The Weak Bruhat Order and Separable Permutations *

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Abstract In this paper we consider the rank generating function of a separable permutation π in the weak Bruhat order on the two intervals $[\mathrm{id}, \pi]$ and $[\pi, w_0]$, where $w_0 = n, (n-1), \ldots, 1$. We show a surprising result that the product of these two generating functions is the generating function for the symmetric group with the weak order. We then obtain explicit formulas for the rank generating functions on $[\mathrm{id}, \pi]$ and $[\pi, w_0]$, which leads to the rank-symmetry and unimodality of the two graded posets.

1 Introduction and Definitions

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $1, 2, \ldots, n$. Define the *length* of the permutation $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ by

$$\ell(\pi) = \#\{1 \le i < j \le n \colon a_i > a_j\},$$

which is the number of inversions of π. One of the fundamental partial orderings of $\mathfrak{S}_n$ is the *weak (Bruhat) order*. A cover relation $\pi \lessdot \sigma$ in weak order, i.e., $\pi < \sigma$ and nothing is in between, is defined by $\sigma = \pi s_i$ for some adjacent transposition $s_i = (i, i+1)$, provided that $\ell(\sigma) > \ell(\pi)$. We are multiplying permutations right-to-left, so for instance $2413 s_2 = 2143$. The weak order makes $\mathfrak{S}_n$ into a graded poset of rank $\binom{n}{2}$. If $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$,

*This research was carried out under the direction of R. Stanley when the author was an undergraduate at M.I.T.

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then the rank function of $\mathfrak{S}_n$ (which will have the weak order unless stated otherwise) is the function ℓ. The rank generating function is then given by

$$F(\mathfrak{S}_n, q) = \sum_{\pi \in \mathfrak{S}_n} q^{\ell(\pi)} = [n]!,$$

where $[n]! = [1][2]\cdots[n]$ and $[i] = 1 + q + q^2 + \cdots + q^{i-1}$.

A permutation $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is *3142-avoiding* and *2413-avoiding* if there do not exist $i < j < k < h$ with $a_j < a_h < a_i < a_k$ or $a_k < a_i < a_h < a_j$. Such permutations are also called *separable*. For a general introduction to pattern avoidance, see [4]. Separable permutations first arose in the work of Avis and Newborn [2] and have subsequently received a lot of attention. A survey of some of their properties appears in [1]. In particular, the number of separable permutations in $\mathfrak{S}_n$ is the (large) Schröder number r_{n-1}. Let id denote the identity element of $\mathfrak{S}_n$ (the unique minimal element in weak order), and let $w_0 = n, n-1, \ldots, 1$, the unique maximal element. For $\pi \in \mathfrak{S}_n$, let Λ_π denote the interval $[\mathrm{id}, \pi]$ (in weak order), and let $V_\pi = [\pi, w_0]$. Thus Λ_π and V_π are themselves graded posets (with $\mathrm{rank}(\pi) = 0$ in V_π). The main result of this paper is the surprising formula

$$F(\Lambda_\pi, q) F(V_\pi, q) = F(\mathfrak{S}_n, q) = [n]!. \tag{1}$$

Equation (1) was conjectured by R. Stanley. It was inspired by an observation of Steven Sam that if π is 231-avoiding, then Λ_π appears to be rank-symmetric and rank-unimodal. These two properties are simple consequences of Theorem 3.5. (See Corollary 3.11.) Figure 1 shows the Hasse diagram of $\mathfrak{S}_4$. If for instance $\pi = 4132$ (which is separable), then $F(\Lambda_\pi, q) = 1 + 2q + 2q^2 + 2q^3 + q^4$ and $F(V_\pi, q) = 1 + q + q^2$. Then multiplying $F(\Lambda_\pi, q)$ and $F(V_\pi, q)$ gives us $[4]!$.

We also give a convenient method to find an explicit formula for $F(\Lambda_\pi, q)$ and $F(V_\pi, q)$. In fact, when $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is 231-avoiding, meaning that there do not exist $i < j < k$ with $a_k < a_i < a_j$, the explicit formula for $F(\Lambda_\pi, q)$ is given by

$$F(\Lambda_\pi, q) = \prod_{i=1}^{n} [c_i], \tag{2}$$

where $a_{c_i + i}$ is the first element to the right of a_i in π satisfying $a_{c_i+i} > a_i$, setting $a_{n+1} = \infty$.

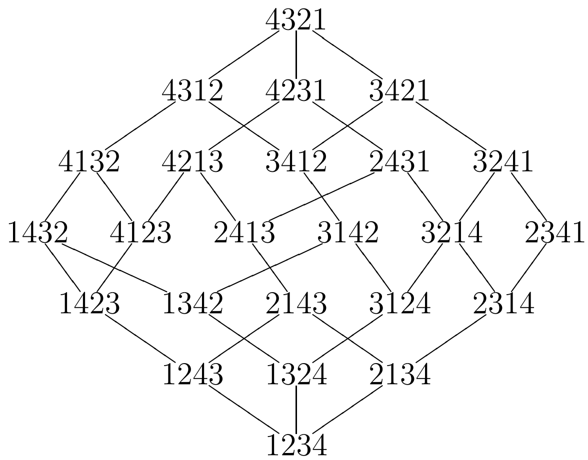

Figure 1. The graded poset $\mathfrak{S}_4$ under weak order

The *inversion poset* P_π of $\pi = a_1a_2 \cdots a_n \in \mathfrak{S}_n$ has the relations $a_i < a_j$ in P if $i < j$ and $a_i < a_j$ in $\mathbb{Z}$. Figure 2 is the diagram of the inversion posets of the permutations 34125 and 31425.

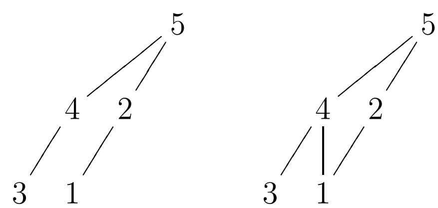

Figure 2. The inversion posets of 34125 (left) and 31425 (right)

Let P and Q be posets on disjoint sets. The *disjoint union* $P + Q$ is the poset on the union $P \cup Q$ such that $s \le t$ in $P + Q$ if either $s, t \in P$ and $s \le t$ in P, or $s, t \in Q$ and $s \le t$ in Q. The *ordinal sum* $P \oplus Q$ is the poset on the union $P \cup Q$ such that $s \le t$ in $P \oplus Q$ if either $s, t \in P$ and $s \le t$ in P, or $s, t \in Q$ and $s \le t$ in Q, or $s \in P$ and $t \in Q$.

The following lemma is easy to prove, so we omit the proof here.

1.1 Lemma. *Let $\pi \in \mathfrak{S}_n$ with $\pi = \pi_A\pi_B$, where π_A is a permutation of size m and π_B is a permutation of size $n - m$ for some $m < n$. Then*

- $P_\pi = P_{\pi_A} + P_{\pi_B}$ *if and only if* π_B *is a permutation of the letters* $\{1, 2, \ldots, m\}$ *and* π_A *is a permutation of the letters* $\{m+1, m+2, \ldots, n\}$.

- $P_\pi = P_{\pi_A} \oplus P_{\pi_B}$ *if and only if* π_A *is a permutation of the letters* $\{1, 2, \ldots, m\}$ *and* π_B *is a permutation of the letters* $\{m+1, m+2, \ldots, n\}$.

A *linear extension* of a poset P on the set $\{1, 2, \ldots, n\}$ is a permutation $\pi = a_1 \cdots a_n \in \mathfrak{S}_n$ such that if $i < j$ in P, then i precedes j in π. We use $\mathcal{L}(P)$ to denote the set of linear extensions of P. Since a linear extension π of a poset P on $\{1, \ldots, n\}$ has been defined as a permutation of $\{1, \ldots, n\}$, it has length $\ell(\pi)$ as defined above. We define

$$F(\mathcal{L}(P), q) = \sum_{\pi \in \mathcal{L}(P)} q^{\ell(\pi)}.$$

We have the following rules for the operation on $F(\mathcal{L}(P), q)$.

1.2 Lemma. *Let* P *and* Q *be two posets, where* P *is on* $\{1, 2, \ldots, m\}$ *and* Q *is on* $\{m+1, \ldots, m+n\}$. *Then*

$$F(\mathcal{L}(P \oplus Q), q) = F(\mathcal{L}(P, q)) F(\mathcal{L}(Q, q)), \tag{3}$$

$$F(\mathcal{L}(P + Q), q) = F(\mathcal{L}(P, q)) F(\mathcal{L}(Q, q)) \begin{bmatrix} m+n \\ m \end{bmatrix}, \tag{4}$$

where $\begin{bmatrix} m+n \\ m \end{bmatrix} = \dfrac{[m+n]!}{[m]![n]!}$.

The proof of (3) is immediate by considering the definition of ordinal sum and counting the number of inversions. The proof of (4) follows from the theory of P-partitions, a straightforward extension of the second proof of Proposition 1.3.17 of [7].

A *reduced decomposition* of a permutation $\pi \in \mathfrak{S}_n$ is a sequence $(i_1, i_2, \ldots, i_\ell)$ such that $\pi = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and ℓ is minimal, viz., $\ell = \ell(\pi)$. If $\pi = \pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_m = \sigma$ is a saturated chain C from π to σ, where $\pi_j = \pi_{j-1} s_{i_j}$, then $r(C) := (i_1, \ldots, i_\ell)$ is a reduced decomposition of $\pi^{-1}\sigma$. Write $R(\pi)$ for the set of reduced decompositions of π. Thus the map $C \mapsto r(C)$ is a bijection between saturated chains from id to π and reduced decompositions of π.

With the definitions above, we proceed to the proofs of the main theorem and the explicit formula for $F(\Lambda_\pi, q)$.

2 Preliminary Results

The following lemma states a property of separable permutations which is of great importance to our proof of the main theorem.

2.1 Lemma. *If $n > 1$ and $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is a separable permutation, then we can write $\pi = \pi_A \pi_B$ (concatenation of words), where π_A and π_B are both separable permutations satisfying one of the two following properties:*

- *π_A is a permutation of $1, 2, \ldots, m$ and π_B is of $m+1, \ldots, n$ for some m with $1 \leq m < n$;*
- *π_A is a permutation of $m+1, \ldots, n$ and π_B is of $1, 2, \ldots, m$ for some m with $1 \leq m < n$.*

Lemma 2.1 is well-known and easy to prove; thus we omit the proof here. The following lemma is an immediate consequence of Lemma 2.1

2.2 Corollary. *If $n > 1$ and $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is a separable permutation, then there exist two disjoint nonempty posets P_{π_A}, P_{π_B} such that $P_\pi = P_{\pi_A} + P_{\pi_B}$ or $P_\pi = P_{\pi_A} \oplus P_{\pi_B}$.*

The following lemma is a special case of a result of Björner and Wachs [3, Thm. 6.8].

2.3 Lemma. *Let π be any permutation in $\mathfrak{S}_n$, then $F(\mathcal{L}(P_\pi), q) = F(\Lambda_\pi, q)$.*

Now we arrive at one of the main preliminary results of this section.

2.4 Proposition. *If $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is a separable permutation, then the following hold:*

(i) *When $a_1 < a_n$, we can write $\pi = \pi_A \pi_B$ where π_A is a permutation of size m for some m with $1 \leq m < n$, and*

$$F(\Lambda_\pi, q) = F(\Lambda_{\pi_A}, q) \cdot F(\Lambda_{\pi_B}, q). \tag{5}$$

(ii) *When $a_1 > a_n$, we can write $\pi = \pi_A\pi_B$, where π_A is a permutation of size m for some m with $1 \leq m < n$, and*

$$F(\Lambda_\pi, q) = \begin{bmatrix} n \\ m \end{bmatrix} F(\Lambda_{\pi_A}, q) \cdot F(\Lambda_{\pi_B}, q). \quad (6)$$

Proof. Let P be the inversion poset of π, P_A be the inversion poset of π_A, and P_B be the inversion poset of π_B.

When $a_1 < a_n$, it follows from Lemma 2.1 that we can write $\pi = \pi_A\pi_B$, where π_A is a permutation of $\{1, 2, \ldots, m\}$ and π_B is a permutation of $\{m+1, m+2, \ldots, n\}$. By Lemma 1.1, we have $P = P_A \oplus P_B$. It follows from Lemma 1.2 that

$$F(\mathcal{L}(P_A \oplus P_B), q) = F(\mathcal{L}(P_A), q)F(\mathcal{L}(P_B), q).$$

Since π, π_A, π_B are all separable permutations, by Lemma 2.3, we have

$$\begin{aligned} F(\Lambda_\pi, q) = F(\mathcal{L}(P, q)) &= F(\mathcal{L}(P_A \oplus P_B), q) \\ &= F(\mathcal{L}(P_A), q)F(\mathcal{L}(P_B), q) \\ &= F(\Lambda_{\pi_A}, q)F(\Lambda_{\pi_B}, q). \end{aligned}$$

The proof of (ii) is similar. □

2.5 Proposition. *If $\pi = a_1a_2\cdots a_n \in \mathfrak{S}_n$ is a separable permutation, then the following hold:*

(i) *If $a_1 < a_n$, then we can write $\pi = \pi_A\pi_B$ where π_A is a permutation of size m for some m with $1 \leq m < n$, and*

$$F(V_\pi, q) = \begin{bmatrix} n \\ m \end{bmatrix} F(V_{\pi_A}, q) \cdot F(V_{\pi_B}, q). \quad (7)$$

(ii) *If $a_1 > a_n$, then we can write $\pi = \pi_A\pi_B$, where π_A is a permutation of size m for some m with $1 \leq m < n$, and*

$$F(V_\pi, q) = F(V_{\pi_A}, q) \cdot F(V_{\pi_B}, q). \quad (8)$$

The proof is similar to that of Proposition 2.4 by using the *complement* π^c of a permutation $\pi = a_1a_2\cdots a_n \in \mathfrak{S}_n$ defined by $\pi^c = a'_1a'_2\cdots a'_n$ where $a'_i = n+1-a_i$ for all $1 \leq i \leq n$.

A standard property of π^c and weak order is stated in the following lemma, and we omit the easy proof here.

2.6 Lemma. *The rank relation between a permutation and its complement is given by*

$$\ell(\pi^c) = \binom{n}{2} - \ell(\pi).$$

In fact, there exists a bijection $\mu : [\pi, w_0] \to [\mathrm{id}, \pi^c]$ *defined by* $\mu(w) = w^c$ *for all* $w \in [\pi, w_0]$.

Proof of Proposition 2.5. For any $\omega \in [\pi, w_0]$, by Lemma 2.6 and the fact that

$$\ell(w^c) = \binom{n}{2} - \ell(w) = \ell(\pi^{-1}w_0) - \ell(\pi^{-1}w),$$

any $q^{\ell(\pi^{-1}\omega)}$ in $F(V_\pi, q)$ corresponds uniquely to a term $q^{\ell(\pi^{-1}w_0)-\ell(\pi^{-1}\omega)}$ in $F(\Lambda_{\pi^c}, q)$. Thus

$$q^{\ell(\pi^{-1}w_0)} F(V_\pi, q^{-1}) = F(\Lambda_{\pi^c}, q). \tag{9}$$

We now consider π^c in the two cases in Proposition 2.5.

(i) When $a_1 < a_n$, by equation (6) we have

$$F(\Lambda_{\pi^c}, q) = \begin{bmatrix} n \\ m \end{bmatrix} F(\Lambda_{\pi_A{}^c}, q) \cdot F(\Lambda_{\pi_B{}^c}, q). \tag{10}$$

Combining (9) and (10) gives us

$$q^{\ell(\pi^{-1}w_0)} F(V_\pi, q^{-1}) = \begin{bmatrix} n \\ m \end{bmatrix} F(V_{\pi_A{}^c}, q^{-1}) F(V_{\pi_B{}^c}, q^{-1}) \cdot q^{\binom{m}{2}-\ell(\pi_A)} \cdot q^{\binom{n-m}{2}-\ell(\pi_B)}. \tag{11}$$

Since the letters in π_A are all smaller than the letters in π_B, we have $\ell(\pi) = \ell(\pi_A) + \ell(\pi_B)$. Substituting q^{-1} for q in (11), which converts $\begin{bmatrix} n \\ m \end{bmatrix}$ into $q^{\binom{n}{2}-\binom{m}{2}-\binom{n-m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}$, completes the proof of (7).

(ii) Since all the letters in π_A are greater than the letters in π_B, we have

$$\ell(\pi) = (n-m)m + \ell(\pi_A) + \ell(\pi_B).$$

The rest of (8) can be proved analogously. □

3 Main Results

3.1 Main Theorem

3.1 Theorem. *Let* $\pi \in \mathfrak{S}_n$*,* $\Lambda_\pi = [\mathrm{id}, \pi]$*, and* $V_\pi = [\pi, w_0]$*. The following equation holds for any separable permutation* π*:*

$$F(\Lambda_\pi, q)F(V_\pi.q) = F(\mathfrak{S}_n, q) = [n]!. \tag{12}$$

Proof. When $n = 2$, it is easy to verify that the expression holds. Suppose the statement holds when $k < n$ for some $n \geq 3$; we want to show that when $k = n$, the statement still holds.

Let π_A and π_B be the same as before. When $a_1 > a_n$ we have by (6) and (8) that

$$F(\Lambda_\pi, q)F(V_\pi.q) = \begin{bmatrix} n \\ m \end{bmatrix} F(\Lambda_{\pi_A}, q)F(\Lambda_{\pi_A}, q) \cdot F(V_{\pi_A}, q)F(V_{\pi_B}, q).$$

Thus by the inductive hypothesis, we have

$$F(\Lambda_\pi, q)F(V_\pi, q) = \begin{bmatrix} n \\ m \end{bmatrix} [m]![n-m]! = [n]!.$$

The proof for $a_1 < a_n$ is similar. □

3.2 A Bijection $\varphi\colon \Lambda_w \times V_w \to \mathfrak{S}_n$

We can also give a bijective proof of Theorem 3.1.

3.2 Theorem. *Let* $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ *be a separable permutation. The map*

$$\phi : \Lambda_\pi \times V_\pi \to \mathfrak{S}_n$$

defined by $\phi(u, v) = u^{-1}v$*, where* $u \leq \pi$ *and* $v \geq \pi$*, is a bijection.*

Since $(u^{-1}v)^{-1} = v^{-1}u$, it is a direct consequence of Theorem 3.2 that the map

$$\phi' : \Lambda_\pi \times V_\pi \to \mathfrak{S}_n$$

defined by $\phi'(u, v) = v^{-1}u$ for $u \leq \pi$ and $v \geq \pi$ is also a bijection.

We use the following lemma to prove this theorem.

3.3 Lemma. *If* $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ *is a separable permutation with* $a_1 < a_n$*, and* $(i_1, \ldots, i_\ell) \in R(\pi)$*, then there exists an integer* m *with* $1 \le m < n$ *such that none of the simple transpositions* s_{i_j} *transposes an element in* $A_\pi = \{1, 2, \ldots, m\}$ *with an element in* $B_\pi = \{m+1, \ldots, n\}$*. In other words, there is no interaction between the sets* A_π *and* B_π*.*

The proof of Lemma 3.3 can be achieved easily from the definition of weak order.

Proof. If the lemma does not hold, then in the sequence of all simple transpositions there exists a nonempty subsequence consisting of simple transpositions between the letters in A_π and the letters in B_π. Suppose the last transposition in this subsequence is between $a \in A_\pi$ and $b \in B_\pi$. From Proposition 2.1 we know that a is to the left of b. Since $a < b$, by the definition of weak order the permutation after the transposition is covered by the permutation before swapping a and b, which leads to a contradiction. $\square$

Proof of Theorem 3.2. When $a_1 < a_n$, by Lemma 2.1 we can write $\pi = \pi_A \pi_B$ where π_A is a separable permutation of $\{1, 2, \ldots, m\}$ for some $m > 0$.

For the injectivity part, we want to show that there do not exist two different pairs $(u_1, v_1), (u_2, v_2) \in \Lambda_\pi \times V_\pi$ such that $u_1^{-1} v_1 = u_2^{-1} v_2$. It is sufficient to show that $u^{-1}\pi \neq \pi^{-1} v$ for all $(u, v) \in \Lambda_\pi \times V_\pi$, and $u, v \neq \pi$.

Let $r_1(C_\Lambda) = (i_1, i_2, \ldots, i_{k_1})$ be the reduced decomposition of $u^{-1}\pi$ and $r_2(C_V) = (j_1, j_2, \ldots, j_{k_2})$ be the reduced decomposition of $\pi^{-1}v$. We need only consider the situation when $k_1 = k_2$.

Since π_A is a permutation of $\{1, 2, \ldots, m\}$ and π_B is a permutation of $\{m+1, \ldots, n\}$, by Lemma 3.3 we can write $u = u_A u_B$ where u_A is a permutation of $\{1, 2, \ldots, m\}$ and u_B is a permutation of $\{m+1, \ldots, n\}$. Furthermore, we can also write the reduced decomposition of $u^{-1}\pi$ as a concatenation of the reduced decompositions of $u_A^{-1}\pi_A$ and $u_B^{-1}\pi_B$. Accordingly, if there exists $v \ge \pi$ such that $\pi v = u^{-1}\pi$, then we can write v as a concatenation of two subpermutations v_A, v_B, and the reduced decomposition for $\pi^{-1}u$ is a concatenation of $\pi_A^{-1} v_A$ and $\pi_B^{-1} v_B$. Hence in order to have $u^{-1}\pi = \pi v$, we must have

$$u_A^{-1}\pi_A = \pi_A^{-1} v_A \text{ and } u_B^{-1}\pi_B = \pi_B^{-1} v_B.$$

Thus we need only consider the case in which the size of the permutation is less than n.

For the surjectivity part, we want to show that, for each permutation $w \in \mathfrak{S}_n$, there exists $(u, v) \in \Lambda_\pi \times V_\pi$ such that $u^{-1}v = w$.

Let $w \in \mathfrak{S}_n$ be as in Proposition 2.5(ii). Let w_1 be the sub-permutation of w which consists of the letters $\{1, 2, \ldots, m\}$, and let w_2 be the sub-permutation of w which consists of $\{m+1, m+2, \ldots, n\}$. By the inductive hypothesis, there exist $(u_1, v_1) \in \Lambda_{\pi_A} \times V_{\pi_A}$ and $(u_2, v_2) \in \Lambda_{\pi_B} \times V_{\pi_B}$ such that

$$u_1^{-1}v_1 = w_1 \text{ and } u_2^{-1}v_2 = w_2.$$

It follows that

$$(u_1u_2)^{-1}(v_1v_2) = w_1w_2,$$

and $(u_1u_2, v_1v_2) \in \Lambda_\pi \times V_\pi$.

We now show that we can find $v' \geq v_1v_2$ such that

$$(v_1v_2)^{-1}v' = (w_1w_2)^{-1}w.$$

Then it follows that for any arbitrary w, there exists a $(u_1u_2, v') \in \Lambda_\pi \times V_\pi$ and

$$(u_1u_2)^{-1}v' = (u_1u_2)^{-1}(v_1v_2)(v_1v_2)^{-1}v' = (w_1w_2)(w_1w_2)^{-1}w = w.$$

We will show an explicit way to find v'.

Let $A_1 < A_2 < \cdots < A_m$ be the positions in π that are occupied by the letters $\{1, 2, \ldots, m\}$. We start by shifting the letters $\{1, 2, \ldots, m\}$ in both v_1v_2 and w_1w_2 to the positions indexed by $A_1, A_2, \ldots, A_m$. That is, we move the letters at the mth position in v_1v_2 and w_1w_2 to the position indexed by A_m, and then move the letter at the $(m-1)$-st position to the position indexed by A_{m-1}, and so on. Finally, we move the letter at the first position to the position indexed by A_1. Recall that v_1 and w_1 are permutations of $\{1, 2, \ldots, m\}$ and v_2 and w_2 are permutations of $\{m+1, \ldots, n\}$. Since $A_1 < A_2 < \cdots < A_m$, it is easy to show that during the shifting process, all the transpositions are between a letter in $\{1, 2, \ldots, m\}$ and a letter in $\{m+1, m+2, \ldots, n\}$, and that after each transposition, the length of the permutation increases by 1. This process thus turns w_1w_2 into w and v_1v_2 into another permutation, which we set to be v'. Accordingly, by the inductive hypothesis and this shifting process, we have an explicit way to find v' such that

$$(u_1u_2)^{-1}v' = w.$$

When $a_1 > a_n$, we use the complement of the permutation, and the rest of the proof is similar. □

3.3 Explicit Formulas for $F(\Lambda_\pi, q)$ and $F(V_\pi, q)$

Based on Proposition 2.4 and Proposition 2.5, we introduce a convenient method to find the explicit formulas for $F(\Lambda_\pi, q)$ and $F(V_\pi, q)$.

The most convenient way is to use a *separating tree*. We define it recursively as follows.

Let $\pi = a_1 a_2 \cdots a_n$ be a separable permutation.

When $n = 2$, its separating tree T_π is an ordered binary tree with the left leaf a_1 and right leaf a_2.

When $n > 2$, by Lemma 2.1 we can write $\pi = \pi_A \pi_B$ where π_A and π_B are separable permutations with size strictly smaller than n. Then T_π is an ordered binary tree, with the subtree rooted at the left child of the root, being T_{π_A}, and the subtree rooted at the right child of the root, being T_{π_B}.

Since there might be more than one way to write $\pi = \pi_A \pi_B$, a separable permutation can have more than one separating tree. Also, only the separable permutations have separating trees.

The definition of the separating tree T_π gives the following lemma, which is easy to prove.

3.4 Lemma. *For any node in T_π, the leaves of the subtree rooted at that node form a subrange, a set of consecutive integers.*

This lemma allows us to classify the nodes in T_π into two categories. A node is *negative* if the subrange of its left child is greater than that of its right child. *Positive* node is defined analogously. Figure 3 shows a separating tree for 4231, which has two negative nodes and one positive node, as labeled in the figure.

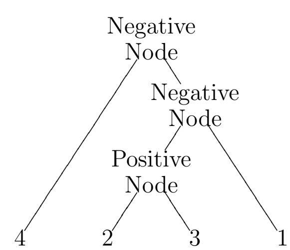

Figure 3. The separating tree for 4231

3.5 Theorem. *Let $S^-(\pi) = \{$all negative nodes V_i in T_π whose parents are not negative$\}$ and $S^+(\pi) = \{$all positive nodes V_j in T_π whose parents are not positive$\}$. Let V_0 be the root of the tree, and V_0 is not in either $S^-(\pi)$ nor $S^+(\pi)$. Let $N(V_k)$ denote the number of leaves in the subtree rooted at V_k. In particular, we define $\prod_{V_i \in \varnothing} [N(V_i)]! = 1$. Then*

$$F(\Lambda_w, q) = \begin{cases} \dfrac{\prod_{V_i \in S^-(\pi)} [N(V_i)]!}{\prod_{V_j \in S^+(\pi)} [N(V_j)]!}, & V_0 \text{ is a positive node;} \\ \\ \dfrac{\prod_{V_i \in S^-(\pi)} [N(V_i)]!}{\prod_{V_j \in S^+(\pi)} [N(V_j)]!}[N(V_0)]!, & V_0 \text{ is a negative node.} \end{cases} \tag{13}$$

$$F(V_w, q) = \begin{cases} \dfrac{\prod_{V_i \in S^+(\pi)} [N(V_i)]!}{\prod_{V_j \in S^-(\pi)} [N(V_j)]!}[N(V_0)]!, & V_0 \text{ is a positive node;} \\ \\ \dfrac{\prod_{V_i \in S^+(\pi)} [N(V_i)]!}{\prod_{V_j \in S^-(\pi)} [N(V_j)]!}, & V_0 \text{ is a negative node.} \end{cases} \tag{14}$$

3.6 Example. Let $w = 4231$. Its separating tree is shown in Figure 3. It has one negative node with no parent, one negative node with a negative parent node, and one positive node with a negative parent node. Thus $F(\Lambda_w, q) = [4]!/[2]!$, and $F(V_w, q) = [2]!/[1]!$.

Proof of Theorem 3.5. Let $\pi = a_1 a_2 \cdots a_n$ be a separable permutation. We can use induction to prove Theorem 3.5.

By the definition of $N(V)$, we have $N(V_0) = n$.

When $a_1 < a_n$, we write $\pi = \pi_A \pi_B$ where π_A is a permutation of $\{1, 2, \ldots, m\}$. The root V of T_π has two children with the left child V_L having leaves $\{1, 2, \ldots, m\}$ and the right child V_R having leaves $\{m+1, m+2, \ldots, n\}$. Thus V is a positive node. Let T_L be the subtree rooted at V_L and T_R be the subtree rooted at V_R. Applying formula (13) to π_A and π_B, together with (5) and (7), we can prove (13) and (14) by induction.

When $a_1 > a_n$, the root of T_π is a negative node. The rest of the proof is similar to the case above when $a_1 < a_n$. ∎

More specifically, when the permutation $\pi = a_1 a_2 \cdots a_n$ is 231-avoiding (a 231-avoiding permutation requires more restrictions than a general separable permutation), a more direct formula for $F(\Lambda_\pi, q)$ can be given.

3.7 Corollary (explicit formula for $F(\Lambda_\pi, q)$ for a 231-avoiding permutation)**.** *Let $\pi = a_1 a_2 \cdots a_n$ be 231-avoiding, and a_{c_i+i} be the first element to the right of a_i in π satisfying $a_{c_i+i} > a_i$, setting $a_{n+1} = \infty$. Then*

$$F(\Lambda_\pi, q) = \prod_{i=1}^{n} [c_i].$$

Before proving this proposition, we give an example to explain the notation in the formula.

3.8 Example. Let $\pi = a_1 a_2 \cdots a_6 = 142365$. We set $a_7 = \infty$. For $a_1 = 1$, letter 4 is the first one greater than 1 and to the right of a_1; the distance between these two integers, c_1, is thus $2-1 = 1$. Similarly, $c_2 = 5-2 = 3, c_3 = 4-3 = 1, c_4 = 5-4 = 1, c_5 = 7-5 = 2, c_6 = 7-6 = 1$. Thus the generating function is $F(\Lambda_{142365}, q) = \prod_{i=1}^{6} [c_i] = [1][3][1][1][2][1] = (q^2+q+1)(q+1)$.

Proof. By Lemma 3.3 we know that when π is 231-avoiding, either π has the greatest letter n at its first position, or n is at the $(m+1)$-st position with $m > 0$. Thus we can write $\pi = \pi_A \pi_B$ where π_A is a 231-avoiding permutation of $\{1, \ldots, m\}$ and π_B is a 231-avoiding permutation of $\{m+1, \ldots, n\}$. Then we can construct the separating tree by repeatedly applying the following steps.

For a separating tree with root V_0, we first decide its left child V_L and right child V_R by identifying the position of the greatest letter in π, i.e., finding m such that $a_{m+1} = n$.

When $m = 0$, the subtree rooted at V_L has only one leaf $a_1 = n$, while the subtree rooted at V_R is the separating tree of the permutation $a_2 a_3 \cdots a_n$, which we will construct similarly.

When $m > 0$, the subtree rooted at V_L is the separating tree for the permutation $a_1 a_2 \cdots a_m$, while the subtree rooted at V_R is the separating tree for the permutation $a_{m+1} \cdots a_n$. We then construct these two separating tree similarly.

In the first case, V_0 is a negative node. We already know that $c_1 = n$ and

$$F(\Lambda_\pi, q) = [n] \cdot F(\Lambda_{a_2 a_3 \cdots a_n}, q),$$

and $c_1 = n$.

In the second case, V_0 is a positive node. We have $c_1 = n$ and

$$F(\Lambda_\pi, q) = F(\Lambda_{a_1 a_2 \cdots a_m}, q) F(\Lambda_{a_{m+1} a_{m+2} \cdots a_n}, q).$$

We also know that, for a letter a in $\{1, 2, \ldots, m\}$, the distance between a and the first letter greater than a and to its right is the same in both π and $a_1 a_2 \cdots a_m$.

The rest of the proof can be completed by induction. □

Since we know that $F(\Lambda_\pi, q)F(V_\pi, q) = [n]!$, as well as the explicit formula for $F(\Lambda_\pi, q)$, we can also obtain an explicit formula for $F(V_\pi, q)$.

By symmetry, we can obtain analogous explicit formulas when the permutation avoids any of the patterns 132, 231, 312, or 213.

The following two lemmas are standard results about unimodality; see for instance [6].

3.9 Lemma. *The q-binomial coefficient* $\begin{bmatrix} n \\ m \end{bmatrix}$ *is rank-unimodal and rank-symmetric.*

3.10 Lemma. *Let* $F(q)$ *and* $G(q)$ *be symmetric unimodal polynomials with nonnegative real coefficients. Then* $F(q)G(q)$ *is also symmetric and unimodal.*

Lemma 3.9 and Lemma 3.10 imply the following corollary.

3.11 Corollary. $F(\Lambda_\pi, q)$ *and* $F(V_\pi, q)$ *are rank symmetric and unimodal.*

Theorem 3.5 determines the number of elements of each rank k of the poset Λ_π when π is separable. We can also determine the number of elements that cover k elements. A *descent* of a permutation $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is a position i with $1 \leq i < n$, such that $a_i > a_{i+1}$. Let $\operatorname{des}(\pi)$ be the number of descents of π. It is easy to see that $\operatorname{des}(\pi)$ is equal to the number of elements that π covers in the weak order on $\mathfrak{S}_n$. If P_π is the inversion poset of π, then the enumeration of linear extensions of P_π by number of descents is the same as the enumeration of elements of Λ_π in weak order by number of covers. Let $\Omega_P(m)$ denote the number of order-preserving maps $f \colon P \to \{1, \ldots, m\}$. Then we have the following theorem which relates $\Omega_P(m)$ with the descent number. The proof can be found in [7, Thm. 4.5.14].

3.12 Theorem. *For any poset* P *on* $\{1, 2, \ldots, n\}$*, we have*

$$\sum_{m \geq 1} \Omega_P(m) x^m = \frac{\sum_{\pi \in \mathcal{L}(P)} x^{\operatorname{des}(\pi)+1}}{(1-x)^{n+1}}.$$

Using the recursive structure of P_π when π is separable (Corollary 2.2) we can give a recursive description of $\Omega_{P_\pi}(m)$ and thus of the number of elements in Λ_π that cover k elements. We do not enter into the details here.

Our results suggest several open problems. For what permutations $\pi \in \mathfrak{S}_n$ is the poset Λ_π rank-symmetric? When is $[n]!$ divisible by the rank generating function $F(\Lambda_\pi, q)$? When is $F(\Lambda_\pi, q)$ a product of cyclotomic polynomials? R. Stanley has verified that for $n \leq 8$, if Λ_π is rank-symmetric then $F(\Lambda_\pi, q)$ is a product of cyclotomic polynomials, but $F(\Lambda_\pi, q)$ need not divide $[n]!$. For instance, when $n = 8$ there are 8558 separable permutations, 10728 permutations π for which Λ_π is rank-symmetric (and hence a product of cyclotomic polynomials), and 961 permutations π for which Λ_π is rank-symmetric but $F(\Lambda_\pi, q)$ does not divide $[8]!$. A further problem is to extend our work to the weak order of other Coxeter groups.

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