i \leq j \leq s-1$ and the exponents of (i) and of (sj) are positive.

Each element of the form $(t\ s-1)^*$, for $t \geq s$, commutes with all elements of the form $(t'\ t'')^*$ for $t' > t$ and $t'' \leq s-1$. Thus, we obtain another form of w:

$$\begin{aligned} w = &(1)^*(2)^* \dots (i)^* \\ &(sj)^*(s\ j+1)^* \dots (s\ s-2)^* \\ &(s+1\ 1)^*(s+1\ 2)^* \dots (s+1\ s-2)^* \\ &\dots \\ &(n1)^*(n2)^* \dots (n\ s-2)^* \\ &(s\ s-1)^*(s+1\ s-1)^* \dots (n\ s-1)^* \\ &(s)^*(s+1)^* \dots (n)^*, \end{aligned}$$

where the first or second row (or both) may disappear. If in both of these rows some elements with non-zero exponents are left, then w may be written in the above form with i, j satisfying the condition $i \leq j \leq s-2$ and the exponents of (i) and of (sj) are positive. If some elements remain only in the first row, then in the above form we have $0 < i \leq s-1$, if only in the second row, then $0 < j \leq s-2$.

Note that this form is not unique, for example $(k\ s-1)l = (l\ s-1)k$ for $k, l > s-1$, where the element $(l\ s-1)$ commutes with all $(l'\ s-1)$ for $l' > s-1$ and the element k commutes with all $k' \geq s$. Thus we can perform all such possible changes for $k > l$, coming to the following form of w:

$$(\star) \qquad \begin{aligned} w =& (1)^*(2)^* \dots (i)^* \\ & (sj)^*(s\ j+1)^* \dots (s\ s-2)^* \\ & (s+1\ 1)^*(s+1\ 2)^* \dots (s+1\ s-2)^* \\ & \dots \\ & (n1)^*(n2)^* \dots (n\ s-2)^* \\ & (s\ s-1)^*(s+1\ s-1)^* \dots (k\ s-1)^* \\ & (l)^*(l+1)^* \dots (n)^*, \end{aligned}$$

where the first or second row (or both) may disappear. If in both of these rows some elements with non-zero exponents are left, then w may be written in the above form with i, j satisfying the condition $i \leq j \leq s-2$ and the exponents of (i) and of (sj) are positive. If some elements remain in only one of the first two rows, in the above form we have $i \leq s-1$ or $j \leq s-2$. Moreover, the last or the second last row (or both of them) may also disappear. If in both of these rows some elements remain with non-zero exponents, w may be written in the above form with k, l satisfying the condition $s \leq k \leq l$. If some elements remain in only one of the last two rows, then in the above form we have $k \geq s$ or $l \geq s$.

For example, for $s = 2$ the above algorithm leads to the following form of w:

$$\begin{aligned} w =& (1)^* \\ & (21)^*(31)^* \dots (k1)^* \\ & (l)^*(l+1)^* \dots (n)^*, \end{aligned}$$

where the conditions concerning the last two rows and the values of k and l are similar to those described above for an arbitrary s.

We shall prove by induction on n that the general form $(\star)$ of w is unique. By assumption $n \geq 3$.

First, consider the case $n = s = 3$. We get

$$\begin{aligned} w =& (1)^*(2)^* \\ & (3\ 1)^* \\ & (3\ 2)^* \\ & (3)^*, \end{aligned}$$

where at least one among the exponents of (2), $(3\ 1)$ is zero. Let

$$w = (1)^x(2)^y(3\ 1)^z(3\ 2)^t(3)^u, \quad W = (1)^X(2)^Y(3\ 1)^Z(3\ 2)^T(3)^U,$$

where either $y = 0$ or $z = 0$ and either $Y = 0$ or $Z = 0$. Assume that $w = W$.

If ρ is of type $\diamondsuit$, the relations introduced by factoring by ρ are $\overline{a_1a_2} = \overline{a_2a_1}$ and $\overline{a_1a_3a_2} = \overline{a_2a_3a_1}$. By introducing in M_ρ a new relation $\overline{a_1} = \overline{a_2}$, we obtain a homomorphism of M_ρ into the Chinese monoid $M_2 = \langle \overline{a_1}, \overline{a_3} \rangle$. So, we can use the canonical forms in M_2 of the images of w and W, by comparing the corresponding exponents. The image of w is $(1)^{x+y}(3\ 1)^{z+t}(3)^u$, where either $y=0$ or $z=0$, the image of W is $(1)^{X+Y}(3\ 1)^{Z+T}(3)^U$, where either $Y=0$ or $Z=0$. Therefore,

$$\begin{cases} x+y = X+Y \\ z+t = Z+T \\ u = U. \end{cases}$$

On the other hand, comparing the degrees of w and W with respect to their generators, we obtain

$$\begin{cases} x+z = X+Z \\ y+t = Y+T \\ z+t+u = Z+T+U. \end{cases}$$

If x, y, z, t, u are known and the equality $Y=0$ holds, we may calculate $U=u$, $T=y+t$, $X=x+y$ oraz $Z=z-y$. Since either $y=0$ or $z=0$ and the exponents are non-negative, the equality $Z=z-y$ yields $y=0$. Therefore $Z=z$, $T=t$, $X=x$, so all exponents in w and W are equal. The case of $Z=0$ is similar. Therefore, for $n=s=3$ the form $(\star)$ is unique, as was claimed.

A similar proof works in the case where $n=3$ and $s=2$.

Thus, assume that $n>3$ and that the above form $(\star)$ is unique for all Chinese monoids of rank $m<n$ and all congruences of type $\diamondsuit$ defined on them.

Assume first that $s \geq 3$. Then, as we already know, w is of the form

$$(\star) \qquad \begin{aligned} w = &(1)^{\alpha_1}(2)^{\alpha_2}\dots(i)^{\alpha_i}\dots(s-1)^{\alpha_{s-1}} \\ &(s1)^{\alpha_{s1}}\dots(sj)^{\alpha_{sj}}(s\ j+1)^{\alpha_{s\ j+1}}\dots(s\ s-2)^{\alpha_{s\ s-2}} \\ &(s+1\ 1)^{\alpha_{s+1\ 1}}(s+1\ 2)^{\alpha_{s+1\ 2}}\dots(s+1\ s-2)^{\alpha_{s+1\ s-2}} \\ &\dots \\ &(n1)^{\alpha_{n1}}(n2)^{\alpha_{n2}}\dots(n\ s-2)^{\alpha_{n\ s-2}} \\ &(s\ s-1)^{\alpha_{s\ s-1}}(s+1\ s-1)^{\alpha_{s+1\ s-1}}\dots(k\ s-1)^{\alpha_{k\ s-1}}\dots(n\ s-1)^{\alpha_{n\ s-1}} \\ &(s)^{\alpha_s}\dots(l)^{\alpha_l}(l+1)^{\alpha_{l+1}}\dots(n)^{\alpha_n}, \end{aligned}$$

where the following conditions $(\star\star)$ and $(\star\star\star)$ hold:

$(\star\star)$ $\alpha_1 = \alpha_2 = \ldots = \alpha_{s-1} = 0$ or

$\alpha_{s1} = \alpha_{s2} = \ldots = \alpha_{s\ s-2} = 0$ or

in the first and in the second row there exist non-zero exponents and

$$\begin{cases} \alpha_{i+1} = \alpha_{i+2} = \ldots = \alpha_{s-1} = 0 \\ \alpha_{s1} = \alpha_{s2} = \ldots = \alpha_{s\ j-1} = 0, \quad \text{where } i \leq j \leq s-2 \end{cases}$$

and

$(\star\star\star)$ $\alpha_{s\ s-1} = \alpha_{s+1\ s-1} = \ldots = \alpha_{n\ s-1} = 0$ or
$\alpha_s = \alpha_{s+1} = \ldots = \alpha_n = 0$ or
in the last and in the second last row there exist non-zero exponents and

$$\begin{cases} \alpha_{k+1\ s-1} = \alpha_{k+2\ s-1} = \ldots = \alpha_{n\ s-1} = 0 \\ \alpha_s = \alpha_{s+1} = \alpha_{l-1} = 0, \quad \text{where } s \leq k \leq l. \end{cases}$$

Let us now introduce a new relation $\overline{a_1} = \overline{a_2}$ in M_ρ. Then the relations in the new monoid $M_\rho/(\overline{a_1} = \overline{a_2})$ are exactly the same as in the Chinese monoid of rank $n-1$ with generators $\overline{a_2}, \ldots, \overline{a_n}$ and with relations of $\diamondsuit$ type for the distinguished elements $\overline{a_{s-1}}, \overline{a_s}$. Using notation of 1.2.1, we get natural isomorphisms $M_\rho/(\overline{a_1} = \overline{a_2}) \simeq \overline{M^{s-1,s}_{2,\ldots,n}} \simeq \overline{M^{s-2,s-1}_{n-1}}$.

In the new monoid $M_\rho/(\overline{a_1} = \overline{a_2})$, the image of w is

$$\begin{aligned}
\tilde{w} =& (2)^{\alpha_1+\alpha_2} \ldots (i)^{\alpha_i} \ldots (s-1)^{\alpha_{s-1}} \\
& (s2)^{\alpha_{s1}+\alpha_{s2}} \ldots (sj)^{\alpha_{sj}} (s\ j+1)^{\alpha_{s\ j+1}} \ldots (s\ s-2)^{\alpha_{s\ s-2}} \\
& (s+1\ 2)^{\alpha_{s+1\ 1}+\alpha_{s+1\ 2}} \ldots (s+1\ s-2)^{\alpha_{s+1\ s-2}} \\
& \ldots \\
& (n2)^{\alpha_{n1}+\alpha_{n2}} \ldots (n\ s-2)^{\alpha_{n\ s-2}} \\
& (s\ s-1)^{\alpha_{s\ s-1}} (s+1\ s-1)^{\alpha_{s+1\ s-1}} \ldots (k\ s-1)^{\alpha_{k\ s-1}} \ldots (n\ s-1)^{\alpha_{n\ s-1}} \\
& (s)^{\alpha_s} \ldots (l)^{\alpha_l} (l+1)^{\alpha_{l+1}} \ldots (n)^{\alpha_n},
\end{aligned}$$

where conditions $(\star\star)$ and $(\star\star\star)$ hold.

Assume we have an element w' such that $w' = w$ in M_ρ, which is also written in the form $(\star)$, with exponents denoted respectively by β_x or β_{xy}. To prove that the form $(\star)$ is unique, we need to show that the corresponding exponents α and β are equal. Let the image of w', after introducing the relation $\overline{a_1} = \overline{a_2}$, be equal to

$$\begin{aligned}
\tilde{w}' =& (2)^{\beta_1+\beta_2} \ldots (i')^{\beta_{i'}} \ldots (s-1)^{\beta_{s-1}} \\
& (s2)^{\beta_{s1}+\beta_{s2}} \ldots (sj')^{\beta_{sj'}} (s\ j'+1)^{\beta_{s\ j'+1}} \ldots (s\ s-2)^{\beta_{s\ s-2}} \\
& (s+1\ 2)^{\beta_{s+1\ 1}+\beta_{s+1\ 2}} \ldots (s+1\ s-2)^{\beta_{s+1\ s-2}} \\
& \ldots \\
& (n2)^{\beta_{n1}+\beta_{n2}} \ldots (n\ s-2)^{\beta_{n\ s-2}} \\
& (s\ s-1)^{\beta_{s\ s-1}} (s+1\ s-1)^{\beta_{s+1\ s-1}} \ldots (k'\ s-1)^{\beta_{k'\ s-1}} \ldots (n\ s-1)^{\beta_{n\ s-1}} \\
& (s)^{\beta_s} \ldots (l')^{\beta_{l'}} (l'+1)^{\beta_{l'+1}} \ldots (n)^{\beta_n},
\end{aligned}$$

where conditions analogous to $(\star\star)$ and $(\star\star\star)$ hold.

Since $w = w'$, also $\tilde{w} = \tilde{w}'$. Now, $\tilde{w}$ and $\tilde{w}'$ are elements of the monoid $\overline{M^{s-1,s}_{2,\ldots,n}}$. As noted before, this monoid is defined by relations of type $\diamondsuit$ on a Chinese monoid on $n-1$ generators. Therefore, by the induction hypothesis, w and w' are uniquely presented in the form $(\star)$. Thus $i = i'$, $j = j'$, $k = k'$, $l = l'$ (if they exist) and

$$
\begin{array}{ccccc}
\alpha_1+\alpha_2=\beta_1+\beta_2 & \dots & \alpha_i=\beta_{i'} & \dots & \alpha_{s-1}=\beta_{s-1}\\
\alpha_{s1}+\alpha_{s2}=\beta_{s1}+\beta_{s2} & \dots & \alpha_{sj}=\beta_{sj'} & \dots & \alpha_{s\ s-2}=\beta_{s\ s-2}\\
\alpha_{s+1\ 1}+\alpha_{s+1\ 2}=\beta_{s+1\ 1}+\beta_{s+1\ 2} & \dots & \dots & \dots & \alpha_{s+1\ s-2}=\beta_{s+1\ s-2}\\
\dots & \dots & \dots & \dots & \dots\\
\alpha_{n1}+\alpha_{n2}=\beta_{n1}+\beta_{n2} & \dots & \dots & \dots & \alpha_{n\ s-2}=\beta_{n\ s-2}\\
\alpha_{s\ s-1}=\beta_{s\ s-1} & \dots & \alpha_{k\ s-1}=\beta_{k'\ s-1} & \dots & \alpha_{n\ s-1}=\beta_{n\ s-1}\\
\alpha_s=\beta_s & \dots & \alpha_l=\beta_{l'} & \dots & \alpha_n=\beta_n,
\end{array}
$$

where conditions $(\star\star)$ and $(\star\star\star)$ hold.

Suppose $s>3$. Then we may introduce the relation $\overline{a_2}=\overline{a_3}$ in M_ρ instead of the relation $\overline{a_1}=\overline{a_2}$. We then obtain a system of equations similar to the one above. It can easily be checked that this system, combined with the one above, leads to the conclusion that all corresponding exponents α and β are equal. Therefore the form $(\star)$ of w is unique.

The last case to consider is $s\leq 3$. Suppose first that $s<n-1$.

We may then introduce a new relation $\overline{a_{n-1}}=\overline{a_n}$ in M_ρ. As in the case considered above, using the induction hypothesis and the commutativity of all elements of the form nx for arbitrary x, we can reach the conclusion that $i=i'$, $j=j'$, $k=k'$, $l=l'$ (if they exist) and that

$$
\begin{array}{ccccc}
\alpha_1=\beta_1 & \dots & \alpha_i=\beta_{i'} & \dots & \alpha_{s-1}=\beta_{s-1}\\
\alpha_{s1}=\beta_{s1} & \dots & \alpha_{sj}=\beta_{sj'} & \dots & \alpha_{s\ s-2}=\beta_{s\ s-2}\\
\alpha_{s+1\ 1}=\beta_{s+1\ 1} & \dots & \dots & \dots & \alpha_{s+1\ s-2}=\beta_{s+1\ s-2}\\
\dots & \dots & \dots & \dots & \dots\\
\alpha_{n-1\ 1}+\alpha_{n1}=\beta_{n-1\ 1}+\beta_{n1} & \dots & \dots & \dots & \alpha_{n-1\ s-2}+\alpha_{n\ s-2}=\beta_{n-1\ s-2}+\beta_{n\ s-2}\\
\alpha_{s\ s-1}=\beta_{s\ s-1} & \dots & \alpha_{k\ s-1}=\beta_{k'\ s-1} & \dots & \alpha_{n\ s-1}=\beta_{n\ s-1}\\
\alpha_s=\beta_s & \dots & \alpha_l=\beta_{l'} & \dots & \alpha_{n-1}+\alpha_n=\beta_{n-1}+\beta_n,
\end{array}
$$

where conditions $(\star\star)$ and $(\star\star\star)$ hold.

We may also introduce the relation $\overline{a_{n-2}}=\overline{a_{n-1}}$ instead of $\overline{a_{n-1}}=\overline{a_n}$. We then obtain a similar system of equations which, combined with the one above, leads to the conclusion that all corresponding exponents α and β are equal. Thus the form $(\star)$ of w is indeed unique.

The last case to consider is $n-1\leq s$ and $s\leq 3$. Then $n\leq 4$. For $n=3$ we know by the induction hypothesis that the claim is true. For $n=4$ we have to consider only the case $4-1\leq s\leq 3$, so $s=3$.

Let us introduce in M_4/ρ the relation $\overline{a_1}=\overline{a_2}$, which leads to the system of equations as above. Also, independently, we introduce the relation $\overline{a_3}=\overline{a_4}$, leading to another system of equations as above. These two systems of equations combined lead to the conclusion that

$$\begin{cases} \alpha_{31}+\alpha_{32}=\beta_{31}+\beta_{32} \\ \alpha_{41}+\alpha_{42}=\beta_{41}+\beta_{42} \\ \alpha_{31}+\alpha_{41}=\beta_{31}+\beta_{41} \\ \alpha_{32}+\alpha_{42}=\beta_{32}+\beta_{42} \end{cases}$$

and all other exponents α, β are, respectively, equal. Furthermore, either $\alpha_2 = 0$ or $\alpha_{31} = 0$ and either $\alpha_3 = 0$ or $\alpha_{42} = 0$, similarly for β.

Now we introduce in M_4/ρ the relations $\overline{a_2a_x} = \overline{a_xa_2}$ and $\overline{a_3a_x} = \overline{a_xa_3}$ for $x = 1, 2, 3, 4$. We obtain

$$\tilde{M} = {}^{M_4/\rho}\Big/_{\left(\substack{\overline{a_2}\ \text{central} \\ \overline{a_3}\ \text{central}}\right)} \simeq M_2 \times \langle \overline{a_2} \rangle \times \langle \overline{a_3} \rangle,$$

where $\overline{a_1}$ and $\overline{a_4}$ are the generators of M_2. In M_2 we have the canonical form of elements, so the elements of $M_2 \times \langle \overline{a_2} \rangle \times \langle \overline{a_3} \rangle$ may be written in the canonical form $(1)^*(41)^*(4)^*(2)^*(3)^*$. Therefore, for the any elements $w = w'$ in M_4/ρ and their images $\tilde{w} = \tilde{w}' \in \tilde{M}$, comparing the exponents in the canonical forms $\tilde{w}$ and $\tilde{w}'$ we obtain in particular $\alpha_{41} = \beta_{41}$. Combined with the system of equations obtained above for exponents α and β, this equality leads to the conclusion that all respective exponents are equal. Therefore the forms w and w' are identical, so the form $(\star)$ is indeed unique also in this case.

Therefore, we have a unique form $(\star)$ of elements in M_ρ. We shall now prove that the element $s\ (s-1)$ is regular. Suppose that for some elements w, w' the equality $s\ (s-1)\ w = s\ (s-1)\ w'$ holds and that the exponents of the element $s\ (s-1)$ in elements w, w' written in the form $(\star)$ are $\alpha_{s\ s-1}$ and $\beta_{s\ s-1}$, respectively. By Definition 1.1.3, the element $\overline{a_sa_{s-1}}$, denoted here by $s\ (s-1)$, is central in M_ρ. Therefore the exponents of the element $s\ (s-1)$ in $s\ (s-1)\ w$ and $s\ (s-1)\ w'$, written in the form $(\star)$, are equal to $\alpha_{s\ s-1}+1$ and $\beta_{s\ s-1}+1$, respectively. Since, by assumption, $s\ (s-1)\ w = s\ (s-1)\ w'$ and the form $(\star)$ is unique, the equality $\alpha_{s\ s-1}+1 = \beta_{s\ s-1}+1$ holds, so $\alpha_{s\ s-1} = \beta_{s\ s-1}$ also holds. All other exponents in the canonical forms of the elements $s\ (s-1)\ w$ and $s\ (s-1)\ w'$ are the same as in the elements w and w', so they are equal. Hence $w = w'$, which means that the element $s\ (s-1)$ is left regular. Since it is central, the assertion follows. ■

Note that Remark 1.2.5 and Lemma 1.2.6 imply that for ρ of $\diamondsuit$ type we can consider the central localization $M_\rho\langle(\overline{a_sa_{s-1}})^{-1}\rangle$.

1.2.7 Lemma. If ρ is a congruence of $\diamondsuit$ type with distinguished generators a_{s-1}, a_s, then there is an isomorphism

$$M_\rho\langle(\overline{a_sa_{s-1}})^{-1}\rangle \simeq \overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z}$$

which is the natural extension of the map $\lambda\colon M_\rho \to \overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z}$ defined by

$$\begin{cases} \lambda(\overline{a_{s-1}}) = (1, p, g) \\ \lambda(\overline{a_s}) = (1, q, 1) \\ \lambda(\overline{a_i}) = (\overline{a_i}, p, 1) \ \text{ for } i < s-1 \\ \lambda(\overline{a_j}) = (\overline{a_j}, q, 1) \ \text{ for } j > s. \end{cases}$$

Proof. Define the transformation $\widehat{\psi}_\diamondsuit \colon M \to \overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z}$ by

$$\begin{cases} \widehat{\psi}_\diamondsuit(a_{s-1}) = (1, p, g) \\ \widehat{\psi}_\diamondsuit(a_s) = (1, q, 1) \\ \widehat{\psi}_\diamondsuit(a_i) = (\overline{a_i}, p, 1) \text{ for } i < s-1 \\ \widehat{\psi}_\diamondsuit(a_j) = (\overline{a_j}, q, 1) \text{ for } j > s. \end{cases}$$

We will prove that $\widehat{\psi}_\diamondsuit$ is a homomorphism by checking each coordinate separately.

In the first coordinate we have the transformation $a_{s-1} \mapsto 1$, $a_s \mapsto 1$, $a_l \mapsto \overline{a_l} = \psi_\diamondsuit(a_l)$ for $l \neq s-1, s$. We shall check that under this map the images of generators satisfy all the relations satisfied by the generators, i.e. the defining relations of M: $a_i a_j a_k = a_i a_k a_j = a_j a_i a_k$ for $i \geq j \geq k$.

If all three indices i, j, k are equal to $s-1$ or s, then all the images of generators are equal 1 and satisfy all the relations.

If exactly two among the indices i, j, k are equal $s-1$ or s, then on both sides of each of the relations there is only the image of the generator with the third index. Therefore all the relations are satisfied.

If only the index i is equal $s-1$ or s, then

$$\begin{aligned} a_i a_j a_k &\mapsto 1\overline{a_j a_k}, \\ a_i a_k a_j &\mapsto 1\overline{a_k a_j}, \\ a_j a_i a_k &\mapsto \overline{a_j} 1 \overline{a_k} \end{aligned}$$

and all the images are equal, because, by the definition of $\psi_\diamondsuit$, $\overline{a_j}$ and $\overline{a_k}$ commute. Similarly, one verifies the case where only the index k is equal to $s-1$ or s.

If only the index j is equal $s-1$ or s, then

$$\begin{aligned} a_i a_j a_k &\mapsto \overline{a_i} 1 \overline{a_k}, \\ a_i a_k a_j &\mapsto \overline{a_i a_k} 1, \\ a_j a_i a_k &\mapsto 1\overline{a_i a_k} \end{aligned}$$

and in this case also all the images are equal.

If none of the indices i, j, k is equal to $s-1$ or s, then the images of the generators satisfy the respective relations, because $\psi_\diamondsuit$ is a homomorphism. This completes the proof of the fact that the first coordinate of $\widehat{\psi}_\diamondsuit$ is a homomorphism.

In the second coordinate we have $a_i \mapsto p$ for $i \leq s-1$, $a_j \mapsto q$ for $j \geq s$. As above, we verify that under this map, the images of the generators satisfy all the defining relations of M. Let $i \geq j \geq k$. If $k \geq s$, then the image of each of the elements $a_i a_j a_k$, $a_i a_k a_j$, $a_j a_i a_k$ is equal to q^3, thus these images are equal. Similarly, if $i \leq s-1$, then the images are equal to p^3. If $i \geq s > s-1 \geq j \geq k$, then

$$\begin{aligned} a_i a_j a_k &\mapsto qpp = p, \\ a_i a_k a_j &\mapsto qpp = p, \\ a_j a_i a_k &\mapsto pqp = p, \end{aligned}$$

thus also all the images are equal. Similarly, for $i \geq j \geq s > s-1 \geq k$ all the images are equal to q. Therefore the second coordinate of $\widehat{\psi_\diamondsuit}$ is indeed a homomorphism.

The third coordinate of $\widehat{\psi_\diamondsuit}$ is a homomorphism as well, because all the relations in M are homogeneous with respect to a_{s-1}. This completes the proof of the fact that $\widehat{\psi_\diamondsuit}$ is a homomorphism.

We shall now prove that $ker(\psi_\diamondsuit) \subseteq ker(\widehat{\psi_\diamondsuit})$. So, assume that $\psi_\diamondsuit(x) = \psi_\diamondsuit(y)$ for some $x, y \in M$. We will show that also $\widehat{\psi_\diamondsuit}(x) = \widehat{\psi_\diamondsuit}(y)$. It suffices to prove that $\widehat{\psi_\diamondsuit}(x) = \widehat{\psi_\diamondsuit}(y)$ for pairs (x, y) generating ρ.

For $i, k < s-1$ and the pair $(a_i a_k, a_k a_i)$, we obtain

$$\widehat{\psi_\diamondsuit}(a_i a_k) = (\overline{a_i}, p, 1) \cdot (\overline{a_k}, p, 1) = (\overline{a_i a_k}, p^2, 1)$$

and similarly $\widehat{\psi_\diamondsuit}(a_k a_i) = (\overline{a_k a_i}, p^2, 1)$. Therefore $\widehat{\psi_\diamondsuit}(a_i a_k) = \widehat{\psi_\diamondsuit}(a_k a_i)$, as claimed. The proof is similar for $j, l > s$ and the pair $(a_j a_l, a_l a_j)$.

For $i < s-1$ and the pair $(a_i a_{s-1}, a_{s-1} a_i)$, we obtain

$$\widehat{\psi_\diamondsuit}(a_i a_{s-1}) = (\overline{a_i}, p, 1) \cdot (1, p, g) = (\overline{a_i}, p^2, g) = \widehat{\psi_\diamondsuit}(a_{s-1} a_i).$$

Similarly for $j > s$ and the pair $(a_j a_s, a_s a_j)$.

For $i, k < s-1$ and the pair $(a_i a_s a_k, a_k a_s a_i)$, we obtain

$$\widehat{\psi_\diamondsuit}(a_i a_s a_k) = (\overline{a_i}, p, 1) \cdot (1, q, 1) \cdot (\overline{a_k}, p, 1) = (\overline{a_i a_k}, pqp, 1) = (\overline{a_i a_k}, p, 1) = \widehat{\psi_\diamondsuit}(a_k a_s a_i).$$

Similarly for $j, l > s$ and the pair $(a_j a_{s-1} a_l, a_l a_{s-1} a_j)$. If in the above cases $i = s-1$, $k = s-1$, $j = s$ or $l = s$, the proof is analogous.

We have thus completed the proof of the fact that $ker(\psi_\diamondsuit) \subseteq ker(\widehat{\psi_\diamondsuit})$.

Therefore, $\widehat{\psi_\diamondsuit}$ can be presented as the composition of the epimorphism $\psi_\diamondsuit$ and some homomorphism λ:

$$\begin{tikzcd}
M \arrow[r, "\widehat{\psi_\diamondsuit}"] \arrow[d, "\psi_\diamondsuit"'] & \overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z} \\
M_\rho \arrow[ur, dashed, "\lambda"'] &
\end{tikzcd}$$

The element $\overline{a_s a_{s-1}}$ is central in M_ρ and from Lemma 1.2.6 we know it is regular, therefore we can consider the localization $M_\rho\langle(\overline{a_s a_{s-1}})^{-1}\rangle$. The image $\lambda(\overline{a_s a_{s-1}}) = (1, 1, g)$ is an invertible element in $\overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z}$. Hence, we may consider the natural extension λ' of λ to the localization $M_\rho\langle(\overline{a_s a_{s-1}})^{-1}\rangle$.

We shall check that λ' is an epimorphism. We have

$$\lambda'(\overline{a_s a_{s-1}}) = (1, 1, g) \quad \text{and} \quad \lambda'((\overline{a_s a_{s-1}})^{-1}) = (1, 1, g^{-1}),$$

so also

$$\lambda'(\overline{a_{s-1}}(\overline{a_s a_{s-1}})^{-1}) = (1, p, g)(1, 1, g^{-1}) = (1, p, 1).$$

Moreover

$$\lambda'(\overline{a_s}) = (1, q, 1),$$

so also

$$\lambda'(\overline{a_s a_i}) = (1, q, 1)(\overline{a_i}, p, 1) = (\overline{a_i}, 1, 1) \quad \text{for} \quad i < s-1$$

and

$$\lambda'(\overline{a_j}(\overline{a_{s-1}}(\overline{a_s a_{s-1}})^{-1})) = (\overline{a_j}, q, 1)(1, p, 1) = (\overline{a_j}, 1, 1) \quad \text{for} \quad j > s.$$

Therefore, in the image of λ' we can obtain any value in each of the three coordinates separately. Thus λ' is an epimorphism.

Next, we prove that λ' determines an isomorphism $\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle \simeq B \times \mathbb{Z}$. Notice that by the definition of M_n and $\psi_\diamondsuit$, we have $\langle \overline{a_{s-1}}, \overline{a_s} \rangle \simeq \langle a_{s-1}, a_s \rangle \simeq M_2$, because $\psi_\diamondsuit|_{\langle a_{s-1}, a_s \rangle}$ is trivial. Let $M_2' = \langle \overline{a_{s-1}}, \overline{a_s} \rangle$, so that $M_2' \simeq M_2$. The localization of $\langle \overline{a_{s-1}}, \overline{a_s} \rangle$ with respect to $\langle (\overline{a_s a_{s-1}})^{-1} \rangle$ is $\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle$, so

$$\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle = M_2' \langle (\overline{a_s a_{s-1}})^{-1} \rangle.$$

Consider the restriction of λ' to $\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle$. We know that

$$\lambda'(\overline{a_{s-1}}) = (1, p, g), \quad \lambda'(\overline{a_s}) = (1, q, 1), \quad \lambda'((\overline{a_s a_{s-1}})^{-1}) = (1, 1, g^{-1}),$$

therefore, a proof similar to the one above shows that this restriction of λ' is an epimorphism onto $\{1\} \times B \times \mathbb{Z}$.

We shall check that it is also an injection. Each element $w \in \langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle = M_2' \langle (\overline{a_s a_{s-1}})^{-1} \rangle$ can be written in the form $w = \overline{a_{s-1}}^k \overline{a_s}^l (\overline{a_s a_{s-1}})^{-m}$, where $k, l \in \mathbb{N}$, $m \in \mathbb{Z}$. We then obtain $\lambda'(w) = (1, p^k q^l, g^{k-m})$. If for some element $v = \overline{a_{s-1}}^{k'} \overline{a_s}^{l'} (\overline{a_s a_{s-1}})^{-m'}$ the equality $\lambda'(v) = \lambda'(w) = (1, p^k q^l, g^{k-m})$ holds, then from the uniqueness of the canonical forms of elements of B and $\mathbb{Z}$ it follows that $k = k'$, $l = l'$, $k - m = k' - m'$, so also $m = m'$ and thus $w = v$.

Therefore the considered restriction of λ' is indeed an injection. Since we know it is a surjection, it is an isomorphism and thus

$$M_2' \langle (\overline{a_s a_{s-1}})^{-1} \rangle = \langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle \overset{\lambda'}{\simeq} \{1\} \times B \times \mathbb{Z} \simeq B \times \mathbb{Z}.$$

We shall now prove that

$$M_\rho \langle (\overline{a_s a_{s-1}})^{-1} \rangle \simeq \overline{C} \times B \times \mathbb{Z},$$

where

$$C \overset{\text{def}}{=} \langle a_s a_i, a_j a_{s-1} \colon i < s-1; s < j \rangle \subseteq M.$$

First we check that

$$M_\rho \langle (\overline{a_s a_{s-1}})^{-1} \rangle = \overline{C} \cdot \langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle.$$

By the relations in the Chinese monoid, the following equalities hold in M:

$$(a_s a_i) a_{s-1} = a_{s-1}(a_s a_i), \ (a_s a_i) a_s = a_s(a_s a_i)$$

for $i \leq s-1$ and similarly

$$(a_j a_{s-1}) a_s = a_s(a_j a_{s-1}), \ (a_j a_{s-1}) a_{s-1} = a_{s-1}(a_j a_{s-1})$$

for $j \geq s$. Analogous equalities hold in M_ρ. Therefore, each element of the set $\overline{C} = \langle \overline{a_s a_i}, \overline{a_j a_{s-1}} \colon i < s-1; \ s < j \rangle$ commutes with all elements of the set $\langle \overline{a_{s-1}}, \overline{a_s} \rangle$.

In the localization $M_\rho \langle (\overline{a_s a_{s-1}})^{-1} \rangle$ the following equalities hold:

$$\overline{a_i} = (\overline{a_s a_{s-1}})^{-1} (\overline{a_s a_i}) \overline{a_{s-1}} \quad \text{for } i < s-1,$$
$$\overline{a_j} = (\overline{a_s a_{s-1}})^{-1} (\overline{a_j a_{s-1}}) \overline{a_s} \quad \text{for } j > s.$$

Hence $M_\rho \subseteq \langle \overline{C}, \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1}\rangle$, thus also $M_\rho\langle(\overline{a_s a_{s-1}})^{-1}\rangle \subseteq \overline{C}\cdot\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1}\rangle$. The opposite inclusion holds by the definition of C, thus we obtain

$$M_\rho\langle(\overline{a_s a_{s-1}})^{-1}\rangle = \overline{C}\cdot\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1}\rangle. \tag{9}$$

Let $\beta\colon \overline{M_{n-2}^{s-1,s}} \to \overline{M}$ be the map defined by

$$\begin{cases} \beta(\overline{a_i}) = \overline{a_s a_i} & \text{for } i < s-1 \\ \beta(\overline{a_j}) = \overline{a_j a_{s-1}} & \text{for } j > s. \end{cases}$$

We shall check that β is a homomorphism. It suffices to check that all relations of $\overline{M_{n-2}}$ hold also for the images of elements in $\overline{M}$. The relations in $\overline{M_{n-2}}$ are the relations of the Chinese monoid M_{n-2} and the relations introduced by ρ. To simplify notation, instead of $\overline{a_x}$ we shall write only x. Then in $\overline{M_{n-2}}$ we have:

1) relations from ρ: commutativity of elements $1,\dots,(s-2)$,

2) relations from ρ: commutativity of elements $(s+1),\dots,n$,

3) relations from the Chinese monoid: $zyx = zxy = yzx$ for $x \le y \le z$.

Notice that if $z < s-1$ or $x > s$, then the relations listed in (3) follow from the relations from (1) and (2). Therefore instead of (3) we can consider only:

3.1) if $x \le y < s < z$, then $zyx = yzx$,

3.2) if $x < s < y \le z$, then $zyx = zxy$.

In $\overline{M}$ we have relations of the Chinese monoid M and the relations introduced by ρ. We shall now check that the images of elements in $\overline{M}$ satisfy relations stated in (1), (2), (3.1) and (3.2).

1) by the definition of β for $i,k < s-1$ we have $\beta i = si$, $\beta(k) = sk$, so using the relations in $\overline{M}$ we obtain $\beta(i)\beta(k) = sisk = s(isk) = s(ksi) = sksi = \beta(k)\beta(i)$; therefore the images of elements $1,\dots,(s-2)$ commute.

2) by an analogous argument.

3.1) for $x \le y \le s-1 < s < z$, using the Chinese relations in $\overline{M}$, we obtain

$$\begin{aligned}\beta(z)\beta(y)\beta(x) = z(s-1)sysx = z(sx)(s-1)sy = (zx)s(s-1)(sy) = \\ = (sy)zsx(s-1) = syz(s-1)sx = \beta(y)\beta(z)\beta(x).\end{aligned}$$

3.2) by an analogous argument.

Thus, we have verified that β is indeed a homomorphism.

From the definitions of β and C we obtain that $\beta\colon \overline{M_{n-2}^{s-1,s}} \twoheadrightarrow \overline{C}$. Therefore $\overline{C}$ is the homomorphic image of the monoid $\overline{M_{n-2}^{s-1,s}}$.

We may now define the natural homomorphism

$$\beta'\colon \overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z} \to \overline{M} \times B \times \mathbb{Z}$$

as β on the first coordinate $\overline{M_{n-2}^{s-1,s}}$ and identity on $B\times\mathbb{Z}$. Therefore, $\beta'(\overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z}) = \overline{C}\times B\times\mathbb{Z}$. Earlier we have shown that $\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1}\rangle \simeq B\times\mathbb{Z}$. Therefore,

$$\overline{C}\times B\times\mathbb{Z} \simeq \overline{C}\times\langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1}\rangle. \tag{10}$$

The composition of the epimorphisms λ' and β' gives a natural epimorphism

$$\beta'\lambda'\colon M_\rho\langle(\overline{a_s a_{s-1}})^{-1}\rangle \twoheadrightarrow \overline{C}\times B\times\mathbb{Z}. \tag{11}$$

We also have a natural epimorphism

$$\overline{C} \times \langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle \twoheadrightarrow \overline{C} \cdot \langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle. \tag{12}$$

Using (9)-(12) we obtain the commutative diagram

$$\begin{tikzcd}
M_\rho\langle (\overline{a_s a_{s-1}})^{-1} \rangle \arrow[d, two heads, "\beta'\lambda'"'] & = & \overline{C} \cdot \langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle \\
\overline{C} \times B \times \mathbb{Z} & \simeq & \overline{C} \times \langle \overline{a_{s-1}}, \overline{a_s}, (\overline{a_s a_{s-1}})^{-1} \rangle \arrow[u, two heads]
\end{tikzcd}$$

Therefore both maps in (11) and (12) must be isomorphisms. Thus in particular

$$M_\rho\langle (\overline{a_s a_{s-1}})^{-1} \rangle \simeq \overline{C} \times B \times \mathbb{Z}.$$

Denote this isomorphism by α, so $\alpha = \beta'\lambda'$. Then we have the commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "\widehat{\psi}_\diamond"] \arrow[d, two heads, "\psi_\diamond"'] & & \overline{M^{s-1,s}_{n-2}} \times B \times \mathbb{Z} \arrow[rrd, two heads, "\beta'"] & & \\
M_\rho \arrow[rru, "\lambda"] & \subseteq & M_\rho\langle (\overline{a_s a_{s-1}})^{-1} \rangle \arrow[u, two heads, "\lambda'"'] & \underset{\alpha}{\simeq} & \overline{C} \times B \times \mathbb{Z}
\end{tikzcd}$$

This means that β' and λ' are isomorphism, which in particular leads to the conclusion that $M_\rho\langle (\overline{a_s a_{s-1}})^{-1} \rangle \simeq \overline{M^{s-1,s}_{n-2}} \times B \times \mathbb{Z}$. This completes the proof. ■

Notice that $\overline{M^{s-1,s}_{n-2}} \subseteq M_\rho$, but the factor $\overline{M^{s-1,s}_{n-2}}$ in the image of λ', i.e. in $\overline{M^{s-1,s}_{n-2}} \times B \times \mathbb{Z}$, is not the same object.

2. Minimal prime ideals in $K[M]$

In this Section, a bijection between the set of minimal prime ideals of $K[M]$ and the set of leaves of a certain tree D is established. More precisely, the elements of D are defined as diagrams of some special type, which correspond in a constructive way to certain homogeneous congruences on M. In particular, it follows that every minimal prime ideal P of $K[M]$ is of the form $P = \mathcal{I}_{\rho_P}$, where ρ_P is the congruence on M defined by $\rho_P = \{(s,t) \in M \times M \colon s - t \in P\}$. Therefore $K[M]/P \simeq K[M/\rho_P]$. The construction implies also that M/ρ_P embeds into the monoid $\mathbb{N}^{c_P} \times (B \times \mathbb{Z})^{d_P}$, where $c_P + 2d_P = n$. In Part 2.1 the tree D is introduced. In Parts 2.2 and 2.3 some intermediate steps are proved. The main result is summarized in Theorem 2.3.2.

Recall that if the rank n of the monoid M is equal to 1 or 2 then the algebra $K[M]$ is prime and semiprimitive, [3]. Hence, as before, we shall assume that $n \geq 3$.

2.1. **Diagrams and the tree D.**

2.1.1 Notation. We start with defining certain auxiliary *diagrams*, built on the set of n generators $a_1, a_2, \ldots, a_n$ of M. Let $\underset{i}{\circ}$ denote the i-th generator. The simplest diagram is of the form

$$\underset{1}{\circ}\ \underset{2}{\circ}\ \cdots\ \underset{n}{\circ}$$

If unambiguous, we omit the indices, denoting the above diagram also by

$$\circ \cdots \circ$$

The next simple diagrams are of the form

$$\circ \cdots \circ \;\bullet\; \circ \cdots \circ$$

with a distinguished generator a_s. A diagram of this type will be called a *dot* a_s or simply a *dot*. We consider such diagrams only for $s = 2, 3, \ldots, n-1$. If the number of generators is $k < n$, such a diagram is called *dot*$_k$.

A diagram of the form

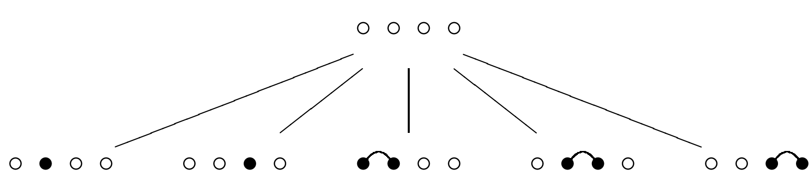
$$\circ \cdots \circ \;\overset{\frown}{\bullet\bullet}\; \circ \cdots \circ$$

with an arc joining generators a_{s-1} and a_s is called an *arc* $a_s a_{s-1}$ or simply an *arc*. Here s can be any of the numbers $2, \ldots, n$. If the number of generators is $k < n$, such a diagram is called an *arc*$_k$.

Next we construct more complicated diagrams. It turns out that all considered diagrams can be organized in a tree D, which indicates the order and the way these diagrams are constructed.

2.1.2 Definition. We construct a finite tree D whose vertices are diagrams. The construction is performed in several steps. We start with defining the root of D, then in the first step we connect it by edges with certain new diagrams, which treated as vertices of D form the *first level of* D. In the next steps we build the subsequent levels of D.

- We start with the vertex corresponding to the first of the diagrams described in 2.1.1; this vertex is called the *root* of D,
- in the first step we connect the root with $2n-3$ vertices: $n-2$ diagrams which are dots and $n-1$ diagrams which are arcs (in the sense of 2.1.1); for example, if $n = 4$, then we get the first level of D:



- generators involved in the construction of the appropriate dots or arcs are marked in black and are called the *used* generators, while the other generators are called *unused*,
- in the next steps we construct the subsequent *levels* of D, in each step adding, as vertices of D, more complicated diagrams constructed according to the following rules $\bigoplus$ and $\bigodot$.

Rules $\bigoplus$:

- in every diagram each generator can be used at most once,

- if a diagram has k unused generators, we connect to it, as vertices of D, all diagrams obtained by adding a dot$_k$ or an arc$_k$ (for these k generators), in a way allowed by the remaining rules; according to 2.1.1, neither a_1 nor a_n can be used as a dot.
- if in a diagram there is an arc using one of the extreme generators a_1 or a_n, then we do not connect any new vertices of D to this vertex and we call such an arc an *extreme arc*, and the corresponding vertex – a *leaf* of D,
- an arc$_k$ (for some k unused generators) can be added only *above*, which means that this arc connects the two generators that are neighbors of some used generators. (As we shall see in Remark 2.1.4, in every step of the construction used generators have indices ranging from j to $j+i$ for some $j > 0$, $i \geq 0$, so that the two neighboring generators are well defined). We get a diagram of the form

  $$\circ \cdots \circ \bullet \; \begin{smallmatrix} n-k \text{ used} \\ \text{generators} \end{smallmatrix} \; \bullet \circ \cdots \circ$$

  We denote i subsequent used generators by $< i >$, so that the above diagram is simply written as

  $$\circ \cdots \circ \bullet < n-k > \bullet \circ \cdots \circ$$

- if in a given step we do not add to some diagram an arc above, and this diagram is not a leaf of D then we have to add a dot obeying rules $\odot$.

Rules $\odot$:

- after an arc$_k$ a dot$_{k-2}$ can only follow next to this arc, in other words, after the diagram whose last step of construction was an arc$_k$

  $$\circ \cdots \circ \bullet < n-k > \bullet \circ \cdots \circ$$

  we can either have the diagram

  $$\circ \cdots \circ \bullet \bullet < n-k > \bullet \bullet \circ \cdots \circ$$

  or one of the following two diagrams

  $$\circ \cdots \circ \bullet < n-k > \bullet \;\bullet \circ \cdots \circ \quad \text{or} \quad \circ \cdots \circ \bullet\; \bullet < n-k > \bullet \circ \cdots \circ$$

- after a dot$_k$, for $k < n$, the next dot$_{k-1}$ can occur only as a direct neighbor of the former dot; in other words, after a diagram

  $$\circ \cdots \circ < n-k > \bullet \circ \cdots \circ$$

  whose last step of construction was the indicated dot$_k$, either a diagram of the following form can follow

  $$\circ \cdots \circ \bullet < n-k > \bullet \bullet \circ \cdots \circ$$

  or the following diagram can follow

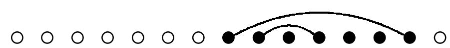
  $$\circ \cdots \circ < n-k > \bullet \bullet \circ \cdots \circ$$

- immediately after a dot in the first level of D only an arc$_{n-1}$ above can be added, so after a diagram

  $$\circ \cdots \circ \bullet \circ \cdots \circ$$

the following diagram can only follow

2.1.3 Example. The following diagrams are vertices of some trees D (for $n = 15$ and 9, respectively)

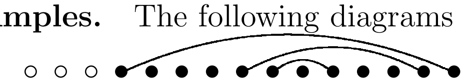


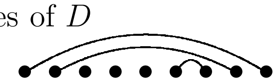
2.1.4 Remark. The tree D is finite. In every step of the above construction the used generators have indices $j, \ldots, j+i$ for some $i \geq 0$, $j > 0$. The order in which all dots and arcs were added can be uniquely determined from the form of a given diagram. The generators a_1 or a_n can only be used as elements of an arc, and such an arc is an extreme arc. A leaf of D is a vertex in which an extreme arc has appeared.

2.1.5 Definition. *A branch* in D is a chain of connected vertices, leading from the root to some vertex d. If d is a leaf then such a branch is called *maximal.*

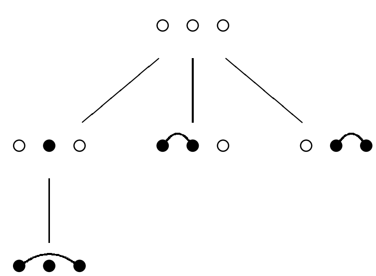
If a vertex d_2 was connected to a vertex d_1 in the process of construction of D, then d_2 is called a *descendant* of the vertex d_1.

2.1.6 Examples. The following diagrams are leaves of D

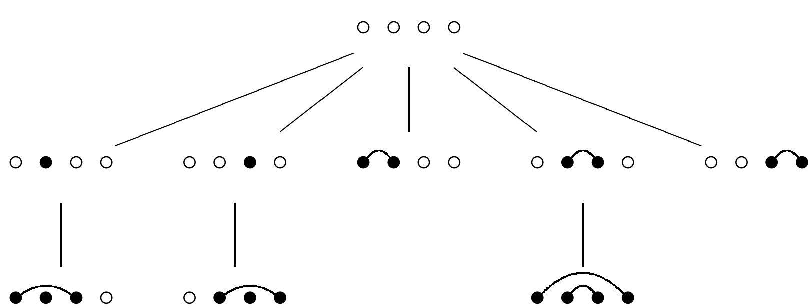




For $n = 3$ the tree D has the form



while for $n = 4$ the tree D has the form



2.2. **Diagrams as congruences on M.**

In this part we show that every leaf of D determines a minimal prime ideal of $K[M]$.

2.2.1 Notation. If $u < v$, by $M_{i_j}^{u,v}$ we denote the Chinese monoid with i_j generators $a_1, \ldots, a_{u-1}, a_{v+1}, \ldots, a_n$; so that $i_j = n - v + u - 1$. Sometimes we denote this monoid simply by M_{i_j}, if it is clear from the context or inessential which of the generators $a_1, \ldots, a_n$ are skipped. This generalizes the notation used earlier: M_{n-1}^{s} and $M_{n-2}^{s-1,s}$. Indices i_j will be helpful because we shall build sequences of congruences ρ_j for $j = 0, 1, 2, \ldots$ and monoids M_{i_j} corresponding to these congruences.

Recall that ρ_0 denotes the trivial congruence on M. For a congruence ρ on M, by M_{i_j}/ρ we mean $M_{i_j}/(\rho|_{M_{i_j}})$.

For a given congruence ρ_j, let $\psi_j \colon M \to M/\rho_j$ be the natural epimorphism. For every $x \in M$ we write $\psi_j(x) = \widehat{x}^j$. In particular, for $x \in M_{i_j}$ by $\widehat{x}^j$ we mean the image of x in $M_{i_j}/\rho_j = M_{i_j}/(\rho_j|_{M_{i_j}})$. With this notation, $M/\rho_0 = M$, $\psi_0 = id$, $\widehat{x}^0 = x$. If ρ_1 is a congruence of type $\heartsuit$ or $\diamondsuit$ on M, then $\widehat{x}^1 = \overline{x} = \psi(x)$, where $\psi = \psi_1 \colon M \to M/\rho_1$ is the natural homomorphism.

2.2.2 Definition. We define inductively the following sequences of pairs (S_t, i_t) for $t \geq 1$, $i_t > 0$. Let $i_0 = n$ and

$$\begin{cases} S_1 = \mathbb{N} \\ i_1 = n-1 \end{cases} \qquad \text{or} \qquad \begin{cases} S_1 = B \times \mathbb{Z} \\ i_1 = n-2 \end{cases}$$

and for every $t > 1$ let

$$\begin{cases} S_t = S_{t-1} \times \mathbb{N} \\ i_t = i_{t-1} - 1 \end{cases} \qquad \text{or} \qquad \begin{cases} S_t = S_{t-1} \times B \times \mathbb{Z} \\ i_t = i_{t-1} - 2. \end{cases}$$

Every such sequence (S_t, i_t) is clearly finite. In each of the pairs, S_t is a direct product of $n - i_t$ factors.

For example, $(\mathbb{N}, n-1)$, $(\mathbb{N} \times B \times \mathbb{Z}, n-3)$, $(\mathbb{N} \times B \times \mathbb{Z} \times \mathbb{N}, n-4)$, $\ldots$ are initial elements of a sequence of pairs.

2.2.3 Construction. With each of the diagrams defined in part 2.1 we associate in a natural way a congruence on M such that if a vertex d_1 is a descendant of a vertex d_2 in the tree D, then the congruence corresponding to the diagram d_1 contains the congruence corresponding to the diagram d_2.

Proof. We proceed by induction. We adopt an induction hypothesis consisting of five parts and we immediately verify the validity of the first inductive step.

Part (I). Consider the diagrams described in 2.1.1. With the diagram $\circ \cdots \circ$ we associate the trivial congruence ρ_0. With a diagram from the first level of the tree D (so a dot or an arc) we associate congruences ρ_1 of type $\heartsuit$ and $\diamondsuit$, respectively, with appropriate values of the distinguished index s. Clearly, all such ρ_1 satisfy $\rho_0 \subseteq \rho_1$.

Hence, assume inductively that we already know congruences corresponding to all diagrams up to the j-th level of the tree D. We wish to define a congruence for some diagram d' from level $j+1$. This diagram was constructed in step $j+1$ of the construction of the tree D from a diagram d in level j, with which a congruence ρ_j is associated, by adding an arc above or adding a dot on one of the sides. So we assume that **(I)**: a diagram d from level j is not a leaf of D and that we already constructed a chain of congruences $\rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \ldots \subseteq \rho_j$ on M, corresponding to the branch of D which leads to a diagram d' from level $j+1$.

Part (II). Let $a_u, a_{u+1}, \ldots, a_v$ be the generators used in our diagram d from level j (we know that $u \neq 1$, $v \neq n$ and in view of 2.1.4 the used generators form a connected segment). Consider $M_{i_j}^{u,v}/\rho_j$.

For $j = 0$ this is $M_n/\rho_0 = M_n$. For $j = 1$ we know from 1.2.1 that in case $\heartsuit$

$$M_{n-1}^s/\rho_1 = \overline{M_{n-1}^s} = M_{n-1}^s \Big/ \left({}^{a_1,\ldots,a_{s-1} \text{ commute}}_{a_{s+1},\ldots,a_n \text{ commute}}\right),$$

while in case $\diamondsuit$

$$M_{n-2}^{s-1,s}/\rho_1 = \overline{M_{n-2}^{s-1,s}} = M_{n-2}^{s-1,s} \Big/ \left({}^{a_1,\ldots,a_{s-2} \text{ commute}}_{a_{s+1},\ldots,a_n \text{ commute}}\right).$$

Hence, assume inductively that **(II)**: the congruence ρ_j is chosen in such a way that $M_{i_j}^{u,v}/\rho_j$ is a Chinese monoid of rank i_j with generators $\widehat{a}_1^j, \ldots, \widehat{a}_{u-1}^j, \widehat{a}_{v+1}^j, \ldots, \widehat{a}_n^j$, and with additional relations making the monoid $\langle \widehat{a}_1^j, \ldots, \widehat{a}_{u-1}^j \rangle$ free commutative and making $\langle \widehat{a}_{v+1}^j, \ldots, \widehat{a}_n^j \rangle$ free commutative.

Part (III). Assume that for every diagram from any level $t \leq j$, which has some number i_t of unused generators, there is an associated pair (S_t, i_t), in accordance with Definition 2.2.2. For $t = 0$, so for the root of D, we have $i_0 = 0$, while S_0 is not defined. With diagrams of the first level of the tree D, so for $t = 1$, we associate the pairs (S_1, i_1) as in Definition 2.2.2.

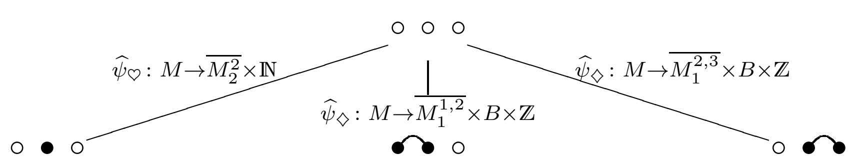
Part (IV). By Corollary 1.2.4 and Lemma 1.2.7 we have, for ρ_1 of type $\heartsuit$ and $\diamondsuit$, respectively, an epimorphism $\widehat{\psi}_\heartsuit$ or a homomorphism $\widehat{\psi}_\diamondsuit$, such that

$$(\heartsuit) \quad M \to M/\rho_1 \simeq \overline{M_{n-1}^s} \times \langle \overline{a_s} \rangle \simeq \overline{M_{n-1}^s} \times \mathbb{N} \simeq M_{n-1}^s/\rho_1 \times \mathbb{N},$$

$$(\diamondsuit) \quad M \to M/\rho_1 \hookrightarrow M/\rho_1 \left\langle (\overline{a_s a_{s-1}})^{-1} \right\rangle \simeq \overline{M_{n-2}^{s-1,s}} \times B \times \mathbb{Z} \simeq M_{n-2}^{s-1,s}/\rho_1 \times B \times \mathbb{Z},$$

with the embedding accomplished by the central localization with respect to $\langle \overline{a_s a_{s-1}} \rangle$.

More precisely, in M_n, for $n \geq 2$, we have $n-2$ possible congruences ρ_1 of type $\heartsuit$, so also $n-2$ possible epimorphisms $\widehat{\psi}_\heartsuit$, and also we have $n-1$ possible congruences ρ_1 of type $\diamondsuit$, so also $n-1$ possible homomorphisms $\widehat{\psi}_\diamondsuit$. These homomorphisms can be associated with the corresponding branches in D, depending on the value of s. For example, if $n = 3$, we get the following first level of the tree D:



Let f be a diagram from the branch of D leading to the considered diagram d from level $j+1$ (for which we want to construct ρ_{j+1}). Assume that f is from level $t+1 \leq j$ in D and it was created from a diagram from level $t < j$, in which there are $0 \leq i_t \leq n$

unused generators, and the used generators have indices $u_t, \ldots, v_t$. The value of i_t can be different for different diagrams from level t, see Definition 2.2.2.

If the diagram f was created by adding a dot then assume inductively that **(IV $\heartsuit$)**: there exists an epimorphism

$$\widehat{\psi}^t_\heartsuit \colon M^{u_t,v_t}_{i_t}/\rho_t \to M_{i_t-1}/\rho_{t+1} \times \mathbb{N}$$

given by

$$\begin{cases} \widehat{\psi}^t_\heartsuit(\widehat{a}^t_s) = (1, g_s) \\ \widehat{\psi}^t_\heartsuit(\widehat{a}^t_l) = (\widehat{a}^{t+1}_l, 1) \qquad \text{for } l \neq s, \end{cases}$$

where $\langle g_s \rangle \simeq \mathbb{N}$ and $s = u_t - 1$ or $s = v_t + 1$, depending on which of the two possible dots was added. We associate this epimorphism with the edge of the tree D which is used to add the diagram f.

If f was created by adding an arc, then assume inductively that **(IV $\diamondsuit$)**: there exists a homomorphism

$$\widehat{\psi}^t_\diamondsuit \colon M^{u_t,v_t}_{i_t}/\rho_t \to M^{u_t-1,v_t+1}_{i_t-2}/\rho_{t+1} \times B \times \mathbb{Z}$$

given by

$$\begin{cases} \widehat{\psi}^t_\diamondsuit(\widehat{a}^t_{u-1}) = (1, p, g) \\ \widehat{\psi}^t_\diamondsuit(\widehat{a}^t_{v+1}) = (1, q, 1) \\ \widehat{\psi}^t_\diamondsuit(\widehat{a}^t_l) = (\widehat{a}^{t+1}_l, p, 1) \qquad \text{for } l < u_t - 1 \\ \widehat{\psi}^t_\diamondsuit(\widehat{a}^t_l) = (\widehat{a}^{t+1}_l, q, 1) \qquad \text{for } l > v_t + 1, \end{cases}$$

where $\mathbb{Z} \simeq \langle g, g^{-1} \rangle$. We associate this homomorphism with the edge of the tree D, which was used to add the diagram f.

Notice that for $t = 0$ the corresponding homomorphisms are $\widehat{\psi}^0_\heartsuit = \widehat{\psi}_\heartsuit$ and $\widehat{\psi}^0_\diamondsuit = \widehat{\psi}_\diamondsuit$.

Part (V). Define for $t < j$ and for $\triangle = \heartsuit$ or $\triangle = \diamondsuit$ the map $\widehat{\kappa}^t_\triangle$ by $\widehat{\kappa}^0_\triangle = \widehat{\psi}_\triangle$ and for $0 < t < j$

$$\widehat{\kappa}^t_\triangle \colon M_{i_t}/\rho_t \times S_t \to M_{i_{t+1}}/\rho_{t+1} \times S_{t+1}, \qquad \widehat{\kappa}^t_\triangle = (\widehat{\psi}^t_\triangle, id).$$

By the induction hypothesis (IV) applied to $\widehat{\psi}^t_\triangle$ we know that for every edge in D one of the maps $\widehat{\kappa}^t_\heartsuit$ or $\widehat{\kappa}^t_\diamondsuit$ exists and $\widehat{\kappa}^t_\heartsuit$ is an epimorphism. Each map $\widehat{\kappa}^t_\triangle = (\widehat{\psi}^t_\triangle, id)$ we associate with the edge in D with which the corresponding $\widehat{\psi}^t_\triangle$ is associated.

Consider the branch of D leading from the root to a diagram f from level $t+1$. The subsequent edges of this branch correspond to some homomorphisms $\widehat{\kappa}^0_*, \widehat{\kappa}^1_*, \ldots, \widehat{\kappa}^t_*$, where each $*$ denotes $\heartsuit$ or $\diamondsuit$. For $\triangle$ equal to $\heartsuit$ or $\diamondsuit$, we define a homomorphism

$$\widehat{\kappa'}^t_\triangle \colon M \to M_{i_{t+1}}/\rho_{t+1} \times S_{t+1}$$

as the composition

$$\widehat{\kappa'}^t_\triangle = \widehat{\kappa}^t_\triangle \circ \widehat{\kappa}^{t-1}_* \circ \ldots \circ \widehat{\kappa}^1_* \circ \widehat{\kappa}^0_*.$$

For $t = 0$ we have $\widehat{\kappa'}^0_\triangle = \widehat{\kappa}^0_\triangle$, while for $0 < t < j$ we have

$$\widehat{\kappa'}^t_\triangle = \widehat{\kappa}^t_\triangle \circ (\widehat{\kappa}^{t-1}_\nabla \circ \widehat{\kappa}^{t-2}_* \circ \ldots \widehat{\kappa}^1_* \circ \widehat{\kappa}^0_*) = \widehat{\kappa}^t_\triangle \circ \widehat{\kappa'}^{t-1}_\nabla,$$

$$\widehat{\kappa'}^t_\triangle \colon M \overset{\widehat{\kappa'}^{t-1}_\nabla}{\to} M^{u,v}_{i_t}/\rho_t \times S_t \overset{\widehat{\kappa}^t_\triangle}{\to} M_{i_{t+1}}/\rho_t \times S_{t+1},$$

where each $*$ denotes $\heartsuit$ or $\diamondsuit$ and $\nabla = \heartsuit$ or $\nabla = \diamondsuit$. The map $\widehat{\kappa'}^t_\triangle$ is an epimorphism if and only if all $*$ are equal to $\heartsuit$ and $\nabla = \heartsuit$. However, by the construction of D we cannot simultaneously have $\widehat{\kappa}^0_* = \widehat{\kappa}^0_\heartsuit$ and $\widehat{\kappa}^1_* = \widehat{\kappa}^1_\heartsuit$. Hence $\widehat{\kappa'}^t_\triangle$ is an epimorphism only for $t = 0$ and $\triangle = \heartsuit$.

In cases where the index $\triangle$ is not important we simply write $\widehat{\psi}^t$ or $\widehat{\kappa}^t$ or $\widehat{\kappa'}^t$, respectively.

Assume inductively that **(V)**: for $t+1 \leq j$ we have $\rho_{t+1} = ker(\widehat{\kappa'}^t)$. For $t = 0$, since $\widehat{\kappa'}^0 = \widehat{\kappa}^0 = \widehat{\psi}^0$, we have $\rho_1 = ker(\psi)$, which agrees with the definition of ρ_1.

Next, we define a congruence ρ_{j+1} and we verify that it satisfies the inductive claim, so conditions (I)-(V) are satisfied. We may assume that the considered diagram d' from level $j+1$ was constructed from a diagram d from level j by adding a dot a_{u-1} (the proof for a dot a_{v+1} is similar) or an arc $a_{v+1}a_{u-1}$. In the former case, we define a map

$$\widehat{\psi}^j_\heartsuit \colon M^{u,v}_{i_j}/\rho_j \to M_{i_j-1}/\rho_j \times \mathbb{N}$$

as the natural extension of the homomorphism defined on generators as follows:

$$\begin{cases} \widehat{\psi}^j_\heartsuit(\widehat{a}^j_{u-1}) = (1, g_{u-1}) \\ \widehat{\psi}^j_\heartsuit(\widehat{a}^j_l) = (\widehat{a}^j_l, 1) \quad \text{for } l \neq u-1, \end{cases}$$

where $\langle g_{u-1} \rangle \simeq \mathbb{N}$, and in the latter case as the homomorphism

$$\widehat{\psi}^j_\diamondsuit \colon M^{u,v}_{i_j}/\rho_j \to M_{i_j-2}/\rho_j \times B \times \mathbb{Z}$$

naturally extending:

$$\begin{cases} \widehat{\psi}^j_\diamondsuit(\widehat{a}^j_{u-1}) = (1, p, g) \\ \widehat{\psi}^j_\diamondsuit(\widehat{a}^j_{v+1}) = (1, q, 1) \\ \widehat{\psi}^j_\diamondsuit(\widehat{a}^j_l) = (\widehat{a}^j_l, p, 1) \ \text{ for } l < u-1 \\ \widehat{\psi}^j_\diamondsuit(\widehat{a}^j_l) = (\widehat{a}^j_l, q, 1) \ \text{ for } l > v+1. \end{cases}$$

Both maps are homomorphisms because they are homomorphisms on each of the components. Moreover $\widehat{\psi}^j_\heartsuit$ is an epimorphism and $\widehat{\psi}^j_\diamondsuit$ is not an epimorphism. This is verified in the same way as for $\widehat{\psi}_\heartsuit$ and $\widehat{\psi}_\diamondsuit$.

Let

$$\begin{cases} S_{j+1} = S_j \times \langle g_{u-1} \rangle \simeq S_j \times \mathbb{N} \\ i_{j+1} = i_j - 1 \end{cases}$$

in case $\heartsuit$ and

$$\begin{cases} S_{j+1} = S_j \times B \times \mathbb{Z} \\ i_{j+1} = i_j - 2 \end{cases}$$

in case $\diamondsuit$. Then i_{j+1} so defined coincides with the number of used generators in the diagram. Moreover, the pairs (S_{j+1}, i_j) defined in this way satisfy conditions of Definition 2.2.2. This completes **the proof of part (III)** of the inductive claim.

Define the homomorphism

$$\widehat{\kappa}^j_\triangle\colon M^{u,v}_{i_j}/\rho_j \times S_j \to M_{i_{j+1}}/\rho_j \times S_{j+1} \quad \text{by} \quad \widehat{\kappa}^j_\triangle = (\widehat{\psi}^j_\triangle, id),$$

so $\widehat{\kappa'}^j_\triangle$ are defined in the same way as $\widehat{\kappa'}^t_\triangle$ for $t < j$.

Similarly as for $\widehat{\kappa'}^t$, let $\widehat{\kappa'}^j_\triangle\colon M \to M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}$ be the homomorphism defined by $\widehat{\kappa'}^j_\triangle = \widehat{\kappa}^j_\triangle \circ \widehat{\kappa}^{j-1}_*$. The homomorphism $\widehat{\kappa'}^j$ corresponds to a congruence $ker(\widehat{\kappa'}^j)$ on M. We define $\rho_{j+1} = ker(\widehat{\kappa'}^j)$. We will show that ρ_{j+1} satisfies the inductive claim.

By the inductive hypothesis we know that $\rho_t \subseteq \rho_{t+1}$ for $0 < t < j$. Also, for $0 < t \leq j$ we have $\widehat{\kappa'}^t = \widehat{\kappa}^t \circ \widehat{\kappa'}^{t-1}$, so that $ker(\widehat{\kappa'}^{t-1}) \subseteq ker(\widehat{\kappa'}^t)$. Thus, by the inductive hypothesis (V) and by the definition of ρ_{j+1} we get

$$\rho_j = ker(\widehat{\kappa'}^{j-1}) \subseteq ker(\widehat{\kappa'}^j) = \rho_{j+1},$$

so that $\rho_j \subseteq \rho_{j+1}$. This completes **the proof of part (I)** of the inductive claim.

Now we will show that $\rho_j|_{M_{i_{j+1}}} \supseteq \rho_{j+1}|_{M_{i_{j+1}}}$. Assume that for some $x, y \in M_{i_{j+1}}$ we have $(x, y) \in \rho_{j+1}|_{M_{i_{j+1}}}$. Similarly as above, by the definition of ρ_{j+1} this means that $(x, y) \in ker(\widehat{\kappa'}^j)$, so that $\widehat{\kappa'}^j(x) = \widehat{\kappa'}^j(y)$. By the definition of $\widehat{\kappa'}^j : M \to M_{i_{j+1}}/\rho_j \times S_{j+1}$, the latter implies that the first components (belonging to $M_{i_{j+1}}/\rho_j$) of elements $\widehat{\kappa'}^j(x)$ and $\widehat{\kappa'}^j(y)$ are equal, so that the images of x, y in $M_{i_{j+1}}/\rho_j$ are equal. Hence $(x, y) \in \rho_j|_{M_{i_{j+1}}}$. So indeed we have $\rho_j|_{M_{i_{j+1}}} \supseteq \rho_{j+1}|_{M_{i_{j+1}}}$, as desired.

Therefore, in view of the opposite inclusion proved before, we get $\rho_j|_{M_{i_{j+1}}} = \rho_{j+1}|_{M_{i_{j+1}}}$, whence also

$$M_{i_{j+1}}/\rho_j = M_{i_{j+1}}/\rho_{j+1}.$$

Thus $M_{i_{j+1}}/\rho_{j+1}$ is the Chinese monoid on i_{j+1} generators, with the same additional relations as $M_{i_{j+1}}/\rho_j$, so with commutativity of the sets of generators that are on the same side of the generators used earlier. This completes **the proof of part (II)** of the inductive claim.

Since $M_{i_{j+1}}/\rho_j = M_{i_{j+1}}/\rho_{j+1}$, it follows that

$$\widehat{\psi}^j_\heartsuit\colon M^{u,v}_{i_j}/\rho_j \to M_{i_j-1}/\rho_j \times \mathbb{N} = M_{i_{j+1}}/\rho_j \times \mathbb{N} = M_{i_{j+1}}/\rho_{j+1} \times \mathbb{N},$$

$$\widehat{\psi}^j_\diamondsuit\colon M^{u,v}_{i_j}/\rho_j \to M_{i_j-2}/\rho_j \times B \times \mathbb{Z} = M_{i_{j+1}}/\rho_j \times B \times \mathbb{Z} = M_{i_{j+1}}/\rho_{j+1} \times B \times \mathbb{Z}.$$

Moreover, for every generator a_l of $M_{i_{j+1}}$ we therefore have $\widehat{a}^j_l = \widehat{a}^{j+1}_l$. By the above, and in view of the definition, $\widehat{\psi}^j$ satisfies all the conditions of the inductive hypothesis for $\widehat{\psi}^t$ with $t < j$. This completes **the proof of part (IV)** of the inductive claim.

Hence, also $\widehat{\kappa'}^j$ satisfies all conditions that hold by the assumption for $\widehat{\kappa'}^t$ with $t < j$. Moreover, by the definition, $\rho_{j+1} = ker(\widehat{\kappa'}^j)$. This completes **the proof of part (V)** of the inductive claim, and hence completes the entire construction. ■

2.2.4 Notation. From now on we adopt the notation used in Construction 2.2.3. We know that $\rho_j \subseteq \rho_{j+1}$ and ψ_j and ψ_{j+1} are epimorphisms. Hence there exists a natural epimorphism φ_j such that the diagram

$$\begin{tikzcd} M \arrow[r, two heads, "\psi_j"] \arrow[dr, "\psi_{j+1}"'] & M/\rho_j \arrow[d, "\varphi_j"] \\ & M/\rho_{j+1} \end{tikzcd}$$

commutes, that is $\psi_{j+1} = \varphi_j \circ \psi_j$.

2.2.5 Lemma. For every j there exists a natural embedding

$$\lambda_{j+1}\colon M/\rho_{j+1} \hookrightarrow M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}.$$

Moreover, the following diagram commutes

$$\begin{tikzcd} M/\rho_j \arrow[r, hook, "\lambda_j"] \arrow[d, "\varphi_j"'] & M_{i_j}/\rho_j \times S_j \arrow[d, "\widehat{\kappa}^j"] \\ M/\rho_{j+1} \arrow[r, hook, "\lambda_{j+1}"] & M_{i_{j+1}}/\rho_{j+1} \times S_{j+1} \end{tikzcd}$$

Proof. First, consider the case $j = 1$. If ρ_1 is of type $\heartsuit$ then Lemma 1.2.3 yields an isomorphism $M/\rho_1 \simeq M_{i_1}/\rho_1 \times S_1$, which we denote by λ_1. We know that the diagram

$$\begin{tikzcd} M \arrow[d, "\psi"'] \arrow[dr, "\widehat{\kappa'}^0 = \widehat{\psi}"] & \\ M/\rho_1 \arrow[r, phantom, "\underset{\lambda_1}{\simeq}"] & M_{i_1}/\rho_1 \times S_1 \end{tikzcd}$$

commutes.

If ρ_1 is of type $\diamondsuit$, then by the proof of Lemma 1.2.7 we have an embedding $\lambda\colon M/\rho_1 \hookrightarrow M_{i_1}/\rho_1 \times \ S_1$, which we denote by λ_1. As in case $\heartsuit$ we know that the diagram

$$\begin{tikzcd} M \arrow[d, "\psi"'] \arrow[dr, "\widehat{\kappa'}^0 = \widehat{\psi}"] & \\ M/\rho_1 \arrow[r, hook, "\lambda_1"] & M_{i_1}/\rho_1 \times S_1 \end{tikzcd}$$

commutes.

For $j > 1$, by the inductive construction of $\widehat{\kappa'}^j$ in 2.2.3, we get $Im(\widehat{\kappa'}^j) \subseteq M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}$. Since $\rho_{j+1} = ker(\widehat{\kappa'}^j)$, we thus get the desired natural embedding

$$\lambda_{j+1}\colon M/\rho_{j+1} \hookrightarrow M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}.$$

Therefore the diagram

$$\begin{tikzcd} M \arrow[d, "\psi_{j+1}"'] \arrow[dr, "\widehat{\kappa'}^j"] & \\ M/\rho_{j+1} \arrow[r, hook, "\lambda_{j+1}"] & M_{i_{j+1}}/\rho_{j+1} \times S_{j+1} \end{tikzcd}$$

commutes.

Since congruences from higher levels of the tree D satisfy analogous conditions, for every $j > 0$ we get a commuting diagram

$$\begin{tikzcd} M \arrow[d, "\psi_j"'] \arrow[dr, "\widehat{\kappa'}^{j-1}"] & \\ M/\rho_j \arrow[r, hook, "\lambda_j"] & M_{i_j}/\rho_j \times S_j. \end{tikzcd}$$

Hence $\widehat{\kappa'}^{j-1} = \lambda_j \circ \psi_j$. Thus $\widehat{\kappa'}^{j-1} \circ \psi_j^{-1}(x) = \lambda_j(x)$ for every $x \in M/\rho_j$.

By the definition of $\widehat{\kappa'}^{j}$ we have $\widehat{\kappa'}^{j} = \widehat{\kappa}^j \circ \widehat{\kappa'}^{j-1}$. Moreover, by the definition of φ_j, we have $\psi_{j+1} = \varphi_j \circ \psi_j$. Hence, for every $x \in M/\rho_j$ and its preimage $\psi_j^{-1}(x) \subseteq M$, it follows that $\psi_{j+1}(\psi_j^{-1}(x)) = \varphi_j(x)$. All the above easily leads to

$$\begin{aligned} \lambda_{j+1} \circ \varphi_j(x) = (\lambda_{j+1} \circ \psi_{j+1})(\psi_j^{-1}(x)) = \widehat{\kappa'}^{j}(\psi_j^{-1}(x)) = \\ = (\widehat{\kappa}^j \circ \widehat{\kappa'}^{j-1})(\psi_j^{-1}(x)) = \widehat{\kappa}^j \circ \lambda_j(x), \end{aligned}$$

which establishes the assertion. ■

2.2.6 Notation. For a fixed diagram d in D, let A_l be the submonoid of M generated by all products $a_y a_x$, corresponding to arcs built in this diagram up to the l-th step (inclusive) of the construction of d. In case ♡ for $l = 1$ (where there are no arcs), we define $A_1 = \{1\}$.

2.2.7 Notation. For simplicity, we sometimes identify M/ρ_l with $\lambda_l(M/\rho_l)$ and we identify $(M/\rho_l) \cdot (\widehat{A}_l^l)^{-1}$ with $\lambda_l(M/\rho_l) \cdot (\lambda_l(\widehat{A}_l^l))^{-1}$.

2.2.8 Lemma. With notation as in 2.2.6, for every $l > 0$ the elements $\widehat{a}_y^l \widehat{a}_x^l$ are central and regular in M/ρ_l. Moreover $(M/\rho_l) \cdot (\widehat{A}_l^l)^{-1} \subseteq M_{i_l}/\rho_l \times S_l$ (identifying M/ρ_l with $\lambda_l(M/\rho_l)$).

Proof. We know that $\widehat{A}_l^l = \psi_l(A_l) \subseteq M/\rho_l$. From Lemma 2.2.5 we have an embedding $\lambda_l \colon M/\rho_l \hookrightarrow M_{i_l}/\rho_l \times S_l$. We will consider the images of elements of A_l in $M_{i_l}/\rho_l \times S_l$ under the map $\lambda_l \circ \psi_l = \widehat{\kappa'}^{l-1} \colon M \to M_{i_l}/\rho_l \times S_l$.

Assume that in some step $k + 1 < l$ of the construction, an arc $a_y a_x$ is built, where $k \geq 0$. We study the images of the generators up to this step $k + 1$.

Consider all the steps of the construction, from step one till step k. By the last part of the proof of part (IV) of the inductive claim in Construction 2.2.3 we know that for every t, in step $t + 1$ we have (depending on the case, respectively)

$$\begin{cases} \widehat{\psi}^t_{\heartsuit}(\widehat{a}^t_x) = (\widehat{a}^{t+1}_x, 1) \\ \widehat{\psi}^t_{\heartsuit}(\widehat{a}^t_y) = (\widehat{a}^{t+1}_y, 1) \end{cases} \qquad \text{or} \qquad \begin{cases} \widehat{\psi}^t_{\diamondsuit}(\widehat{a}^t_x) = (\widehat{a}^{t+1}_x, p, 1) \\ \widehat{\psi}^t_{\diamondsuit}(\widehat{a}^t_y) = (\widehat{a}^{t+1}_y, q, 1). \end{cases}$$

This follows from the definition of the maps $\widehat{\psi}^t_{\triangle}$ and from the fact that in step k an arc $a_y a_x$ is built, whence in the previous steps generators with indices between x and y are used.

This implies that in step k the images of the generators $a_x, a_y \in M$ under $\widehat{\kappa'}^{k-1}$ have the form

$$\begin{cases} \widehat{\kappa'}^{k-1}(a_x) = (\widehat{a}_x^k, [1,p]) \\ \widehat{\kappa'}^{k-1}(a_y) = (\widehat{a}_y^k, [1,q]), \end{cases}$$

where $\widehat{a}_x^k, \widehat{a}_y^k \in M_{i_k}^{x+1,y-1}/\rho_k$, and $[1,p],[1,q] \in S_k = \mathbb{N}^c \times (B \times \mathbb{Z})^d$ denote sequences of length $c+2d$ consisting of $(c+d)$ elements 1 and d elements p or (respectively) $(c+d)$ elements 1 and d elements q, and p in $[1,p]$ occurs in exactly the same places as q in $[1,q]$.

In step $(k+1)$ an arc $a_y a_x$ is built, so according to the definition of $\widehat{\psi}_\diamondsuit^k$ we get

$$\begin{cases} \widehat{\psi}_\diamondsuit^k(\widehat{a}_x^k) = (1,p,g) \\ \widehat{\psi}_\diamondsuit^k(\widehat{a}_y^k) = (1,q,1), \end{cases}$$

so that

$$\begin{cases} \widehat{\kappa'}_\diamondsuit^k(a_x) = \widehat{\kappa}_\diamondsuit^k \circ \widehat{\kappa'}^{k-1}(a_x) = \widehat{\kappa}_\diamondsuit^k(\widehat{a}_x^k, [1,p]) = (\widehat{\psi}_\diamondsuit^k(\widehat{a}_x^k), [1,p]) = (1,p,g,[1,p]) \\ \widehat{\kappa'}_\diamondsuit^k(a_y) = (1,q,1,[1,q]) \qquad \text{(analogously).} \end{cases}$$

In the next steps of the construction, from step $k+2$ till step j, elements 1 occurring as the first components of the above images of a_x and a_y yield in the image $(1,1)$ in case $\heartsuit$ and $(1,1,1)$ in case $\diamondsuit$, respectively. More precisely, since $\widehat{\psi}_\heartsuit^t$ and $\widehat{\psi}_\diamondsuit^t$, are homomorphisms, for the element $1 \in M_{i_t}/\rho_t$ we get the equalities

$$\begin{cases} \widehat{\psi}_\heartsuit^t(1) = (1,1) \in M_{i_t-1}/\rho_{t+1} \times \mathbb{N} \\ \widehat{\psi}_\diamondsuit^t(1) = (1,1,1) \in M_{i_t-2}/\rho_{t+1} \times B \times \mathbb{Z}, \end{cases}$$

respectively. Hence, for every $z \in S_t$ we get (respectively)

$$\begin{cases} \widehat{\kappa}_\heartsuit^t(1,z) = (1,1,z) \\ \widehat{\kappa}_\diamondsuit^t(1,z) = (1,1,1,z), \end{cases}$$

respectively. Therefore, in step l of the construction, with $l > k+1$ we get

$$\begin{cases} \widehat{\kappa'}^{l-1}(a_x) = (\widehat{a}_x^l, [1,p]) = (1,\dots,1,1,p,g,[1,p]) \\ \widehat{\kappa'}^{l-1}(a_y) = (\widehat{a}_y^l, [1,q]) = (1,\dots,1,1,q,1,[1,q]). \end{cases}$$

Since p occurs in $[1,p]$ in the same components as q occurs in $[1,q]$, it follows that

$$\begin{aligned} \widehat{\kappa'}^{l-1}(a_y a_x) = \widehat{\kappa'}^{l-1}(a_y)\widehat{\kappa'}^{l-1}(a_x) = (\widehat{a}_y^l \widehat{a}_x^l, [1,qp]) = \\ = (1,\dots,1,1,qp,g,[1,qp]) = (1,\dots,1,g,1,\dots,1) \in M_{i_l}/\rho_l \times S_l. \end{aligned}$$

Moreover, the above implies that g occurs in the components $\mathbb{N}$ occurring in S_l.

Thus in step l for $l > k+1$ the image of the element $a_y a_x$ (corresponding to any previously built arc) in $M_{i_l}/\rho_l \times S_l$ is of the form $(\widehat{a}_y^l \widehat{a}_x^l, [1,1]) = (1,\dots,1,g,1,\dots,1)$. This is a central element. It is also invertible in $M_{i_l}/\rho_l \times S_l$. So it is also central and regular in $M/\rho_l \subseteq M_{i_l}/\rho_l \times S_l$ (where M/ρ_l is identified with $\lambda_l(M/\rho_l)$).

In particular, we may consider the localization $(M/\rho_l)\cdot(\widehat{A}_l^l)^{-1}$ with respect to the submonoid generated by all such elements. Moreover, with identifications as in 2.2.7, since we have inclusions $M/\rho_l \subseteq M_{i_l}/\rho_l \times S_l$ and $(\widehat{A}_l^l)^{-1} \subseteq M_{i_l}/\rho_l \times S_l$, we also get $(M/\rho_l)\cdot(\widehat{A}_l^l)^{-1} \subseteq M_{i_l}/\rho_l \times S_l$. This completes the proof. ■

2.2.9 Theorem. With identifications as in 2.2.7, we have

$$(M/\rho_l)(\widehat{A}_l^l)^{-1} = M_{i_l}/\rho_l \times S_l.$$

Proof. By Lemma 2.2.8 we know that $(M/\rho_l)(\widehat{A}_l^l)^{-1} \subseteq M_{i_l}/\rho_l \times S_l$.

For ρ_1 of type $\heartsuit$ we have $A_1 = \emptyset$ by the definition, and $i_1 = n-1$, $S_1 = \mathbb{N}$, so that the claim takes the form $M/\rho_1 \simeq M_{n-1}/\rho_1 \times \mathbb{N}$, which holds by Lemma 1.2.3. For ρ_1 of type $\diamondsuit$, the set A_1 consists of one element $a_s a_{s-1}$, and $i_1 = n-2$, $S_1 = B \times \mathbb{Z}$, so the claim takes the form $(M/\rho_1)\cdot\langle(\overline{a_s a_{s-1}})\rangle^{-1} = M_{n-2}/\rho_1 \times B \times \mathbb{Z}$, which follows from Lemma 1.2.7.

Assume by induction that the claim holds for all congruences ρ corresponding to diagrams in level j of the tree D. In particular, it holds for ρ_j corresponding to the diagram from which the considered diagram from level $j+1$ was constructed (the one for which the congruence ρ_{j+1} was constructed). Then the embedding $\lambda_j \colon M/\rho_j \hookrightarrow M_{i_j}/\rho_j \times S_j$ leads to an isomorphism, so we get

$$(M/\rho_j)\cdot(\widehat{A}_j^j)^{-1} = M_{i_j}/\rho_j \times S_j$$

and the inductive claim takes the form $(M/\rho_{j+1})\cdot(\widehat{A}_{j+1}^{j+1})^{-1} = M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}$.

Consider the commuting diagram (see Lemma 2.2.5)

(#)
$$\begin{tikzcd} M/\rho_j \arrow[r, hook, "\lambda_j"] \arrow[d, "\varphi_j"'] & M_{i_j}/\rho_j \times S_j \arrow[d, "\widehat{\kappa}^j"] \\ M/\rho_{j+1} \arrow[r, hook, "\lambda_{j+1}"] & M_{i_{j+1}}/\rho_{j+1} \times S_{j+1} \end{tikzcd}$$

By the inductive hypothesis, $(M/\rho_j)(\widehat{A}_j^j)^{-1} = M_{i_j}/\rho_j \times S_j$ (under identification as in 2.2.7), while by Lemma 2.2.8 the map λ_{j+1} yields an embedding $(M/\rho_{j+1})(\widehat{A}_{j+1}^{j+1})^{-1} \hookrightarrow M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}$. All elements $\lambda_j(\widehat{A}_j^j)^{-1}$ are invertible in $M_{i_j}/\rho_j \times S_j$, by the last part of the proof of Lemma 2.2.8. Let for any $i = 1, 2, \ldots$, the homomorphism λ'_i be the unique extension of λ_i to the localization $(M/\rho_{j+1})(\widehat{A}_{j+1}^{j+1})^{-1}$, built as in the proof of Lemma 1.2.7. Consider the diagram

(##)
$$\begin{tikzcd} (M/\rho_j)(\widehat{A}_j^j)^{-1} \arrow[r, phantom, "\overset{\lambda'_j}{=}"] \arrow[d, "\varphi'_j"'] & M_{i_j}/\rho_j \times S_j \arrow[d, "\widehat{\kappa}^j"] \\ (M/\rho_{j+1})(\widehat{A}_{j+1}^{j+1})^{-1} \arrow[r, hook, "\lambda'_{j+1}"] & M_{i_{j+1}}/\rho_{j+1} \times S_{j+1} \end{tikzcd}$$

where φ'_j is the natural extension of φ_j. We know that φ_j is an epimorphism.

We will show that for ρ_j of type $\heartsuit$ the map φ'_j also is an epimorphism. We have $A_j = A_{j+1}$, because in case $\heartsuit$ there is no new arc added. Consider the image $\widehat{A}^j_j \subseteq M/\rho_j$ under the map $\varphi_j \colon M/\rho_j \to M/\rho_{j+1}$. We get $\widehat{A}^{j+1}_j \subseteq M/\rho_{j+1}$, and the elements of $\widehat{A}^{j+1}_j$ are central in M/ρ_{j+1} (because so are the elements of $\widehat{A}^j_j$ in M/ρ_j). Moreover, we know that $\widehat{A}^{j+1}_j = \widehat{A}^{j+1}_{j+1}$, because $A_j = A_{j+1}$, so that the image $\widehat{A}^j_j \subseteq M/\rho_j$ under the map φ_j is equal to $\widehat{A}^{j+1}_{j+1}$. The image of M/ρ_j is equal to M/ρ_{j+1}, whence indeed φ'_j is an epimorphism.

Next we show that the diagram (##) commutes. We know that the diagram (#) commutes, so

$$(\lambda'_{j+1}\varphi'_j)|_{M/\rho_j} = \lambda_{j+1}\varphi_j = \widehat{\kappa}^j\lambda_j = (\widehat{\kappa}^j\lambda'_j)|_{M/\rho_j}.$$

Under this map, the images of elements of $\widehat{A}^j_j$ are invertible in $M_{i_{j+1}}/\rho_{j+1}\times S_{j+1}$, because $\varphi_j(\widehat{A}^j_j) = \widehat{A}^{j+1}_j \subseteq \widehat{A}^{j+1}_{j+1}$ and we know from the last part of the proof of Lemma 2.2.8 that the elements $\lambda_{j+1}((\widehat{A}^{j+1}_{j+1})^{-1})$ are invertible in $M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}$. Hence, there exists a unique extension to the localization $(M/\rho_j)(\widehat{A}^j_j)^{-1}$. It is equal to $\lambda'_{j+1}\varphi'_j$ and also equal to $\widehat{\kappa}^j\lambda'_j$, so that $\lambda'_{j+1}\varphi'_j = \widehat{\kappa}^j\lambda'_j$. In other words, the diagram (##) commutes.

We know that in case $\heartsuit$ the maps $\widehat{\kappa}^j$, λ'_j and φ'_j are epimorphisms. Hence, $\lambda'_{j+1}\varphi'_j = \widehat{\kappa}^j\lambda'_j$ is an epimorphism, so that the embedding λ'_j is an epimorphism. Thus, in case $\heartsuit$, λ'_j is an isomorphism. Then we get $(M/\rho_{j+1})(\widehat{A}^{j+1}_{j+1})^{-1} = M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}$, as desired. This completes the inductive step in case $\heartsuit$.

In case $\diamondsuit$, $\widehat{\kappa}^j$ and φ'_j are not epimorphisms and $A_j \subsetneq A_{j+1}$. We have $Im(\varphi'_j) = (M/\rho_{j+1})(\widehat{A}^{j+1}_j)^{-1}$. It also follows that $Im(\widehat{\kappa}^j) \subseteq Im(\lambda'_{j+1})$, because

$$\begin{aligned} Im(\widehat{\kappa}^j) = \lambda'_{j+1}(Im(\varphi'_j)) = \lambda'_{j+1}((M/\rho_{j+1})(\widehat{A}^{j+1}_j)^{-1}) \subseteq \\ \subseteq \lambda'_{j+1}\left((M/\rho_{j+1})(\widehat{A}^{j+1}_{j+1})^{-1}\right) = Im(\lambda'_{j+1}). \end{aligned}$$

Since $\widehat{\kappa}^j|_{S_j} = id$, we also have

$$S_j = \widehat{\kappa}^j(S_j) \subseteq Im(\widehat{\kappa}^j) \subseteq Im(\lambda'_{j+1}) = \lambda'_{j+1}\left((M/\rho_{j+1})(\widehat{A}^{j+1}_{j+1})^{-1}\right).$$

Consider $M^{u,v}_{i_j}/\rho_{j+1} = M_{i_j}/\rho_{j+1}$. This is a Chinese monoid $M^{u,v}_{i_j}$ with the additional relations of type $\diamondsuit$, so with relations corresponding to the arc $a_v a_u$. Hence, we are in a case as in Lemma 1.2.7, where relations of type $\diamondsuit$ are imposed on the Chinese monoid M. Moreover, notice that $\lambda'_{j+1}|_{M_{i_j}/\rho_{j+1}}$ coincides with the map λ from Lemma 1.2.7. Hence we may apply Lemma 1.2.7 to M_{i_j}/ρ_{j+1}. We then get $i_{j+1} = i_j - 2$, and $M_{i_{j+1}}/\rho_{j+1}$ corresponds to $\overline{M_{n-2}}$, and more generally

$$\lambda'_{j+1}\left((M_{i_j}/\rho_{j+1})\langle(\widehat{a}^j_v\widehat{a}^j_u)^{-1}\rangle\right) = M_{i_{j+1}}/\rho_{j+1} \times B \times \mathbb{Z}.$$

Since $M_{i_j}/\rho_{j+1} \subseteq M/\rho_{j+1}$ and $\langle(\widehat{a}^j_v\widehat{a}^j_u)^{-1}\rangle \subseteq (\widehat{A}^{j+1}_{j+1})^{-1}$, we thus get

$$\begin{aligned} M_{i_{j+1}}/\rho_{j+1} \times B \times \mathbb{Z} = \lambda'_{j+1}\left((M_{i_j}/\rho_{j+1})\langle(\widehat{a}^j_v\widehat{a}^j_u)^{-1}\rangle\right) \subseteq \\ \subseteq \lambda'_{j+1}\left((M/\rho_{j+1})(\widehat{A}^{j+1}_{j+1})^{-1}\right) = Im(\lambda'_{j+1}). \end{aligned}$$

Moreover we know that $S_j \subseteq Im(\lambda'_{j+1})$. This leads to

$$M_{i_{j+1}}/\rho_{j+1} \times S_{j+1} = M_{i_{j+1}}/\rho_{j+1} \times B \times \mathbb{Z} \times S_j \subseteq Im(\lambda'_{j+1}).$$

The opposite inclusion holds by the assumption. Hence the embedding λ'_{j+1} is an epimorphism, which implies that

$$\lambda'_{j+1}\left((M/\rho_{j+1})(\widehat{A}^{j+1}_{j+1})^{-1}\right) = M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}.$$

Hence, as in case $\heartsuit$, λ'_{j+1} is an isomorphism, so that $(M/\rho_{j+1})(\widehat{A}^{j+1}_{j+1})^{-1} = M_{i_{j+1}}/\rho_{j+1} \times S_{j+1}$. This completes the inductive step in case $\diamondsuit$, proving the assertion. ■

Notice that Construction 2.2.3 assigns ideals of the form $\mathcal{I}_{\rho_i} \lhd K[M]$ to the vertices of the tree D.

2.2.10 Theorem. Under the correspondence described in Construction 2.2.3, ideals of $K[M]$ corresponding to the leaves of D are prime.

Proof. Construction 2.2.3 describes an inductive interpretation of all diagrams in D as congruences on M. In particular, by Theorem 2.2.9, for every leaf from level r of D and for the corresponding congruence ρ_r on M, we have

$$(M/\rho_r)(\widehat{A}^r_r)^{-1} \simeq M_{i_r}/\rho_r \times S_r,$$

and S_r is of the form $\mathbb{N}^k \times (B \times \mathbb{Z})^m$ for some exponents k, m.

If the extreme arc occurring in the given diagram does not join generators a_1, a_n then there are $i_r > 0$ unused generators. The congruence ρ_r introduces the commutativity of these remaining generators, so that $M_{i_r}/\rho_r \simeq \mathbb{N}^{i_r}$. Hence

$$M_{i_r}/\rho_r \times S_r \simeq \mathbb{N}^{i_r} \times \mathbb{N}^k \times (B \times \mathbb{Z})^m \simeq \mathbb{N}^k \times (B \times \mathbb{Z})^m.$$

On the other hand, if the extreme arc joins a_1 and a_n then $i_r = 0$ and

$$M_{i_r}/\rho_r \times S_r \simeq M_0/\rho_r \times S_r \simeq S_r \simeq \mathbb{N}^k \times (B \times \mathbb{Z})^m.$$

Therefore, we have

$$K[M/\rho_r](\widehat{A}^r_r)^{-1} \simeq K[(M/\rho_r)(\widehat{A}^r_r)^{-1}] \simeq K[\mathbb{N}^k \times (B \times \mathbb{Z})^m].$$

It is well known that, for every field L, the algebra $L[B]$ is primitive, see [15]. From [14] it then follows that $R[B]$ is prime for every prime algebra K. This easily implies that $K[\mathbb{N}^k \times (B \times \mathbb{Z})^m]$ is prime. Thus $K[M/\rho_r](\widehat{A}^r_r)^{-1}$ is prime. Since it is a central localization of $K[M/\rho_r]$, also $K[M/\rho_r] \simeq K[M]/\mathcal{I}_{\rho_r}$ is prime. Hence $\mathcal{I}_{\rho_r}$ is a prime ideal of $K[M]$. ■

2.2.11 Definition. By the *middle of a diagram* $d \in D$ we mean
– the first generator used as a dot, if the construction of d starts with a dot,
– the middle of the first arc, if the construction of d starts with an arc.

An argument similar to that in the proof of Lemma 2.2.8 can be used to prove the following result.

2.2.12 Lemma. Let ρ be the congruence on M corresponding to a diagram d. Consider the images of the generators of M in M/ρ, interpreted as a submonoid in the appropriate $M_{i_l}/\rho \times S_l$, where S_l is the product of some copies of $B \times \mathbb{Z}$ and some copies of $\mathbb{N}$ (as in the proof of Lemma 2.2.8). Then in the images of the used generators on the left from the middle of d, elements 1 and p occur and at most one element g (where $\langle g, g^{-1}\rangle \simeq \mathbb{Z}$, if the given generator was used in an arc) or one g_s (if a_s was used as a dot; then $\langle g_s\rangle \simeq \mathbb{N}$ as in Construction 2.2.3).

On the other hand, the images of the used generators on the right of the middle of d contain components 1 and q and at most one g_s (if the generator a_s was used as a dot). If a dot a_s is the middle of d then $(1, \dots, 1, g_s)$ is the image of this generator.

The images in M/ρ of all generators used in d have elements p and q in components corresponding to the arcs built during the construction of d. Moreover, from the construction it follows that for every component B in S_l there exist generators a_i, a_j, whose images in M/ρ have in this component elements p and q, respectively. Hence, if the image in M/ρ of some $w \in M$ has p^+ or q^+ in this component, then it is not central in M/ρ (here $+$ denotes an arbitrary positive integer).

2.2.13 Lemma. If ρ and ρ' are congruences corresponding to diagrams d and d', and d, d' are in different branches of D, then $\mathcal{I}_\rho \nsubseteq \mathcal{I}_{\rho'}$.

Proof. First, consider the case where the root of D is the only vertex of D that is contained in both branches leading from the root to d and from the root to d'. This means that these diagrams start with

(1a) two different dots: d with a_s, and d' with a_t, where $t \neq s$, or

(1b) two different arcs: d with $a_s a_{s-1}$, and d' with $a_t a_{t-1}$, where $t \neq s$, or

(1c) one of them, say d, starts with an arc $a_s a_{s-1}$, and d' starts with a dot a_t.

Clearly, the middle of d is different than the middle of d'.

In case (1a) the image of a_s in M/ρ is central. If d' consists of a single dot a_t, then the image of a_s in M/ρ' is not central because it does not commute with the images of generators lying on the other side of a_t. Otherwise, in d', directly after the initial dot a_t, according to the rules, the arc $a_{t+1}a_{t-1}$ was built. Hence, Lemma 2.2.12 implies that in M/ρ' in the images of all generators on the left of a_t there is a component p, while on the right there is q. Since $a_s \neq a_t$, some component of the image of a_s is equal to p or q, hence also in this case this image is not central in M/ρ'. Therefore $\rho \nsubseteq \rho'$.

In the same way we see that the image of a_t is central in M/ρ', but it is not central in M/ρ, whence $\rho' \nsubseteq \rho$. This proves the assertion in case (1a).

Similarly, in case (1b), the image of $a_s a_{s-1}$ is central in M/ρ. Assume, with no loss of generality, that $s > t$. Then, by Lemma 2.2.12, in the images of a_s and a_{s-1} in M/ρ' there are components equal to q and there are no components equal to p, so the image of $a_s a_{s-1}$ is not central in M/ρ'. In the same way we see that the image of $a_t a_{t-1}$ is central in M/ρ', but is not central in M/ρ. This yields the assertion in case (1b).

In case (1c), similarly, assume that $s > t$. The image of $a_s a_{s-1}$ is central in M/ρ. If d' consists of the single dot a_t, then in M/ρ' we have the same relations as in M and additionally the images of $a_1, \dots, a_t$ commute and the images of $a_t, \dots, a_n$ commute. Therefore, the image of $a_s a_{s-1}$ is not central in M/ρ', because it does not commute with the image of a_{t-1} (since a_t is a dot, we must have $t > 1$).

On the other hand, if d' is not a single dot, then as in case (1a), in the diagram d' directly after the initial dot a_t the arc $a_{t+1}a_{t-1}$ must have been built. Hence, by

Lemma 2.2.12, in the image of a_s in M/ρ' one of the components is equal to q, while in the image of a_{s-1} there are no components equal to p. Hence, the image of $a_s a_{s-1}$ is not central in M/ρ'.

Similarly, the image of a_t is central in M/ρ', but it is not central in M/ρ, because one of its components is equal to p. Hence, again $\mathcal{I}_\rho \nsubseteq \mathcal{I}_{\rho'}$ and $\mathcal{I}_{\rho'} \nsubseteq \mathcal{I}_\rho$.

This completes the proof in the case where the root of D is the only common vertex of the branches containing d and d'.

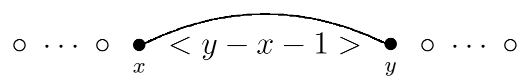
Now, consider the opposite case. So, up to a certain step in the construction of D the diagrams d and d' are equal. Assume that generators $a_{x+1}, \dots, a_{y-1}$, where $1 < x+1 < y-1 < n$, were used in this common part of the construction of d and d' (so the diagram obtained in this step is not a leaf of D). We may assume that $x+1 < y-1$, so the number of generators used in the common part of the construction of d and d' exceeds 1, because if two diagrams start with the same dot then in both of them the same arc must follow.

Hence, one of the following cases must occur.

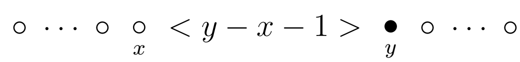
(2a) In one of the diagrams, say in d, an arc $a_y a_x$ was built, while in d' a dot was built (say, to the right of the previously used generators, so a_y, and then $y < n$).

(2b) In one of the diagrams, say in d, a dot a_x was built on one side, while in d' a dot a_y was built on the other side (then $x > 1$ and $y < n$).

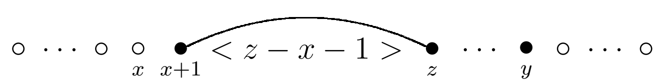
First, consider case (2a). Recall that $< i >$ denotes i consecutive used generators, so an initial step in the construction of d in this case looks like



while an initial step in the construction of d' is of the form

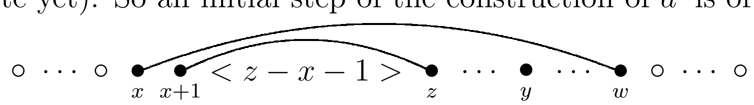


In d', before the dot a_y, some dots might have been added on the same side of the previously used generators, and before this an arc $a_z a_{x+1}$ must had been added, for some $x+1 < z < y$. Therefore, the following diagram is an initial step in the construction of d'

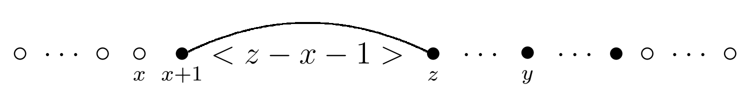


In the next steps of the construction of d', after the dot a_y some dots might have been added on the same side as previously used generators, which was followed by one of the following two steps.

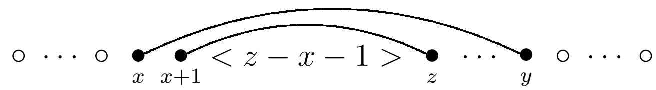
(2a.1) An arc $a_w a_x$ was added for some $w > y$ (and perhaps the construction of d' was not complete yet). So an initial step of the construction of d' is of the form



(2a.2) The construction of d' was completed, so d' is of the form



Since the generators $a_{x+1}, \ldots, a_{y-1}$ are in both diagrams d and d' used in the same way, an initial step of the construction of d must be of the form



In d the generator a_{x+1} is not the middle, because if it were the initial dot then an arc $a_{x+2}a_x$ would follow. However, we know that $y \neq x+2$, because we assume that $y - x - 1 > 1$.

In view of Lemma 2.2.8, the image of $a_y a_x$ is not central in M/ρ.

First, consider case (2a.1). Since the construction of ρ' involves a dot a_y, and at a later stage an arc $a_w a_x$, where $x < y < w$, it follows that the image of a_y in M/ρ' is of the form $(\ldots, 1, 1, \ldots)$, while the image of a_x is of the form $(\ldots, p, g, \ldots)$, where the distinguished two components $B \times \mathbb{Z}$ result from adding the arc $a_w a_x$ in the construction of M/ρ', and $\ldots$ denote the values of the remaining components. The forms of these sequences are derived as in the proof of Lemma 2.2.8, by representing M/ρ' as a submonoid of an appropriate monoid $M_{i_l}/\rho' \times S_l$, where S_l is a direct product of some copies of $B \times \mathbb{Z}$ and some copies of $\mathbb{N}$. Therefore, the image of $a_y a_x$ is of the form $(\ldots, p, g, \ldots)$, whence – because of the component p – it is not central in M/ρ'. Hence, in case (2a.1), $\rho \nsubseteq \rho'$.

Next, consider case (2a.2). Using an argument and notation as in the proof of Lemma 2.2.8, and applying Lemma 2.2.12 and the fact that a_y appears in d' to the right of the middle of d', one can get a more detailed description of the image of a_y in M/ρ' as $(1, \ldots, 1, g_y, [1, q])$, where g_y appears in a component corresponding to $\mathbb{N}$ in M/ρ'. Similarly, the image of a_x has the form $(\widehat{a}^l_x, 1, \ldots, 1, [1, p])$, where elements p in $[1, p]$ occur in the same components as elements q occur in $[1, q]$ in the image of a_y. Hence, the image of $a_y a_x$ has the form $(\widehat{a}^l_x, 1, \ldots, 1, g_y, [1, 1])$. Moreover, $\widehat{a}^l_x \in M_{i_l}/\rho'$ does not commute with $\widehat{a}^l_{y+1} \in M_{i_l}/\rho'$, so the image of $a_y a_x$ is not central in M/ρ'. Therefore, in case (2a.2) we also have $\rho \nsubseteq \rho'$.

Hence, in both subcases of case (2a) we get $\rho \nsubseteq \rho'$ and in the rest of the proof we treat both these cases together.

We claim that the image of $a_y a_{x+1}$ in M/ρ' is central, but its image in M/ρ is not central, which will yield $\rho' \nsubseteq \rho$. As above, we know that the image of a_y in M/ρ' looks like $(1, \ldots, 1, g_y, [1, q])$, where g_y occurs in a component corresponding to $\mathbb{N}$ w M/ρ'. On the other hand, a_{x+1} appears in d' to the left of the middle of d', so in the image of a_{x+1} the components are 1, p and a single g (in the component corresponding to an arc with the left end in a_{x+1}). Moreover, the elements p occur in the same components in which the elements q occur in the image of a_y. Therefore all components of the image of $a_y a_{x+1}$ are equal to 1 except for a single component g and a single component g_y. Hence, this element is central in M/ρ'.

Similarly, the element a_y in the diagram d is to the right of the middle of d, so the image of a_y in M/ρ has the form $(1, \ldots, 1, q, 1, [1, q])$. As noticed before, the element a_{x+1} is in d to the left of the middle of d, whence in the image of a_{x+1} in M/ρ there are components 1, p and either a single component g, if the generator a_{x+1} was used in an arc, or a single g_{x+1}, if it was used as a dot. Moreover, all components different than 1 occur in the components corresponding to the part of d, which was built before the construction of the arc $a_y a_x$, so in some of the components covered by $[1, q]$ in the image of a_y. The remaining components of the image of a_{x+1} are all equal to 1. This implies that the image of $a_y a_{x+1}$ is not central in M/ρ, because q is not central. Hence, we indeed get $\rho' \nsubseteq \rho$, as desired.

It follows that $\mathcal{I}_\rho \nsubseteq \mathcal{I}_{\rho'}$ and $\mathcal{I}_{\rho'} \nsubseteq \mathcal{I}_\rho$, which completes the proof in case (2a).

Finally, we deal with case (2b). This case can occur only if the common initial part of d and d' is „covered" with an arc (only in this case a dot can be added both on the left and on the right of the arc). Hence, an initial part of the construction of d' has the form

$$\circ \cdots \circ \ \underset{x}{\circ} \ \underset{x+1}{\bullet} \overbrace{< y-x-2 >} \underset{y-1}{\bullet} \ \underset{y}{\bullet} \ \circ \cdots \circ$$

which is a special case of the diagram d' described in case (2a). So we know that the image of $a_y a_{x+1}$ is central in M/ρ'. In this case

$$\circ \cdots \circ \ \underset{x}{\bullet} \ \underset{x+1}{\bullet} \overbrace{< y-x-2 >} \underset{y-1}{\bullet} \ \underset{y}{\circ} \ \circ \cdots \circ$$

is an initial step in the construction of the diagram d, whence a_{x+1} is on the left of the middle of d. Hence, as in case (2a), one can show that the image of $a_y a_{x+1}$ is not central in M/ρ. By symmetry, the image of $a_{y-1}a_x$ is central in M/ρ, but is not central in M/ρ'. Then, again $\mathcal{I}_\rho \nsubseteq \mathcal{I}_{\rho'}$ and $\mathcal{I}_{\rho'} \nsubseteq \mathcal{I}_\rho$, which completes the proof in case (2b) and therefore the proof of the lemma. ■

2.3. **Minimal prime ideals as the leaves of D.**

In this part we prove that every prime ideal of $K[M]$ contains an ideal corresponding to a leaf of the tree D, and more generally that there is a bijection between the set of leaves of D and the set of minimal prime ideals of $K[M]$.

2.3.1 Theorem. Every prime ideal of $K[M]$ contains a prime ideal $\mathcal{I}_{\rho_r}$, where ρ_r is the congruence corresponding to a leaf of D.

Proof. Let P be a fixed minimal prime ideal of $K[M]$. By Theorem 1.1.4, P contains an ideal of the form $\mathcal{I}_\rho$, where ρ is a congruence coming from a diagram in the first level of the tree D, so it is of one of the following types:

♡) ρ corresponds to a diagram $\circ\cdots\circ \ \bullet \ \circ\cdots\circ$ where the indicated dot is neither a_1 nor a_n,

♢) ρ corresponds to a diagram $\circ\cdots\circ \ \overbrace{\bullet\bullet} \ \circ\cdots\circ$.

Moreover, in cases ♡ and ♢ respectively, the homomorphisms from Lemmas 1.2.3 and 1.2.7 can be extended to maps of the respective semigroup algebras. This leads to homomorphisms

$$\begin{tikzcd} K[M] \arrow[r] & K[M_\rho] \arrow[d] & \simeq K[\overline{M^s_{n-1}} \times \langle \overline{a_s} \rangle] \simeq K[M^s_{n-1}/\rho][\mathbb{N}] \\ & K[M]/P & \end{tikzcd}$$

$$\begin{tikzcd} K[M] \arrow[r] & K[M_\rho] \arrow[d] \arrow[r, hook] & K[M_\rho \langle (\overline{a_s a_{s-1}})^{-1} \rangle] \simeq K[\overline{M^{s-1,s}_{n-2}} \times B \times \mathbb{Z}] \simeq K[M^{s-1,s}_{n-2}/\rho][B \times \mathbb{Z}] \\ & K[M]/P & \end{tikzcd}$$

in cases ♡ and ♢, respectively, where the embedding is accomplished via the central localization with respect to $\langle\overline{a_s a_{s-1}}\rangle$.

Consider all chains of congruences $\rho_0 \subsetneq \rho_1 \subsetneq \rho_2 \subsetneq \ldots \subsetneq \rho_j$ on M, corresponding to a fragment of a branch of the tree D, such that $\mathcal{I}_{\rho_j} \subseteq P$. By Lemma 2.2.5 we then have natural homomorphisms

$$\begin{tikzcd} K[M] \arrow[r] & K[M/\rho_j] \arrow[r, hook] \arrow[d] & K[M_{i_j}/\rho_j][S_j] \\ & K[M]/P & \end{tikzcd}$$

where S_j and i_j are defined as in 2.2.2.

In the set of all such chains we choose a chain for which i_j is minimal. We will show that the corresponding ρ_j is the congruence assigned to a leaf of D. Suppose otherwise. Then $i_j > 0$ and M_{i_j}/ρ_j is not a free abelian monoid of rank i_j, because none of the generators a_1, a_n have been used. More precisely,

$$M_{i_j}/\rho_j = M_{i_j}^{u,v}/(\rho_j|_{M_{i_j}^{u,v}}) = \langle a_1, \ldots, a_{u-1}, a_{v+1}, \ldots, a_n : \\ \underbrace{a_1, \ldots, a_{u-1}}_{\text{commute}}, \underbrace{a_{v+1}, \ldots, a_n}_{\text{commute}}, \text{ and the relations of a Chinese monoid hold}\rangle. \tag{13}$$

Consider an equality $\alpha_1 K[M_{i_j}]\beta_1 = 0$ of type ⊞ (see Notation 1.1.2), where $\alpha_1, \beta_1 \in K[M_{i_j}]$. Then, in $K[M_{i_j}/\rho_j][S_j]$ we get $\widehat{\alpha}_1^j K[M_{i_j}/\rho_j][S_j]\widehat{\beta}_1^j = 0$, where $\widehat{x}^j$ denotes the image of x in $K[M_{i_j}/\rho_j][S_j]$. Notice that $K[M/\rho_j]$ embeds into $K[M_{i_j}/\rho_j][S_j]$, because $M/\rho_j \hookrightarrow M_{i_j}/\rho_j \times S_j$. Thus we can identify M/ρ_j with its image under this embedding. Let

$$\mathcal{I}_{\alpha_1} = \left(K[M_{i_j}/\rho_j]\,\widehat{\alpha}_1^j\,K[M_{i_j}/\rho_j]\right)[S_j] \cap K[M/\rho_j],$$
$$\mathcal{I}_{\beta_1} = \left(K[M_{i_j}/\rho_j]\,\widehat{\beta}_1^j\,K[M_{i_j}/\rho_j]\right)[S_j] \cap K[M/\rho_j].$$

Then

$$\mathcal{I}_{\alpha_1} \cdot \mathcal{I}_{\beta_1} \subseteq \left(K[M_{i_j}/\rho_j]\,\widehat{\alpha}_1^j\,K[M_{i_j}/\rho_j]\,\widehat{\beta}_1^j\,K[M_{i_j}/\rho_j]\right)[S_j] \cap K[M/\rho_j] = 0.$$

Moreover $\mathcal{I}_{\alpha_1}, \mathcal{I}_{\beta_1} \lhd K[M/\rho_j]$, because $\mathcal{I}_{\alpha_1}, \mathcal{I}_{\beta_1} \lhd K[M_{i_j}/\rho_j][S_j]$.

Let $\widehat{P}^j$ be the image of P in $K[M/\rho_j]$. Since $\mathcal{I}_{\rho_j} \subseteq P$, there exists a natural map $K[M/\rho_j] \to K[M]/P$ whose kernel is $\widehat{P}^j$. Moreover, $K[M/\rho_j]/\widehat{P}^j \simeq K[M]/P$. In particular, $\widehat{P}^j$ is a prime ideal in $K[M/\rho_j]$. So, for every pair of ideals $\mathcal{I}_{\alpha_1}, \mathcal{I}_{\beta_1}$, since $\mathcal{I}_{\alpha_1} \cdot \mathcal{I}_{\beta_1} = 0$, we get $\mathcal{I}_{\alpha_1} \subseteq \widehat{P}^j$ or $\mathcal{I}_{\beta_1} \subseteq \widehat{P}^j$. Let $\gamma_1 = \alpha_1$ if $\mathcal{I}_{\alpha_1} \subseteq \widehat{P}^j$ and let $\gamma_1 = \beta_1$ otherwise (then we must have $\mathcal{I}_{\beta_1} \subseteq \widehat{P}^j$). Since $\mathcal{I}_{\gamma_1} \subseteq \widehat{P}^j$, there exists a natural homomorphism ${K[M/\rho_j]}/{\mathcal{I}_{\gamma_1}} \to K[M]/P$.

Now, consider another pair $\alpha_2, \beta_2 \in K[M_{i_j}]$ of type ⊞. The equalities $\alpha_2 K[M_{i_j}]\beta_2 = 0$ hold in $\left({K[M_{i_j}/\rho_j]}/{\widehat{\gamma}_1^j = 0}\right)[S_j]$. It follows that

$$\widehat{\alpha}_2^{j,1}\left({K[M_{i_j}/\rho_j]}/{\widehat{\gamma}_1^j = 0}\right)[S_j]\,\widehat{\beta}_2^{j,1} = 0,$$

where $\widehat{x}^{j,1}$ denotes the image of x in $\left(K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0\right)[S_j]$. Let $\widehat{P}^{j,1}$ be the image of P in the algebra $K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0$. We define $\mathcal{I}_{\alpha_2}$ and $\mathcal{I}_{\beta_2}$ by

$$\mathcal{I}_{\alpha_2} = \left(\left(K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0\right)\widehat{\alpha}_2^{j,1}\left(K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0\right)\right)[S_j] \cap K[M/\rho_j]\big/\mathcal{I}_{\gamma_1},$$
$$\mathcal{I}_{\beta_2} = \left(\left(K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0\right)\widehat{\beta}_2^{j,1}\left(K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0\right)\right)[S_j] \cap K[M/\rho_j]\big/\mathcal{I}_{\gamma_1}.$$

As above, we see that $\mathcal{I}_{\alpha_2}, \mathcal{I}_{\beta_2}$ are ideals in $K[M/\rho_j]\big/\mathcal{I}_{\gamma_1}$, the ideal $\widehat{P}^{j,1}$ is prime in $K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0$ and either $\mathcal{I}_{\alpha_2} \subseteq \widehat{P}^{j,1}$ or $\mathcal{I}_{\beta_2} \subseteq \widehat{P}^{j,1}$. Let $\gamma_2 = \alpha_2$, if $\mathcal{I}_{\alpha_2} \subseteq \widehat{P}^{j,1}$ and let $\gamma_2 = \beta_2$ otherwise (then we must have $\mathcal{I}_{\beta_2} \subseteq \widehat{P}^{j,1}$). Since $\mathcal{I}_{\gamma_2} \subseteq \widehat{P}^{j,1}$, there exists a homomorphism $K[M/\rho_j] \,/\, \mathcal{I}_{\gamma_1}\Big/\mathcal{I}_{\gamma_2} \to K[M]/P$.

Similarly one shows that the image $\widehat{P}^{j,2}$ of P in $K[M/\rho_j] \,/\, \mathcal{I}_{\gamma_1}\Big/\mathcal{I}_{\gamma_2}$ is a prime ideal and

$$K[M/\rho_j] \,/\, \mathcal{I}_{\gamma_1}\Big/\mathcal{I}_{\gamma_2} \hookrightarrow \left(K[M_{i_j}/\rho_j] \,/\, \widehat{\gamma}_1^j = 0\Big/\widehat{\gamma}_2^{j,1} = 0\right)[S_j].$$

By the hypothesis, the above construction yields

$$\begin{tikzcd}
P \arrow[d] & \lhd & K[M] \arrow[d] & \\
\widehat{P}^j \arrow[d] & \lhd & K[M/\rho_j] \arrow[r, hook] \arrow[d] & K[M_{i_j}/\rho_j][S_j] \arrow[d] \\
\widehat{P}^{j,1} \arrow[d] & \lhd & K[M/\rho_j]\big/\mathcal{I}_{\gamma_1} \arrow[r, hook] \arrow[d] & \left(K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0\right)[S_j] \arrow[d] \\
\widehat{P}^{j,2} & \lhd & K[M/\rho_j] \,/\, \mathcal{I}_{\gamma_1}\Big/\mathcal{I}_{\gamma_2} \arrow[r, hook] \arrow[d] & \left(K[M_{i_j}/\rho_j] \,/\, \widehat{\gamma}_1^j = 0\Big/\widehat{\gamma}_2^{j,1} = 0\right)[S_j] \\
 & & K[M]/P &
\end{tikzcd}$$

where the ideals in the first column are prime and the kernels of the three homomorphisms from $K[M]$ to the subsequent three algebras in the second column are contained in P, because $\mathcal{I}_{\rho_j} \subseteq P$, and also $\mathcal{I}_{\gamma_1} \subseteq \widehat{P}^j$ and $\mathcal{I}_{\gamma_2} \subseteq \widehat{P}^{j,1}$.

Let $\widehat{x}^{j,2}$ denote the image of x in $\left(K[M_{i_j}/\rho_j] \,/\, \widehat{\gamma}_1^j = 0\Big/\widehat{\gamma}_2^{j,1} = 0\right)[S_j]$. Similarly, we define also $\widehat{x}^{j,m}$ for $m \geq 3$, using other pairs of elements $\alpha, \beta \in K[M_{i_j}]$ of type $\boxplus$. Let $\widehat{x}^{j,0}$ denote $\widehat{x}^j$.

Notice that each of the elements $\alpha, \beta \in K[M_{i_j}]$ of type $\boxplus$ is a difference of two elements of M_{i_j}, see Theorem 1.1.1. Hence, all considered elements γ are also of this type. Put $\gamma_k = l_k - p_k$, where $l_k, p_k \in M_{i_j}$. Then it is clear that

$$\left(K[M_{i_j}/\rho_j]\big/\widehat{\gamma}_1^j = 0\right)[S_j] = K\left[M_{i_j}/\rho_j\big/\widehat{l}_1^j = \widehat{p}_1^j\right][S_j].$$

We also get

$$\left(K[M_{i_j}/\rho_j]\ /\ \widehat{\gamma}_1^j = 0\Big/\widehat{\gamma}_2^{j,1} = 0\right) = K[M_{i_j}/\rho_j]\big/_{(\widehat{\gamma}_1^j, \widehat{\gamma}_2^j)} = K[M_{i_j}/\rho_j]\big/_{\begin{cases}\widehat{\gamma}_1^j=0\\ \widehat{\gamma}_2^j=0\end{cases}},$$

which, as above, leads to

$$\left(K[M_{i_j}/\rho_j]\ /\ \widehat{\gamma}_1^j = 0\Big/\widehat{\gamma}_2^{j,1} = 0\right)[S_j] = K\left[M_{i_j}/\rho_j\ /\ \begin{cases}\widehat{l}_1^j=\widehat{p}_1^j\\ \widehat{l}_2^j=\widehat{p}_2^j\end{cases}\right][S_j].$$

Proceeding in this way, until all t pairs $\alpha, \beta \in K[M_{i_j}]$ of type $\boxplus$ are used, we extend the above diagram by adding more rows. As above, we get the following form of the last two rows of this diagram:

$$\begin{tikzcd}
\widehat{P}^{j,t} \lhd & K[M/\rho_j]\ /\ \mathcal{I}_{\gamma_1}\ /\ \mathcal{I}_{\gamma_2}\ /\ \ldots/\mathcal{I}_{\gamma_t} \arrow[r, hook] \arrow[d] & K\left[M_{i_j}/\rho_j\ \Big/\ \begin{cases}\widehat{l}_1^j=\widehat{p}_1^j\\ \widehat{l}_2^j=\widehat{p}_2^j\\ \vdots\\ \widehat{l}_t^j=\widehat{p}_t^j\end{cases}\right][S_j] \\
& K[M]/P &
\end{tikzcd}$$

Let η be the congruence on M_{i_j} generated by the set $\{(l_1, p_1), (l_2, p_2), \ldots, (l_t, p_t)\}$. Then we get

$$K\left[M_{i_j}/\rho_j\ \Big/\ \begin{cases}\widehat{l}_1^j=\widehat{p}_1^j\\ \widehat{l}_2^j=\widehat{p}_2^j\\ \vdots\\ \widehat{l}_t^j=\widehat{p}_t^j\end{cases}\right][S_j] = \left(K[M_{i_j}]\ /\ \mathcal{I}_{\rho_j}\Big/_{(\widehat{\gamma}_1^j, \widehat{\gamma}_2^j, \ldots, \widehat{\gamma}_t^j)}\right)[S_j] =$$

$$= \left(K[M_{i_j}]/\left(\mathcal{I}_{\rho_j} \cup (\gamma_1, \gamma_2, \ldots, \gamma_t)\right)\right)[S_j] = K[M_{i_j}/(\rho_j \vee \eta)][S_j],$$

where $\lambda_1 \vee \lambda_2$ denotes the congruence generated by λ_1 and λ_2.

The congruence η is defined by a set containing one element from each pair (α, β) of type $\boxplus$ for $K[M_{i_j}]$, so by Theorem 1.1.4 it contains a congruence η_0 of type $\heartsuit_{i_j}$ or $\diamondsuit_{i_j}$ on M_{i_j}. Therefore, $\eta_0 \nsubseteq \rho_j|_{M_{i_j}}$, and so $\rho_j|_{M_{i_j}} \subsetneq \rho_j|_{M_{i_j}} \vee \eta_0$ (see the description of M_{i_j}/ρ_j in (13).

We know that $M_{i_j}/\rho_j \times S_j = M_{i_j}^{u,v}/\rho_j \times S_j$, so the generators $a_u, \ldots, a_v$ have been used, for some $1 < u \leq v < n$. Let ω be the kernel of the map $M \to \left(M_{i_j}/(\rho_j \vee \eta_0)\right) \times S_j$. The above construction implies that ω satisfies $\mathcal{I}_\omega \subseteq P$.

Let ρ_{j-1} be the congruence corresponding to the diagram d_{j-1} of level $j-1$ in the tree D, which is connected to the diagram d_j corresponding to ρ_j. We will show that one of the following cases holds.

(A) There exists a congruence ρ_{j+1} on M such that $\rho_j \subsetneq \rho_{j+1} \subseteq \omega$, and ρ_{j+1} corresponds to a diagram d_{j+1} in D, which is connected to the diagram d_j. In this case, since $\mathcal{I}_\omega \subseteq P$, we get $\mathcal{I}_{\rho_{j+1}} \subseteq \mathcal{I}_\omega \subseteq P$ and $i_{j+1} < i_j$.

For this, we will find a congruence χ on M_{i_j}, of type $\heartsuit_{i_j}$ or $\diamondsuit_{i_j}$, such that $\rho_j|_{M_{i_j}} \subsetneq \chi \subseteq \rho_j|_{M_{i_j}} \vee \eta_0$ and the congruence $\ddot{\chi}$ on M which is the kernel of the natural homomorphism $M \to M_{i_j}/\rho_j \times S_j \to M_{i_j}/\chi \times S_j$ corresponds to a diagram in D, lying below the diagram corresponding to ρ_j; then we will put $\rho_{j+1} = \ddot{\chi}$.

(B) There exists a congruence ρ'_j such that $\rho_{j-1} \subsetneq \rho'_j$, where $i'_j < i_j$, and i'_j is the number of unused generators in ρ'_j. Moreover, the congruence ρ'_j corresponds to a diagram in D, which is connected to the diagram d_{j-1} and $\rho'_j \subseteq \omega$. In this case, since $\mathcal{I}_\omega \subseteq P$, we get $\mathcal{I}_{\rho'_j} \subseteq \mathcal{I}_\omega \subseteq P$.

Both cases contradict the choice of i_j, which will complete the proof of the fact that ρ_j corresponds to a leaf of D.

We know that η_0 is a congruence of type $\heartsuit$ or $\diamondsuit$ on $M_{i_j} = M^{u,v}_{i_j}$. If η_0 corresponds to a diagram (on M_{i_j}) which is an arc $a_{v+1}a_{u-1}$, then ω corresponds to a diagram in D (on M) and we put $\chi = \rho_j|_{M_{i_j}} \vee \eta_0$. Then $\omega = \ddot{\chi}$ and we define $\rho_{j+1} = \ddot{\chi}$. We thus get case (A).

We consider the remaining possibilities.

If η_0 corresponds to a diagram (on M_{i_j}) which is an arc $a_s a_{s-1}$ for some $s > v+1$, then a_{v+1} becomes central in the monoid $M_{i_j}/(\rho_j \vee \eta_0)$ (it commutes with a_j for $j < v+1$ because $s > v+1$ and commutes with all a_j for $j > v+1$ because of the congruence ρ_j). A symmetric argument shows that, if $s < u$, then a_{u-1} becomes central.

Similarly, if η_0 corresponds to a diagram (on M_{i_j}) which is a dot a_s for some $s > v$, then the elementu a_{v+1} becomes central in $M_{i_j}/(\rho_j \vee \eta_0)$. A symmetric argument shows that, if $s < u$, then a_{u-1} becomes central.

Therefore, in both considered cases, the congruence $\rho_j|_{M_{i_j}} \vee \eta_0$ induces centrality of a_{u-1} or of a_{v+1} in the image of M_{i_j}, so ω is a congruence corresponding to the diagram containing a dot neighboring the previously used generators. If these used generators are covered with an arc, then the new diagram obtained by adding the dot neighboring this arc is allowed by Definition 2.1.2, so it is an element of the tree defined for M_{i_j}. Let χ denote the congruence on M_{i_j} corresponding to this new diagram; so $\chi \subseteq \rho_j|_{M_{i_j}} \vee \eta_0$. The second case is when the new dot is on the same side as some recently added dot. Then it is also easy to see that we get a congruence $\chi \subseteq \rho_j|_{M_{i_j}} \vee \eta_0$ that corresponds to a diagram on M_{i_j}.

In both cases we may thus define $\rho_{j+1} = \ddot{\chi} \supsetneq \rho_j$ and conditions in case (A) are satisfied, in particular $i_{j+1} < i_j$.

It remains to consider the case where the diagram d_j corresponding to ρ_j contains dots on one side and the considered „new dot” (coming from η_0) is on the other side. In this case, we construct a congruence ρ'_j that satisfies conditions in (B).

Assume that the last step in the construction of d_j was the dot a_u, while the new dot is the dot a_{v+1}. Then, by Lemma 1.2.3, we get $M_{i_{j-1}}/\rho_{j-1} \simeq M_{i_j}/\rho_j \times \langle a_u \rangle$. This corresponds to replacing $M_{i_{j-1}}/\rho_{j-1} \times S_{j-1}$ by $M_{i_j}/\rho_j \times \langle a_u \rangle \times S_{j-1}$ (in the process of constructing ρ_j from ρ_{j-1}; see Construction 2.2.3).

Let e be the diagram in D obtained from d_{j-1} by adding the arc $a_{v+1}a_u$. We will show that the congruence ρ'_j, corresponding to e, is contained in ω.

Let M' be the image of $M_{i_j}/\rho_j \times \langle a_u\rangle$ obtained by making the generator a_{v+1} central in the first component, in other words

$$M' = \Big(M_{i_j}/\rho_j/_{(a_{v+1}\ \text{central})}\Big) \times \langle a_u\rangle.$$

We have to check that the following relations hold in M':
– the image of $a_{v+1}a_u$ is central,
– the images of $a_w a_{v+1} a_z$ and $a_z a_{v+1} a_w$ are equal for $w, z < u$,
– the images of $a_w a_u a_z$ and $a_z a_u a_w$ are equal for $w, z > v+1$.
These are the relations that are imposed on M_{i_j}/ρ_{j-1} in the process of constructing ρ'_j from ρ_{j-1} by adding the arc $a_{v+1}a_u$ (see the definition of an ideal of type $\diamondsuit$ in Definition 1.1.3 and Construction 2.2.3).

The image of $a_{v+1}a_u$ in $M_{i_j}/\rho_j \times \langle a_u\rangle$ is equal to $(\widehat{a}^j_{v+1}, a_u)$. In M', the element a_{v+1} becomes central in the first component. Hence the image of $a_{v+1}a_u$ is central in M'.

The image of $a_w a_{v+1} a_z$ in $M_{i_j}/\rho_j \times \langle a_u\rangle$ is equal to $(\widehat{a}^j_w \widehat{a}^j_{v+1} \widehat{a}^j_z, 1)$, while $(\widehat{a}^j_z \widehat{a}^j_{v+1} \widehat{a}^j_w, 1)$ is the image of $a_z a_{v+1} a_w$. Since a_{v+1} is central in the first component and the images of a_z and a_w commute for $w, z < u$, we get $\widehat{a}^j_w \widehat{a}^j_{v+1} \widehat{a}^j_z = \widehat{a}^j_{v+1}(\widehat{a}^j_w \widehat{a}^j_z) = \widehat{a}^j_{v+1}(\widehat{a}^j_z \widehat{a}^j_w) = \widehat{a}^j_z \widehat{a}^j_{v+1} \widehat{a}^j_w$. So in M' the images of $a_w a_{v+1} a_z$ and $a_z a_{v+1} a_w$ are equal.

Similarly, the image of $a_w a_u a_z$ in $M_{i_j}/\rho_j \times \langle a_u\rangle$ is equal to $(\widehat{a}^j_w \widehat{a}^j_z, \widehat{a}^j_u)$, and the image of $a_z a_u a_w$ is equal to $(\widehat{a}^j_z \widehat{a}^j_w, \widehat{a}^j_u)$. In $M_{i_j}/\rho_j \times \langle a_u\rangle$ we get $(\widehat{a}^j_w \widehat{a}^j_z, \widehat{a}^j_u) = (\widehat{a}^j_z \widehat{a}^j_w, \widehat{a}^j_u)$ for $w, z > v+1$, because $\widehat{a}^j_w \widehat{a}^j_z = \widehat{a}^j_z \widehat{a}^j_w$ in M_{i_j}/ρ_j for $w, z > v+1$. Thus, also the images of $a_w a_u a_z$ and $a_z a_u a_w$ are equal in M'.

Hence, all the relations corresponding to adding the arc $a_{v+1}a_u$ are satisfied. It follows that the congruence ρ'_j, corresponding to e is contained in ω. Since the diagram e has $i'_j = i_j - 1$ unused generators, case (B) holds.

This completes the proof of the fact that ρ_j corresponds to a leaf of D. The ideal $\mathcal{I}_{\rho_j}$ is prime in $K[M]$ by Theorem 2.2.10. This proves the assertion. ∎

2.3.2 Theorem. There exists a bijection between the set of leaves of the tree D and the set of minimal prime ideals of $K[M]$. Namely, if d is a leaf of D and ρ is the congruence corresponding to d, then $\mathcal{I}_\rho$ is the minimal prime ideal assigned to d.

Proof. Let P be a minimal prime ideal of $K[M]$. By Theorem 2.3.1, P contains a prime ideal of the form $\mathcal{I}_\rho$, where ρ corresponds to a leaf of D. Therefore $\mathcal{I}_\rho = P$. Let $f(P) = \rho$.

Let d be an arbitrary leaf of D and let η be the corresponding congruence on M. Then, by Theorem 2.2.10, $\mathcal{I}_\eta$ is a prime ideal of $K[M]$. Hence, there exists a minimal prime ideal Q of $K[M]$ contained in $\mathcal{I}_\eta$. Then, again by Theorem 2.3.1, Q contains an ideal of the form $\mathcal{I}_{\eta'}$ for a congruence η' corresponding to a leaf d' of D. Then $\mathcal{I}_{\eta'} \subseteq Q \subseteq \mathcal{I}_\eta$, while by Lemma 2.2.13 we have $\mathcal{I}_{\eta'} \subseteq \mathcal{I}_\eta$ if and only if the vertices of D corresponding to congruences η and η' are in the same branch of D. Since d, d' are leaves, we get $\eta = \eta'$. Then $\mathcal{I}_{\eta'} = Q = \mathcal{I}_\eta$, so $\mathcal{I}_\eta$ is a minimal prime ideal of $K[M]$. We define $g(\eta) = \mathcal{I}_\eta$. Therefore

$$gf(P) = g(\rho) = \mathcal{I}_\rho = P.$$

Clearly, $\rho_P = \rho_{\mathcal{I}_\rho}$. It is easy to see that $x - y \in \mathcal{I}_\rho$ if and only if $(x, y) \in \rho$. Therefore $f(P) = \rho_P$, and hence

$$fg(\rho) = f(\mathcal{I}_\rho) = \rho_{\mathcal{I}_\rho} = \rho.$$

It follows that f and g establish the desired bijection. ■

2.3.3 Notation. If P is a minimal prime ideal of $K[M]$ then the congruence corresponding to P is denoted by ρ_P. Notice that this is a homogeneous congruence, because minimal prime ideals of a $\mathbb{Z}$-graded ring are homogeneous, see for example [9] (this also is a consequence of our construction of ρ_P).

A careful analysis of the proof of Theorem 2.2.10 leads to the following description of the monoids M/ρ_P for minimal prime ideals P of $K[M]$. Recall that $M_P = M_{\rho_r}$, where $\rho_r = \rho_P$, in other words this is the last of the congruences used in the chain $\rho_1 \subsetneq \rho_2 \subsetneq \ldots \subsetneq \rho_r$, constructed in 2.2.3.

2.3.4 Corollary. For every minimal prime ideal P of $K[M]$ there exists an embedding

$$M/\rho_P \hookrightarrow \mathbb{N}^{c_P} \times (B \times \mathbb{Z})^{d_P},$$

where $c_P + 2d_P = n$. Moreover,

($\heartsuit$) if ρ_1 is of type $\heartsuit$, then

$$M_P \simeq T \times \langle \widehat{a}^r_s \rangle \simeq T \times \mathbb{N},$$

where $K[T] \simeq K[M_{n-1}]/Q$ for some minimal prime ideal Q of $K[M_{n-1}]$;

($\diamondsuit$) if ρ_1 is of type $\diamondsuit$, then

$$M_P \subseteq M_P(\widehat{A}^r_j)^{-1} \simeq T \times \mathbb{N}^t \times B \times \mathbb{Z},$$

where $1 \leq j \leq r$ and $K[T] \simeq K[M_{n-2-t}]/Q$ for some $0 \leq t \leq n-2$ and a minimal prime ideal Q in $K[M_{n-2-t}]$. For $t = n-2$ we put $K[M_0] = K$, $Q = 0$ and $T = \{1\}$.

Proof. Using the notation of the proof of Theorem 2.2.10, we know that $\rho_r = \rho_P$ and $(M/\rho_r)(\widehat{A}^r_r)^{-1} \simeq \mathbb{N}^* \times (B \times \mathbb{Z})^*$. Hence there is an embedding

$$M/\rho_P \hookrightarrow \mathbb{N}^{c_P} \times (B \times \mathbb{Z})^{d_P}$$

for some positive integers c_P, d_P. From the algorithm used in the process of building the latter direct product we know that a factor $\mathbb{N}$ appears each time a single generator is used (as a dot), while a factor $B \times \mathbb{Z}$ appears each time a pair of generators is used (as an arc). After the extreme arc is added to a diagram, the submonoid generated by the unused generators is free abelian. Hence $c_P + 2d_P = n$.

We keep the notation used in Construction 2.2.3 and in 2.2.4. For ρ_1 of type $\heartsuit$ we have a commuting diagram

$$\begin{tikzcd}
M \arrow[r, two heads, "\psi_1=\psi_\heartsuit"] \arrow[dr, "\psi_r"'] & M/\rho_1 \arrow[r, phantom, "\overset{\lambda_1}{\simeq}"] \arrow[d, "\varphi_i\circ\ldots\circ\varphi_1"] & \overline{M^s_{n-1}} \times \langle \widehat{a}^r_s \rangle \arrow[r, phantom, "\simeq"] \arrow[d, "\widehat{\kappa}^{r-1}\circ\ldots\circ\widehat{\kappa}^1"] & \overline{M^s_{n-1}} \times \mathbb{N} \arrow[d, "\mu"] \\
& M/\rho_r \simeq M_P \arrow[r, hook, "\lambda_r"'] & M_{i_r}/\rho_r \times S_r \arrow[r, hook] & (\mathbb{N}^* \times (B \times \mathbb{Z})^*) \times \mathbb{N}
\end{tikzcd}$$

where λ_r is as in Lemma 2.2.5 and the last embedding is identity on S_r, while μ is a homomorphism that makes the diagram commute.

From the construction we know that $\widehat{\kappa}^{r-1} \circ \ldots \circ \widehat{\kappa}^1$ is identity on $\langle \widehat{a}_s^r \rangle \simeq \mathbb{N}$. Hence, μ has the form $\theta \times id$, where θ acts on $\overline{M_{n-1}^s}$, and id acts on $\mathbb{N}$. Let

$$T \stackrel{\text{def}}{=} \theta(\overline{M_{n-1}^s}) \subseteq \mathbb{N}^* \times (B \times \mathbb{Z})^*.$$

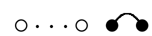
Then T is a homomorphic image of M_{n-1} and $M_P \simeq T \times \mathbb{N}$, where $\mathbb{N}$ is an isomorphic image of $\langle \widehat{a}_s^r \rangle$.

Denote by d the diagram corresponding to the ideal P. In the second step of the construction of d, the dot a_s, corresponding to ρ_1 of type $\heartsuit$, must have been followed by the arc $a_{s+1}a_{s-1}$, corresponding to ρ_2. We know that $M/\rho_1 \simeq \overline{M_{n-1}^s} \times \mathbb{N}$ and the congruence $\rho_r = \rho_P$ corresponds to the homomorphism $\overline{M_{n-1}^s} \times \mathbb{N} \stackrel{\mu}{\twoheadrightarrow} T \times \mathbb{N}$, so that $M/\rho_P = M_P \simeq T \times \mathbb{N}$.

We remove the dot a_s from the diagram d. Then we get a diagram d' in the tree built for the Chinese monoid on $n-1$ generators. Such d' corresponds to a leaf of this new tree, whence to a minimal prime ideal of $K[M_{n-1}]$. On the other hand, d' corresponds to the kernel of the homomorphism $M_{n-1} \to \overline{M_{n-1}^s} \to T$, which is a consequence of the construction of $\rho_1, \rho_2, \ldots, \rho_r$. So T is a homomorphic image of M_{n-1}^s. Let Q be the kernel of the epimorphism $K[M_{n-1}^s] \twoheadrightarrow K[T]$. Then $K[T] \simeq K[M_{n-1}^s]/Q$. Since d' corresponds to a minimal prime ideal of $K[M_{n-1}]$, Q is a minimal prime ideal. This completes the proof in case ρ_1 is of type $\heartsuit$.

Assume now that ρ_1 is of type $\diamondsuit$. We consider two cases.

(a) $r = 1$, so that $\rho_1 = \rho_P$ corresponds to a diagram

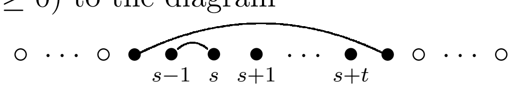
 or 

Then $\overline{M_{n-2}} \simeq \mathbb{N}^{n-2}$, so Lemma 1.2.7 yields

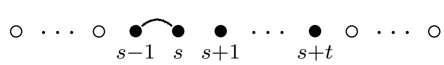
$$M_P = M/\rho_P \hookrightarrow \overline{M_{n-2}} \times B \times \mathbb{Z} \simeq \mathbb{N}^{n-2} \times B \times \mathbb{Z}$$

and the assertion follows with $t = n-2$, $K[M_0] = K$, $Q = 0$ and $T = \{1\}$.

(b) $r > 1$, so in the construction of the diagram d corresponding to the ideal P, after an initial arc corresponding to the congruence ρ_1, there were more steps leading to the leaf d of D. Recall that such a construction must finish with an extreme arc (see Definition 2.1.5). Hence, in d, after the initial arc $a_s a_{s-1}$, a number $t \geq 0$ of dots have been built, followed by another arc. Hence, for some $1 < j+1 \leq r$ the congruence ρ_{j+1} corresponds (for some $t \geq 0$) to the diagram



or to an analogous diagram with t dots on the left of the arc $a_s a_{s-1}$. Then ρ_j corresponds to the diagram



or to the analogous diagram with t dots on the left of the arc $a_s a_{s-1}$. Then the number of unused generators is equal to $i_j = n-2-t$ and $S_j = \mathbb{N}^t \times B \times \mathbb{Z}$, while Lemma 2.2.5 yields a natural embedding

$$M/\rho_j \hookrightarrow M_{i_j}/\rho_j \times S_j = M_{i_j}/\rho_j \times \mathbb{N}^t \times B \times \mathbb{Z}.$$

Moreover, $S_r = Y \times S_j = Y \times \mathbb{N}^t \times (B \times \mathbb{Z})$, where $Y = \mathbb{N}^* \times (B \times \mathbb{Z})^*$ and the construction of M/ρ_P yields natural homomorphisms

$$\begin{tikzcd}
M/\rho_j \arrow[r, hook] \arrow[d, "\varphi_{r-1}\circ\ldots\circ\varphi_j"'] & M_{i_j}/\rho_j \times S_j \arrow[d, "\widehat{\kappa}^{r-1}\circ\ldots\circ\widehat{\kappa}^j"] & = & M_{i_j}/\rho_j \times \mathbb{N}^t \times B \times \mathbb{Z} \arrow[d] \\
M/\rho_r \simeq M_P \arrow[r, hook] & M_{i_r}/\rho_r \times Y \times S_j & = & M_{i_r}/\rho_r \times Y \times \mathbb{N}^t \times B \times \mathbb{Z}
\end{tikzcd}$$

where $\widehat{\kappa}^{r-1} \circ \ldots \circ \widehat{\kappa}^j$ is identity on S_j, so it is of the form $\theta \times id$, with $\theta\colon M_{i_j}/\rho_j \to M_{i_r}/\rho_r \times Y$ and $id\colon S_j \to S_j$. Let

$$T \stackrel{\text{def}}{=} \theta(M_{i_j}/\rho_j),$$

so $T \times S_j$ is the image of $M_{i_j}/\rho_j \times S_j$ under $\widehat{\kappa}^{r-1} \circ \ldots \circ \widehat{\kappa}^j = \theta \times id$.

By Theorem 2.2.9, $M_{i_j}/\rho_j \times S_j = (M/\rho_j)(\widehat{A}^j_j)^{-1}$ (under an appropriate identification). Consider the following diagram, similar to (##) used in Theorem 2.2.9:

(###)
$$\begin{tikzcd}
 & (M/\rho_j)(\widehat{A}^j_j)^{-1} \arrow[d, "\varphi'_{r-1}\circ\ldots\circ\varphi'_j"'] & = & M_{i_j}/\rho_j \times S_j \arrow[d, "\widehat{\kappa}^{r-1}\circ\ldots\circ\widehat{\kappa}^j=\theta\times id"] & \\
M_P(\widehat{A}^r_j)^{-1} \;=& (M/\rho_r)(\widehat{A}^r_j)^{-1} \arrow[rr, hook, "\lambda''_r"] & & M_{i_r}/\rho_r \times Y \times S_j & = \; M_{i_r}/\rho_r \times S_r
\end{tikzcd}$$

where λ''_r is the restriction of λ'_r to $M_P(\widehat{A}^r_j)^{-1}$, and every φ'_k, for $k = j, \ldots, r-1$, is the natural extension of φ_k to the appropriate localization. Then $\varphi'_{r-1} \circ \ldots \circ \varphi'_j$ maps M/ρ_j onto M/ρ_r, while $\widehat{A}^j_j$ is mapped onto $\widehat{A}^r_j$. Therefore, this is an epimorphism onto $(M/\rho_r)(\widehat{A}^r_j)^{-1}$.

We know that $T \times S_j \subseteq M_{i_r}/\rho_r \times S_r$ is the image of $(M/\rho_j)(\widehat{A}^j_j)^{-1} = M_{i_j}/\rho_j \times S_j$ under $\widehat{\kappa}^{r-1} \circ \ldots \circ \widehat{\kappa}^j = \theta \times id$. Since diagram (###) commutes, this image must be equal to

$$\lambda''_r \circ \varphi'_{r-1} \circ \ldots \circ \varphi'_j((M/\rho_j)(\widehat{A}^j_j)^{-1}) = \lambda''_r(M_P(\widehat{A}^r_j)^{-1}) \simeq M_P(\widehat{A}^r_j)^{-1} \subseteq M_{i_r}/\rho_r \times S_r.$$

Therefore

$$M_P \subseteq M_P(\widehat{A}^r_j)^{-1} \simeq T \times S_j = T \times \mathbb{N}^t \times B \times \mathbb{Z},$$

which proves the first part of the assertion in case ρ_1 is of type $\diamondsuit$.

Removing from d the dots $s-1, s, s+1, \ldots, s+t$ leads to a diagram d' in the tree constructed for the Chinese monoid M_{n-2-t}. This diagram d' starts with an arc $a_{s+t+1}a_{s-2}$. Hence, as in the last part of the above proof in case $\heartsuit$, the diagram d' corresponds to the kernel of the homomorphism $M_{n-2-t} \to M_{n-2-t}/\rho_j \to T$ and we get $K[T] \simeq K[M_{n-2-t}]/Q$ for a minimal prime ideal Q in M_{n-2-t}. This completes the proof in case $\diamondsuit$, and hence the proof of the proposition. ■

3. Applications

Our final goal is to derive certain important consequences of the main result of Section 2. First, in Part 3.1, we show that the prime radical of the Chinese algebra $K[M]$ coincides with its Jacobson radical. Next, in Part 3.2, we obtain a formula for the number of minimal primes of $K[M]$. A surprising new representation of the monoid M as a submonoid of the direct product $B^d \times \mathbb{Z}^e$ for some $d, e \geq 1$ is found in Part 3.3. In particular, the latter implies that M satisfies a nontrivial identity.

3.1. $B(K[M]) = J(K[M])$.

We start with the following result.

3.1.1 Theorem. For every minimal prime ideal P of $K[M]$ the algebra $K[M]/P$ is semiprimitive.

Proof. Let n denote the rank of M. If $n = 1$ then $K[M_1] = K[x]$. If $n = 2$ then from [3] we know that $K[M_2]$ is also prime and semiprimitive. Hence, we may assume that $n \geq 3$. By induction, we may also assume that the assertion is satisfied for all Chinese algebras of rank less than n. We shall consider the two cases, denoted by $\heartsuit$ and $\diamondsuit$, as in Corollary 2.3.4.

First, consider case $\heartsuit$. From Corollary 2.3.4 we know that $K[M]/P \simeq K[M_P] \simeq K[T][x]$, where $K[T]$ is an algebra of the form $K[M_{n-1}]/Q$ for some minimal prime ideal $Q \lhd K[M_{n-1}]$. By the inductive hypothesis, we get $J(K[M_{n-1}]/Q) = 0$. Since $K[M_{n-1}]/Q \simeq K[T]$, this implies that $J(K[M]/P) \simeq J(K[T][x]) = 0$, as desired.

Next, consider case $\diamondsuit$. Suppose that $J(K[M_P]) \neq 0$ and choose some nonzero $a \in J(K[M_P])$. From Corollary 2.3.4 we know that $M_P \hookrightarrow T \times \mathbb{N}^t \times B \times \mathbb{Z}$ for an appropriate T. Hence $K[M_P]$ can be viewed as a $\mathbb{Z}$-graded algebra (according to the last component of the above direct product) or as an $\mathbb{N}$-graded algebra (for each of the t components $\mathbb{N}$). Therefore, from Theorem 30.28 in [12] we know that $J(K[M_P])$ is homogeneous. Thus we may assume that a is homogeneous with respect to each of the gradations coming from components $\mathbb{Z}$ or $\mathbb{N}$. Let $a = \sum_{i=1}^{k} \lambda_i s_i$ for some $k \geq 1$, $0 \neq \lambda_i \in K$, $s_i \in M_P$. Then all s_i coincide when restricted to each of these components. This means that there exist elements $m \in \mathbb{N}^t$, $z \in \mathbb{Z}$ (independent of i) such that $s_i = (t_i, m, b_i, z) \in T \times \mathbb{N}^t \times B \times \mathbb{Z}$. Since $a \neq 0$, also $\sum_{i=1}^{k} \lambda_i (t_i, b_i) \neq 0$.

Consider the natural projection

$$\Pi \colon T \times \mathbb{N}^t \times B \times \mathbb{Z} \to T \times B$$

and the induced map of semigroup algebras. Clearly $\Pi(a) = \sum_{i=1}^{k} \lambda_i (t_i, b_i) \neq 0$.

We know that $\Pi(M_P) \subseteq \Pi(T \times \mathbb{N}^t \times B \times \mathbb{Z}) = T \times B$. We will show that the opposite inclusion $\Pi(M_P) \supseteq T \times B$ also holds.

The monoid $\Pi(M_P)$ contains $(1, p)$ and $(1, q)$, because under the homomorphism $\psi_r \colon M \to M_P \subseteq T \times \mathbb{N}^t \times B \times \mathbb{Z}$ we have $a_{s-1} \mapsto (1, 1, p, g)$, $a_s \mapsto (1, 1, q, 1)$. Therefore, for every $b \in B$ we have $(1, b) \in \Pi(M_P)$.

From the proof of Corollary 2.3.4 and from the commuting diagrams used in this proof (in case $\diamondsuit$) it follows that the following diagram commutes:

$$\begin{tikzcd} M/\rho_j \arrow[rrr, hook] \arrow[d, "\varphi_{r-1}\circ\ldots\circ\varphi_j"'] & & & M_{i_j}/\rho_j \times S_j \arrow[d, "\theta\times id"] & & \\ M_P \arrow[r, hook] & M_P(\widehat{A}_j^r)^{-1} & \simeq & T \times S_j & = T \times \mathbb{N}^t \times B \times \mathbb{Z} \arrow[r, two heads, "\Pi"] & T \times B \end{tikzcd}$$

The embedding in the first row, composed with the projection $M_{i_j}/\rho_j \times S_j \to M_{i_j}/\rho_j$, maps M/ρ_j onto M_{i_j}/ρ_j. By the definition $T = \theta(M_{i_j}/\rho_j)$ and Π is a projection, whence the homomorphism

$$M/\rho_j \hookrightarrow M_{i_j}/\rho_j \times S_j \overset{\theta\times id}{\twoheadrightarrow} T \times S_j \overset{\Pi}{\twoheadrightarrow} T \times B,$$

composed with the projection onto T, is a map onto T. Commutativity of the above diagram implies now that also the homomorphism in the second row

$$M_P \hookrightarrow T \times S_j \overset{\Pi}{\twoheadrightarrow} T \times B,$$

composed with $T \times B \to T$, is a map onto T.

It follows that the image of $\Pi(M_P)$ under $T \times B \to T$ coincides with T. Hence, for every $t \in T$ there exists $b = p^i q^j \in B$ such that $(t, b) \in \Pi(M_P)$. Multiplying by $(1, q^i) \in \Pi(M_P)$ on the left and by $(1, p^j) \in \Pi(M_P)$ on the right, we get $(t, 1) \in \Pi(M_P)$. This and the fact that $(1, b) \in \Pi(M_P)$ for every $b \in B$ imply that $\Pi(M_P) \supseteq T \times B$, as desired.

Therefore, $\Pi(M_P) = T \times B$, so that $\Pi|_{M_P}$ is surjective and so its natural extension to $K[M_P]$ is also surjective. Therefore we get $\Pi(J(K[M_P])) \subseteq J(K[T \times B])$. Since $0 \neq a \in J(K[M_P])$ and $\Pi(a) \neq 0$, this implies that

$$0 \neq \Pi(a) \in \Pi(J(K[M_P])) \subseteq J(K[T \times B]). \tag{14}$$

Moreover, $K[T\times B] \simeq K[T][B]$ and from [3] we know that $K[T][B]$ contains an ideal $\mathcal{I} \simeq \mathcal{M}_\infty(K[T])$ such that $K[T][B]/\mathcal{I} \simeq K[T][x, x^{-1}]$. Here $\mathcal{M}_\infty(K[T])$ stands for the algebra of $\mathbb{N} \times \mathbb{N}$ matrices over $K[T]$ with finitely many nonzero entries.

As in case $\heartsuit$, from the inductive hypothesis it follows in view of Corollary 2.3.4 that $J(K[T]) = 0$. Hence, the above implies that $J(K[T][B]/\mathcal{I}) \simeq J(K[T][x, x^{-1}]) = 0$. Moreover, $J(K[T]) = 0$ yields

$$J(\mathcal{I}) \simeq J(\mathcal{M}_\infty(K[T])) \simeq \mathcal{M}_\infty(J(K[T])) = 0.$$

Since $J(\mathcal{I}) = 0$ and $J(K[T][B]/\mathcal{I}) = 0$, it follows that $J(K[T \times B]) \simeq J(K[T][B]) = 0$. This contradicts (14), completing the proof in case $\diamondsuit$. ■

As a direct consequence we get

3.1.2 Corollary. $B(K[M]) = J(K[M])$.

Notice that the properties of the algebra $K[M_3]$ are different than those of the plactic algebra of rank 3, which is not prime but is semiprimitive, see [3]. Namely, if $n \geq 3$ then the Chinese algebra $K[M]$ of rank n is not semiprime, [8].

3.2. **Number of minimal prime ideals of $K[M]$.**

In order to get a formula for the number of minimal primes of $K[M]$ we use the construction of the tree D and the bijection between the leaves of D and the minimal primes in $K[M]$, established in Theorem 2.3.2.

The following analogue of the Fibonacci sequence will be crucial.

3.2.1 Definition. *The Tribonacci sequence* is the sequence defined by the linear recurrence

$$\begin{cases} T_0 = T_1 = T_2 = 1 \\ T_{n+1} = T_n + T_{n-1} + T_{n-2} \text{ for } n \geq 2. \end{cases}$$

The properties of this sequence are described in [19, A000213]. The initial elements

of T_n are $T_0 = T_1 = T_2 = 1,\ T_3 = 3,\ T_4 = 5,\ T_5 = 9,\ T_6 = 17,\ T_7 = 31,\ T_8 = 57,$ $T_9 = 105,\ T_{10} = 193.$

3.2.2 Theorem. Let M be the Chinese monoid of rank n. Then T_n is the number of minimal prime ideals in $K[M]$.

Proof. Recall that, if the rank n of the Chinese monoid M is 1 or 2, then the algebra $K[M]$ is prime. Hence, we may assume that $n \geq 3$.

By Theorem 2.3.2, it is enough to enumerate the leaves of the tree D. From the construction of D in Definition 2.1.2 we also know that a diagram f is a leaf of D if and only if the last step in the construction of f is an arc containing one of the generators a_1, a_n, in other words an extreme arc. Hence, we will count the number of such diagrams.

Let k be the number of generators used in the construction of f before constructing the respective extreme arc (that is, the number of generators under this arc). Let U_k denote the number of all possible configurations of k generators under an arc in a diagram. For $k = 0$ we put $U_0 = 1$. If $k = 1$, then $U_1 = 1$, because the only possibility is a single dot under the arc. If $k = 2$, clearly there is also a single possibility, so that $U_2 = 1$. For $k = 3$ there are 3 possibilities. For example, if a_1a_5 is the given arc, then under this arc we can have: either the dot a_3 and the arc a_2a_4, or the arc a_3a_4 and the dot a_2, or the arc a_2a_3 and the dot a_4. Hence $U_3 = 3$. Similarly, one can easily see that $U_4 = 5$.

In general, for every $k \geq 3$ there are two different types of configurations of exactly k generators under an arc A. The first type occurs when there is another arc A' directly under A. Then there are $k-2$ generators under A', so the number of such configurations is the same as for $k-2$, that is U_{k-2}. The second type occurs when directly under A there is a number $i > 0$ of consecutive dots (on one of the sides, right or left) and another arc covering all other generators. In this case, the interior arc covers $k-2-i$ generators, and the number of such configurations is twice the number of configurations for $k-2-i$, so $2U_{k-2-i}$.

The above implies that $U_0 = U_1 = U_2 = 1$ and $U_k = U_{k-2} + 2 \cdot \sum_{i=1}^{k-2} U_{k-2-i}$ for $k \geq 3$. Notice that $\sum_{i=1}^{k-2} U_{k-2-i} = \sum_{i=0}^{k-3} U_i$, so that

$$U_k = U_{k-2} + 2 \cdot \sum_{i=0}^{k-3} U_i.$$

Therefore

$$U_{k+1} = U_{k-1} + 2 \cdot \sum_{i=0}^{k-2} U_i = U_{k-1} + 2 \cdot \sum_{i=0}^{k-3} U_i + 2U_{k-2}$$

and subtracting one of these equalities from the other one we get $U_{k+1} - U_k = U_{k-1} + U_{k-2}$. So, for $k \geq 3$,

$$U_{k+1} = U_k + U_{k-1} + U_{k-2}.$$

Let T'_n denote the number of all minimal prime ideals of $K[M]$. Then we may assume $T'_0 = T'_1 = T'_2 = 1$ and from Example 2.1.6 we know that $T'_3 = 3$ and $T'_4 = 5$. Recall that $n \geq 3$. If the last step in the construction of a leaf of D is the arc a_1a_n then there are $n-2$ generators under this arc, hence there are U_{n-2} leaves of this type. On the other hand, if the extreme arc used in the construction of a leaf contains only one of the generators a_1, a_n, then there are $k \leq n-3$ generators under it, so the number of such leaves is $2U_k$. Therefore, for $n \geq 3$ we get $T'_n = U_{n-2} + 2 \cdot \sum_{k=0}^{n-3} U_k$. Notice that $T'_n = U_n$. The number of minimal prime ideals of $K[M]$ is therefore given be the linear recurrence

$$\begin{cases} T'_0 = T'_1 = T'_2 = 1 \\ T'_{n+1} = T'_n + T'_{n-1} + T'_{n-2}. \end{cases}$$

The assertion follows. ■

3.3. **An embedding $M \hookrightarrow \mathbb{N}^c \times (B \times \mathbb{Z})^d$.**

The construction of the monoids M/ρ_P, for all minimal prime ideals $P \lhd K[M]$ and the associated congruences ρ_P, allows us to find an entirely new faithful representation of M as a submonoid of the direct product $\mathbb{N}^c \times (B \times \mathbb{Z})^d$, with $c + 2d = nT_n$, where T_n is the n-th element of the Tribonacci sequence.

Let $\mathcal{P}_k$ be the set of all minimal prime ideals of $K[M_k]$, for any $1 \leq k \leq n$. If $k = n$, we will simply write $\mathcal{P} = \mathcal{P}_k$. By Theorem 3.2.2, we know that $|\mathcal{P}| = T_n$.

3.3.1 Lemma. $\bigcap_{P \in \mathcal{P}} \rho_P = \rho_0$, where ρ_0 stands for the trivial congruence on M.

Proof. If $n = 1$ then $K[M_1] = K[x]$, while for $n = 2$ the algebra $K[M_2]$ is also prime by [3]. Hence, we may assume that $n \geq 3$.

If $n = 3$ then there are 3 minimal primes in $K[M]$, say P_1, P_2 and P_3, see Example 2.1.6 and Theorem 2.3.2, or [3]. We prove that if two elements $w, v \in M$ are such that $(w, v) \in \rho_{P_i}$ for $i = 1, 2, 3$, then $w = v$. Let $a = a_1, b = a_2, c = a_3$. Let $w = (a)^{\alpha_a}(ba)^{\alpha_{ba}}(b)^{\alpha_b}(ca)^{\alpha_{ca}}(cb)^{\alpha_{cb}}(c)^{\alpha_c}$ and $v = (a)^{\beta_a}(ba)^{\beta_{ba}}(b)^{\beta_b}(ca)^{\beta_{ca}}(cb)^{\beta_{cb}}(c)^{\beta_c}$ be the canonical forms of w, v, respectively.

For simplicity, we write $(x), (xy)$ for any non-negative powers of x and xy, if $x, y \in \{a, b, c\}$. Let $\widetilde{u}$ denote the image of $u \in M$ in M/ρ_{P_i}, for a fixed i.

We know that ρ_{P_1} corresponds to imposing on M the additional relations $ab = ba$ and $acb = bca$. By the proof of Lemma 1.2.6, the canonical form of the element $\widetilde{w} \in M/\rho_{P_1}$ is $(\widetilde{a})(\widetilde{b})(\widetilde{c}\widetilde{a})(\widetilde{c}\widetilde{b})(\widetilde{c})$, where the exponent of $(\widetilde{b})$ or of $(\widetilde{c}\widetilde{a})$ is equal to 0. Clearly, all the exponents are determined by those in the element w, in particular the exponent of $(\widetilde{c})$ is equal to the exponent of (c) in w. Since $\widetilde{w} = \widetilde{v}$, all exponents in the canonical forms of these two elements of M/ρ_{P_1} are equal, so in particular we get $\alpha_c = \beta_c$.

Similarly, the congruence ρ_{P_2} corresponds to imposing relations $bc = cb$ and $bac = cab$ on M. So $(\widetilde{a})(\widetilde{b})(\widetilde{b}\widetilde{a})(\widetilde{c})(\widetilde{c}\widetilde{a})$ is the canonical form of elements of M/ρ_{P_2}, with the exponent of $(\widetilde{b}\widetilde{a})$ or of $(\widetilde{c})$ equal 0 and the exponent of $(\widetilde{a})$ equal to the exponent of (a) in the original element of M. This and the equality $\widetilde{w} = \widetilde{v}$ imply that $\alpha_a = \beta_a$.

The congruence ρ_{P_3} corresponds to the relations $ab = ba$ and $bc = cb$ and it leads to the canonical form $(\widetilde{a})(\widetilde{b})(\widetilde{c})(\widetilde{c}\widetilde{a})$ in M/ρ_{P_3}. For $\widetilde{w} = \widetilde{v}$ this yields equalities of the corresponding exponents of $(\widetilde{a}), (\widetilde{b}), (\widetilde{c})$ and $(\widetilde{c}\widetilde{a})$:

$$\begin{cases} \alpha_a + \alpha_{ba} = \beta_a + \beta_{ba} \\ \alpha_b + \alpha_{ba} + \alpha_{cb} = \beta_b + \beta_{ba} + \beta_{cb} \\ \alpha_c + \alpha_{cb} = \beta_c + \beta_{cb} \\ \alpha_{ca} = \beta_{ca}. \end{cases}$$

These equalities, together with the earlier ones: $\alpha_c = \beta_c$ and $\alpha_a = \beta_a$ easily imply that every exponent in the canonical form of w is equal to the corresponding exponent in the form of v. Hence $w = v$, which finishes the proof in case $n = 3$.

Let $n \geq 4$. Proceeding by induction we assume that the assertion is true for the monoid M_{n-1}. Let $w, v \in M$ and let $\widetilde{w}, \widetilde{v}$ be their images under some fixed epimorphism $M \twoheadrightarrow M_{n-1}$.

If $(w, v) \in \bigcap_{P\in\mathcal{P}} \rho_P$, then $w - v \in \bigcap_{P\in\mathcal{P}} P = B(K[M])$. Hence $\widetilde{w} - \widetilde{v} \in \bigcap_{P\in\mathcal{P}_{n-1}} P = B(K[M_{n-1}])$. This means that for every $P \in \mathcal{P}_{n-1}$ one has $\widetilde{w} - \widetilde{v} \in P$, so that $(\widetilde{w}, \widetilde{v}) \in \rho_P$. By the induction hypothesis the latter implies that $\widetilde{w} = \widetilde{v}$.

Using the canonical forms of elements of M_{n-1}, as in the case $n = 3$, from such equalities we get equalities of the corresponding exponents. For simplicity, the k-th generator of M and its image will be denoted by k, for $k = 1, 2, \ldots, n$.

Consider the maps f_k, for $k = 1, \ldots, n-1$, defined on the generators of M by:

$$k, k+1 \mapsto k \quad \text{and} \quad i \mapsto i \text{ for } i \neq k, k+1.$$

It is easy to see that every such map transforms the defining relations of M into the relations defining the Chinese monoid of rank $n-1$ with generators $1, \ldots, k, k+2, \ldots, n$. Hence, every f_k defines a surjective homomorphism $M \to M_{n-1}$. Notice that for every

$w \in M$ and every fixed k, the image $f_k(w)$ is of the form

$$\begin{aligned}\widetilde{w} =& (1)(21)(2)\dots(k-1)\\ & (k1)(k2)\dots(k)\\ & (k1)(k2)\dots(kk)(k)\\ & (k+2\ 1)\dots(k+2\ k)(k+2\ k)\\ & \dots\\ & (n1)(n2)\dots(n\ k-1)(nk)(nk)(n\ k+2)\dots(n).\end{aligned}$$

Since (ki) and (kj) commute for $i,j \le k$, the latter leads to

$$\begin{aligned}\widetilde{w} =& (1)(21)(2)\dots(k-1)\\ & (k1)(k2)\dots(k\ k-1)(k)\\ & (k+2\ 1)\dots(k+2\ k)\\ & \dots\\ & (n1)(n2)\dots(n\ k-1)(nk)(n\ k+2)\dots(n),\end{aligned}$$

with the (non-indicated) exponents depending on the exponents in the canonical form of w. Moreover, the above is the canonical form of $\widetilde{w}$ in the corresponding Chinese monoid of rank $n-1$ (see (2)). Hence, from $\widetilde{w} = \widetilde{v}$ we derive the following system of equalities

$$\begin{cases} \alpha_{ij} = \beta_{ij} & \text{for } i<k \text{ and every } j\\ \alpha_{kj} + \alpha_{k+1\ j} = \beta_{kj} + \beta_{k+1\ j} & \text{for } j<k\\ \alpha_k + 2\alpha_{k+1\ k} + \alpha_{k+1} = \beta_k + 2\beta_{k+1\ k} + \beta_{k+1} & \\ \alpha_{ij} = \beta_{ij} & \text{for } i>k \text{ and } j \neq k, k+1\\ \alpha_{ik} = \beta_{ik} & \text{for every } i, \end{cases}$$

where α's, β's are the exponents in the canonical form of w, v, respectively, and with the convention that $\alpha_k = \alpha_{kk}$ and $\beta_k = \beta_{kk}$. The homomorphism of the above type for $k = n-1$, with $n \ge 4$, leads in particular to the following equalities

$$\begin{cases} \alpha_{ij} = \beta_{ij} & \text{for } i<n-1 \text{ and every } j\\ \alpha_{n-1\ 1} + \alpha_{n1} = \beta_{n-1\ 1} + \beta_{n1} & \\ \alpha_{n-1\ 2} + \alpha_{n2} = \beta_{n-1\ 2} + \beta_{n2}. & \end{cases}$$

On the other hand, for $k=1$ we get in particular

$$\begin{cases} \alpha_{n-1\ j} = \beta_{n-1\ j} & \text{for } j \neq 1,2\\ \alpha_{nj} = \beta_{nj} & \text{for } j \neq 1,2, \end{cases}$$

while for $k=3$, with $n \ge 4$, we get

$$\begin{cases} \alpha_{n1} = \beta_{n1}\\ \alpha_{n2} = \beta_{n2}. \end{cases}$$

It is easy to see that the above three systems of equalities lead to the conclusion that all exponents in the canonical form of w are equal to the corresponding exponents in v. Hence $w = v$, which completes the proof. ∎

3.3.2 Theorem. There exists an embedding $M \hookrightarrow \prod_{P\in\mathcal{P}} M/\rho_P$.

Proof. Let $m \in M$. Let m_P denote the image of m in M/ρ_P, for $P \in \mathcal{P}$. Then $m \mapsto (m_P)_{P\in\mathcal{P}}$ determines a homomorphism, which is injective by Lemma 3.3.1. ■

3.3.3 Corollary. There exists an embedding $M \hookrightarrow \mathbb{N}^c \times (B \times \mathbb{Z})^d$, where $c + 2d = nT_n$.

Proof. From Corollary 2.3.4 we know that for every $P \in \mathcal{P}$ there is an embedding $M/\rho_P \hookrightarrow \mathbb{N}^{c_P} \times (B\times\mathbb{Z})^{d_P}$ such that $c_P + 2d_P = n$. In view of Theorem 3.3.2 this yields an embedding

$$M \hookrightarrow \mathbb{N}^c \times (B \times \mathbb{Z})^d,$$

with $c + 2d = n \cdot |\mathcal{P}| = nT_n$. ■

It is well known that the bicyclic monoid B satisfies the identity $xy^2xxyxy^2x = xy^2xyxxy^2x$, [1]. The following surprising result is an immediate consequence.

3.3.4 Corollary. The Chinese monoid M satisfies the identity

$$xy^2xxyxy^2x = xy^2xyxxy^2x.$$

3.3.5 Corollary. $B(K[M])$ is not of the form $\mathcal{I}_\rho$ for any congruence ρ on M.

Proof. Suppose that $B(K[M]) = \mathcal{I}_\rho$ for a congruence ρ on M. Then $\rho = \rho_{B(K[M])} \subseteq \rho_P$ for every prime ideal P of $K[M]$. Thus, $\rho \subseteq \bigcap_{P\in\mathcal{P}} \rho_P$. From Lemma 3.3.1 we know that $\bigcap_{P\in\mathcal{P}} \rho_P = \rho_0$, where ρ_0 is the trivial congruence. Hence $\rho = \rho_0$ and $B(K[M]) = \mathcal{I}_{\rho_0} = 0$. As recalled after Corollary 3.1.2, this contradicts [8]. The assertion follows. ■

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