GAMMA-CONVERGENCE RESULTS FOR PHASE-FIELD APPROXIMATIONS OF THE 2D-EULER ELASTICA FUNCTIONAL

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ABSTRACT. We establish some new results about the Γ-limit, with respect to the L^1-topology, of two different (but related) phase-field approximations of the so-called Euler's Elastica Bending Energy for curves in the plane.

1. Introduction

In this paper we present some new results about the sharp interface limit of two sequences of phase-field functionals involving the so-called Cahn-Hilliard energy functional and its L^2-gradient. The study of this kind of problems is motivated by applications in different fields ranging from image processing (*e.g.*, [17, 25, 8, 5]), to the diffuse interface approximation of elastic bending energies (*e.g.*, [14, 15, 16, 26, 3, 7]), to the study of singular limits of partial differential equations and systems (*e.g.*, [32, 29, 24, 34]), up to the study of rare events for stochastic perturbations of the so-called Allen-Cahn equation (*e.g.*, [22, 30]).

Let us now introduce the two sequences of energies we wish to study. Given $\Omega \subset \mathbb{R}^d$ open, bounded and with smooth boundary, we define the so-called Cahn-Hilliard energy by

$$\mathcal{P}_\varepsilon(u) := \begin{cases} \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\, dx & \text{if } u \in W^{1,2}(\Omega), \\ +\infty & \text{otherwise on } L^1(\Omega) \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter representing the typical "diffuse interface width", and $W \in C^3(\mathbb{R}, \mathbb{R}^+ \cup \{0\})$ is a double-well potential with two equal minima (throughout the paper we make the choice $W(s) := (1 - s^2)^2/4$, though most of the results we obtain hold true for a wider class of potentials). The sequences of functionals $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$, $\{\mathcal{E}_\varepsilon\}_\varepsilon$ we consider in this paper are respectively defined by

$$\widetilde{\mathcal{E}}_\varepsilon := \mathcal{P}_\varepsilon + \mathcal{W}_\varepsilon : L^1(\Omega) \to [0, +\infty], \tag{1.2}$$

$$\text{where } \mathcal{W}_\varepsilon(u) := \begin{cases} \frac{1}{\varepsilon} \int_\Omega \left(\varepsilon \Delta u - \frac{W'(u)}{\varepsilon}\right)^2 dx & \text{if } u \in W^{2,2}(\Omega) \\ +\infty & \text{elsewhere on } L^1(\Omega), \end{cases}, \tag{1.3}$$

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and

$$\mathcal{E}_\varepsilon := \mathcal{P}_\varepsilon + \mathcal{B}_\varepsilon : \, L^1(\Omega) \to \, [0, +\infty], \tag{1.4}$$

$$\text{where } \mathcal{B}_\varepsilon(u) := \begin{cases} \frac{1}{\varepsilon}\int_\Omega \left|\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u \otimes \nu_u\right|^2 dx & \text{if } u \in W^{2,2}(\Omega) \\ +\infty & \text{elsewhere on } L^1(\Omega) \end{cases}, \tag{1.5}$$

and ν_u is a unit vector-field such that

$$\nu_u = \frac{\nabla u}{|\nabla u|} \text{ on } \{\nabla u \neq 0\} \text{ and } \nu_u \equiv \mathit{const.} \text{ on } \{\nabla u = 0\}.$$

We remark that $\mathcal{W}_\varepsilon(u)$ represents the (rescaled) norm of the L^2-gradient of $\mathcal{P}_\varepsilon$ at u, and that $\mathcal{W}_\varepsilon$ and $\mathcal{B}_\varepsilon$ are linked by the relation

$$\operatorname{tr}\Big[\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u \otimes \nu_u\Big] = \varepsilon\Delta u - \frac{W'(u)}{\varepsilon}.$$

Hence, denoted by $\{\lambda_1, \dots, \lambda_d\}$ the eigenvalues of the symmetric $d \times d$-matrix $\varepsilon\nabla^2 u - (W'(u)/\varepsilon)\nu_u \otimes \nu_u$, we have

$$d\Big(\sum_{i=1}^d \lambda_i^2\Big) = d\left|\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u \otimes \nu_u\right|^2 \geq \left(\varepsilon\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\right)^2 = \Big(\sum_{i=1}^d \lambda_i\Big)^2. \tag{1.6}$$

Next, we briefly summarize the known results about the sharp interface limit of $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$ and $\{\mathcal{E}_\varepsilon\}_\varepsilon$. The starting point for the analysis of the asymptotic behavior, as $\varepsilon \to 0$, of the sequences $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$, $\{\mathcal{E}_\varepsilon\}_\varepsilon$ is a well-known result, due to Modica and Mortola, establishing the Γ-convergence of $\mathcal{P}_\varepsilon$ to the area functional. More precisely in [28] it has been proved that the $\Gamma(L^1(\Omega))$-limit of the sequence $\{\mathcal{P}_\varepsilon\}_\varepsilon$ is given by

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{P}_\varepsilon(u) = \mathcal{P}(u) := \begin{cases} \frac{c_0}{2}\int_\Omega d|\nabla u| & \text{if } u \in BV(\Omega, \{-1, 1\}), \\ +\infty & \text{elsewhere in } L^1(\Omega) \end{cases}$$

where $c_0 := \int_{-1}^1 \sqrt{2W(s)}\, ds$ (see Section 2.5 and Section 3 for further details). We remark that for every $u \in BV(\Omega, \{-1, 1\})$ we can write $u = 2\chi_E - 1 =: \mathbb{1}_E$, where χ_E denotes the characteristic function of the finite perimeter set $E := \{u \geq 1\}$. Hence $\mathcal{P}(u) = c_0\mathcal{H}^{d-1}(\partial^* E)$ where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$ and $\partial^* E$ denotes the reduced boundary of E (see [35]).

The main result concerning the Γ-convergence of $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$ has been established, for $d = 2$ and $d = 3$, by Röger and Schätzle in [31] and independently, but only in the case $d = 2$, by Tonegawa and Yuko in [36], partially answering to a conjecture of De Giorgi (see [11]). In particular in [31] the authors proved that for $d = 2$ or 3 and $u = \mathbb{1}_E \in BV(\Omega, \{-1, 1\})$ such that $E \subset \Omega$ is open and $\Omega \cap \partial E \in C^2$, we have

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u) = c_0 \int_{\Omega\cap\partial E} \big[1 + |\mathbf{H}_{\partial E}(x)|^2\big]\, d\mathcal{H}^{d-1}(x), \tag{1.7}$$

where $\mathbf{H}_{\partial E}(x)$ denotes the mean curvature vector of ∂E in the point $x \in \partial E$. When $d = 2$ we call the functional on the right hand side of (1.7) the Euler's Elastica Functional.

The sequence of functionals $\{\mathcal{E}_\varepsilon\}_\varepsilon$ has been introduced in [3] in connection with the problem of finding a diffuse interface approximation of the Gaussian curvature. As a straightforward consequence of the results established in [3] it follows that, again for $d = 2, 3$ and $u = \mathbb{1}_E \in BV(\Omega, \{-1, 1\})$ such that $E \subset \Omega$ is open and $\Omega \cap \partial E \in C^2$, we have

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u) = c_0 \int_{\Omega\cap\partial E} [1 + |\mathbf{B}_{\partial E}(x)|^2] \, d\mathcal{H}^{d-1}(x), \tag{1.8}$$

where this time $\mathbf{B}_{\partial E}(x)$ denotes the second fundamental form of ∂E in the point $x \in \partial E$.

In the present paper we restrict to the case $d = 2$, and investigate the behavior of $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$ and $\{\mathcal{E}_\varepsilon\}_\varepsilon$ along sequences $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ such that

$$L^1(\Omega) - \lim_{\varepsilon\to 0} u_\varepsilon = \mathbb{1}_E \in BV(\Omega, \{-1, 1\}), \tag{1.9}$$

removing the regularity assumption on the limit set E. In other words we aim to prove a full Γ-convergence result, on the whole space $L^1(\Omega)$.
We recall that if a sequence of functionals Γ-converges, and a certain equicoercivity property holds, then the minimizers of such sequence converge to the minimizers of the Γ-limit. Therefore, besides its possible mathematical interest, we expect that a description of the Γ-limit may be of some relevance at least for those applications, such as [5, 13, 14, 15, 3, 26], where the sequences $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$ and $\{\mathcal{E}_\varepsilon\}_\varepsilon$ are introduced to formulate, and solve numerically, a "diffuse interface" variational problem whose solutions are expected to converge, as $\varepsilon \to 0$, to the solutions of a given sharp interface minimum problem.

In synthesis our result says that the sharp interface limits of $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$ and $\{\mathcal{E}_\varepsilon\}_\varepsilon$ in general do not coincide out of "smooth points", although in two space dimensions, by (1.7) and (1.8) and

$$|\mathbf{B}_{\partial E}(x)|^2 = |\mathbf{H}_{\partial E}(x)|^2, \tag{1.10}$$

we have, for $u = \mathbb{1}_E$ and $E \subset \Omega$ open such that $\Omega \cap \partial E \in C^2$,

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u) = \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u).$$

More precisely we prove that, in accordance with (1.6), we have

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u) \le \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u)$$

and that there are functions $u \in BV(\Omega, \{-1, 1\})$ for which the above inequality is strict. In fact we show that, on one hand, a uniform bound on $\mathcal{E}_\varepsilon(u_\varepsilon)$ implies that the energy density measures

$$\mu_\varepsilon := c_0^{-1}[\varepsilon/2|\nabla u_\varepsilon|^2 + W(u_\varepsilon)/\varepsilon]\mathcal{L}^d_{\llcorner\Omega}$$

(here $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$) concentrate on a set whose tangent cone in *every* point is given by an unique tangent line. On the other hand we show that a uniform bound on $\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon)$ allows the energy measures to concentrate on cross-shaped sets. This difference in regularity between the support of the two limit measures is related to the existence of so called "saddle shaped solutions" to the semilinear elliptic equation $-\Delta U + W'(U) = 0$ on $\mathbb{R}^2$ (see [10, 6, 12], and the proof of Theorem 4.5 in this paper).

To give a better description of our results, let us briefly explain the role played by the regularity assumption on the limit set E in the proofs of (1.7) and (1.8), and discuss the obstructions to remove such an assumption. To this aim, for the readers convenience, we briefly recall the backbone of the proof of (1.7) (and point out that the proof (1.8) follows the same line of arguments).

To prove (1.7) one has to find a lower-bound for $\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon)$ proving the so-called $\Gamma - \liminf$ inequality; and to show that such lower-bound is in a way "optimal" via the so-called $\Gamma - \limsup$ inequality. (See Section 2.5 for a precise definition of Γ-convergence).

Let us begin recalling how the $\Gamma - \liminf$ inequality has been proved. Suppose that $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ verifies (1.9) and

$$\sup_{\varepsilon>0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) < +\infty. \tag{1.11}$$

Thanks to the bound (uniform in ε) on $\mathcal{P}_\varepsilon(u_\varepsilon)$ and the convergence of u_ε to u, applying the results of [28] it can be easily deduced that (up to subsequences) the energy-density measures μ_ε defined above converge to a Radon measure μ in Ω such that $\mathcal{H}^{d-1}_{\llcorner \partial^* E} << \mu$. That is, roughly speaking, the support of μ contains the (reduced) boundary of E. In case only a bound on $\mathcal{P}_\varepsilon(u_\varepsilon)$ is available, there is no much hope to obtain more informations about the measure μ, since this latter may be quite irregular (for example it may contain parts that are absolutely continuous with respect to $\mathcal{L}^d$). However when (1.11) holds, the bound on $\mathcal{W}_\varepsilon(u_\varepsilon)$ implies that μ has some "weak" regularity properties. In fact the first crucial step in the derivation of a lower bound for $\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon)$ consists in proving that (1.11) guarantees that:

- the measure μ has the form $\mu = \theta\, \mathcal{H}^{d-1}_{\llcorner M}$, where M is a generalized hypersurface of $\mathbb{R}^d$, and $\theta : M \to \mathbb{N}$ is an integer valued $\mathcal{H}^{d-1}_{\llcorner M}$ -measurable function;
- a generalized mean curvature vector $\mathbf{H}_\mu \in L^2(\mu)$ is well defined μ-a.e.;
- the following relation holds

$$\liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) \geq c_0 \int [1 + |\mathbf{H}_\mu|^2]\, d\mu. \tag{1.12}$$

(Namely, μ is the weight measure of an integral rectifiable varifold with L^2-bounded first variation, see Section 2.3 for some basic facts and terminology about varifolds theory).

The next step in the proof of the $\Gamma - \liminf$-inequality consists in relating the generalized mean curvature vector $\mathbf{H}_\mu$ of μ with the (generalized) mean curvature vector of the phase-boundary $\partial^* E$. Since $\mathcal{H}^{d-1}_{\llcorner \partial^* E} << \mu$, by the results of [27] (see also [33, 23]) it follows that $\partial^* E$ can be covered with the union of a countable family of $(d-1)$-dimensional C^2-manifolds embedded in $\mathbb{R}^d$, and with a set of $\mathcal{H}^{d-1}$-measure zero. Hence the mean curvature vector $\mathbf{H}_{\partial^* E}$ of $\partial^* E$ is well defined $\mathcal{H}^{d-1}_{\llcorner \partial^* E}$-a.e. Furthermore by [27] (see also [23, 33] and Remark 2.2) we have $\mathbf{H}_\mu(x) = \mathbf{H}_{\partial^* E}(x)$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E$. Eventually, being $\mathcal{H}^{d-1}_{\llcorner \partial^* E} << \mu$ and $\theta(x) \geq 1$ for μ-a.e. $x \in M$, by (1.12) it follows that

$$\liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) \geq c_0 \int [1 + |\mathbf{H}_\mu|^2]\, d\mu \geq c_0 \int_{\Omega\cap\partial^* E} [1 + |\mathbf{H}_{\partial^* E}|^2]\, d\mathcal{H}^{d-1}. \tag{1.13}$$

It remains to establish if (or when) such a lower bound is "optimal". More precisely, it remains to understand for which $u = \mathbb{1}_E \in BV(\Omega, \{-1, 1\})$ is it possible to find a " recovery sequence" $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$, that is a sequence such that

$$\lim_{\varepsilon\to 0} u_\varepsilon = u \text{ in } L^1(\Omega) \text{ and } \limsup_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) \leq c_0 \int_{\Omega\cap\partial^* E} [1 + |\mathbf{H}_{\partial^* E}|^2]\, d\mathcal{H}^{d-1}. \tag{1.14}$$

For those u it follows that

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) = c_0 \int_{\partial^* E\cap\Omega} [1 + |\mathbf{H}_{\partial^* E}|^2]\, d\mathcal{H}^{d-1} \tag{1.15}$$

(in fact (1.13) and (1.14) respectively represent the $\Gamma - \liminf$ and the $\Gamma - \limsup$-inequality). When $E \subset \Omega$ and $\Omega \cap \partial E \in C^2$, it is relatively easy to construct a sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ verifying (1.14) (see [4]), and this concludes the proof of (1.7).

Actually, by a simple diagonal argument, we can construct a sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ verifying (1.14) (and consequently obtain that (1.15) holds true) for all those functions $u = \mathbb{1}_E \in BV(\Omega, \{-1, 1\})$ for which there exists a sequence $\{E_h\}_h$ such that $E_h \subset \Omega$ is open and $\Omega \cap \partial E_h \in C^2$ for every $h \in \mathbb{N}$, and such that

$$\lim_{h\to\infty} \mathbb{1}_{E_h} = u \text{ in } L^1(\Omega), \quad \lim_{h\to\infty} \int_{\Omega\cap\partial E_h} [1+|\mathbf{H}_{\partial E_h}|^2]\, d\mathcal{H}^{d-1} = \int_{\Omega\cap\partial E} [1+|\mathbf{H}_{\partial E}|^2]\, d\mathcal{H}^{d-1}$$

(*e.g.*, if $E \subset \Omega$ is open and $\Omega \cap \partial E \in W^{2,2}$, see Remark 2.14).

As we already said (1.8) is obtained following a similar line of arguments.

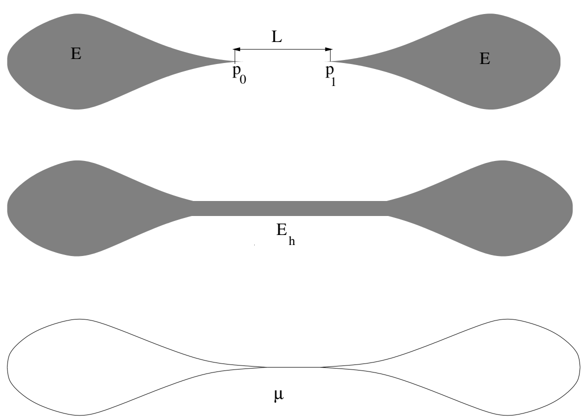

FIGURE 1. The set $E \subset\subset \Omega$ has smooth boundary out of the two sharp cusps p_0, p_1, that are aligne and placed at a distance $L > 0$. The first variation of the rectifiable varifold $\widehat{V} = \mathbf{v}(\partial^* E, 1)$ associated to $\widehat{\mu} := \mathcal{H}^1_{\llcorner \partial^* E}$ is given by $\delta\widehat{V} = \mathbf{H}_{\partial^* E}\mathcal{H}^1_{\llcorner \partial^* E} + 2\sum_{j=0,1} \mathbf{e}_1(-1)^j \delta_{p_j}$, where $\mathbf{e}_1 = (1,0) \in \mathbb{R}^2$ and δ_{p_j} is the Dirac-delta at p_j. Hence $\delta\widehat{V} \notin [L^2(\widehat{\mu})]^*$, see Section 2.3. For every $h \in \mathbb{N}$ the set E_h, such that $E_h \subset\subset \Omega$ and $\partial E_h \in C^2$, is obtained replacing p_0 and p_1 with a flat tubular neighborhood of height $1/h$ that approximates the segment connecting p_0 and p_1 as $h \to \infty$.

Yet we do not expect neither (1.15) nor its analogue for the $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u)$ to be always true, as the following example suggests. Suppose that $E \subset\subset \Omega$ is as in Figure 1. We then have $u = \mathbb{1}_E \in BV(\Omega, \{-1, 1\})$. Moreover if we consider the sequence $\{E_h\}_h$ of smooth sets represented in Figure 1, we have

$$L^1(\Omega) - \lim_{h\to\infty} \mathbb{1}_{E_h} = u,$$

$$\lim_{h\to\infty} \int_{\partial E_h} [1 + |\mathbf{H}_{\partial E_h}|^2]\, d\mathcal{H}^1 = \int [1 + |\mathbf{H}_\mu|^2]\, d\mu = \int_{\partial^* E} [1 + |\mathbf{H}_{\partial^* E}|^2]\, d\mathcal{H}^1 + 2L.$$

Hence, for every $u_h := \mathbb{1}_{E_h}$ ($h \in \mathbb{N}$), by [4, 3], we can construct a recovery sequence $\{u_{h,\varepsilon}\}_\varepsilon \subset C^2(\Omega)$. Then, by a diagonal argument, we can select a sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ such that $\lim_{\varepsilon\to 0} u_\varepsilon = u$ in $L^1(\Omega)$ and

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = c_0 \int_{\Omega\cap\partial^* E} [1 + |\mathbf{H}_{\partial^* E}|^2]\, d\mathcal{H}^1 + 2c_0 L < +\infty.$$

Therefore we can conclude that

$$\max\{\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u),\, \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u)\} < +\infty.$$

For this choice of $u \in BV(\Omega, \{-1, 1\})$ we expect that neither (1.15), nor its analogue for $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u)$ hold. In fact: on the one hand (1.7) and (1.8) hold as soon as we localize the functionals $\widetilde{\mathcal{E}}_\varepsilon$ and $\mathcal{E}_\varepsilon$ on any open subset ω such that $\overline{\omega} \cap \{p_0, p_1\} = \emptyset$; on the other hand we cannot have

$$\lim_{\varepsilon\to 0} \mu_\varepsilon = \mathcal{H}^1 \llcorner_{\partial^* E} \text{ as Radon measures on } \Omega,$$

as this would contradict the fact (established in [31, 36, 3] and recalled above) that the rectifiable varifold associated with the limit of the μ_ε has L^2-bounded first variation. Hence we expect that for every sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ we have

$$\frac{1}{c_0} \min\{\liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon),\, \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon)\} \geq \int [1 + |\mathbf{H}_\mu|^2]\, d\mu > \int_{\partial^* E} [1 + |\mathbf{H}_{\partial^* E}|^2]\, d\mathcal{H}^1,$$

that is the last term on the right hand side is a too rough (or "non-optimal") lower-bound for both $\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon)$ and $\mathcal{E}_\varepsilon(u_\varepsilon)$. It is thus rather natural to try to answer the question: what are the Γ-limits of $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$ and $\{\mathcal{E}_\varepsilon\}_\varepsilon$ out of "smooth sets"?

We try to answer this question in the case $d = 2$ only, and from now on, throughout the paper we will assume that $d = 2$, unless otherwise specified.

Since Γ-limits are necessarily lower semi-continuous functionals (see [9, Proposition 4.16]), in view of (1.7), (1.8) and (1.10), a natural candidate for the Γ-limit of both $\widetilde{\mathcal{E}}_\varepsilon$ and $\mathcal{E}_\varepsilon$ is the lower semi-continuous envelope (with respect to the $L^1(\Omega)$-topology) of the functional

$$\mathcal{F}_o : L^1(\Omega) \to [0, +\infty], \quad u \mapsto \begin{cases} \int_{\Omega\cap\partial E} [1 + |\mathbf{H}_{\partial E}|^2]\, d\mathcal{H}^1 & \text{if } u = \mathbb{1}_E \text{ and } \Omega \cap \partial E \in C^2, \\ +\infty & \text{otherwise on } L^1(\Omega), \end{cases}$$

that is the functional

$$\begin{aligned} \overline{\mathcal{F}}_o(u) &:= \inf\{\liminf_{k\to\infty} \mathcal{F}_o(u_k) :\ L^1(\Omega) - \lim_{k\to\infty} u_k = u\} \\ &= \sup\{\mathcal{G}(u) :\ \mathcal{G} \leq \mathcal{F}_o \text{ on } L^1(\Omega),\, \mathcal{G} \text{ is lower semi-continuous on } L^1(\Omega)\}. \end{aligned}$$

Since by [1, Theorem 3.2] we have $\overline{\mathcal{F}}_o(u) = \mathcal{F}_o(u)$ for every $u = \mathbb{1}_E$ such that $\Omega \cap \partial E \in W^{2,2}$ (see also [33] for a more general statement), by (1.7), (1.8) and the definition of $\overline{\mathcal{F}}_o$, we can conclude that

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon \leq c_0 \overline{\mathcal{F}}_o \text{ on } L^1(\Omega), \quad \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon \leq c_0 \overline{\mathcal{F}}_o \text{ on } L^1(\Omega).$$

We can now rephrase the results we obtain as follows: we prove that $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon = c_0\overline{\mathcal{F}}_o$ (at least under suitable boundary conditions for the phase-field variable), and we show that in general $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\widetilde{\mathcal{E}}_\varepsilon < c_0\overline{\mathcal{F}}_o$.

The outline of the paper is the following. In Theorem 4.1, we show that the assumption

$$\sup_{\varepsilon>0}\mathcal{E}_\varepsilon(u_\varepsilon) < +\infty,$$

implies additional "regularity" on the support of the measure $\mu := \theta\mathcal{H}^1_{\llcorner M}$ arising as limit of the energy density measures μ_ε defined above. Namely we establish that in every point of $M\cap\Omega$ a (unique) tangent-line to M is well defined. Moreover, in Corollary 4.2, we show that if $\{u_\varepsilon\}_\varepsilon \subset X$ where

$$X := \{u \in C^2(\Omega) : \ u(x) \equiv 1,\ \partial_{\nu_\Omega}u(x) = 0,\ \forall x \in \partial\Omega\},$$

then the set $M \cup \partial\Omega$ has an uniquely defined tangent line in every point. In view of [2, 3] (see also Theorem 2.13 and Theorem 3.3 in this paper) this allows us to conclude that $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\mathcal{E}_{\varepsilon\llcorner X} = c_0\overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ denotes the $L^1(\Omega)$-lower semi-continuos envelope of $\mathcal{F}_{o\llcorner\mathscr{K}}$ and

$$\mathscr{K} := \{E \subset \Omega : \ E \text{ is open, compactly contained in } \Omega \text{ and } \partial E \in C^2\}$$

(the restriction of $\mathcal{F}_o$ to $\mathscr{K}$ is a consequence of the fact that here we are considering the Γ-limit of $\mathcal{E}_{\varepsilon\llcorner X}$).
We remark that we do not expect that an analogoue of Theorem 4.1 holds in space dimensions $d > 2$. In fact, to prove Theorem 4.1 we make use of a blow-up argument and of some regularity results obtained in [18], that are valid only for generalized $(d-1)$-dimensional hypersurfaces (namely, Hutchinson's curvature varifolds) with p-integrable (generalized) second fundamental form for some $p > (d-1)$. Moreover, though we expect that an analogue of Corollary 4.2 holds also (at least) when $d = 3$, to prove such a result we would probably need a different approach. In fact, in the proof of Corollary 4.2 we make an essential use of an "explicit" representation of $\overline{\mathcal{F}}$, that has been established in [2] and is available only in two space dimensions.

For what concerns the sequence $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$, in Theorem 4.5 we show that in general the support of the limit measure does not necessarily have an unique tangent line in *every* point, and we obtain the existence of a function $u = \mathbb{1}_E \in BV(\Omega, \{-1,1\})$ such that

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\widetilde{\mathcal{E}}_\varepsilon(u) < c_0\overline{\mathcal{F}}_0(u) = \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u) = +\infty.$$

This means that the sharp interface limit of $\mathcal{E}_\varepsilon$ and $\widetilde{\mathcal{E}}_\varepsilon$ do not coincide as functionals on $L^1(\Omega)$, although as we already remarked $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\widetilde{\mathcal{E}}_\varepsilon(u) = \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u)$ whenever $u = \mathbb{1}_E$ and $\Omega\cap\partial E \in C^2$.

Although we are not able to completely identify the $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\widetilde{\mathcal{E}}_\varepsilon(u)$ we shortly discuss how the results of [12] can be applied to obtain the value of the Γ-limit in a certain number of cases.

The paper is organized as follows. In Section 2 we fix some notation, and recall some results about varifolds and the lower semi-continuous envelope of $\mathcal{F}_o$. In Section 3, for the readers convenience, we state the main results of [31, 36, 3]. In Section 4 we state our main results, the proofs of which are presented in Sections 6-8. Finally, in Section 5 we collect some preliminary lemmata needed in the proof of our main results.

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2. Notation and preliminary results

2.1. **General Notation.** Throughout the paper we adopt the following notation. By Ω we denote an open bounded subset of $\mathbb{R}^2$ with smooth boundary. By $B_R(x) := \{z \in \mathbb{R}^2 : |z| < R\}$ we denote the euclidean open ball of radius R centered in x.

By $\mathcal{L}^2$ we denote the 2-dimensional Lebesgue-measure, and by $\mathcal{H}^1$ the one-dimensional Hausdorff measure.

For every set $E \subseteq \mathbb{R}^2$ we denote by χ_E the characteristic function of E, that is $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ if $x \notin E$. Moreover we define the function $\mathbb{1}_E$ by $\mathbb{1}_E := 2\chi_E - 1$. We denote by $\overline{E}$ and ∂E respectively the closure and the topological boundary of E. All sets we consider are assumed to belong to $\mathcal{M}$, the class of all measurable subsets of $\mathbb{R}^2$.

We say that $E \subset \mathbb{R}^2$ is of class $W^{2,2}$ (resp. C^k, $k \geq 1$) in Ω, and write $\Omega \cap \partial E \in W^{2,2}$ (resp. $\Omega \cap \partial E \in C^k$) if $E \cap \Omega$ is open and can be locally represented as the subgraph of a function of class $W^{2,2}$ (resp. C^k).

We say that a set $E \subset \mathbb{R}^2$ has finite perimeter in Ω if $\chi_E \in BV(\Omega)$, moreover if E has finite perimeter by $\partial^* E$ we denote its reduced boundary (see [35]).

We endow the space of the (2×2) matrices $M = (m_{ij}) \in \mathbb{R}^{2\times 2}$ (resp. 2^3 tensors $T = (t_{ijk}) \in \mathbb{R}^{2^3}$) with the norm

$$|M|^2 := \mathrm{tr}(M^T M) = \sum_{i,j=1}^{2} (m_{ij})^2 \qquad \left(\text{resp. } |T|^2 := \sum_{i,j,k=1}^{2} (t_{ijk})^2\right), \tag{2.1}$$

where M^T is the transposed of M.

If $P \in \mathbb{R}^{2\times 2}$ is a (symmetric) orthogonal projection matrix onto some subspace of $\mathbb{R}^2$ and M is symmetric, then

$$|P^T M P|^2 \leq |M|^2. \tag{2.2}$$

2.2. **Differential Geometry.** Let Σ be a smooth, compact oriented curve without boundary embedded in $\mathbb{R}^2$. If $x \in \Sigma$, we denote by $P_\Sigma(x)$ the orthogonal projection onto the tangent line $T_x\Sigma$ to Σ at x. Often we identify the linear operator $P_\Sigma(x)$ with the symmetric (2×2)-matrix $\mathrm{Id} - \nu_x \otimes \nu_x$ where $x \to \nu_x \in (T_x\Sigma)^\perp$ is a smooth unit covector field orthogonal to $T_x\Sigma$.

Let us recall that, when Σ is given as a level surface $\{v = t\}$ of a smooth function v such that $\nabla v \neq 0$ on $\{v = t\}$, we can take at $x \in \{v = t\}$

$$\nu_x = \frac{\nabla v(x)}{|\nabla v(x)|}, \qquad P_\Sigma(x) = \mathrm{Id} - \frac{\nabla v(x) \otimes \nabla v(x)}{|\nabla v(x)|^2}.$$

The second fundamental form $\mathbf{B}_\Sigma$ of Σ has the expression

$$\mathbf{B}_\Sigma = \Big(P_\Sigma^T \frac{\nabla^2 v}{|\nabla v|} P_\Sigma\Big) \otimes \frac{\nabla v}{|\nabla v|},$$

where $P_\Sigma^T = (P_\Sigma)^T$. The definition of $\mathbf{B}_\Sigma$ depends only on Σ and not on the particular choice of the function v. Moreover $\mathbf{B}_\Sigma(x)$, if restricted to $T_x\Sigma$ and considered as a bilinear map from $T_x\Sigma \times T_x\Sigma$ with values in $(T_x\Sigma)^\perp$, coincides with the usual notion of second fundamental form. By

$$\mathbf{H}_\Sigma(x) = (H_1(x), H_2(x)) = \mathrm{tr}\Big(P_\Sigma^T \frac{\nabla^2 v}{|\nabla v|} P_\Sigma\Big)\nu_x,$$

we denote the curvature vector of Σ at x.

Let us also define $A^\Sigma(x) := (A^\Sigma_{ijk}(x))_{1\le i,j,k\le 3} \in \mathbb{R}^{2^3}$ as

$$A^\Sigma_{ijk} = \delta^\Sigma_i P_{\Sigma jk} \qquad \text{on } \Sigma, \tag{2.3}$$

where $\delta^\Sigma_i := P_{\Sigma ij}\frac{\partial}{\partial x_j}$.

To better understand definition (2.3), it is useful to recall the links between $\mathbf{B}_\Sigma$ and A^Σ (see [19, Proposition 5.2.1]).

Proposition 2.1. *Set $A = A^\Sigma$, $\mathbf{B} = \mathbf{B}_\Sigma$ and $\mathbf{H} = \mathbf{H}_\Sigma$. For $i, j, k \in \{1, 2\}$ the following relations hold:*

$$B^k_{ij} = P_{jl}A_{ikl}, \tag{2.4}$$

$$A_{ijk} = B^k_{ij} + B^j_{ik}, \tag{2.5}$$

$$\mathbf{H}_i = A_{jij} = B^j_{ji} + B^i_{jj}. \tag{2.6}$$

Let $u \in C^2(\Omega)$. We will often look at geometric properties of the *ensemble of the level sets* of u. We define

$$\nu_u := \frac{\nabla u}{|\nabla u|}, \qquad P^u := \mathrm{Id} - \nu_u \otimes \nu_u, \qquad \text{on } \{\nabla u \neq 0\}, \tag{2.7}$$

and $\nu_u := \mathbf{e}_3$, $P^u := \mathrm{Id} - \mathbf{e}_3 \otimes \mathbf{e}_3$ on $\{\nabla u = 0\}$. Moreover we define the second fundamental form of the ensemble of the level sets of u by

$$\mathbf{B}_u = \frac{(P^u)^T \nabla^2 u P^u}{|\nabla u|} \otimes \nu_u, \tag{2.8}$$

on $\{\nabla u \neq 0\}$ and $\mathbf{B}_u := \otimes^3 \mathbf{e}_3$ on $\{\nabla u = 0\}$. Similarly we define

$$A^u_{ijk} := -P^u_{il}\big[\partial_l((\nu_u)_j(\nu_u)_k)\big], \tag{2.9}$$

on $\{\nabla u \neq 0\}$ and $A^u := \otimes^3 \mathbf{e}_3$ on $\{\nabla u = 0\}$.

It will be convenient to consider $\mathbf{B}_u$ and A^u as maps defined on $G_2(\Omega)$ by $\mathbf{B}_u(x, S) := \mathbf{B}_u(x)$, $A^u(x, S) := A^u(x)$.

2.3. **Geometric Measure Theory: varifolds.** Let us recall some basic fact in the theory of varifolds, the main bibliographic sources being [35] and [19].

Let $M \subset \mathbb{R}^2$ be a Borel-set. We say that M is 1-rectifiable if there exists a countable family of graphs (suitably rotated and translated) $\{\Gamma_n\}_{n\in\mathbb{N}}$ of Lipschitz functions f_n of one variable such that $\mathcal{H}^1(M \setminus \cup_{n\in\mathbb{N}}\Gamma_n) = 0$ and $\mathcal{H}^1(M) < +\infty$.

By $G_{1,2}$ we denote the Grasmannian of 1-subspaces of $\mathbb{R}^2$. We identify $T \in G_{1,2}$ with the projection matrix $P_T \in \mathbb{R}^{2\times 2}$ on T, and endow $G_{1,2}$ with the relative distance as a compact subset of $\mathbb{R}^{2\times 2}$. Moreover, given $\Omega \subset \mathbb{R}^2$ open, we define the product space $G_1(\Omega) := \Omega \times G_{1,2}$, and endow it with the product distance.

We call *varifold* any positive Radon measure on $G_1(\Omega)$. In this paper we are confined to curves, hence we use the terms varifold to mean a 1-varifold in Ω.

By *varifold convergence* we mean the convergence as Radon measures on $G_1(\Omega)$.

For any varifold V we define μ_V to be the Radon measure on Ω obtained by projecting V onto Ω.

Let M be a 1-rectifiable subset of $\mathbb{R}^2$ and let $\theta : M \to \mathbb{R}^+$ be a $\mathcal{H}^1 \llcorner M$-measurable functions. We define the *rectifiable varifold* $V = \mathbf{v}(M, \theta)$, by

$$V(\phi) := \int_M \phi(x, T_x M)\, \theta(x) d\mathcal{H}^2 \qquad \forall \phi \in C_c^0(G_2(\Omega)).$$

When θ takes values in $\mathbb{N}$ we say that $V = \mathbf{v}(M, \theta)$ is a *rectifiable integral varifold* and we write $V \in \mathbf{IV}_1(\Omega)$.

Let V be a varifold on Ω. We define *the first variation of* V as the linear operator

$$\delta V : C_c^1(\Omega, \mathbb{R}^3) \to \mathbb{R}, \qquad Y \to \int \operatorname{tr}(S \nabla Y(x))\, dV(x, S).$$

We say that V has *bounded first variation* if δV can be extended to a linear continuous operator on $C_c^0(\Omega, \mathbb{R}^2)$. In this case by $|\delta V|$ we denote the total variation of δV. Whenever the varifold V has bounded first variation we call *generalized mean curvature vector of* V the vector field

$$\mathbf{H}_V = \frac{d\delta V}{d\mu_V},$$

where the right-hand side denotes the Radon-Nikodym derivative of δV with respect to μ_V.

Remark 2.2. Let us recall that if $V \in \mathbf{IV}_1(\Omega)$ and V has bounded first variation then, by the results recently proved in [27, 23], we have: the support of μ_V is a 1-dimensional C^2-rectifiable subset of Ω; $\mathbf{H}_V$ depends only on the local structure of the varifold V, that is for every V_1, $V_2 \in \mathbf{IV}_1(\Omega)$ we have $\mathbf{H}_{V_1}(x) = \mathbf{H}_{V_2}(x)$ for $\mathcal{H}^1$-a.e. $x \in \operatorname{spt}(\mu_{V_1}) \cap \operatorname{spt}(\mu_{V_2})$.

We say that a varifold V is stationary if $\delta V \equiv 0$.

We say that $V \in \mathbf{IV}_1(\Omega)$ has L^p*-bounded first variation* ($p > 1$) if

$$\sup_{\substack{Y \in C_c^1(\Omega), \\ \|Y\|_{L^p(\mu_V)} \leq 1}} \delta V(Y) < +\infty.$$

It can be easily checked that every $V \in \mathbf{IV}_1(\Omega)$ with L^p-bounded first variation verifies $|\delta V| << \mu_V$ (as Radon measures), so that

$$\delta V(Y) = \int \mathbf{H}_V \cdot Y \, d\mu_V, \quad \mathbf{H}_V \in L^p(\mu_V).$$

For every $V \in \mathbf{IV}_1(\Omega)$ with L^p-bounded first variation we set

$$\mathcal{F}_p(V) := \int [1 + |\mathbf{H}_V|^p]\, d\mu_V = \mu_V(\Omega) + \Big(\sup_{\substack{Y \in C_c^1(\Omega), \\ \|Y\|_{L^p(\mu_V)|} \leq 1}} \delta V(Y) \Big)^p.$$

Remark 2.3. If $V \in \mathbf{IV}_1(\Omega)$ has L^p-bounded first variation for some $p > 1$, by [35, Corollary 17.8], the 1-density of μ_V in x

$$\Theta(\mu_V, x) := \lim_{\rho \to 0} \frac{\mu_V(B_\rho(x))}{\pi \rho}, \tag{2.10}$$

is well defined everywhere on $\operatorname{spt}(\mu_V)$, $\Theta(\mu_V, x) \in \mathbb{N}$ and $\Theta(\mu_V, x) < C$, where $C > 0$ is a constant that depends only on $\|\mathbf{H}_{\mu_V}\|_{L^p(\mu_V)}$. Moreover we can write $V = v(M, \theta)$ where $M = \operatorname{spt}(\mu_V) \cap \Omega$ and $\theta(x) = \Theta(\mu_V, x)$. In the rest of the paper we will always assume that varifolds with L^p-bounded first variation are represented in this manner. Eventually let us also recall that for $\mathcal{H}^1$-almost every $x_0 \in \mathrm{spt}(\mu_V)$, there exists $P \in G_{1,2}$ such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho}\int \phi(\rho x + x_0, S)\, dV(x,S) = \theta(x_0)\int_P \phi(y,P)\, d\mathcal{H}^1, \quad \forall \phi \in C^0_c(G_1(\Omega).$$

Moreover P is a classical tangent line to M at x_0 in the sense that

$$\limsup_{\rho\to 0}\left\{\frac{\mathrm{dist}(x, P+x_0)}{\rho} : \ x \in M \cap B_\rho(x_0)\right\} = 0.$$

For our purposes we also need to introduce a further class of varifolds. Following [19] we define the notion of Hutchinson's curvature varifold with generalized second fundamental form.

Definition 2.4. *Let $V \in \mathbf{IV}_1(\Omega)$. We say that V is a curvature varifold with generalized second fundamental form in L^p $(p > 1)$, if there exists $A_V = A^V_{ijk} \in L^p(V, \mathbb{R}^{2^2})$ such that for every function $\phi \in C^1_c(G_1(\Omega))$ and $i = 1, 2$,*

$$\int_{G_2(\Omega)} (S_{ij}\partial_j\phi + A^V_{ijk} D_{m_{jk}}\phi + A^V_{jij}\phi)\, dV(x,S) = 0, \tag{2.11}$$

where $D_{m_{jk}}\phi$ denotes the derivative of $\phi(x,\cdot)$ with respect to its jk-entry variable.

Moreover we define the generalized second fundamental form $\mathbf{B}_V = (B^k_{ij})_{1\le i,j,k\le 3}$ of V as

$$B^k_{ij}(x,S) := S_{jl}A^V_{ikl}(x,S). \tag{2.12}$$

Eventually by $\mathscr{CV}^p_1(\Omega)$ we denote the calss of Hutchinson's curvature varifolds with p-integrable second fundamental form in Ω.

Remark 2.5. If $V = \mathbf{v}(\Sigma, 1)$, where Σ is a smooth, compact surface without boundary, the generalized second fundamental form as well as the mean curvature and the tensor A_V coincide with the classical quantities defined in Section 2.2. Moreover, for every $V \in \mathscr{CV}^p_1(\Omega)$ $(p > 1)$ the functions A^V_{ijk}, B^k_{ij} verify V-a.e.the identities stated in Proposition 2.1.

Remark 2.6. Every curvature varifold V with generalized second fundamental form in L^p has also L^p-bounded first variation and

$$\mathbf{H}_V(x) = (A_{212}(x, T_x\mu_V), A_{121}(x, T_x\mu_V)) \in L^p(\mu_V, \mathbb{R}^2), \tag{2.13}$$

for μ_V almost every $x \in \Omega$ (see [19]). Moreover if $V \in \mathscr{CV}^p_1(\Omega)$, by Proposition 2.1, we have

$$\mathcal{F}_p(V) = \int [1 + |\mathbf{H}_V|^p]\, d\mu_V = \int [1 + |\mathbf{B}_V|^p]\, dV = \int [1 + |A_V|^p]\, dV. \tag{2.14}$$

Let us also recall that there are, however, varifolds $V \in \mathbf{IV}_1(\Omega)$ with L^p-bounded first variation for every $p > 1$, that do not belong to $\mathscr{CV}^p_1(\Omega)$. An example is given by the (stationary) varifold $\mathbf{v}(M, 1) \in B_R$ where M is given by the union of three line segments of length R having one end point in the origin, and forming angles of $2\pi/3$ radiants one with the other.

Eventually we introduce the set $\mathscr{D}(\Omega) \subsetneq \mathscr{CV}^2_1(\Omega)$ of Hutchinson's curvature varifolds that can be approximated (in the varifolds topology) by a sequence of C^2-smooth embedded curves in Ω, having uniformly L^2-bounded second fundamental form. More precisely we give the following

Definition 2.7. *We define the set $\mathscr{D}(\Omega)$ as the set of $V \in \mathscr{CV}_1^2(\Omega)$ for which there exists a sequence $\{E_k\}_k$ of open, bounded subsets with smooth boundary such that $E_k \subset\subset \Omega$ and such that*

$$\lim_{k\to\infty} \mathbf{v}(\partial E_k, 1) = V, \text{ as varifolds in } \Omega,$$

$$\sup_{k\in\mathbb{N}} \mathcal{F}_2(V_k) = \sup_{k\in\mathbb{N}} \int_{\partial E_k} [1 + |\mathbf{H}_{\partial E_k}|^2]\, d\mathcal{H}^1 = \sup_{k\in\mathbb{N}} \int_{\partial E_k} [1 + |\mathbf{B}_{\partial E_k}|^2]\, d\mathcal{H}^1 < +\infty.$$

As a straightforward consequence of the results proved in [2] we have the following characterization

$$\mathscr{D}(\Omega) = \Big\{ V = \mathbf{v}(M,\theta) \in \mathscr{CV}_1^2(\mathbb{R}^2) : \ M \cup \partial\Omega \text{ has an unique tangent line in every point} \}. \tag{2.15}$$

Remark 2.8. If in Definition 2.7 we drop the assumption $E_k \subset\subset \Omega$ on the sequence of smooth sets approximating $V = \mathbf{v}(M,\theta)$, then (2.15) ceases to hold. In fact, in this case M has a unique tangent line in every point belonging to $M \cap \Omega$ (see Proposition 2.9 below), but there might be points $p \in M \cap \partial\Omega$ where the tangent line to $M \cup \partial\Omega$ is not unique. As a consequence, though $V \in \mathscr{CV}_1^2(\Omega)$, in general we have $V \notin \mathscr{CV}_1^2(\mathbb{R}^2)$.

We conclude this section with a further easy consequence of [2], that we need in the proof of Theorem 4.5.

Proposition 2.9. *Let $V = \mathbf{v}(M,\theta)$ be an integrable, rectifiable varifold with L^2-bounded first variation in Ω. Suppose we can find a sequence of manifolds M_k smooth, embedded and without boundary in Ω, such that $V = \lim_{k\to\infty} \mathbf{v}(M_k, 1)$ with respect to varifolds convergence in Ω and such that*

$$\sup_{k\in\mathbb{N}} \int_{M_k} 1 + |\mathbf{H}_{M_k}|^2\, d\mathcal{H}^1 < +\infty. \tag{2.16}$$

Then $\mathrm{spt}(\mu_V) = M$ has an unique tangent line in every point of $M \cap U$ for every $U \subset\subset \Omega$.

Remark 2.10. Let us mention that we expect that the arguments used in [2] can be adapted to prove also the converse of Proposition 2.9. That is, if $\mathbf{v}(M,\theta) \in \mathscr{CV}_1^2(\Omega)$ is such that M has an unique tangent line in every point of $M \cap \Omega$ then there exists a sequence of manifolds M_k smooth, embedded and without boundary in Ω, such that $V = \lim_{k\to\infty} \mathbf{v}(M_k, 1)$ with respect to varifolds convergence in Ω, and such that V_k verify (2.16).

Remark 2.11. Let $V = \mathbf{v}(M,\theta) \in \mathscr{CV}_1^2(\Omega)$. In [2] it has been proved that to say that in every point of $M \cap \Omega$ an unique tangent line is well defined, is equivalent to say that $M \cap \Omega$ can be locally (and up to rigid motions) represented as a finite union of graphs of $W^{2,2}$-functions that do not cross each other.

2.4. **Preliminary Results on the Relaxed elastica Functional.**

Let us define the functional

$$\mathcal{F} := \mathcal{F}_{o \llcorner \mathscr{K}} : L^1(\Omega) \to [0, +\infty],$$

$$u \mapsto \begin{cases} \int_{\partial E} [1 + |\mathbf{H}_{\partial E}|^2]\, d\mathcal{H}^1 & \text{if } u = \mathbb{1}_E, \text{ and } E \subset\subset \Omega,\ \partial E \in C^2, \\ +\infty & \text{otherwise on } L^1(\Omega), \end{cases} \tag{2.17}$$

and its L^1-lower-semicontinuous-envelope

$$\overline{\mathcal{F}}(u) := \inf\{\liminf_{k\to\infty} \mathcal{F}(u_k) : \lim_{k\to\infty} u_k = u \text{ in } L^1(\Omega)\}. \tag{2.18}$$

Remark 2.12. We remark that if $E \subset \Omega$ is open and with smooth boundary, $u := \mathbb{1}_E$ and $V = \mathbf{v}(\partial E, 1) \in \mathbf{IV}_1(\Omega)$, we have $\mathcal{F}_2(V) = \mathcal{F}(u)$.

As a straightforward consequence of [2, Theorem 4.3] we have the following

Theorem 2.13. *Let $E \subset \Omega$ and $u = \mathbb{1}_E \in L^\infty(\Omega, \{-1, 1\})$. Then $\overline{\mathcal{F}}(u) < +\infty$ if and only if $u \in BV(\Omega, \{-1, 1\})$ and the set*

$$\begin{aligned}\mathscr{A}(E) := \{V = \mathbf{v}(M, \theta) \in \mathscr{D}(\Omega) : M \supset \partial^* E \neq \emptyset, \\ \theta(x) \equiv 1 \operatorname{mod} 2, \forall x \in \partial^* E, \\ \theta(x) \equiv 0 \operatorname{mod} 2, \forall x \in \operatorname{spt}(\mu_V) \setminus \partial^* E\},\end{aligned}$$

is not empty. Moreover, if $\mathscr{A}(E) \neq \emptyset$, the following representation formula holds

$$\overline{\mathcal{F}}(u) = \min_{V \in \mathscr{A}(E)} \mathcal{F}_2(V).$$

In particular if ∂E is $W^{2,2}$-smooth in Ω and, if $\partial E \cap \partial\Omega \neq \emptyset$, ∂E touches $\partial\Omega$ tangentially, then

$$\overline{\mathcal{F}}(u) = \mathcal{F}(u).$$

Remark 2.14. If $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of open smooth subsets of Ω (that do not necessarily verify $E_k \subset\subset \Omega$) such that $L^1(\Omega) - \lim_{k\to\infty} u_k = \mathbb{1}_E$, where E is a subset with smooth boundary, by [1, 33] we still can conclude that

$$\liminf_{k\to\infty} \int_{\Omega\cap\partial E_k} 1 + |\mathbf{H}_{\partial E_k}|^2 \, d\mathcal{H}^1 \geq \int_{\Omega\cap\partial E} 1 + |\mathbf{H}_{\partial E}|^2 \, d\mathcal{H}^1.$$

2.5. **Γ-convergence.** Let X be a topological space and $F_\varepsilon : X \to [0, +\infty]$ a sequence of functionals on X. We say that F_ε Γ-converge to the Γ-limit $F : X \to [0, +\infty]$ in X, and we write $\Gamma(X) - \lim_{\varepsilon\to 0} F_\varepsilon = F$, if the following two conditions hold:

- Lower bound inequality (or $\Gamma - \liminf$-inequality): For every sequence $\{x_\varepsilon\} \subset X$ such that $\lim_{\varepsilon\to 0} x_\varepsilon = x$ in X,
$$\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \geq F(x).$$
- Upper bound inequality (or $\Gamma - \limsup$-inequality): For every $x \in X$, there exists a recovery sequence $\{x_\varepsilon\}_\varepsilon \subset X$ such that
$$\lim_{\varepsilon\to 0} x_\varepsilon = x \text{ in } X, \quad \limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \leq F(x).$$

3. Preliminary known Results on Diffuse Interfaces Approximations of $\mathcal{F}$

We begin this section specifying some further notation needed in the sequel. We set

$$W(r) := \frac{1}{4}(1 - r^2)^2, \qquad r \in \mathbb{R},$$

and

$$c_0 := \int_{-1}^{1} \sqrt{2W(s)} \, ds. \tag{3.1}$$

If $\gamma(s) := \tanh(s)$ we have $\ddot{\gamma} = \frac{d}{ds}(W(\gamma))$,

$$\int_{\mathbb{R}} |\dot{\gamma}|^2 \, ds = \int_{\mathbb{R}} 2W(\gamma) \, ds = c_0,$$

and

$$c_0 = \min \left\{ \int_{\mathbb{R}} \left(\frac{|\dot{v}|^2}{2} + W(v) \right) \, ds : \, v \in H^1_{\mathrm{loc}}(\mathbb{R}), \, \lim_{s \to \pm\infty} v(s) = \pm 1 \right\}. \tag{3.2}$$

To every sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ we associate

- the sequences of Radon measures

$$\mu_\varepsilon := \Big(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} \Big) \mathcal{L}^2_{\llcorner \Omega}, \quad \widetilde{\mu}_\varepsilon := \varepsilon |\nabla u_\varepsilon|^2 \mathcal{L}^2_{\llcorner \Omega}; \tag{3.3}$$

- the sequence of diffuse varifolds

$$V^\varepsilon_{u_\varepsilon}(\phi) := c_0^{-1} \int \phi(x, P^{u_\varepsilon}(x)) \, d\widetilde{\mu}_\varepsilon(x), \quad \forall \phi \in C^0_c(G_1(\Omega)), \tag{3.4}$$

  where $P^{u_\varepsilon}(x)$ denotes the projection on the tangent space to the level line of u_ε passing through x (see (2.7)).

The next result has been proved in [31, 36]

Theorem 3.1. *Let $\{u_\varepsilon\} \subset C^2(\Omega)$ be a sequence such that*

$$\sup_{0<\varepsilon} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) = \sup_{0<\varepsilon} \mathcal{P}_\varepsilon(u_\varepsilon) + \mathcal{W}_\varepsilon(u_\varepsilon) < +\infty. \tag{3.5}$$

There exists a subsequence (still denoted by $\{u_\varepsilon\}$) converging to $u = \mathbb{1}_E$ in $L^1(\Omega)$, where E is a finite perimeter set. Moreover

(A) *$\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0^+$ weakly* in Ω as Radon measures and μ verifies*

$$\mu \geq c_0 \mathcal{H}^1 \llcorner \partial E.$$

In addition

$$\lim_{\varepsilon \to 0^+} \int_\Omega |\xi_\varepsilon| \, dx = 0, \tag{3.6}$$

where

$$\xi_\varepsilon := \Big(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{W(u_\varepsilon)}{\varepsilon} \Big),$$

and hence

$$\mu = \lim_{\varepsilon \to 0^+} \mu^\varepsilon_{u_\varepsilon} = \lim_{\varepsilon \to 0^+} \widetilde{\mu}^\varepsilon_{u_\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{2W(u_\varepsilon)}{\varepsilon} \mathcal{L}^2 \llcorner \Omega \text{ as Radon measures.} \tag{3.7}$$

(B) *The sequence $\{V^\varepsilon_{u_\varepsilon}\}$ converges in the varifolds sense to an integral-rectifiable varifold $V = \mathbf{v}(M, \theta) \in \mathbf{IV}_1(\Omega)$ with L^2-bounded first variation, and such that $\mu_V = c_0^{-1} \mu$. Moreover the function θ assumes odd (respectively even) values on $\partial^* E$ (respectively $M \setminus \partial^* E$).*

(C) *For any $Y \in C^1_c(\Omega; \mathbb{R}^n)$ we have*

$$c_0 \lim_{\varepsilon \to 0^+} \delta V^\varepsilon_{u_\varepsilon}(Y) = \lim_{\varepsilon \to 0^+} \int_\Omega \Big(\frac{W'(u_\varepsilon)}{\varepsilon} - \varepsilon \Delta u_\varepsilon \Big) \nabla u_\varepsilon \cdot Y \, dx = - \int \mathbf{H}_V \cdot Y \, d\mu, \tag{3.8}$$

and

$$c_0 \mathcal{F}_2(V) = c_0 \int_\Omega |\mathbf{H}_V|^2 \, d\mu_V \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_\Omega \left(\varepsilon \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} \right)^2 \, dx. \tag{3.9}$$

As a straightforward consequence of (3.9), Remark 2.2, and [4] we obtain the following

Corollary 3.2. *For every $u = \mathbb{1}_E \in BV(\Omega, \{-1,1\})$ such that $\Omega \cap \partial E \in W^{2,2}$, we have*

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u) = c_0 \int_{\Omega\cap\partial E} [1 + |\mathbf{H}_{\partial E}|^2]\, d\mathcal{H}^1 = c_0 \mathcal{F}(u).$$

Next we recall some of the main results obtained in [3] concerning the Γ-convergence of the sequence $\mathcal{E}_\varepsilon := \mathcal{P}_\varepsilon + \mathcal{B}_\varepsilon$.

Theorem 3.3. *Let $\{u_\varepsilon\} \subset C^2(\Omega)$ be such that*

$$\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(u_\varepsilon) := \sup_{\varepsilon>0} \mathcal{P}_\varepsilon(u_\varepsilon) + \mathcal{B}_\varepsilon(u_\varepsilon) < +\infty. \tag{3.10}$$

Then there exists a subsequence (still denoted by $\{u_\varepsilon\}$) converging to $u = \mathbb{1}_E$ in $L^1(\Omega)$, where E is a finite perimeter set. Moreover

(A1) *$\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0^+$ weakly* in Ω as Radon measures and μ verifies*

$$\mu \geq c_0 \mathcal{H}^1 \llcorner \partial E.$$

In addition

$$\lim_{\varepsilon\to 0^+} \nabla\xi_\varepsilon\, \mathcal{L}^2_{\llcorner\Omega} = 0 \text{ as Radon measures}, \quad \lim_{\varepsilon\to 0} \|\xi_\varepsilon\|_{L^p(\Omega)}, \text{ for every } 1 < p < 2,$$

and (3.7) holds.

(B1) *The sequence $\{V^\varepsilon_{u_\varepsilon}\}$ converges to a varifold $V = \mathbf{v}(M,\theta) \in \mathscr{CV}^2_1(\Omega)$, such that: $\mu_V = c_0^{-1}\mu$; and such that the function θ assumes odd (respectively even) values on $\partial^* E$ (respectively $M \setminus \partial^* E$).*

(C1) *Let A^u_{ijk} $(i,j,k = 1,2)$ be as in (2.9). For every $\phi \in C^1_c(G_1(\Omega);\mathbb{R}^2)$ we have*

$$\lim_{\varepsilon\to 0^+} \int_\Omega A^{u_\varepsilon}_{ijk}(x,S)\phi(x,S)\, dV^\varepsilon_{u_\varepsilon} = \int A^V_{ijk}(x,S)\phi(x,S)\, dV(x,S), \tag{3.11}$$

for every $i,j,k = 1,2$. Moreover

$$c_0\mathcal{F}_2(V) = c_0 \int |\mathbf{B}_V|^2\, dV \leq \liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_\Omega \left| \varepsilon\nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} \nu_{u_\varepsilon} \otimes \nu_{u_\varepsilon} \right|^2 dx. \tag{3.12}$$

Remark 3.4. We notice that, in view of (1.6), the main assumption of Theorem 3.3, that is (3.10), is stronger than the main assumption of Theorem 3.1, that is (3.5). However also the conclusions of Theorem 3.3 are stronger than those of Theorem 3.1. In fact, in Theorem 3.3-(A1) the convergence to zero of the discrepancies ξ_ε is proved to hold with respect to a topology that is stronger than the one with respect to which the vanishing of the discrepancies is obtained in Theorem 3.1-(A). Moreover in Theorem 3.3-(B1) the limit varifold V belongs to the set $\mathscr{CV}^2_1(\Omega)$, which is strictly contained in the set of varifolds $V \in \mathbf{IV}_1(\Omega)$ having L^2-bounded first variation (see Remark 2.6).

Eventually we notice that, by a straightforward adaptation of the proof of Corollary 3.2 and [3, Theorem 4.2], we obtain

Corollary 3.5. *For every $u = \mathbb{1}_E \in BV(\Omega, \{-1,1\})$ such that $\Omega \cap \partial E \in W^{2,2}$, we have*

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u) = c_0 \int_{\partial E} [1 + |\mathbf{B}_{\partial E}|^2]\, d\mathcal{H}^1 = c_0 \int_{\partial E} [1 + |\mathbf{H}_{\partial E}|^2]\, d\mathcal{H}^1 = c_0\mathcal{F}(u).$$

4. Main Results

The first of our main results shows that every varifold $V = \mathbf{v}(M,\theta) \in \mathscr{CV}^2_1(\Omega)$ arising as the limit of diffuse interface varifolds veryfing (3.10) (see Theorem 3.3-(B1)) is more regular than a generic element of $\mathscr{CV}^2_1(\Omega)$. In fact we show that M has an unique tangent line at *every* point $p \in M \cap \Omega$ (consequently M can be represented, locally and up to rigid motions, as the finite union of the graphs of $W^{2,2}$-functions, see Remark 2.11).

Theorem 4.1. *Let* $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ *satisfy* (3.10). *Let* $V^\varepsilon_{u_\varepsilon}$ *be as in* (3.4) *and suppose* $\lim_{\varepsilon\to 0} V^\varepsilon_{u_\varepsilon} = V = \mathbf{v}(M,\theta) \in \mathscr{CV}^2_1(\Omega)$. *Then* M *has an unique tangent line in every* $p \in M \cap \Omega$.

As a consequence of Theorem 4.1 we obtain the following full $\Gamma(L^1)$-convergence result

Corollary 4.2. *Let*

$$X := \{u \in C^2(\Omega) : \ u(x) \equiv 1, \, \partial_{\nu_\Omega} u(x) = 0, \, \forall x \in \partial\Omega\}.$$

Define (with a small abuse of notation)

$$\mathcal{E}_{\varepsilon \llcorner X} : L^1(\Omega) \to [0,+\infty], \quad u \mapsto \begin{cases} \mathcal{P}_\varepsilon(u) + \mathcal{B}_\varepsilon(u) & \textit{if } u \in X, \\ +\infty & \textit{otherwise on } L^1(\Omega) \end{cases}.$$

Then $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_{\varepsilon \llcorner X} = \overline{\mathcal{F}}$, *where* $\overline{\mathcal{F}}$ *is as in* (2.18).

Remark 4.3. We remark that from the proof of Corollary 4.2 it follows that the Γ-limit of the sequence $\{\mathcal{E}_{\varepsilon \llcorner X}\}_\varepsilon$ with respect to the varifold convergence of $V^\varepsilon_{u_\varepsilon}$ is given by the functional

$$V \mapsto \begin{cases} \mathcal{F}_2(V) & \text{if } V \in \mathscr{D}(\Omega), \\ +\infty & \text{otherwise on } \mathscr{CV}^2_1(\Omega), \end{cases}$$

where $\mathcal{F}_2$ has been defined in (2.14).

Remark 4.4. In Corollary 4.2 we need to introduce the space X in order to constrain the "diffuse interfaces"

$$\Sigma_{\varepsilon,\delta} := \{x \in \Omega : \ |u_\varepsilon| < 1 - \delta\},$$

to be compactly contained in Ω for every ε and δ positive. This fact, together with the results of Theorem 3.3, enables us to conclude that the measure $\mu_V = \theta\mathcal{H}^1_{\llcorner M}$ can be approximated by a sequence obtained restrcting the $\mathcal{H}^1$-measure to the boundaries of a sequence of open subsets compactly contained in Ω, and in turn to apply Theorem 2.13. Proving a full Γ-convergence result when the functionals $\{\mathcal{E}_\varepsilon\}_\varepsilon$ are defined on a more general functions' space that allows the diffuse interfaces $\Sigma_{\varepsilon,\delta}$ to hit the boundary, seems to be merely a technical point that can be solved by proving that the "conjecture" stated in Remark 2.10 is true.

Eventualy we also obtain some results concerning the *Gamma*-limit of the sequence of functionals $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$, and its relation with $\overline{\mathcal{F}}$. More precisely in Theorem 4.5, as a quite direct consequence of the results proved in [10] (see also [6, 12]), we prove the following

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^2$. There exists a sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ such that*

$$\begin{gathered} L^1(\Omega) - \lim_{\varepsilon\to 0} u_\varepsilon = u = \mathbb{1}_E \in BV(\Omega, \{-1,1\}) \textit{ for some } E \neq \emptyset; \\ \lim_{\varepsilon\to 0} V^\varepsilon_{u_\varepsilon} = V = \mathbf{v}(\partial E, 1) \in \mathscr{C}\mathscr{V}^2_1(\Omega), \\ \sup_{\varepsilon>0} \mathcal{P}_\varepsilon(u_\varepsilon) < +\infty, \quad \mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0, \end{gathered}$$

and such that $\partial E = \operatorname{spt}(\mu_V)$ does not have an unique tangent line in every point.
Moreover

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) = \mathcal{F}_2(V) < \overline{\mathcal{F}}_0(u) = \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u) = +\infty. \tag{4.1}$$

Remark 4.6. Although we are not able to identify the Γ-limit of the sequence $\{\widetilde{\mathcal{E}}_\varepsilon\}_\varepsilon$, we believe that for any given varifold $V \in \mathscr{C}\mathscr{V}^2_1(\Omega)$, combining the results of [18] with those of [12] and [4], it is possible to construct (with some additional work) a sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ such that

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) = \mathcal{F}_2(V).$$

5. Preliminary Lemmata

In order to prove Theorem 4.1 we need the following Lemmata.

Lemma 5.1. *Let $\{V_k := \mathbf{v}(M_k, 1)\}_k \subset \mathbf{IV}_1(B_{2R})$. Suppose that $M_k \cap B_{2R}$ are smooth C^2-embedded 1-manifolds without boundary in B_{2R}, and*

$$\begin{gathered} 0 < \liminf_{k\to\infty} \mu_{V_k}(B_R) \leq \limsup_{k\to\infty} \mu_{V_k}(B_{2R}) = K < +\infty, \\ \lim_{k\to\infty} |\delta V_k|(B_{2R}) = \lim_{k\to\infty} \int_{M_k} |\mathbf{H}_{M_k}|\, d\mathcal{H}^1 = 0. \end{gathered} \tag{5.1}$$

There exist a finite collection of 1-dimensional affine subspaces $T_1, \ldots, T_N$ of $\mathbb{R}^2$ such that

$$T_i \cap T_j \cap B_R = \emptyset, \quad \textit{for } i \neq j,\ i,j \in \{1,\ldots,N\}, \tag{5.2}$$

and a subsequence (not relabelled) $\{V_k\}_k \subset \mathbf{IV}_1(B_{2R})$ such that

$$\lim_{k\to\infty} V_k(\phi) = \sum_{j=1}^N \Theta_j \int_{T_j} \phi(x, T_j)\, d\mathcal{H}^1 =: V(\psi), \quad \forall \psi \in C^0_c(G_1(B_R)), \tag{5.3}$$

where $\Theta_j \in \mathbb{N}$ are constants.

Proof. By (5.1) we can apply Allard's compactness Theorem (see [35, Theorem 42.7]), and extract a subsequence such that $V_k \to V$, where $V \in \mathbf{IV}_1(B_{2R})$ is stationary in B_{2R}, and $\mu_V(B_R) > 0$.

Next we claim that (up to subsequences):

(i) there are no closed curves between the connected components of $M_k \cap B_{2R}$;
(ii) the connected components $M_k \cap B_{2R}$ intersecting $B_{3R/2}$ are in a fixed number.

In fact, suppose that along a subsequence $\{M_{k'}\}_{k'}$ we can find a closed curve $\widetilde{M}_{k'} \subset M_{k'}$ such that $\widetilde{M}_{k'} \subset\subset B_R$ for every $k' \in \mathbb{N}$. Then

$$\begin{aligned}|\delta V_{k'}|(B_{2R}) = \int_{M_{k'}\cap B_{2R}} |\mathbf{H}_{M_{k'}}|\, d\mathcal{H}^1 \geq \int_{\widetilde{M}_{k'}} |\mathbf{H}_{\widetilde{M}_{k'}}|\, d\mathcal{H}^1 \\ \geq \Big| \int_{\widetilde{M}_{k'}} \mathbf{H}_{\widetilde{M}_{k'}}\, d\mathcal{H}^1 \Big| = 2\pi,\end{aligned}$$

which is in contradiction with (5.1). Hence (i) holds.

Let us now prove (ii). Any C^2-embedded, non-closed curve without boundary in B_{2R} intersecting $B_{3R/2}$ has a length of at least $R/2$. Hence the number of connected components of M_k such that $M_k \cap B_R \neq \emptyset$ is smaller or equal than $2K/R$. Therefore, possibly passing to a further subsequence, we can suppose that the number of connected components of $M_k \cap B_{3R/2}$ equals a certain $N \in \mathbb{N}$ for every $k \in \mathbb{N}$.

In view of the results establised above and the assumption (5.1), we can find a constant $C > 0$, a collection of N intervals $I_j \subset \mathbb{R}$ $j = 1, \ldots, N$ and N sequence of maps $\{\alpha_{j,k}\}_{k\in\mathbb{N}} \subset C^2(I_j, B_{3R/2})$ such that, for $j = 1, \ldots, N$ and $k \in \mathbb{N}$, we have

$$C < |\dot{\alpha}_{j,k}| = const. \text{ on } I_j, \quad M_k \cap B_R = \bigcup_{j=1}^{N} (\alpha_{j,k})(I_j) \cap B_R.$$

Since

$$\lim_{k\to\infty} |\delta V_k(B_R)| = \lim_{k\to\infty} \sum_{j=1}^{N} \frac{1}{l(\alpha_{j,k})} \int |\ddot{\alpha}_{j,k}|\, dt = 0,$$

we have (up to the extraction of a further subsequence) $\alpha_{j,k} \to \alpha_j$ strongly in $W^{2,1}(I_j)$, for every $j = 1, \ldots, N$. Moreover by

$$\sup_{s,t\in I_j} |\dot{\alpha}_{j,k}(s) - \dot{\alpha}_{j,k}(t)| \leq \sup_{s,t\in I_j} \int_s^t |\ddot{\alpha}_{j,k}(\tau)|\, d\tau \leq \int_{I_j} |\ddot{\alpha}_{j,k}(\tau)|\, d\tau \to 0,$$

we also have (again up to a subsequence) $\alpha_{j,k} \to \alpha_j$ uniformly and $\ddot{\alpha}_j \equiv 0$ on I_j. Therefore $\alpha_j \in C^1(I_j)$, being $\dot{\alpha}_j$ constant on I_j for every $j = 1, \ldots, N$. By

$$\begin{aligned} V(\phi) &= \lim_{k\to\infty} V_k(\phi) = \int_{M_k} \phi(x, T_x M_k)\, d\mathcal{H}^1 \\ &= \lim_{k\to\infty} \sum_{j=1}^{N} \int_{I_j} \phi\Big(\alpha_{j,k}(s), Id - \frac{\dot{\alpha}_{j,k}(s) \otimes \dot{\alpha}_{j,k}(s)}{|\dot{\alpha}_{j,k}(s)|^2}\Big) |\dot{\alpha}_{j,k}(s)|\, ds \\ &= \sum_{j=1}^{N} \int_{I_j} \phi\Big(\alpha_j(s), Id - \frac{\dot{\alpha}_j(s) \otimes \dot{\alpha}_j(s)}{|\dot{\alpha}_j(s)|^2}\Big) |\dot{\alpha}_j(s)|\, ds \end{aligned}$$

we conclude that (5.3) holds.

In order to prove (5.2) we proceed by contradiction. Suppose, without loss of generality, that $T_1 \neq T_2$, and $T_1 \cap B_R$, $T_2 \cap B_R \subset \operatorname{spt}(\mu_V \cap B_R)$, and $T_1 \cap T_2 \cap B_R \neq \emptyset$. We can find $\alpha_{j_l,k} \in C^2(I_{j_l})$, parametrizing a connected components of $M_k \cap B_{3R/2}$, uniformly convergent to a constant speed paramatrization $\alpha_{j_l} \in C^1(I_{J_l})$ of $T_l \cap B_R$ $(l = 1, 2)$. Since $|\dot{\alpha}_{j_l,k}|$ is constant for $l = 1, 2$ and every $k \in \mathbb{N}$, by the uniform convergence of $\alpha_{j_1,k}$, $\alpha_{j_2,k}$ and by $T_1 \cap T_2 \cap B_R \neq \emptyset$ we can conclude that $\alpha_{j_1,k}(I_{j_1}) \cap \alpha_{j_2,k}(I_{j_2}) \neq \emptyset$ for every k big enough. But this contradicts the embededdness assumption made on M_k. Hence (5.2) holds too, and the proof is complete. $\square$

Lemma 5.2. *Let $\widetilde{u}_\varepsilon \in C^2(B_{2R})$ be such that*

$$0 < \liminf_{\varepsilon\to 0} \int_{B_{2R}} \frac{\varepsilon}{2}|\nabla \widetilde{u}_\varepsilon|^2 + \frac{W(\widetilde{u}_\varepsilon)}{\varepsilon}\, dx \le \limsup_{\varepsilon\to 0} \int_{B_{2R}} \frac{\varepsilon}{2}|\nabla \widetilde{u}_\varepsilon|^2 + \frac{W(\widetilde{u}_\varepsilon)}{\varepsilon}\, dx < +\infty, \tag{5.4}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{B_{2R}} \left| \varepsilon \nabla^2 \widetilde{u}_\varepsilon - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} \nu_{\widetilde{u}_\varepsilon} \otimes \nu_{\widetilde{u}_\varepsilon} \right|^2 dx = 0. \tag{5.5}$$

Then, being $V^\varepsilon_{\widetilde{u}_\varepsilon}$ as in (3.4)*, up to a subsequence we have*

$$\lim_{\varepsilon\to 0} V^\varepsilon_{\widetilde{u}_\varepsilon} = \widetilde{V} \ \textit{as varifolds in } \Omega,$$

where $\widetilde{V} \in \mathbf{IV}_1(B_{2R})$ is stationary and verifies (5.3) *and* (5.2)*.*

Proof. We begin by selecting a subsequence (not relabeled) such that

$$0 < \lim_{\varepsilon\to 0} \int_{B_{2R}} \frac{\varepsilon}{2}|\nabla \widetilde{u}_\varepsilon|^2 + \frac{W(\widetilde{u}_\varepsilon)}{\varepsilon}\, dx < +\infty.$$

We fix Ω' such that $B_R \subset\subset \Omega' \subset\subset B_{2R}$. By Sard's Lemma and [3, Lemma 7.1] we can find a subsequence $\{V^{\varepsilon_k}_{\widetilde{u}_{\varepsilon_k}}\}_k$ and a subset $J \subset [-1,1]$, with $\mathcal{L}^1(J) = 0$, such that for every $s \in [-1,1] \setminus J$,

$$\begin{gathered} \{\widetilde{u}_{\varepsilon_k} = s\} \text{ is a smooth embedded surface without boundary in } \Omega' \\ \{\widetilde{u}_{\varepsilon_k} = s\} \cap \{\nabla \widetilde{u}_{\varepsilon_k} = 0\} = \emptyset, \\ \lim_{k\to\infty} \mathbf{v}(\{\widetilde{u}_{\varepsilon_k} = s\}, 1) = \widetilde{V} \ \text{ as varifolds on } \Omega'. \end{gathered}$$

For every $x \in \Omega'$ such that $\widetilde{u}_{\varepsilon_k}(x) = s \in [-1,1] \setminus J$ we set

$$\mathbf{B}_{\widetilde{u}_{\varepsilon_k}} := \frac{(P^{\widetilde{u}_{\varepsilon_k}})^T \nabla^2 \widetilde{u}_{\varepsilon_k} P^{\widetilde{u}_{\varepsilon_k}}}{|\nabla \widetilde{u}_{\varepsilon_k}|} \otimes \nu_{\widetilde{u}_{\varepsilon_k}},$$

that is $\mathbf{B}_{\widetilde{u}_{\varepsilon_k}}(x)$ is the second fundamental form of $\{\widetilde{u}_{\varepsilon_k} = s\}$ at the point x (see (2.8)). Let us also recall that (see [3, Lemma 5.3])

$$|\mathbf{B}_{\widetilde{u}_{\varepsilon_k}}|\, \varepsilon_k |\nabla \widetilde{u}_{\varepsilon_k}| \le \left| \varepsilon \nabla^2 \widetilde{u}_\varepsilon - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} \nu_{\widetilde{u}_\varepsilon} \otimes \nu_{\widetilde{u}_\varepsilon} \right|. \tag{5.6}$$

Next we fix $\delta > 0$ and set $I_\delta := [-1+\delta, 1-\delta]$. By (5.6) we have

$$\begin{aligned} &\int_{I_\delta \setminus J} |\delta \mathbf{v}(\{\widetilde{u}_{\varepsilon_k} = s\}, 1)|\,(\Omega')\, ds = \int_{I_\delta \setminus J} \int_{\{\widetilde{u}_{\varepsilon_k} = s\}\cap\Omega'} \left|\operatorname{div}\left(\nu_{\widetilde{u}_{\varepsilon_k}}\right)\right| d\mathcal{H}^1\, ds \\ \le &\frac{1}{(2\delta - \delta^2)} \int_{\Omega'} \left|\operatorname{div}\left(\nu_{\widetilde{u}_{\varepsilon_k}}\right)\right| \sqrt{2W(\widetilde{u}_{\varepsilon_k})}|\nabla \widetilde{u}_{\varepsilon_k}|\, dx = \frac{2}{(2\delta - \delta^2)} \int_{\Omega'} |\mathbf{B}_{\widetilde{u}_{\varepsilon_k}}| \sqrt{2W(\widetilde{u}_{\varepsilon_k})}|\nabla \widetilde{u}_{\varepsilon_k}|\, dx \\ \le &\frac{2}{(2\delta - \delta^2)} \left(\int_{B_{2R}} |\mathbf{B}_{\widetilde{u}_{\varepsilon_k}}|^2\, d\widetilde{\mu}^\varepsilon_{\widetilde{u}_{\varepsilon_k}} \right)^{1/2} \left(\int_{B_{2R}} \frac{W(\widetilde{u}_{\varepsilon_k})}{\varepsilon_k}\, dx \right)^{1/2} \\ \le &\frac{2}{(2\delta - \delta^2)} \left(\frac{1}{\varepsilon_k} \int_{B_{2R}} \left| \varepsilon \nabla^2 \widetilde{u}_\varepsilon - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} \nu_{\widetilde{u}_\varepsilon} \otimes \nu_{\widetilde{u}_\varepsilon} \right|^2 dx \right)^{1/2} \left(\int_{B_{2R}} \frac{W(\widetilde{u}_{\varepsilon_k})}{\varepsilon_k}\, dx \right)^{1/2}. \end{aligned}$$

By the choice of ε_k and of the set J, and by (5.5), we can conclude that there exists $s_{\varepsilon_k} \in I_\delta \setminus J$ such that, setting $V_k := \mathbf{v}(\{\widetilde{u}_{\varepsilon_k} = s_{\varepsilon_k}\} \cap \Omega', 1)$, we have

$$\limsup_{k\to\infty} \mu_{V_k}(\Omega') < +\infty,$$
$$\limsup_{k\to\infty} \Big|\delta V_k\Big|(\Omega') = 0.$$

therefore we are in a position to apply Lemma 5.1 to the sequence $\{V_k\}_k \subset \mathbf{IV}_1(B_2R)$. Hence we can conclude the proof by [3, Theorem 4.1] and

$$\lim_{k\to\infty} \mathbf{v}(\{\widetilde{u}_{\varepsilon_k} = s_{\varepsilon_k}\}, 1) = \widetilde{V} \;\text{ as varifolds on } \Omega'.$$

□

6. Proof of Theorem 4.1

Since V is a Hutchinson's varifold with square-integrable second fundamental form by Remark 2.6 the conclusions of Remark 2.3 hold.

For $x \in \mathbb{R}^2$ and $\lambda > 0$ we define

$$\eta_{x,\lambda} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad y \mapsto \frac{y-x}{\lambda}.$$

and consider, for $x \in \operatorname{spt}(\mu_V)$, the Radon measure

$$\widetilde{\mu}_{x,\rho}(\psi) := \frac{1}{\rho}\int_{\eta_{x,\rho}(M)} \psi(y)\,\theta(\rho y + x) d\mathcal{H}^1(y), \quad \forall \psi \in C^0_c(\mathbb{R}^2).$$

By [18, Theorem 3.4] (see also [20]) we can conclude that for *every* $x \in \operatorname{spt}(\mu_V)$ there exists a Radon measure $\widetilde{\mu}_x$ on $\mathbb{R}^2$ such that

$$\lim_{\rho\to 0^+} \widetilde{\mu}_{x,\rho}(\psi) = \widetilde{\mu}_x(\psi), \quad \forall \psi \in C^0_c(\Omega),$$

and moreover that the measure $\widetilde{\mu}_x$ satisfies

$$\widetilde{\mu}_x = \sum_{i=1}^{N_x} \Theta_i(x) \mathcal{H}^1 \llcorner_{\tilde{T}_i(x)},$$

where $N_x \in \mathbb{N}$, $\tilde{T}_1(x), \ldots, \tilde{T}_{N_x}(x) \in G_{1,2}$, and $\Theta_1(x), \ldots, \Theta_{N_x}(x) \in \mathbb{N}$. In order to prove the existence of an unique tangent line in every point of $\operatorname{spt}(\mu_V)$ we show that $N_x = 1$ for every $x \in \operatorname{spt}(\mu_V)$.
Without loss of generality we suppose that $x = 0$. In view of the Hutchinson's regularity result cited above, to conclude that $N_0 = 1$ it is enough to prove that for every sequence $\{\rho_k\}_k \subset \mathbb{R}^+$ such that $\lim_{k\to\infty} \rho_k = 0$, setting

$$\widetilde{\mu}_k(\psi) := \frac{1}{\rho_k}\int_{\eta_{0,\rho_k}(M)} \psi(y)\,\theta(\rho_k y) d\mathcal{H}^1(y), \quad \forall \psi \in C^0_c(\mathbb{R}^2),$$

we have

$$\widetilde{\mu}(\psi) = \Theta^1(\mu, 0) \int_T \psi(y)\, d\mathcal{H}^1, \tag{6.1}$$

where $T \in G_{1,2}$ is a linear 1-dimensional subspace of $\mathbb{R}^2$.

Since $\widetilde{\mu}_k \to \widetilde{\mu}$ as Radon-measures on $\mathbb{R}^2$ and $\mu_{\varepsilon_k} \to \mu_V$ as Radon measures in Ω, for every open bounded subset $U \subset \mathbb{R}^2$, we can find a sequence $\{\varepsilon_k\}_k$ such that

$$\lim_{k\to\infty} \varepsilon_k = \lim_{k\to\infty} \frac{\varepsilon_k}{\rho_k} = 0,$$

and such that, setting $\widetilde{u}_k(y) := u_{\varepsilon_k}(\rho_k y)$, $\widetilde{\varepsilon}_k := \varepsilon_k/\rho_k$, the following hold

$$\int_B \frac{\widetilde{\varepsilon}_k}{2}|\nabla \widetilde{u}_k|^2 + \frac{W(\widetilde{u}_k)}{\widetilde{\varepsilon}_k}\,dx = \frac{\mu_{\varepsilon_k}(\rho_k B)}{\rho_k} \to \widetilde{\mu}(B),\ \forall B \subset\subset U \text{ Borel},$$
$$0 < \lim_{k\to\infty}\int_U \widetilde{\varepsilon}_k|\nabla \widetilde{u}_k|^2 + \frac{W(\widetilde{u}_k)}{\widetilde{\varepsilon}_k}\,dx < +\infty.$$

Moreover by the definition of $\widetilde{u}_k$ and $\widetilde{\varepsilon}_k$ and (2.14), we have

$$\begin{aligned}
&\frac{1}{\widetilde{\varepsilon}_k}\int_U \left|\widetilde{\varepsilon}_k \nabla^2 \widetilde{u}_k - \frac{W'(\widetilde{u}_k)}{\widetilde{\varepsilon}_k}\nu_{\widetilde{u}_k}\otimes\nu_{\widetilde{u}_k}\right|^2 dy \\
&= \frac{\rho_k}{\varepsilon_k}\int_U \left|\varepsilon_k\nabla^2 u_{\varepsilon_k}(\rho_k y) - \frac{W'(u_{\varepsilon_k}(\rho_k y))}{\varepsilon_k}\frac{\nabla u_{\varepsilon_k}(\rho_k y)\otimes\nabla u_{\varepsilon_k}(\rho_k y)}{|\nabla u_{\varepsilon_k}(\rho_k y)|^2}\right|^2 \rho_k^2\,dy \qquad (6.2)\\
&= \frac{\rho_k}{\varepsilon_k}\int_{\rho_k U}\left|\varepsilon_k\nabla^2 u_{\varepsilon_k}(x) - \frac{W'(u_{\varepsilon_k}(x))}{\varepsilon_k}\nu_{u_{\varepsilon_k}}(x)\otimes\nu_{u_{\varepsilon_k}}(x)\right|^2 dx \le C\rho_k.
\end{aligned}$$

We can thus apply Lemma 5.2 and extract a sequence (not relabelled) such that

$$V_{\widetilde{u}_k}^{\widetilde{\varepsilon}_k} \to \widetilde{V} = \sum_{j=1}^N \Theta_j \mathcal{H}^1\llcorner_{T_j\cap U} \text{ as varifolds in } U,$$
$$\mu_{\widetilde{V}} = \widetilde{\mu}\llcorner_U,$$

and $T_i \cap T_j \cap B_R = \emptyset$ for every $B_{2R} \subset\subset U$.

However since $\widetilde{\mu}$ verifies

$$\widetilde{\mu}(B_R) = R\lim_{k\to\infty}\frac{\widetilde{\mu}_k(B_{\rho_k R})}{R\rho_k} = R\,\Theta(\mu_V, 0) =: R\,\Theta_0,$$

we have $\widetilde{V} = \Theta_0\mathcal{H}^1\llcorner_{T\cap U}$ that is (6.1). □

7. Proof of Corollary 4.2

We begin proving the so-called $\Gamma-\liminf$-inequality. We suppose that $\{u_\varepsilon\}_\varepsilon \subset X$ satisfies (3.10) (otherwise we have nothing to prove). By Theorem 3.3 we can find a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\varepsilon_k = 0$ and

$$\lim_{k\to\infty}\mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}) = \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon),$$
$$L^1(\Omega) - \lim_{\varepsilon\to 0} u_{\varepsilon_k} \to u = \mathbb{1}_E \in BV(\Omega, \{-1,1\}),$$
$$\lim_{k\to\infty} V_{u_{\varepsilon_k}}^{\varepsilon_k} = V \in \mathscr{CV}_1^2(\Omega) \text{ as varifolds.}$$

If we prove that $V \in \mathscr{D}$, by Theorem 2.13 and Theorem 3.3-(C1), we obtain

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon) = \lim_{k\to\infty}\mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}) \ge c_0\int(1+|\mathbf{H}_V|^2)\,d\mu_V = \mathcal{F}_2(V) \ge c_0\overline{\mathcal{F}}(E).$$

That is the $\Gamma-\liminf$ inequality holds. In order to prove that $V = \mathbf{v}(M,\theta) \in \mathscr{D}(\Omega)$, by (2.15), it is enough to show that: (i) $\mathbf{v}(M,\theta)$ is actually a Hutchinson's curvature varifold in the whole of $\mathbb{R}^2$; (ii) $M\cup\partial\Omega$ has an unique tangent-line in *every* point.

We begin establishing that (i) holds. To this purpose we fix $\Omega_1 \subset\subset \mathbb{R}^2$ such that $\Omega_1 \supset\supset \Omega$, and define $\Omega_\delta := \{x \in \Omega_1\setminus\Omega :\ \operatorname{dist}(x,\Omega) < \delta\}$ for $\delta > 0$. Next we notice that, since $\{u_\varepsilon\}_\varepsilon \subset X$, we can extend u_ε to $u'_\varepsilon \in W^{2,2}(\Omega_1)$ simply setting

$u'_\varepsilon \equiv 1$ on $\Omega_1 \setminus \overline{\Omega}$. Since u'_ε satisfies (3.10) on Ω_1, by Theorem 3.3 we can extract a subsequence such that $\lim_{\varepsilon\to 0} V^\varepsilon_{u'_\varepsilon} = V' \in \mathscr{CV}^2_1(\Omega_1)$. However, since

$$\int_{\Omega'} \frac{\varepsilon}{2}|\nabla u'_\varepsilon|^2 + \frac{W(u'_\varepsilon)}{\varepsilon}\,dx = \int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\,dx,$$

we can conclude that $\mathrm{spt}(\mu_{V'}) \subset \overline{\Omega}$. Hence we obtain that $V = V'$ is a Hutchinson varifold in Ω_1 whose support is compactly contained in Ω_1, and therefore $\mathbf{v}(M,\theta) \in \mathscr{CV}^2_1(\mathbb{R}^2)$.

We now pass to prove (ii). By [3, Theorem 4.2], we can find an infinitesimal, strictly decreasing sequence $\{\delta_\varepsilon\}_\varepsilon \subset \mathbb{R}^+$, and a sequence $g_\varepsilon \in C^2(\Omega_1 \setminus \overline{\Omega})$ such that

$$g_\varepsilon \equiv -1 \text{ on } \Omega_1 \setminus \Omega_{\delta_\varepsilon}, \quad g_\varepsilon \equiv 1 \text{on } \Omega_{2\varepsilon} \setminus \Omega,$$
$$\lim_{\varepsilon\to 0} V^\varepsilon_{g_\varepsilon} = \mathbf{v}(\partial\Omega, 1) \text{ as varifolds }, \quad L^1(\Omega_1 \setminus \overline{\Omega}) - \lim_{\varepsilon\to 0} g_\varepsilon \equiv -1,$$
$$\int_{\Omega_1\setminus\overline{\Omega}} \frac{\varepsilon}{2}|\nabla g_\varepsilon|^2 + \frac{W(g_\varepsilon)}{\varepsilon}\,dx = c_0\mathcal{H}^1(\partial\Omega) + O(\varepsilon),$$
$$\frac{1}{\varepsilon}\int_{\Omega_1\setminus\overline{\Omega}} \left|\varepsilon\nabla^2 g_\varepsilon - \frac{W'(g_\varepsilon)}{\varepsilon}\nu_{g_\varepsilon}\otimes\nu_{g_\varepsilon}\right|^2 dx = \int_{\partial\Omega} |\mathbf{B}_{\partial\Omega}|^2\,d\mathcal{H}^1 + O(\varepsilon).$$

Hence, again by the assumption $\{u_\varepsilon\} \subset X$, we can conclude that setting

$$u''_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in \Omega, \\ g_\varepsilon & \text{if } x \in \Omega_1 \setminus \overline{\Omega}, \end{cases}$$

the sequence $\{u''_\varepsilon\}_\varepsilon \subset W^{2,2}(\Omega_1)$ satisfies (3.10), and moreover (up to subsequences) as $\varepsilon \to 0$, we have

$$V^\varepsilon_{u''_\varepsilon} \to \mathbf{v}(M,\theta) + \mathbf{v}(\partial\Omega, 1) \in \mathscr{CV}^2_1(\mathbb{R}^2) \text{ as varifolds.}$$

Applying Theorem 4.1 to the sequence $\{u''_\varepsilon\}_\varepsilon$ we obtain that $M \cup \partial\Omega$ has an unique tangent line in every point. Hence $V \in \mathscr{D}(\Omega)$ and the $\Gamma - \liminf$ inequality holds.

Finally, the $\Gamma - \limsup$ inequality now follows by [3, Theorem 4.2] and a standard density argument. In fact, by the previous step we can conclude that for every $u = \mathbb{1}_E \in L^1(\Omega)$ such that $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_{\varepsilon,\llcorner X}(u) < +\infty$ we also have $\overline{\mathcal{F}}(u) < +\infty$, and therefore we can find a sequence $\{E_h\}_h$ such that $E_h \subset\subset \Omega$, and $\Omega \cap \partial E_h \in C^2$, and

$$L^1(\Omega) - \lim_{h\to\infty} \mathbb{1}_{E_h} = u, \quad \lim_{h\to\infty} \mathcal{F}(\mathbb{1}_{E_h}) = \overline{\mathcal{F}}(u).$$

□

8. Proof of Theorem 4.5

Without loss of generality we suppose that $\Omega = B_1$. In order to prove Theorem 4.5 we begin by showing the existence of a sequence $\{u_\varepsilon\}_\varepsilon \subset C^3(\Omega)$ such that

$$\varepsilon\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} = 0, \qquad \forall\varepsilon > 0 \tag{8.1}$$

$$\sup_{\varepsilon>0}\int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\,dx \le C, \tag{8.2}$$

and

(a) $L^1(\Omega) - \lim_{\varepsilon\to 0} u_\varepsilon = u = \mathbb{1}_{E_0} \in BV(\Omega, \{-1,1\})$, where

$$E_0 = \{(x_1,x_2) \in \Omega : \; x_1 > 0,\, x_2 > 0\} \cup \{(x_1,x_2) \in \Omega : \; x_1 < 0,\, x_2 < 0\};$$

(b) $\lim_{\varepsilon\to 0} V_\varepsilon = V_{\mathfrak{C}} := \mathbf{v}(\mathfrak{C} \cap \Omega, \theta) \in \mathbf{IV}_1(\Omega)$ as varifolds, where

$$\mathfrak{C} := \{x := (x_1,x_2) \in \mathbb{R}^2 : \; x_1 \text{ vel } x_2 \text{ equals } 0\} = \partial E_0.$$

The fact that showing the existence of a sequence $\{u_\varepsilon\}_\varepsilon$ with the above properties is enough to conclude the proof of the first part of Theorem 4.5 is pretty easy to see. In fact, since $0 \in \mathfrak{C} = \mathrm{spt}(\mu_{V_{\mathfrak{C}}})$ and the tangent cone in 0 to $\mathfrak{C}$ coincides with $\mathfrak{C}$ itself , we have that $\mu_{V_{\mathfrak{C}}}$ can not have an uniquely defined tangent line in $0 \in \mathrm{spt}(\mu_{V_{\mathfrak{C}}}) \cap \Omega$.

We construct the sequence $\{u_\varepsilon\} \subset C^3(\Omega)$ verifying (8.1), (8.2) via the blow-down of a particular entire solution of the Allen-Cahn equation in the plane. More precisely, let $U \in C^3(\mathbb{R}^2)$ be a "saddle solution" of the Allen-Cahn equation, that is

$$\Delta U = W'(U) \qquad \text{on } \mathbb{R}^2, \tag{8.3}$$

and U is such that

- $\|U\|_{L^\infty(\mathbb{R}^2)} \le 1$, $\{U = 0\} = \mathfrak{C}$ and $U > 0$ (respectively $U < 0$) in the I and III (respectively II and IV) quadrant of $\mathbb{R}^2$;
- there exists $C > 0$ such that for every $R > 0$

$$\int_{B_R} \frac{1}{2}|\nabla U|^2 + W(U)\, dy \le C\, R. \tag{8.4}$$

The existence of such a solution has been proved in [6, Theorem 1.3] (see also [10, 12]).

We define $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ by $u_\varepsilon(x) := U(x/\varepsilon)$. By (8.3), (8.4) we then have

$$\varepsilon \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} = 0,$$
$$\int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\, dx = \varepsilon \int_{B_{\varepsilon^{-1}}} \frac{1}{2}|\nabla U|^2 + W(U)\, dy \le C,$$

that is (8.1) and (8.2) hold. Hence we are in a position to apply the results proved in [21], and obtain that

(HT1) (see [21, Proposition 2.2]) for every $r < 1$ there exists $c := c(r) > 0$ such that $\sup_{B_r} \xi_\varepsilon^+ \le c$ for every ε small enough ;

(HT2) (see [21, Proposition 3.4]) for every $x \in \Omega$, $0 < \sigma < \rho$ such that $B_\rho(x) \subset\subset B_r$ $(r < 1)$, and ε small enough we have

$$\frac{\mu_\varepsilon(B_\rho(x))}{\rho} \ge \frac{\mu_\varepsilon(B_\sigma(x))}{\sigma} - c\rho, \tag{8.5}$$

where $c = c(r)$ is defined in (HT1);

(HT3) (see [21, Theorem 1]) from the sequence $\{V_\varepsilon\}_\varepsilon$ (see (3.4)) we can extract a subsequence (not relabeled) such that

$$\lim_{\varepsilon\to 0} V_\varepsilon = V := \mathbf{v}(M, \theta) \text{ as varifolds in } \Omega,$$

and $V \in \mathbf{IV}_1(\Omega)$ is stationary.

Next we show that $\mu_V(\Omega) > 0$ and $M = \mathrm{spt}(\mu_V) = \mathfrak{C} \cap \Omega$.

Let $x_0 \in \mathfrak{C} \cap \Omega$. We choose ε small enough that $B_\varepsilon(x_0) \subset\subset B_{(1-|x_0|)/2} \subset\subset \Omega$, and define

$$\tilde{U}_\varepsilon \in C^2(B_1), \quad \tilde{U}_\varepsilon(z) := u_\varepsilon(\varepsilon z + x_0) = U(z + \varepsilon^{-1} x_0).$$

We then have

$$\Delta \tilde{U}_\varepsilon = W'(\tilde{U}_\varepsilon) \text{ in } B_1, \text{ and } \tilde{U}_\varepsilon(0) = U(\varepsilon^{-1} x_0) = 0.$$

Hence, by standard elliptic estimates, we have $\|\tilde{U}_\varepsilon\|_{C^1(B_{1/2})} < \tilde{C}$, where $\tilde{C} > 0$ is uniform with respect to ε, and therefore we can find $\delta > 0$ (independent of ε) such that $\sup_{z \in B_\delta} |\tilde{U}_\varepsilon(z)| < 1/2$. Hence

$$\begin{aligned} \frac{\mu_\varepsilon(B_{\delta\varepsilon}(x_0))}{\delta\varepsilon} &= \frac{1}{\delta\varepsilon} \int_{B_{\delta\varepsilon}(x_0)} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} \, dx \\ &= \int_{B_\delta} \frac{1}{2} |\nabla \tilde{U}_\varepsilon|^2 + W(\tilde{U}_\varepsilon) \, dz \geq \int_{B_\delta} W(\tilde{U}_\varepsilon) \, dz \geq C_W \end{aligned}$$

where $C_W := C_W(\delta) = \pi\delta^2 \min\{W(s) : s \in (-1/2, 1/2)\}$.

We now choose $\rho < \rho_0$ where ρ_0 is such that $C_W - c\rho_0 > C_W/2$. By (8.5) for every ε small enough we have

$$\frac{\mu_\varepsilon(B_\rho(x_0))}{\rho} \geq \frac{\mu_\varepsilon(B_{\delta\varepsilon}(x_0))}{\delta\varepsilon} - c\rho > \frac{C_W}{2},$$

from which we deduce $\mu_V(\Omega) > 0$ and $\mathrm{spt}(\mu_V) \supseteq \mathfrak{C} \cap \Omega$.

However, in view of [10, Lemma 5], we can find a constant $K > 0$, independent of ε, such that that for every $\eta \in (0, 1/2)$ there exists $\varepsilon_0 := \varepsilon_0(\eta)$ such that for $\varepsilon < \varepsilon_0$ we have

$$\{x = (x_1, x_2) \in \Omega : \ |x_1|, |x_2| > \varepsilon k\} \subset \{x \in \Omega : \ |u_\varepsilon(x)| \geq 1 - \eta\}.$$

By this latter estimate and [21, Proposition 5.1], we can conclude that

$$\lim_{\varepsilon \to 0} \mu_\varepsilon(\overline{A}) = 0, \quad \forall A \subset\subset \Omega \setminus \mathfrak{C}.$$

Hence $\mathrm{spt}(\mu_V) \subseteq \mathfrak{C}$ and this concludes the proof of the part of Theorem 4.5.

It remains to prove that (4.1) holds. To this aim it is enough to remark that, being $\{u_\varepsilon\}_\varepsilon$ and u as above, by Proposition 2.9 and Theorem 4.1 we have

$$\overline{\mathcal{F}}_o(u) = \Gamma(L^1(\Omega)) - \lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u) = +\infty.$$

□

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