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Some Theorems on the Algorithmic Approach to Probability Theory and Information Theory.

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Contents

The dissertation uses the terms and notation in paper [6] that is attached to [Russian original of] this text. The paper also contains figures referred to in the text of the dissertation as well as the index.

The author is deeply grateful to his advisor A. N. Kolmogorov, to A. K. Zvonkin who helped a lot in presenting the results, to V. N. Agafonov, Ya. M. Barzdin', R. L. Dobrushin, A. G. Dragalin, M. I. Kanovich, A. N. Kolodiy, P. Martin-Löf, L. B. Medvedovsky, N. Y. Petri, A. B. Sosinsky, V. A. Uspensky, J. T. Schwartz, and to all participants of A. A. Markov's seminar for discussion.

Some definitions and notation

We consider strings in the alphabet $\{0,1\}$, i.e., finite sequences of zeroes and ones in a 1-1 correspondence with natural numbers:

$$
\begin{array}{rcl}
\Lambda & \leftrightarrow & 0 \\
0 & \leftrightarrow & 1 \\
1 & \leftrightarrow & 2 \\
00 & \leftrightarrow & 3 \\
01 & \leftrightarrow & 4 \\
10 & \leftrightarrow & 5 \\
11 & \leftrightarrow & 6 \\
000 & \leftrightarrow & 7 \\
001 & \leftrightarrow & 8 \\
\vdots & \vdots & \vdots
\end{array}
$$

(Λ is the empty string). We do not distinguish strings and number and use the terms interchangeably. They are usually denoted by lower case Latin letters. The set of all strings-numbers is denoted by S. The result of adding (concatenating) the string y to the string x is denoted by xy. We need also to encode the ordered pair (x, y) of strings by one string. To avoid introducing a special separator (such as a comma) let us agree that for $x = x_1x_2 \dots x_n$ $(x_i \in \{0, 1\})$

$$\overline{x} = x_1x_1x_2x_2 \dots x_nx_n01. \tag{0.1}$$

Then one can recover both x and y from the string $\overline{x}y$. Denote by $\pi_1(z)$ and $\pi_2(z)$ functions such that $\pi_1(\overline{x}y) = x$, $\pi_2(\overline{x}y) = y$. If the string z is not representable as $\overline{x}y$ then $\pi_1(z) = \Lambda$, $\pi_2(z) = \Lambda$.[1]

The length $l(x)$ of a string x is the number of its digits; $l(\Lambda) = 0$. Obviously

$$l(xy) = l(x) + l(y), \tag{0.2}$$

$$l(\overline{x}) = 2l(\overline{x}) + 2. \tag{0.3}$$

Let $d(A)$ be the number of elements in a set A. Obviously

$$d\{x : l(x) = n\} = 2^n, \tag{0.4}$$

$$d\{x : l(x) < n\} = 2^n - 1. \tag{0.5}$$

We also consider the space Ω of infinite binary sequences, denoting them with lower-case Greek letters. $\Omega^* = \Omega \bigcup S$ is the set of all finite and infinite sequences. Let $\omega \in \Omega^*$. The n-prefix of ω, denoted $(\omega)_n$, is the string of its first n digits. If $\omega \in S$ with $l(\omega) \leq n$ then $(\omega)_n = \omega$ by definition. An $\omega \in \Omega$ is a characteristic sequence for the set $S_\omega = \{n_1, n_2, \dots\}$ of positive integers if ω has 1 at the places $n_1, n_2, \dots$ and zeroes everywhere else. Denote Γ_x the set of all sequences (from Ω or Ω^*, as follows from the context) that have prefix x:

$$\Gamma_x = \{\omega : (\omega)_{l(x)} = x\}. \tag{0.6}$$

Notation $x \subset y$ means $\Gamma_x \supseteq \Gamma_y$, that is the string x is a prefix of y. The relation $\subset$ is a partial order on S (Figure 1).

Functions defined on the Cartesian product $S^n = S \times \dots \times S$ (n times) are denoted by capital Latin letters (except some standard functions). Sometimes a superscript n denoting the number of variables is added: $F^n = F^n(x_1, \dots, x_n)$. The sentence *for all admissible values of variables $y_1, \dots, y_n$ there exists a constant C such that for all admissible values $x_1, \dots, x_n$*

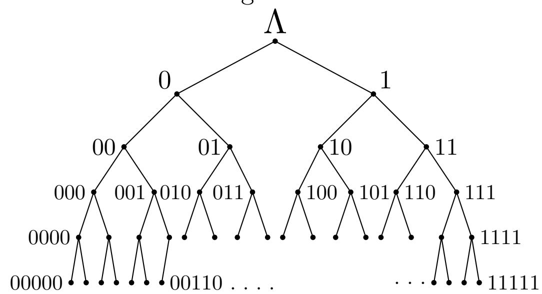
$$F^{n+m}(x_1, \dots, x_n; y, \dots, y_m) \leq G^{n+m}(x_1, \dots, x_n; y, \dots, y_m) + C \tag{0.7}$$

is abbreviated as follows: with parameters $(y_1, \dots, y_n)$,

$$F^{n+m}(x_1, \dots, x_n; y, \dots, y_m) \preccurlyeq G^{n+m}(x_1, \dots, x_n; y, \dots, y_m). \tag{0.8}$$

[1] More common enumerations of pairs may violate the property (0.11) important below.

Figure 1:



The relation $\succcurlyeq$ is defined similarly. $F \asymp G$ means both $F \preceq G$ and $G \preccurlyeq F$ hold. Obviously, the relations $\preccurlyeq$, $\succcurlyeq$, $\asymp$ are transitive and

$$l(x) \asymp \log_2(x) \text{ for } x > 0, \tag{0.9}$$

$$l(\overline{x}) \asymp 2l(x), \tag{0.10}$$

$$l(\overline{x}y) \asymp l(y) \qquad \text{(with } x \text{ as a parameter).} \tag{0.11}$$

1 Introduction

1.1 The general construction of complexity

The topics studied here were introduced in 1964 when A.N.Kolmogorov defined complexity of constructive objects. (Similar concepts were independently considered by A.A.Markov and R.J.Solomonoff.)

A.N. Kolmogorov defines the complexity of a string x for an algorithm A as the least length of binary strings p encoding x, i.e., such that $A(p) = x$. The value so defined depends strongly on the choice of A. The central result that prompted all further investigations was a theorem established by A.N. Kolmogorov and independently (in slightly different terms) by R.J. Solomonoff. It states the existence of an optimal algorithm A providing the smallest (compared to any other algorithm B) value of complexity up to an additive constant C_B (independent of x). Complexity for an arbitrary optimal A is thus sufficiently invariant to be a fundamental characteristics of x. It found many applications, and quickly generated a rich theory (cf. for example, a survey [6]).

In the development of this theory, several other quantities similar to complexity (though different from it) turned out to be useful. For example, A.A.Markov and D.Loveland considered the decision complexity of binary strings, P.Martin-Löf defined their deficiency of "randomness," the present author introduced "universal probability," etc. At present, about ten such functions are known. The need exists for some organization of this diversity of quantities from a unified standpoint.

Definition 1. *A* finitary function *is a table defining a function from a finite set $A \subset S$ to S. (We assume it has value ∞ on $A \backslash S$.)*

Definition 2. *A* volume restriction *is an enumerable family V of finitary functions such that*

1. *If $f \geq g$ and $g \in V$ then $f \in V$;*
2. *$\exists C \forall f, g{\in}V \ (C + \min\{f, g\}) \in V$.*

We assume for simplicity that $C = 1$.

Definition 3. *Let V be a volume restriction. By V-*majorant*, we call any function $F(x)$ such that*

1. *the set of points over its graph is enumerable and*
2. *for every finitary function g, if $g \geq F$ then $g \in V$.*

Theorem 1. *For any volume restriction V, there exists a V-majorant $K_V(x)$, that is smallest (up to an additive constant), i.e., such that $K_V(x) \preccurlyeq L(x)$ for every V-majorant $L(x)$.*

Proof. For a finite set $\mathcal{M}$ of pairs of numbers, we get a graph of a finitary function by taking the lowest point of $\mathcal{M}$ on each vertical line intersecting $\mathcal{M}$. Let us call this function a lower boundary of $\mathcal{M}$.

Let partial recursive function $U(i,t)$ enumerate the i-th enumerable set U_i of pairs (x,a) for every i. Let's define $U'(i,t)$ enumerating $U'_i \subset U_i$ for each i, but slower than U. Namely, U' generates the next element only after verifying that the lower bound of the set enumerated so far belongs to V.

Obviously U'_i is a V-majorant for each i, and no majorant is "forgotten." Now let $\mathcal{M}$ be the set of pairs situated above pairs $(x, a+Ci)$ where C is the constant in the definition of volume restriction, and $(x,a) \in U'_i$.

Let us prove that $\mathcal{M}$ defines an (obviously optimal) V-majorant. In other words, every finitary function f whose graph is contained in $\mathcal{M}$, belongs to the family V. By definition of $\mathcal{M}$, $f \geq \min(g_i + Ci)$ for some family of functions $g_i \in V$, $i \leq n$. This implies that $f \in V$. Indeed let $h_k = \min_{i>k}(g_i + C(i-k))$ Then $f \geq h_0$, $h_{k-1} = C + \min\{h_k, g_k\}$ and induction on k from n down completes proving the theorem.

For any decidable volume restriction V, one can compute a common lower bound $m_V(x) = \min_{f\in V} f(x)$ for all V-majorants. It is simpler to study differences $K_V(x) - m_V(x)$ instead of the majorants $K_V(x)$. These differences will be V'-majorants where $f \in V' \leftrightarrow (f + m_V) \in V$. Obviously $m_{V'}(x) = 0$. We call such V' "reduced." There is no need to study non-reduced decidable V.

Theorem 2. *Among reduced volume restriction, there is the one that is most "narrow." The universal majorant $p(x)$ corresponding to it will be the largest*[2]*. This restriction is given by the condition $f \in V \Leftrightarrow \sum_x 2^{-f(x)} \leq 1$.*

Proof. Clearly, V is a reduced volume restriction. If V' is any volume restriction and $f \in V$, then finitely many applications of item 2 of the definition of volume restrictions yield $f \in V'$. This "extreme" majorant $p(x)$ turns out to be not far from the complexity $K(x)$ of [9] (which hence is close to the limit).

Theorem 3. $K(x) \preccurlyeq p(x) \preccurlyeq K(x) + 2\log_2 K(x)$.

Proof. $K(x) \preccurlyeq p(x)$ by Theorem 2 (see also Theorem 4a). To prove the second inequality, we show that any finitary function $f(x) \geq K(x) + 2\log_2 K(x)$ belongs to volume restriction V (from Theorem 2). Indeed,

$$\sum_x 2^{-f(x)} = \sum_a \sum_{x:K(x)=a} 2^{-f(x)} \leq \sum_a \sum_{x:K(x)=a} 2^{-K(x)-2\log_2 K(x)} =$$

$$= \sum_a d\{x : K(x) = a\} \cdot \frac{1}{2^a \cdot a^2}.$$

Since $d\{x : K(x) = a\} \leq 2^a$, this $\leq \sum_a \frac{2^a}{2^a \cdot a^2} \preccurlyeq 1$, which completes the proof.

1.2 Examples of majorants

Definition 4. *(*A. N. Kolmogorov*)*
The complexity *of x with respect to a p.r. function F^1 is*

$$K_{F^1}(x) \stackrel{\text{def}}{=} \begin{cases} \min_{F^1(p)=x} l(p) \\ \infty, \text{ if there is no such } p. \end{cases}$$

We call a word p with $F^1(p) = x$ a code or program for F^1 to restore x.

Definition 5. *(*A. N. Kolmogorov*) The* conditional complexity *of x for known y with respect to a p.r. function $F^2(p,y)$ is*

$$K_{F^2}(x|y) \stackrel{\text{def}}{=} \begin{cases} \min l(p) : F^2(p,y) = x \\ \infty, \text{ if } \forall p\, F^2(p,y) \neq x. \end{cases}$$

[2]This majorant is a logarithm of the largest (up to a constant) semicomputable probability distribution on natural numbers

Definition 6. *(D. Loveland, A. A. Markov)*
The decision complexity *of a word x with respect to a p.r. function F^2 is*

$$K_{F^2}(x) \stackrel{\text{def}}{=} \begin{cases} \min l(p) : \forall i < l(x)\, F^2(p,i) = x_i \\ \infty, \text{ if there is no such a } p, \end{cases}$$

here x_i is the i-th letter of word x.

We defined three quantities important in complexity theory. Let us show that all three are special cases of the general concept of V-majorant. Then, in particular, Theorem 1 will imply the famous optimality theorems discovery of which by A. N.Kolmogorov and R.J.Solomonoff started complexity theory.

Let V_1 be the set of finitary functions with $\le 2^a$ of x having $f(x) < a$.

Let V_2 be the set of finitary functions $f(x,y)$ on pairs (x,y) (more precisely, on codes of such pairs) with every a, y having $\le 2^a$ of x with $f(x,y) < a$.

Let V_3 be the set of finitary functions with $\le 2^a$ branches in the tree of words x with $f(x) < a$. It is easy to verify that V_1, V_2, V_3 are volume restrictions.

We call classes A and B *equivalent* if for every function f from one of them there is a function $g \preccurlyeq f$ from the other.

Theorem 4. *a) The class of V_1-majorants is equivalent to the complexity class with respect to any algorithms.*

b) The class of V_2-majorants is equivalent to the conditional complexity class with respect to any algorithms.

c) The class of V_3-majorants is equivalent to the class of decision complexities.

Proof. We prove Theorem 4a. Theorems 4b and 4c can be proven similarly. It is easy to see that for any A, $K_A(x)$ is a V_1-majorant. Conversely, for any V_1 majorant F one can enumerate all points (x,n) above its graph and map them to different $p \in \{0,1\}^n$, with pairs (p,x) forming the graph of an algorithm A. $F \in V_1$ assures enough codes p for that. $K_A(x)$ may exceed $F(x)$ by ≤ 1, QED.

1.3 Invariant functions and complexity

Complexity has an important property of invariance, namely

Remark 1. *Under any p.r. isomorphism between two recursively enumerable sets, the complexities of their elements differ at most by a constant.*

Besides, complexity has "informational correctness," namely

Remark 2. *There exists a computable enumeration of pairs $f(x,y){=}i$, $\pi(i){=}x$, $\pi_2(i){=}y$ such that the complexity of $f(x,y)$ is at least the complexity of x and y up to an additive constant (true even for every computable enumeration).*

Complexity is bounded by a logarithm of its argument, i.e. contains only a very limited amount of information about the words. But it turns out that even among functions of arbitrary nature, no "richer" invariants exist.

Theorem 5. *Every invariant informationally correct (in the above sense) function $F(x)$ is at most $K(x)$ up to a multiplicative constant. The constant cannot be made additive because even changing the alphabet changes a constant factor.*

Proof. The algorithm A in $K_A(x)$ can be represented as the composition of function $\pi_1(x)$ and an invertible function. Then the theorem's assumptions imply $F(x) \preccurlyeq F(p)$ when $A(p) = x$. It remains to show $F(p) \le C \cdot l(p)$. This follows from constructing four isomorphisms of natural numbers, combining which we can, in n steps, obtain every n-bit word from 0. QED.

1.4 Computable complexity majorants

Clearly, knowing a word x and its complexity, we can find efficiently (at least, by exhaustive search) a shortest program coding x. Moreover, knowing x and any bound $S > K(x)$, we can find an S-bit program, possibly not a shortest one. Since complexity isn't a computable function, in practice one has to be content with its computable majorants giving the length of an effectively computable code, not necessary the shortest one. Barzdin', Petri, Kanovich showed all such majorants to be very coarse in some cases. However, we have

Theorem 6. *Every "informationally correct" (in the sense of Sec. 1.3) function which is less (up to an additive constant) than every computable complexity majorant is also less (up to an additive constant) than the complexity itself.*

Proof. Every algorithm can be represented as a composition of an invertible algorithm assuming all values in a recursive set and function $\pi_1(x)$. Complexity with respect to an invertible function is just the logarithm of its inverse and hence it is computable. The theorem follows.

1.5 Decision complexity

By many reasons, $K(x)$ or $K(x|l(x))$ are not quite natural to use for studying the complexity of sequences (rather than terminated words). Thus A. A. Markov and D. Loveland introduced $KR(x)$, which proved to be very fruitful. E.g.,

Remark 3. *A sequence ω is computable if and only if $KR((\omega)_n)$ is bounded.*

Evident for $KR(x)$, this isn't true for $K(x)$, and for $K(x\big|l(x))$ isn't evident and remained an open problem for some time. An affirmative answer was given by the author independently of Kolodiy, Loveland (USA) and Mishin. This is implied by the following theorem relating $KR(x)$ with $K(x|l(x))$.

Theorem 7. *For every ω, $KR((\omega)_n)$ is bounded if and only if $K((\omega)_n|n)$ is.*[3]

Proof. One direction is obvious: a computable ω has a general recursive function $F^1(n) = (\omega)_n$. Let $F^2(p,n) = F^1(n)$, then $K_{F_2}((\omega)_n\big|n) = l(\Lambda) = 0$ because $F^2(\Lambda,n) = (\omega)_n$, hence $K((\omega)_n\big|n) \preccurlyeq 0$, $K((\omega)_n\big|n) \leq C$.

Let us prove the other direction. Suppose $K((\omega)_n\big|n) \leq C$. We want to establish the existence of a procedure which, for given n, produces ω_n, the n-th digit of ω. Consider all words p with length at most C and construct a table as shown on Figure in Sec. 2.2 of [6]: at the p-th row of the n-th column we place $F_0^2(p,n)$ (see (1.6) from [6]) provided it halts. The set of all words $F_0^2(p,n)$ in the n-th column we denote A_n. Each A_n has at most 2^{C+1} words, and $(\omega)_n \in A_n$.

Let $l = \overline{\lim}_{n\to\infty} d(A_n)$. Clearly, the set $U = \{n : d(A_n) \geq l\}$ is recursively enumerable and infinite. Moreover, there are only finitely many n with $d(A_n) > l$; the largest of such n we denote m_1. Let $k < 2^{C+1}$ be the number of sequences ω with $K((\omega)_n\big|n) \leq C$. Let m_2 be the smallest length of prefixes distinct for all these sequences (by the way, all columns starting from m_2-th should contain at least k prefixes of these sequences, hence $k \leq l$). Let $m = \max(m_1, m_2)$.[4]

Let U' be an infinite decidable subset of U and $V = U' \cap \{n : n > m\}$.

The algorithm deciding the i-th (in lexicographical order) of our sequences proceeds as follows. To find its j-th digit, we select the smallest $n_r > j$ in V and start filling in the n_r-th column (with words $F_2(p, n_r)$, $l(p) \leq C$). When l words are found, we stop: there are no more. Denote B_{n_r} the set of all n_r-bit words from A_{n_r}. Then we similarly construct the set $B_{n_{r+1}}$ and take from it all words with prefixes from B_{n_r}; this set is denoted $C_{n_{r+1}}$. Then, words from $B_{n_{r+2}}$ with prefixes from $C_{n_{r+1}}$ form the set $C_{n_{r+2}}$; $C_{n_{r+3}}$ is be the set of words from $B_{n_{r+3}}$ with prefixes from $C_{n_{r+2}}$, and so on. We stop when the current set C_{n_s} contains exactly k words: they all are n_s-prefixes of sequences with $K((\omega)_n\big|n) \leq C$. Selecting the i-th lowest of them we take its j-th digit; it is what is required.

[3] However, it was shown by Petri that there is no effective way to calculate a bound on $KR((\omega)_n)$ from a bound on $K((\omega)_n|n)$, that is, the former can be very large.

[4] Our construction uses numbers l, k, m but isn't effective, giving no procedure to find them. We only prove that the required algorithm exists (an intuitionist would say: "cannot but exist"), so only need the mere existence of l, k, m.

2 Measures and Processes

This chapter considers deterministic and non-deterministic processes generating sequences. The central result is introducing a universal semi-computable measure and establishing its relation with complexity. At the end of the chapter, these results are applied to the study of capacities of probabilistic machines.

2.1 Definitions. Equivalence of measures.

Definition 7. *Algorithmic process* or simply *process* is a partial recursive function F mapping words into words, and such that if $F(x)$ is defined for a word x and $y \subset x$ then $F(y)$ is also defined and $F(y) \subset F(x)$.

Let us and apply a process F to all prefixes of $\omega \in \Omega$ while F is defined. It outputs prefixes of a sequence $\rho \in \Omega^*$. [5] This ρ is the result $F(\omega)$ of applying F to ω (i.e., F maps Ω into Ω^*).

Remark 4. There exists a universal process, i.e., a partial recursive function H, such that $H(i,x)$ is a process for all i, and for any process F an i exists such that $H(i,x) \equiv F(x)$. Such H is easily constructed from a universal p.r. function. Without loss of generality we assume (and use later) $H(\Lambda,\Lambda) = \Lambda$.

Processes F and H are said to be *equivalent* if $F(\omega) = G(\omega)$ for any $\omega \in \Omega$.

Remark 5. *Any process has an equivalent one that is primitive recursive.*

Definition 8. We say a process F is *applicable* to ω if $F(\omega)$ is infinite.

Remark 6. Any process is a continuous function on the set of sequences to which it is applicable (with the natural topology on Ω).[6]

Definition 9. A process is *fast growing* (*fast applicable to* ω) if a monotone unbounded total recursive function $\Phi(n)$ exists such that for all x (respectively, for all prefixes x of ω) for which F is defined, $\ell(F(x)) \geq \Phi(\ell(x))$. In this case we say *the speed of growth (of the application to* ω*) of process* F *is* $\geq \Phi(n)$.

Remark 7. One can easily show that a process applicable to all ω is total recursive and fast growing. Clearly, the reverse is also true.

Definition 10. Let P be a probability measure over Ω. We say that process P is *regular* if the set of sequences to which it is applicable has P-measure 1.

In order to define an arbitrary measure on a Borel σ-algebra of subsets of Ω, it suffices to define it on sets Γ_x.

Definition 11. A measure P on Ω is *computable* if there exist total recursive functions $F(x,n)$ and $G(x,n)$ such that the rational number $\alpha_P(x,n) = \frac{F(x,n)}{G(x,n)}$ is a 2^{-n}-approximation of $P(\Gamma_x)$.

Remark 8. Obviously then, $\alpha_P(x,n+1) + 2^{-n+1}$ is a 2^{-n}-approximation of $P(\Gamma_x)$ from above. Hence, without loss of generality, we always assume $\alpha_P(x,n)$ to be an upper bound, and take $\alpha_P(x,n) - 2^{-n}$ as a lower bound.

Denote by L the measure $L(\Gamma_x) = 2^{-l(x)}$, and call it the *uniform measure*. It corresponds to Bernoulli trials with probability 1/2; it is also a Lebesgue measure on the interval $[0,1]$. Obviously, L is computable.

Theorem 8. [7] *a) For any computable measure* P *and any* P*-regular process* F *the measure* $Q(\Gamma_y) = P(\bigcup \Gamma_x : (F(x) \supset y))$ *(i.e., the measure with which the outputs of* F *are distributed) is computable.*

b) For any computable measure Q *there exists an* L*-regular process* F*, generating* Q*-distributed outputs from* L*-distributed inputs. Moreover,* F *has an inverse* G *(i.e.* $F(G(\omega)) = \omega$ *when* G *is applicable) applicable to all non-recursive sequences except maybe some in intervals of* Q*-measure* 0.

[5] If $F((\omega)_n)$ is defined, and for all $m > n$, $F((\omega)_m)$ coincides with $F((\omega)_n)$ or is undefined, then $F(\omega) = F((\omega)_n)$. $F(\omega) = \Lambda$ if $F((\omega)_n)$ is undefined or empty for all n.

[6] In this topology, Ω is equivalent to Cantor perfect set.

[7] A somehow weaker result was independently proven by Mann (USA).

Proof. a) We compute a 2^{-n}-approximation (from above or below; making it an upper bound is easy) $\alpha_Q(y,n)$ to $Q(\Gamma_x)$. Choose m such that $P(\{\omega : l(F((\omega)_m)) > l(y)\}) > 1-2^{-(n+1)}$ (Such an m exists as process F is P-regular, moreover one can effectively find such an m). Take all words $x\in\{0,1\}^m$ such that $y\subset F(x)$, and compute $\alpha_Q(y,n)$ as the sum of $2^{-(m+n+1)}$-approximations to measures $P(\Gamma_x)$ of all these x. Then the error is $\alpha_Q(y,n)-Q(\Gamma_y)\le 2^{-(n+1)}+2^m\cdot 2^{-(m+n+1)}=2^{-n}$ (as there are $<2^m$ of x).

b) We consider sequences $\omega\in\Omega$ as reals in [0, 1] (with binary expansions ω; the cases of binary rationals, where such expansions have ambiguity, will be specially noted). Figure 3 in [6] shows a distribution function g that corresponds to measure Q. As is well known, the random variable $g^{-1}(\xi)$ is Q-distributed with ξ uniformly distributed over [0, 1]. Our construction is based on this idea.

I. A process $F((\alpha)_n)$ generates Q-distributed $g^{-1}(\alpha)$ from L-distributed inputs. It takes upper 2^{-2n}-approximations $\alpha_Q(y,2n)$ of $Q(\Gamma_y)$ for each $y\in\{0,1\}^n$ and outputs the longest common prefix of those $z\in\{0,1\}^n$ for which

$$\sum_{y\le z}\alpha_Q(y,2n)\ge(\alpha)_n\ge 1-2^{-n}-\sum_{y\ge z}\alpha_Q(y,2n). \tag{2.1}$$

II. Due to (2.1), the intervals $\cup\Gamma_z$ contain (for each n) g-image of α. Hence, $F(\alpha)$, if applicable, generates $g^{-1}(\alpha)$ (treating γ in Figure 3 of [6] as a pre-image of $\alpha\in[\sigma',\sigma'']$). To prove F is L-regular suffices to show it being what we need.

1) Let $[\sigma',\sigma'']$ correspond to a single γ with $Q(\gamma)>0$. If $\sigma'<\alpha<\sigma''$ then once $\sigma'\le(\alpha)_n-2^{-n}\le(\alpha)_n+2^{1-n}\le\sigma''$, only a single z satisfies (2.1) and so is output. Thus, F is applicable to such α, though not always to the ends σ', σ''.

2) Now let α not be of such types. Then $Q(\cup\Gamma_z)\to 0$ as $n\to\infty$. Hence, if α is not of type ρ corresponding to a measure 0 interval, then $\cup\Gamma_z$ shrink to a point $\beta=g^{-1}(\alpha)$; their longest common prefix grows infinitely.

3) A notable case of type ρ is $\alpha=g(\beta)$ with a binary rational β: its two binary expansions may form a measure 0 interval mentioned above.

In sum, F is applicable to all sequences except some of types ρ, σ', σ'' of Figure 3 in [6]. This set is clearly countable, so F is L-regular.

III. The inverse process G just computes g. It may be non-applicable only to (computable by Corollary to Theorem 11) γ with $Q(\gamma)>0$, and β, with binary rational $\alpha=g(\beta)$. If $F(\alpha)$ is applicable, it computes β. If not, and β is not of mentioned type γ, it lies on an interval $[\tau',\tau'']$ of zero Q-measure. Q.E.D.

2.2 Semi-computable measures

Definition 12. A semi-computable *(the term is justified by Theorem 9) measure is the distribution of the outputs of an arbitrary (not necessarily regular) process on inputs distributed according to a computable measure.*

Remark 9. *Semi-computable measures are concentrated on Ω^* since a non-regular process can have finite outputs with positive probability. In this section, we assume Γ_x is a set of all finite and infinite sequences with prefix x.*

Remark 10. *The distribution of outputs of any process on inputs with an arbitrary semi-computable distribution is also semi-computable (as a composition of two processes is again a process). Any semi-computable measure can be obtained from a uniform measure by some process (see Theorem 8b).*

Theorem 9. *A measure P is semi-computable iff total recursive functions F,G exist such that $\beta_P(x,t)=\frac{F(x,t)}{G(x,t)}$ is a monotone non-decreasing in t function, and*

$$\lim_{t\to\infty}\beta_P(x,t)=P(\Gamma_x). \tag{2.2}$$

This Theorem implies that the class of semi-computable measures (more accurately, of their logarithms) is equivalent to the class of V-majorant, where V is a set of finitary functions f for which $\sum_{x\in M}2^{-f(x)}\le 1$ for all sets M whose words are not prefixes of each other.

Proof. Let P be a semi-computable measure. Then there exists a process F generating this measure from L. Let it make t steps on all words y with $\ell(y)\le t$ and, denoting the result by $F_t(y)$ (if no results are achieved yet then $F_t(y)=\Lambda$), set $\beta_P(x,t)=L(\cup\Gamma_y : x\subset F_t(y))$.

Inversely, suppose a measure P has a function $\beta_P(x,t)$ satisfying the terms of the Theorem. We wish to construct a process F generating P from L.

The idea is simple: we need to partition the interval [0, 1] into disjoint subsets of measure $P(\Gamma_x)$, and to output x when our uniformly distributed input falls into a corresponding set. Now we describe the construction precisely. Clearly, $P(\Gamma_x) \geq P(\Gamma_{x0}) + P(\Gamma_{x1})$. Moreover, without loss of generality, we assume $\beta_P(x,t) \geq \beta_P(x0,t) + \beta_P(x1,t)$ for all t: each time this fails, we delay growth of $\beta_P(x0,t)$ and $\beta_P(x1,t)$ proportionally to restore the inequality. It is easy to construct subsets of interval $[0,1]$ with the following conditions: to each pair (x,t) there corresponds a union $I_{x,t}$ of a finite number of intervals with binary rational ends and combined length $\beta_P(x,t)$. Within this procedure for any words $x \neq y$ of equal length, I_{x,t_1} and I_{y,t_2} are disjoint for all t_1 and t_2; for any words $x \subset y$ and any t, $I_{y,t} \subset I_{x,t}$; for any $t_1 < t_2$ and any x, $I_{x,t_1} \subset I_{x,t_2}$.

Our $F(z)$ constructs $I_{x,t}$ for all x,t such that $l(x) \leq l(z)$ and $t \leq l(z)$, and outputs a longest x such that $z \in I_{x,t}$ for some t. Obviously, such x is unique as the sets corresponding to divergent x are disjoint, and $x' \subset x''$ for $z' \subset z''$.

2.3 Universal semi-computable measure

Theorem 10. *There exists a semi-computable measure R that is* universal, *i.e., such that for any semi-computable measure Q, there is a constant C such that $C \cdot R(\Gamma_x) \geq Q(\Gamma_x)$ for all x.*[8]

Proof. By a remark in Section 2.1, a universal process $H(i,x)$ exists. Obviously, $F(z) \stackrel{\text{def}}{=} H(\pi_1(z), \pi_2(z))$ is a process. Applied to uniformly distributed sequences, it generates the desired measure. Indeed, let a process $G(x)$ ($= H(i,x)$ for some i and all x) transform some set of sequences into Γ_x. Then $F(x)$ transforms into Γ_x these same sequences with added prefix $\overline{i}$ – (maybe some others as well). Thus, the measure of Γ_x cannot decrease by a factor $> C2^{\ell(\overline{i})}$ ($\approx i^2$).

Remark 11. *This result does not extend to computable measures: no measure is universal among all computable measures. This is one of the reasons for introducing the notion of a semi-computable measure.*

The measure R, being (within a constant factor) "larger" than any other measure, is concentrated on the widest subset of Ω^*.

The following issue is considered in mathematical statistics: find out what distribution P can randomly generate a given sequence ω. If we know nothing a priori about ω, then the only (= the weakest) statement we can make about it is that it can be generated under distribution R. In this sense, R reflects our intuition about "prior probability." The following is of interest:

a) For a constant C, the probability (under measure R) of having a 1 after n zeros is $> \frac{1}{n} \cdot \frac{1}{C \log^2 n}$.

b) For every constant C, at most $\frac{1}{C}$ fraction of n on any interval $[0,N]$, has the probability (under R) of a 1 falling after n zeros to exceed $\frac{1}{n} \cdot C \log^2 n$.

Thus, $R(0^n 1)$ is typically around $\frac{1}{n}$.[9]

The proof easily follows from Theorem 11, taking into account that the complexity $KR(0^n 1)$ does not exceed $\log_2 n + c$, and for the majority of these words this complexity is almost equal to $\log_2 n$.

One can see an analogy between constructing the complexity KR and the universal semi-computable measure. It turns out these two quantities also have a quantitative connection:

Theorem 11. $|KR(x) - (-\log_2 R(\Gamma_x))| \preccurlyeq 2 \log_2 KR(x)$.

Proof. Let $KR(x) = i$, thus, for some $p \in \{0,1\}^i$ and all $n \leq \ell(x)$, we have $G_0^2(p,n) = x_n$ (here G_0^2 is from Theorem 2.1 of [6]). Then, one can easily construct a process transforming each sequence with prefix $\overline{\ell(p)}\, p$ into a sequence with prefix x: this process first separates the prefix $\overline{\ell(p)}$, recovers $\ell(p)$, "reads" p, and

[8] In other words, Q is absolutely continuous with respect to R with Radon-Nikodym derivative bounded by C from above.

[9] Note that this statement is true only for the universal (prior) probability. For example, if we know that the Sun has risen for 10,000 years, this does not mean that the probability of the Sun not rising tomorrow is approximately equal to 1/3,650,000. This statement would be true if the above fact was the only information that we have about the Sun.

sequentially generates $G_0^2(\overline{p,n})$ for $n = 1, 2, \ldots$. From a uniformly distributed input, this generates sequences in Γ_x with probability $\geq 2^{-\ell(\overline{\ell(p)}\,p)}$. Thus, by Theorem 10, $R(\Gamma_x) \geq c \cdot 2^{-\ell(\overline{\ell(p)}\,p)}$, hence

$$-\log_2 R(\Gamma_x) \preccurlyeq \ell\left(\overline{\ell(p)}\,p\right) = \ell(p) + 2\ell(\ell(p)) = i + 2\ell(i) = KR(x) + 2\ell(KR(x)).$$

Now, assume $R(\Gamma_x) = q$. Let us denote $\ell(q) = \lfloor -\log_2 q \rfloor$. To estimate the complexity $KR(x)$, we reconstruct every symbol of x from the triple $\ell(q), k, i$ (i.e., from $\overline{\ell(q)}\,\overline{k}\,i$), where $k \in \{0,1\}$ and $i \leq 2^{\ell(q)+1}$. Our algorithm works as follows: based on $\ell(q)$, it builds the tree (see Fig. 4 in [6]) of all words y with $R(\Gamma_y) > 2^{-\ell(q)-1}$. For this, we compute $\beta_R(y,t)$ for more and more values t and y, and add y to the tree when we get $\beta_R(y,t) > 2^{-\ell(q)-1}$, for some t.

The word x belongs to this tree. We keep only "maximal" words, i.e., words that are not prefixes of other words in the current tree. Clearly, the number of such "maximal" words will neither decrease nor exceed $2^{\ell(q)+1}$. Let A (see Fig. 4 from [6]) be the word from which the last branching from the word x occurs; after this, x continues without branching. To find x, it suffices to have the first digit k of x extending A, and the number i of maximal words at the moment when the tree being constructed branches at A (incrementing the number of maximal words to i). As $i \leq 2^{\ell(q)+1}$, hence $\ell(i) \leq \ell(q) + 1$. Thus,

$$KR(x) \preccurlyeq \ell\left(\overline{\ell(q)}\,\overline{k}\,i\right) \asymp 2\ell(\ell(q)) + \ell(i) \preccurlyeq 2\ell(\ell(q)) + \ell(q) \asymp$$

$$-\log_2 R(\Gamma_x) + 2\log_2(-\log_2 R(\Gamma_x)).$$

But, as proven earlier, $2\log_2(-\log_2 R(\Gamma_x)) \preccurlyeq 2\log_2[KR(x) + 2\ell(KR(x))] \preccurlyeq 2\log_2 KR(x)$, so $KR(x) \preccurlyeq -\log_2 R(\Gamma_x) + 2\log_2 KR(x)$. The theorem is proved.

Remark 12. *It is worth mentioning that the usual measure-theoretic arguments assure that each measure P (not necessarily semi-computable) is almost fully concentrated on the set of all sequences ω for which $\exists c \forall n$ $P((\omega)_n) \geq c \cdot R((\omega)_n)$.*

Similarly, for R-almost all sequences, the inverse inequality also holds. If P is absolutely continuous with respect to R, then the inverse inequality also holds for P-almost all sequences. This implies a statement similar to Theorem 11 for an arbitrary semi-computable measure P and prefixes of P-almost any sequence (of course, the constants may vary with sequences).

As a corollary, we get the well-known de Leeuw-Moore-Shapiro-Shannon theorem about probabilistic machines:

Corollary. *A sequence is computable if and only if some semi-computable measure (hence, also the universal measure) is positive on it.*

Proof. By Theorem 11, the measure of all prefixes is larger than a positive number if and only if their "complexity of solution" KR is uniformly bounded.

2.4 Probabilistic machines

The above Shannon et al. result is sometimes interpreted as the impossibility for probabilistic machines to solve problems that are unsolvable deterministically. However, not all problems require constructing a specific unique object; some allow many solutions, being satisfied by producing any of them[10]. This class clearly has problems that are unsolvable by deterministic machines but solvable if a machine can use a random number generator. An example of such problems is: to generate an uncomputable sequence.

We say a problem of constructing a sequence with a property Π is *solvable on a probabilistic machine* if the universal measure R of the set of all such sequences is positive. The following theorem shows that such problems can indeed be solved on machines with an access to random number generators; moreover, they can be solved with an arbitrary given reliability, and quite efficiently, i.e., with small number of calls to the random number generator. We call functions $f(n)$ and $g(n)$ *asymptotically equal,* denoted $f(n) \sim g(n)$, if $\frac{f(n)}{\log_2 f(n)} \asymp \frac{g(n)}{\log_2 g(n)}$; in this section, inequalities $\preccurlyeq$, $\succcurlyeq$ are understood in a similar "asymptotic" way.

[10] The corresponding concept of a mass problem was formulated in [31].

Theorem 12. *Let $A \subseteq \Omega$ with $R(A) > 0$. Then, for every $\varepsilon > 0$, there exists a fast-growing process F (i.e., one with $\ell(F(x)) \succcurlyeq \ell(x)$) transforming L-distributed sequences into sequences in A with probability $> 1 - \varepsilon$.*[11]

Clearly, one cannot solve, e.g., the problem of obtaining a very complex sequence by using a process which grows faster, since the process cannot increase the complexity of words. If sequences from the desired set A have small complexity, then short programs can generate prefixes of these sequences. However, one can imagine that some A could make such programs so special that short random inputs cannot be used instead, forcing the slow growth of processes solving A. Interestingly, "fast" processes are also possible in all such cases.

Theorem 13. *Let g be a monotonic total recursive function. Then the problem of generating a sequence from a set A is solvable by a random process that grows as $g(n')$ or faster ($n' \sim n$), if and only if there exists a set*[12] *$B \subseteq A$ such that $R(B) > 0$ and all $\omega \in B$, have $KR((\omega)_{g(n)}) \preccurlyeq n$.*

Proof. The previous paragraph makes one direction obvious; we prove the other. Let $B \subseteq A$, $R(B) > 0$ and $KR((\omega)_{g(n)}) < n + c \log n$ for all $\omega \in B, n$.

We first construct a semi-computable measure P with $P(B) > 0$ and **integer** $P((\omega)_{g(n)}) 2^{t_n}$, for $t_n = n + \lceil O(1) + (c+4) \log n \rceil$. For that, we round down $R((\omega)_{g(n)})$ (cut proportionally to satisfy $P(x0) + P(x1) \leq P(x)$). Each rounding cuts the measure by $< 2^{-t_n}$. But $KR(x) < n + c \log n$ for $g(n)$-prefixes of $\omega \in B$, hence by Theorem 11, $R(x) = 2^{-n}/O(n^{c+2})$. Thus, each rounding cuts $O(1)/n^2$ fraction of their measure, leaving $R((\omega)_{g(n)})/O(1)$ for $\omega \in B$, and $P(B) > 0$.

We then construct a process generating P as in the proof of Theorem 9, but select sets corresponding to pairs (x, t) (where $\ell(x) = g(n)$) to consist of intervals of length multiple of 2^{-t_n}. Clearly this process is the desired one.

Let us describe another result about solvability of standard algorithmic problems on probabilistic machines. The first interesting result of this type was proven by Janis Barzdin'. An infinite set of natural numbers is called *immune* if it does not contain any infinite recursively enumerable subset.

Proposition. *(Barzdin')* There exists an immune set for which the problem of constructing a characteristic sequence of its infinite subset is solvable by a probabilistic machine.

The class of all immune sets contains an interesting subclass of hyper-immune sets. For these sets, the following result holds.

Theorem 14. [13] *For any hyper-immune set M, the problem of constructing a characteristic sequence of its infinite subset cannot be solved by a probabilistic machine.*

Proof. Assume a machine can solve this problem with a positive probability. Then, by Theorem 12, a machine can solve it with probability $p > 2/3$. Construct a function $f(i)$ computed by the following algorithm: run this machine on the tree of sequences until it generates, on measure $\geq \frac{2}{3}$, some sequences that have at least i ones; return the maximum of the positions of these ones. Clearly, this function dominates the direct enumeration of M. Q.E.D.

Petri subsequently showed that if M is not fixed, the problem of generating the sequence characteristic for a hyper-immune set is solvable on probabilistic machines. However, the following holds:

Theorem 15. *Let us call a set* strongly hyper-immune *if its direct enumeration dominates, from some place on, each computable function. The problem of generating sequences characteristic for strongly hyper-immune sets is not solvable on probabilistic machines.*

The proof is similar to the one above.

[11] Note that first, the construction of this process is not always efficient and second, as shown by N. Petri, this process sometimes cannot be replaced by a table-based one (i.e., by a total recursive fast-growing process).

[12] This set B can always be selected to be closed.

[13] Also proven by V.N. Agafonov independently of the author.

3 Information Theory

3.1 Definition and basic properties

The complexity $K(x)$ denotes, intuitively, the amount of information required for restoring a text x. The conditional complexity $K(x|y)$ — the amount of information needed in addition to the information in text y, for restoring text x. The difference between these two can be naturally called the amount of information in y about x.

Definition 13. *(*A.N.Kolmogorov*)*
The amount of information in y about x *is* $I(y:x) \stackrel{\text{def}}{=} K(x) - K(x|y)$.

Remark 13. *(a)* $I(x:y) \succcurlyeq 0$*; (b)* $I(x:x) \asymp K(x)$.

Proof. (a) Let $F^2(p,x) = F_0^1(p)$. Then, if $F_0^1(p_0) = y$ and $K(y) = \ell(p_0)$, from $F^2(p_0,x) = y$, we conclude: $K(y|x) \preccurlyeq K_{F_2}(y|x) = K(y)$.

(b) Let $F^2(p,x) = x$. Then $F^2(\Lambda,x) = x$, and consequently $K(x|x) \preccurlyeq K_{F^2}(x|x) = \ell(\Lambda) = 0$. It remains to observe that $I(x:x) = K(x) - K(x|x)$.

3.2 Commutativity of information

Shannon's classical definition of the amount of mutual information in two random variables is commutative, that is, $J(\xi:\eta) = J(\eta:\xi)$. For Kolmogorov's concept of the amount of information in one text about another, such a precise equality, in general, will not hold.

Example. Clearly, some words x of each length have $K(x|\ell(x)) \geq \ell(x){-}1$.
By Theorem 4(b), there exist arbitrarily large values of l with $K(l) \geq \ell(l) - 1$. For so chosen pairs x, $l{=}\ell(x)$,

$$\begin{aligned} I(x:l) &= K(l) - K(l|x) \succcurlyeq \ell(l), \\ I(l:x) &= K(x) - K(x|l) \preccurlyeq l - l = 0. \end{aligned}$$

Thus, in some cases, $I(x:y)$ and $I(y:x)$ differ by order of the logarithm of the complexities of x, y. But A.N. Kolmogorov and L. Levin showed independently in 1967 that this is the largest possible order of magnitude for this difference. So, disregarding the smaller order quantities, $I(x,y)$ is commutative. Specifically, A.N. Kolmogorov and L. Levin proved the following:

Theorem 16. [14]

(a) $|I(x:y) - I(y:x)| \leq 12\ell(K(\overline{x}y))$*;*

(b) $|I(x:y) - [K(x) + K(y) - K(\overline{x}y)]| \leq 12\ell(K(\overline{x}y))$.

Proof. (a) We prove the inequality in one direction:

$$I(x:y) \succcurlyeq I(y:x) - 12\ell(K(xy)). \tag{3.1}$$

The other follows by swapping x and y.

We construct two auxiliary functions. Let the partial recursive function $F^4(n,b,c,x)$ enumerate without repetitions the words y such that $K(y) \leq b$, $K(x|y) \leq c$. The existence of such a function follows from [6, Theorem 0.4] (taking into account the remark). Let j (an uncomputable function of x, b, c) be the number of such words y. Function F^4 halts for all $n \leq j$ and only for them. Hence the predicate $\Pi(b,c,d,x)$, asserting that j defined above is $\geq 2^d$, is equivalent to halting of $F^4(2^d,b,c,x)$ and so is partial recursive. Similarly, there exists a function $G^5(m,b,c,d)$ enumerating without repetitions all words x with $\Pi(b,c,d,x)$. Denote by i (an uncomputable function of b,c,d) the number of such x. Obviously $G^5(m,b,c,d)$ halts for all $m \leq i$ and only for them.

[14] With more careful estimates, the bound can be tightened. For instance, 12 can be replaced by $5{+}\epsilon$. It is not known whether it can be brought down to 1.

We now start the proof. Let $a = K(x)$, $b = K(y)$, $c = K(x|y)$. Then

$$I(y : x) = a - c.$$

With $j, d{=}\ell(j)$ and i so defined, clearly $i \cdot 2^d$ does not exceed the number of pairs (x', y') with $K(y') \le b$, $K(x'|y') \le c$. That number is $\le 2^{b+c+2}$, so

$$\ell(i) + \ell(j) \preccurlyeq b + c. \tag{3.2}$$

Since $F^4(n, b, c, x)$ returns y for some $n \le j$,

$$K(y|x) \preccurlyeq \ell(\overline{b}\overline{c}n) \preccurlyeq 2\ell(b) + 2\ell(c) + \ell(j). \tag{3.3}$$

Furthermore, since $G^5(m, b, c, d)$ returns x for $d = \ell(j)$ and some $m \le i$,

$$a = K(x) \preccurlyeq \ell(\overline{b}\overline{c}\overline{d}m) \preccurlyeq 2\ell(b) + 2\ell(c) + 2\ell(d) + \ell(i). \tag{3.4}$$

Inequalities (3.2)–(3.4) and the $\ell(K(\overline{x}y))$ bound on each of $\ell(b), \ell(c), \ell(d)$ imply $K(y|x) \preccurlyeq b + c - a + 12\ell(K(\overline{x}y))$. Claim (a) follows.

(b) Clearly, $K(\overline{x}\,\overline{x}y) \preccurlyeq K(\overline{x}y)$. So, by claim (a), $|I(\overline{x}y : x) - I(x : \overline{x}y)| \preccurlyeq 12\ell(K(\overline{x}y))$, that is, $|K(\overline{x}y) - K(\overline{x}y|x) - K(x) + K(x|\overline{x}y)| \preccurlyeq 12\ell(K(\overline{x}y))$, or

$$|[K(\overline{x}y) - K(x) - K(y)] + K(y) - K(\overline{x}y|x) + K(x|\overline{x}y)| \preccurlyeq 12\ell(K(\overline{x}y)).$$

Now claim (b) follows by noting that $K(x|\overline{x}y) \asymp 0$, $K(\overline{x}y|x) \asymp K(y|x)$.

3.3 Entropy of arbitrary dynamic systems (stationary stochastic processes) and algorithmic amount of information

A.N.Kolmogorov showed that for processes of independent trials the algorithmic amount of information is asymptotically equal to the classical (probabilistic) one (see [6, Theorem 5.3]). In view of Theorem 16(b), this follows from the connection between algorithmic complexity and probabilistic entropy.

J.T.Schwartz posed the question of whether a similar fact holds for an arbitrary ergodic stationary process (that is, a process for which entropy is defined). We give a positive answer to this question in the following theorem.

Theorem 17. *Let $\{\xi_i\}$, $i = 1, 2, \ldots$, be an arbitrary ergodic stationary stochastic process with values $\xi_i \in \Omega$, let P be the measure on its trajectories $u \in \Omega^{\mathcal{N}}$ that defines this process, and let H be its entropy. Denote by $\alpha_n^i(\omega)$ the word $(\overline{\xi_1})_n(\overline{\xi_2})_n \cdots (\overline{\xi_i})_n$. Then for P-almost all u*

$$\lim_{n\to\infty} \lim_{i\to\infty} \frac{K(\alpha_n^i(\omega))}{i} = H.$$

Clearly, the ergodicity requirement here is not essential. For non-ergodic processes, instead of their average entropy H, one would take "entropy at a point": a function measurable with respect to the σ-algebra of invariant sets, averaging on any such set to its average entropy on the set. This easily follows "decomposition" of arbitrary stationary stochastic processes into ergodic ones.

Returning to the ergodic case, it suffices to prove the theorem for processes with discrete values $(\xi)_n$. The general case will follow by taking the limit on n.

Consider the set of 2^n-ary sequences u – trajectories of our stochastic process. Defined on this set is a transformation T shifting the time by 1 and a T-invariant ergodic measure describing the process. Within k time steps, $2^{n\cdot k}$ different sequences X_i^k of length k can appear. Clearly, for every ϵ, a k exists such that

$$-\sum_{i<2^{n\cdot k}} P(X_i^k) \log_2 P(X_i^k) \le k \cdot (H + \epsilon).$$

Since T^k, as well as T, preserves the measure P, it follows from the Central Ergodic Theorem (C.E.T.) that for P-almost every sequence there exists, for every l, a limit of the frequency of the values of m for which the sequence $T^{mk+l}(\omega)$ begins with X_i^k.

Take any such ω and denote these limits for it by $P_{i,l}$. From C.E.T. for T and the ergodicity of T, it follows that almost always $\sum_{l\leq k} \frac{P_{i,l}}{k} = P(X_i^k)$. Hence we have here k probability distributions on the finite set X_i^k and their average with an entropy $\leq k(H+\epsilon)$.

By convexity of entropy, at least one of the summand distributions has entropy $\leq K(H+\epsilon)$. So for our u, an l exists such that the entropy of the frequencies $P_{i,l}$ with which number m satisfies the condition "$T^{mk+l}(\omega)$ begins with X_i^k" is $\leq K(H+\epsilon)$. By a theorem of Kolmogorov (see [6, Theorem 5.1]), it follows that the "unit complexity" of almost all u is $\leq H$, which gives a "half" of our theorem.

To prove the unit complexity to be $\geq H$, we use some results from Section 2. Consider the collection X_i^k of values of some realization ω of the process over the first k time steps and compare four quantities: the entropy H; the logarithm of the probability of that collection divided by k, that is, $\frac{\log P(X_i^k)}{k}$; the logarithm of its a priori probability (see the definition of R) divided by k also, that is, $\frac{\log R(X_i^k)}{k}$; and the unit complexity $\frac{K(X_i^k)}{k}$. Their limits as $k\to\infty$ are equal. For the first two quantities, this follows from the Shannon-McMillan-Breiman theorem; for the last two, from Theorem 11 of this dissertation; and for the two in the middle, from the last remark of Section 2.3. The theorem is proved.

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