ASYMPTOTICS OF COEFFICIENTS OF MULTIVARIATE GENERATING FUNCTIONS: IMPROVEMENTS FOR MULTIPLE POINTS

ALEXANDER RAICHEV AND MARK C. WILSON

Abstract. Let $F(x) = \sum_{\nu \in \mathbb{N}^d} F_\nu x^\nu$ be a multivariate power series with complex coefficients that converges in a neighborhood of the origin. Assume $F = G/H$ for some functions G and H holomorphic in a neighborhood of the origin. For example, F could be a rational combinatorial generating function. We derive asymptotics for the ray coefficients $F_{n\alpha}$ as $n \to \infty$ for α in a permissible subset of d-tuples of positive integers. More specifically, we give an algorithm for computing arbitrary terms of the asymptotic expansion for $F_{n\alpha}$ when the asymptotics are controlled by a transverse multiple point of the analytic variety $H = 0$. We have implemented our algorithm in Sage and apply it to several examples. This improves upon earlier work on analytic combinatorics in several variables by R. Pemantle and M. C. Wilson.

1. Introduction

In [PW02, PW04] Pemantle and Wilson began a program of analytic combinatorics in several variables to derive asymptotic expansions for coefficients of combinatorial generating functions. In this article we continue that program by improving upon several of their results.

Let $F(x) = \sum_{\nu \in \mathbb{N}^d} F_\nu x_1^{\nu_1} \cdots x_d^{\nu_d}$ be a power series with complex coefficients that converges in a neighborhood of the origin. Assume $F = G/H$ for some functions G and H holomorphic in a neighborhood of the origin. For example, F could be a rational function. We derive asymptotics for the ray coefficients $F_{n\alpha}$ as $n \to \infty$ for α in a permissible subset of d-tuples of positive integers.

The general form of the asymptotic expansion of $F_{n\alpha}$ was determined in [PW02, PW04] for tame local geometries of the singular variety $\mathcal{V} = \{x \in \mathbb{C}^d : H(x) = 0\}$. However, until now, explicit computation of higher-order terms (for numerical approximation for small n or for computing variances of random variables, for instance) has not been attempted.

Our Contribution. In this article we give an algorithm for computing arbitrary terms of the asymptotic expansion for $F_{n\alpha}$ when the asymptotics are controlled by a multiple point of $\mathcal{V}$ of order $r \geq 1$. We do this by first deriving an explicit formula in Section 3 for the special case where $r \leq d$ and the ideal generated by the germ of H in the ring of germs of holomorphic functions is radical. This generalizes the formula for the smooth point case $r = 1$ in [RW08, Theorem 3.2] and improves upon the formula in [PW04, Thm 3.5], which gave an explicit formula for only the leading term. We then show in Section 5 how to reduce the general multiple point case to the special case. This gives a unified method for the computation of higher-order asymptotics that works for any value of r and d. Our method of derivation uses Fourier-Laplace integrals as in [PW04], but avoids

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the complications of infinite stationary phase sets. We have implemented our algorithm in Sage and apply it to examples in Section 6. Section 7 contains most of our proofs.

2. Preliminaries

Throughout this article we make use of basic theorems from local analytic geometry, good references for which are [dJP00, Tay02].

For brevity we write a power series $\sum_{\nu\in\mathbb{N}^d} a_\nu(x_1-c_1)^{\nu_1}\cdots(x_d-c_d)^{\nu_d}$ as $\sum_\nu a_\nu(x-c)^\nu$ and use the multi-index notation $\nu! = \nu_1!\cdots\nu_d!$, $n\nu = (n\nu_1,\dots,n\nu_d)$, $\nu+1 = (\nu_1+1,\dots,\nu_d+1)$, and $\partial^\nu = \partial_1^{\nu_1}\cdots\partial_d^{\nu_d}$, where ∂_j is partial differentiation with respect to component j.

Let $\mathcal{O}(\Omega)$ denote the $\mathbb{C}$-algebra of holomorphic functions on an open set $\Omega\subseteq\mathbb{C}^d$ and $\mathcal{O}_c$ the $\mathbb{C}$-algebra of germs of holomorphic functions at $c\in\mathbb{C}^d$. The latter algebra is a local Noetherian factorial ring whose unique maximal ideal is the set $\{f\in\mathcal{O}_c : f(c)=0\}$ of non-units.

We refer often to both d-tuples and $(d-1)$-tuples and write $\hat{a} = (a_1,\dots,a_{d-1})$ given a tuple $a=(a_1,\dots,a_d)$. For simplicity we assume $d\geq 2$, though our formulas below also apply in the case $d=1$ of univariate functions after making the simple changes described in [RW08, Remark 3.6].

Let $\Omega\subseteq\mathbb{C}^d$ be an (open) neighborhood of the origin and $F(x)=\sum_\nu F_\nu x^\nu\in\mathcal{O}(\Omega)$. Assume $F=G/H$ for some relatively prime $G,H\in\mathcal{O}(\Omega)$. Let $\mathcal{V}$ be the set of singularities of F, that is, the holomorphic variety/analytic set $\{x\in\Omega : H(x)=0\}$ determined by H. We will derive asymptotics for the ray coefficients $F_{n\alpha}$ as $n\to\infty$ with α in a permissible subset of $\mathbb{N}_+^d$, the set of d-tuples of positive integers. For asymptotics of F_ν when $d=2$ and $\nu\to\infty$ along more general paths see [Lla06].

To begin we recall several key definitions from [PW02, PW04].

Just as in the univariate case, asymptotics for the coefficients of F are determined by the location and type of singularities of F, that is, by the geometry of $\mathcal{V}$. Generally the singularities closest to the origin are the most important. We define 'closest to the origin' in terms of polydiscs. For $c\in\mathbb{C}^d$, let $D(c)=\{x\in\mathbb{C}^d : \forall j\ |x_j|<|c_j|\}$ and $C(c)=\{x\in\mathbb{C}^d : \forall j\ |x_j|=|c_j|\}$ be the respective polydisc and polycircle centered at the origin with polyradius determined by c.

Definition 2.1. We say that a point $c\in\mathcal{V}$ is **minimal** if there is no point $x\in\mathcal{V}$ such that $|x_j|<|c_j|$ for all j. We say that $c\in\mathcal{V}$ is **strictly minimal** if there is a unique $x\in\mathcal{V}$ such that $|x_j|\leq|c_j|$ for all j, namely $x=c$, and we say that c is **finitely minimal** if there are finitely many such values of x.

In other words, a point $c\in\mathcal{V}$ is minimal if $\mathcal{V}\cap D(c)\subseteq T(c)$, finitely minimal if it is minimal and $\mathcal{V}\cap D(c)$ is finite, and strictly minimal if $\mathcal{V}\cap D(c)=\{c\}$.

Note that $\mathcal{V}$ always contains minimal points. To see this, let $c\in\mathcal{V}$ and define $f:\mathcal{V}\cap\overline{D(c)}\to\mathbb{R}$ by $f(x)=\sqrt{x_1^2+\cdots+x_d^2}$. Since f is a continuous function on a compact space, it has a minimum, and that minimum is a minimal point of $\mathcal{V}$.

The singularities of F with the simplest geometry are the regular/smooth points of $\mathcal{V}$, that is, points $c\in\mathcal{V}$ with $\nabla H(c)\neq 0$. Asymptotics for $F_{n\alpha}$ dependent on smooth points were derived in [PW02, RW08]. Here we focus on asymptotics dependent on points with the next simplest geometry, that is, multiple points.

Definition 2.2. Let $c\in\mathcal{V}$ and consider the unique factorization of the germ of H in $\mathcal{O}_c$ into irreducible germs. Choosing representatives for these germs gives a factorization

$H = H_1^{a_1} \cdots H_r^{a_r}$ valid in a neighborhood of c. We say that c is a **multiple point** of order r if

- for all j we have $H_j(c) = 0$,
- for some k and for all j we have $c_k \partial_k H_j(c) \neq 0$, and
- every subset $S \subseteq \{\nabla H_1(c), \ldots, \nabla H_r(c)\}$ spans a subspace of $\mathbb{C}^d$ of maximal vector space dimension $\min\{|S|, d\}$.

The first two conditions imply that c is a smooth point for each H_j, hence $\mathcal{V}$ is locally a union of complex manifolds that intersect at c. So a multiple point of order $r = 1$ is a smooth point of $\mathcal{V}$, and in this way multiple points are generalizations of smooth points. The last condition says that the manifolds generated by the H_j intersect transversely at c^*. Notice that this definition depends only on information about H in an arbitrarily small neighborhood of c and so it is independent of the germ representatives chosen.

Lastly, we will need to consider the singularities of F relevant to α that arise in the integrals used to approximate $F_{n\alpha}$. These singularities are called critical points.

Definition 2.3. Let $\alpha \in \mathbb{N}_+^d$ and let $c \in \mathcal{V}$ be a multiple point with $c_k \partial_k H_j(c) \neq 0$ for all $j = 1, \ldots, r$. For each j let

$$\gamma_j(c) = \left(\frac{c_1 \partial_1 H_j(c)}{c_k \partial_k H_j(c)}, \ldots, \frac{c_d \partial_d H_j(c)}{c_k \partial_k H_j(c)} \right).$$

We say that c is **critical** for α if

$$\left(\frac{\alpha_1}{\alpha_k}, \ldots, \frac{\alpha_d}{\alpha_k} \right) = \sum_{j=1}^{r} s_j \gamma_j(c)$$

for some $s_j \geq 0$, that is, if α lies in the conical hull of the $\gamma_j(c)$, which we call the **critical cone** of c.

For a Morse theoretic explanation of the relevance of critical points and logarithmic gradients, which we omit to maintain a relatively simple presentation, see the survey [PW08].

3. The full asymptotic expansion: special case

Let $c \in \mathcal{V}$ be a multiple point of order r, and let $H = H_1^{a_1} \cdots H_r^{a_r}$ be a local factorization of H about c as above. For concreteness and ease of notation, suppose $c_d \partial_d H_j(c) \neq 0$ for all j.

Applying the Weierstrass preparation theorem applied to each H_j, we get

$$H_j(w, y) = U_j(w, y) \left(y - \frac{1}{h_j(w)} \right)$$

* In keeping with [PW08] we are simplifying matters by assuming transversality. For a more general definition of 'multiple point' see [PW04].

in a neighborhood of c, where U_j is holomorphic and nonzero at c, h_j is holomorphic in a neighborhood of $\widehat{c}$ with $1/h_j(\widehat{c}) = c_d$, and $\partial_d H_j(w, 1/h_j(w)) \neq 0$. Thus

$$\begin{aligned} H(w,y) =& U(w,y) \prod_{j=1}^{r} \left(y - \frac{1}{h_j(w)} \right)^{a_j} \\ =& U(w,y) \prod_{j=1}^{r} \left(\frac{-y}{h_j(w)} \right)^{a_j} \prod_{j=1}^{r} \left(\frac{1}{y} - h_j(w) \right)^{a_j} \end{aligned}$$

in a neighborhood of c, where $U = U_1 \cdots U_r$. We use reciprocals, because they turn out to be convenient for proving Lemma 4.5 later on.

Remark 3.1. For the remainder of this section we assume the special case of $a_1 = \ldots = a_r = 1$ and $r \leq d$. Thus

$$H = U(w,y) \prod_{j=1}^{r} \frac{-y}{h_j(w)} \prod_{j=1}^{r} \left(\frac{1}{y} - h_j(w) \right)$$

To express our results we use the following sets and functions, most of which are derived from G and H.

For $r \geq 2$, let

$$\Delta = \{ s \in \mathbb{R}^{r-1} : s_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1}^{r-1} s_j \leq 1 \},$$

the standard orthogonal simplex of dimension $r - 1$.

Let W be a neighborhood of $\widehat{c}$ on which the h_j are defined. For $j = 0, \ldots, r-1$ and $\alpha \in \mathbb{N}_+^d$ define the functions $h : W \times \Delta \to \mathbb{C}$, $A_j : \mathrm{dom}(U) \to \mathbb{C}$, $e : [-1,1]^{d-1} \to \mathbb{C}^{d-1}$, $\widetilde{A}_j, \widetilde{h}, \widetilde{\Phi} : e^{-1}(W \cap C(\widehat{c})) \times \Delta \to \mathbb{C}$, and $P_j : \mathbb{N} \to \mathbb{N}$ by

$$\begin{aligned} \check{G}(w,y) &= \frac{G(w,y)}{U(w,y)} \prod_{j=1}^{r} \frac{-h_j(w)}{y} \\ h(w,s) &= s_1 h_1 + \cdots + s_{r-1} h_{r-1} + (1 - \sum_{j=1}^{r-1} s_j) h_r \\ A_j(w,y) &= (-1)^{r-1} y^{-r+j} \left(\frac{\partial}{\partial y} \right)^j \check{G}(w, y^{-1}) \\ e(t) &= (c_1 \exp(\mathrm{i} t_1), \ldots, c_{d-1} \exp(\mathrm{i} t_{d-1})) \\ \widetilde{h}(t,s) &= h(e(t), s) \\ \widetilde{A}_j(t,s) &= A_j(e(t), \widetilde{h}(t,s)) \\ \widetilde{\Phi}(t,s) &= -\log(c_d \widetilde{h}(t,s)) + \mathrm{i} \sum_{m=1}^{d-1} \frac{\alpha_m}{\alpha_d} t_m \\ P_j(n) &= \binom{r-1}{j} (\alpha_d n - 1)^{\underline{r-1-j}}. \end{aligned}$$

Note that $F(w,y) = \check{G}(w,y)/\prod_{j=1}^{r}(y^{-1} - h_j(w))$ and that $\widetilde{h}$, $\widetilde{A}_j$, and $\widetilde{\Phi}$ are all C^∞ functions. The falling factorial powers in P_j are defined by $a^{\underline{k}} = a(a-1)\cdots(a-k+1)$ and $a^{\underline{0}} = 1$ for $a \in \mathbb{R}$ and $k \in \mathbb{N}$. So the degree of P_j in n is $r-1-j$.

Let $J_{\log}(H,c)$ denote the $r \times d$ logarithmic Jacobian matrix, the jth row of which is the logarithmic gradient vector $\nabla_{\log} H_j(c) = (c_1\partial_1 H_j(c), \ldots, c_d\partial_d H_j(c))$. Notice that if the multiple point c has all nonzero coordinates, then every subset $S \subseteq \{\nabla_{\log} H_1(c), \ldots, \nabla_{\log} H_r(c)\}$ spans a subspace of $\mathbb{C}^d$ of dimension $|S|$.

If α is critical for c, then

$$\alpha = \left(\frac{\alpha_d s_1^*}{c_d\partial_d H_1(c)}, \ldots, \frac{\alpha_d s_r^*}{c_d\partial_d H_r(c)}\right) J_{\log}(c)$$

for some nonnegative tuple s^* with $\sum_{j=1}^{r} s_j^* = 1$. Moreover, if c has all nonzero coordinates, then the tuple s^* is unique since $J_{\log}(c)$ has rank $r \le d$. Let $\theta^* = (0, \ldots, 0, s_1^*, \ldots, s_{r-1}^*) \in \mathbb{R}^{d-1} \times \Delta \subset \mathbb{R}^{d+r-2}$.

If the Hessian $\det \widetilde{\Phi}''(\theta^*)$ is nonzero, then c is called **nondegenerate** for α.

Remark 3.2. In the smooth point case $r = 1$ we can simplify the definitions above. In that case $H = H_1^{a_1}$ with $a_1 = 1$ (in this section) and we set

$$\begin{aligned}
h(w) &= h_1(w) \\
A_0(w) &= y^{-1}\check{G}(w, y^{-1})\Big|_{y=h(w)} \\
\widetilde{A}_0(t) &= A_0(e(t)) \\
\widetilde{h}(t) &= h(e(t)) \\
\widetilde{\Phi}(t) &= -\log(c_d \widetilde{h}(t)) + \mathrm{i}\sum_{m=1}^{d-1} \frac{\alpha_m}{\alpha_d} t_m \\
\theta^* &= t^* = 0.
\end{aligned}$$

Our main theorem is a more explicit form of the following general formula.

Theorem 3.3 [PW04, Theorem 3.5]. Let $\alpha \in \mathbb{N}_+^d$ and $c \in \mathcal{V}$ be a strictly minimal multiple point with all nonzero coordinates that is critical and nondegenerate for α. Then there is a nonsingular matrix $M(c)$ and coefficients $b_q(\alpha)$ such that

$$F_{n\alpha} \sim c^{-n\alpha}\left[(2\pi)^{(r-d)/2} \det M(c)^{-1/2} \sum_{q \ge 0} b_q(\alpha)(\alpha_d n)^{(r-d)/2-q}\right]$$

as $n \to \infty$.

Theorem 3.4. In the situation of Theorem 3.3 we have

$$
(\star) \qquad F_{n\alpha} = c^{-n\alpha}\Bigg[(2\pi)^{(r-d)/2}\det\widetilde{\Phi}''(\theta^*)^{-1/2}\sum_{q=0}^{N-1}(\alpha_d n)^{(r-d)/2-q}
\times \sum_{\substack{0\le j\le\min\{r-1,q\}\\ \max\{0,q-r\}\le k\le q\\ j+k\le q}} L_k(\widetilde{A}_j,\widetilde{\Phi})\binom{r-1}{j}\begin{bmatrix} r-j \\ r+k-q\end{bmatrix}(-1)^{q-j-k}
+ O\left(n^{(r-d)/2-N}\right)\Bigg],
$$

as $n\to\infty$.

Here

$$
L_k(\widetilde{A}_j,\widetilde{\Phi}) = \sum_{l=0}^{2k}\frac{\mathcal{H}^{k+l}(\widetilde{A}_j\underline{\widetilde{\Phi}}^l)(\theta^*)}{(-1)^k 2^{k+l} l!(k+l)!},
$$
$$
\underline{\widetilde{\Phi}}(\theta) = \widetilde{\Phi}(\theta) - \widetilde{\Phi}(\theta^*) - \frac{1}{2}(\theta-\theta^*)\widetilde{\Phi}''(\theta^*)(\theta-\theta^*)^T,
$$

the differential operator $\mathcal{H}$ is given by

$$
\mathcal{H} = -\sum_{1\le a,b\le d+r-2}(\widetilde{\Phi}''(\theta^*)^{-1})_{a,b}\partial_a\partial_b,
$$

and $\left[\begin{smallmatrix} a \\ b\end{smallmatrix}\right]$ denotes the Stirling numbers of the first kind. In every term of $L_k(\widetilde{A}_j,\widetilde{\Phi})$ the total number of derivatives of $\widetilde{A}_j$ and of $\widetilde{\Phi}''$ is at most $2k+j$.

Moreover, for each $N\in\mathbb{N}$ the big-oh constant of $(\star)$ stays bounded as α varies within a compact subset of $\mathbb{R}^d_+$ of the critical cone of c.

Remark 3.5. Regarding the $-1/2$ power occurring in the determinant in $(\star)$, we let $z^{-1/2} = |z|^{-1/2}\exp(-\mathrm{i}\arg z/2)$ for $z\in\mathbb{C}\setminus\{0\}$ with $\arg z\in[-\pi/2,\pi/2]$.

In the smooth point case $r=1$, $(\star)$ agrees with the formula in [RW08, Theorem 3.2]. Moreover, in that case we can allow coordinates of c to be zero as long as $c_k\partial_k H(c)\neq 0$ for some k. Also, when $r=1$ and $d=2$ we can drop the nondegeneracy hypothesis ([RW08, Theorem 3.3]).

In case $r=d$, it can be shown by Leray residue arguments that the asymptotic formula in Theorem 3.4 simplifies: all higher-order terms are zero.

Theorem 3.6 [PW08, Corollary 3.24]. Let $\alpha\in\mathbb{N}^d_+$ and $c\in\mathcal{V}$ be a strictly minimal multiple point with all nonzero coordinates that is critical and nondegenerate for α. If $r=d$ and $G(c)\neq 0$, then there exists $\epsilon\in(0,1)$ such that

$$
F_{n\alpha} = c^{-n\alpha}\left[\frac{G(c)}{|\det J(H,c)|} + O(\epsilon^n)\right],
$$

as $n\to\infty$. Here $J(H,c)$ is the $r\times d$ Jacobian matrix of H.

Moreover, the big-oh constant stays bounded as α varies within a compact subset of $\mathbb{R}^d_+$ of the critical cone of c.

4. Proving Theorem 3.4

To prove Theorem 3.4 we follow an approach similar to that of [PW02, PW04, RW08]. However, in contrast to those articles, here we first assume H has no repeated factors and then show in Section 5 how to reduce to this case via a cohomological trick. We take the following steps.

Step 1: Use Cauchy's integral formula to express $c^{n\alpha}F_{n\alpha}$ as a d-variate integral over a contour C in Ω.

Step 2: Expand the contour C across c_d and use Cauchy's residue theorem to express the innermost integral as a residue.

Step 3: Rewrite the residue as an r-variate integral over the simplex Δ.

Step 4: Rewrite the resulting integral as a Fourier-Laplace integral.

Step 5: Approximate the integral asymptotically.

Starting at step 1, we use Cauchy's integral formula to write

$$c^{n\alpha}F_{n\alpha} = c^{n\alpha}\frac{1}{(2\pi \mathrm{i})^d}\int_C \frac{G(w)dw}{w^{n\alpha+1}H(w)},$$

where C is a contour in Ω. We then follow steps 2–5 by applying the following lemmas, the proofs of which can be found in Section 7.

Lemma 4.1 [PW02, proof of Lemma 4.1], for step 2. Let $\alpha \in \mathbb{N}_+^d$ and $c \in \mathcal{V}$ be a strictly minimal multiple point with nonzero coordinates. There exists $\epsilon \in (0,1)$ and a polydisc neighborhood D of $\widehat{c}$ such that

$$c^{n\alpha}F_{n\alpha} = c^{n\alpha}(2\pi \mathrm{i})^{1-d}\int_X \frac{-R_n(w)}{w^{n\hat{\alpha}+1}}dw + O\left(\epsilon^n\right)$$

as $n \to \infty$, where $X = D \cap C(\widehat{c})$ and $R_n(w)$ is the sum over j of the residues of $y \mapsto y^{-\alpha_d n-1}F(w,y)$ at $h_j(w)$.

The next lemma conveniently expresses the residue sum $R_n(w)$ as an integral.

Lemma 4.2 for step 3. In the previous lemma for $r \geq 2$ we have

$$R_n(w) = \int_\Delta \left(\frac{\partial}{\partial y}\right)^{r-1}(-1)^{r-1}f_n(w,y)\Big|_{y=h(w,s)} ds,$$

where $f_n(w,y) = -y^{\alpha_d n-1}\check{G}(w,y^{-1})$ and ds is the standard volume form $ds_1 \wedge \cdots \wedge ds_{r-1}$ For the smooth case $r=1$ we have $R_n(w) = f_n(w,h(w))$.

Lemma 4.3 for step 4. For $r \geq 2$,

$$c^{n\alpha}F_{n\alpha} = (2\pi)^{1-d}\sum_{j=0}^{r-1}P_j(n)\int_{\widetilde{X}}\int_\Delta \widetilde{A}_j(t,s)\exp(-\alpha_d n\widetilde{\Phi}(t,s))ds\,dt + O(\epsilon^n),$$

as $n \to \infty$, where $\widetilde{X} = e^{-1}(X)$. For $r=1$,

$$c^{n\alpha}F_{n\alpha} = (2\pi)^{1-d}\int_{\widetilde{X}} \widetilde{A}_j(t)\exp(-\alpha_d n\widetilde{\Phi}(t))\,dt + O(\epsilon^n),$$

as $n \to \infty$

The next lemma on Fourier-Laplace integrals provides our key approximation. The function spaces mentioned are complex valued. A stationary and nondegenerate point of a function g is a point θ^* such that $g'(\theta^*) = 0$ and $\det g''(\theta^*) \neq 0$, respectively.

Lemma 4.4 [Hör83, Theorem 7.7.5], for step 5. Let $\mathcal{E} \subset \mathbb{R}^m$ be open, $N \in \mathbb{N}_+$, and $p = N + \lceil m/2 \rceil$. If $A \in C^{2p}(\mathcal{E})$ with compact support in $\mathcal{E}$, $\Phi \in C^{3p+1}(\mathcal{E})$, $\Re\Phi \geq 0$, $\Re\Phi(\theta^*) = 0$, Φ has a unique stationary point $\theta^* \in \operatorname{supp} A$, and θ^* is nondegenerate, then

$$\int_{\mathcal{E}} A(\theta) \exp(-\omega\Phi(\theta)) d\theta = \exp(-\omega\Phi(\theta^*)) \det \left(\frac{\omega\Phi''(\theta^*)}{2\pi} \right)^{-1/2} \sum_{k=0}^{N-1} \omega^{-k} L_k(A, \Phi) + O\left(\omega^{-m/2-N}\right),$$

as $\omega \to \infty$.

Here L_k is the function defined in Theorem 3.4 with $m = d + r - 2$. Moreover, the big-oh constant is bounded when the partial derivatives of Φ up to order $3p + 1$ and the partial derivatives of A up to order $2p$ all stay bounded in supremum norm over $\mathcal{E}$.

The final lemma ensures that the hypotheses of Lemma 4.4 are satisfied in our setting.

Lemma 4.5 for step 5. Let $\alpha \in \mathbb{N}_+^d$ and c be a strictly minimal multiple point that is critical and nondegenerate for α. Then on $\widetilde{X} \times \Delta$, we have $\Re\widetilde{\Phi} \geq 0$ with equality only at points of the form $(0, s)$ (and only at zero for $r = 1$), and $\widetilde{\Phi}$ has a unique stationary point at θ^*.

We can now prove Theorem 3.4.

Proof of Theorem 3.4. By Lemmas 4.1 and 4.3 there exists $\epsilon \in (0, 1)$ and an open bounded neighbourhood $\widetilde{X}$ of 0 such that

$$c^{n\alpha} F_{n\alpha} = (2\pi)^{1-d} \sum_{j=0}^{r-1} P_j(n) I_{j,n} + O(\epsilon^n)$$

as $n \to \infty$, where $I_{j,n} = \int_{\mathcal{E}} \widetilde{A}_j(\theta) \exp(-\alpha_d n \widetilde{\Phi}(\theta)) d\theta$ and $\mathcal{E} = \widetilde{X} \times \Delta^\circ$, where Δ° is the interior of Δ.

Choose $\kappa \in C^\infty(\mathcal{E})$ with compact support in $\mathcal{E}$ (a bump function) such that $\kappa = 1$ on a neighbourhood Y of θ^*. Then

$$I_{j,n} = \int_{\mathcal{E}} \kappa(\theta) \widetilde{A}_j(\theta) \exp(-\alpha_d n \widetilde{\Phi}(\theta)) d\theta + \int_{\mathcal{E}} (1 - \kappa(\theta)) \widetilde{A}_j(\theta) \exp(-\alpha_d n \widetilde{\Phi}(\theta)) d\theta.$$

The second integral decreases exponentially as $n \to \infty$ since $\Re\widetilde{\Phi}$ is strictly positive on the compact set $\overline{\mathcal{E} \setminus Y}$ by Lemma 4.5. By Lemma 4.5 again and the nondegeneracy hypothesis, we we may apply Lemma 4.4 to the first integral. Noting that $L_k(\kappa\widetilde{A}_j, \widetilde{\Phi}) = L_k(\widetilde{A}_j, \widetilde{\Phi})$ because the derivatives are evaluated at θ^* and $\kappa = 1$ in a neighborhood of θ^*, we get

$$\begin{aligned} I_{j,n} &= \exp(-n_d \widetilde{\Phi}(\theta^*)) \det \left(\frac{\alpha_d n \widetilde{\Phi}''(\theta^*)}{2\pi} \right)^{-1/2} \sum_{k=0}^{N-1} (\alpha_d n)^{-k} L_k(\widetilde{A}_j, \widetilde{\Phi}) + O((\alpha_d n)^{-(d-1+r-1)/2-N}) \\ &= (2\pi)^{(d+r-2)/2} \det \widetilde{\Phi}''(\theta^*)^{-1/2} \sum_{k=0}^{N-1} L_k(\widetilde{A}_j, \widetilde{\Phi}) (\alpha_d n)^{-(d+r-2)/2-k} + O\left(n^{-(d+r-2)/2-N}\right) \end{aligned}$$

as $n \to \infty$.

Notice that for $j = 0, \ldots, r - 1$ each $I_{j,n}$ has error $O(n^{-(d+r-2)/2-N})$ and each $P_j(n)$ has degree $r - j - 1$ in n. Thus the error in the asymptotic expansion for $c_{n\alpha} F_{n\alpha}$ will be a sum of terms of the form $O(n^{(r-d)/2-N-j})$ which is $O(n^{(r-d)/2-N})$. So

$$\begin{aligned}
c^{n\alpha}F_{n\alpha} &= (2\pi)^{(r-d)/2}\det\widetilde{\Phi}''(\theta^*)^{-1/2}\sum_{q=0}^{N-1} b_q(\alpha)(\alpha_d n)^{(r-d)/2-q} + O\left(n^{(r-d)/2-N}\right) \\
&= (2\pi)^{(r-d)/2}\det\widetilde{\Phi}''(\theta^*)^{-1/2}\sum_{j=0}^{r-1}\sum_{k=0}^{N-1} P_j(n)L_k(\widetilde{A}_j,\widetilde{\Phi})(\alpha_d n)^{-(d+r-2)/2-k} + O\left(n^{-(r-d)/2-N}\right).
\end{aligned}$$

Let us expand $P_j(n)$ and collect like powers to find the coefficients $b_q(\alpha)$.

The falling factorial powers satisfy $(a-1)^{\underline{m}} = (a-1)\dots(a-1-k) = \frac{1}{a}a^{\underline{m+1}}$ and are related to regular powers and Stirling numbers of the first kind via

$$a^{\underline{m}} = \sum_{l=0}^{p}\begin{bmatrix} m \\ l \end{bmatrix}(-1)^{m-l}a^l;$$

see [GKP94, (6.13)] for instance. Thus

$$P_j(n) = \binom{r-1}{j}\frac{1}{\alpha_d n}\sum_{l=0}^{r-j}\begin{bmatrix} r-j \\ l \end{bmatrix}(-1)^{r-j-l}(\alpha_d n)^l,$$

and so

$$\sum_{q=0}^{N-1} b_q(\alpha)(\alpha_d n)^{(r-d)/2-q} = \sum_{j=0}^{r-1}\sum_{k=0}^{N-1}\sum_{l=0}^{r-j} L_k(\widetilde{A}_j,\widetilde{\Phi})\binom{r-1}{j}\begin{bmatrix} r-j \\ l \end{bmatrix}(-1)^{r-j-l}(\alpha_d n)^{-(d+r)/2-k+l}.$$

The coefficient $b_q(\alpha)$ is found by imposing the constraint $(r-d)/2-q = -(d+r)/2-k+l$. Thus $l = r+k-q$, and we can eliminate the l-sum to arrive at formula $(\star)$.

Lastly, regarding uniformity, we may assume that the $\widetilde{A}_j$ and $\widetilde{\Phi}$ are defined and hence C^∞ on a neighborhood of the closure of $\mathcal{E}$, so that their derivatives up to any given order all stay bounded in supremum norm over $\mathcal{E}$. Now suppose α varies within a compact subset $K \subset \mathbb{R}^d_+$ of the critical cone of c. Since $J_{\log}(H,c)$ has rank $r \le d$ it is a bijective linear transformation from $\mathbb{R}^r$ to its image in $\mathbb{R}^d$ and therefore a bicontinuous function. Thus its inverse maps K to a compact set K' of θ^*s in $\mathcal{E}$. Choose the neighborhood Y in the argument above to contain K' so that one bump function κ works for all θ^*. Since the derivatives of the $\kappa\widetilde{A}_j$ and $\widetilde{\Phi}$ up to any given order all stay bounded in supremum norm over $\mathcal{E}$ and since only $\widetilde{\Phi}$ and $\widetilde{\Phi}'$ depend on α but continuously, we conclude by Lemma 4.4 that for any given N, the big-oh constant in $(\star)$ remains bounded as α varies within K. □

5. The full asymptotic expansion: general case

Again let $c \in \mathcal{V}$ be a strictly minimal multiple point of order r with all coordinates nonzero and let $H = H_1^{a_1}\cdots H_r^{a_r}$ be a local factorization of H. We deal now with the case of arbitrary a_j and r.

In step 2 of the previous section the Cauchy integral can be manipulated to reduce to the special case $a_1 = \ldots = a_r = 1$ and $r \le d$. More specifically, we amend our plan by inserting these three steps after step 2:

(2a) If $r > d$, then decompose F as a sum of fractions whose denominators are of type $\prod_{j\in J} H_j^{b_j}$ where J is a size d subset of $\{1,..,r\}$ and each b_j is an integer with $b_j \le a_j$. So each denominator in the sum has only d irreducible factors of H.

(2b) If some irreducible factor of H is repeated, then treat each resulting integral as the integral of a holomorphic form, and rewrite each integral as the sum of integrals whose denominators are of type $w^{n\alpha+1}\prod_{j\in J} H_j$ where J is a size at most d subset of $\{1,..,r\}$. So each holomorphic form has a denominator with at most d unrepeated irreducible factors of H.

(6) Add up all the asymptotic expansions.

The following two lemmas prove that these additional steps are possible.

Lemma 5.1 [Pem00, Theorem 4.5], for step 2a. Suppose $r > d$, G and $H_1, \dots, H_r$ are holomorphic in a neighborhood U of c, $a_1, \dots, a_r$ are positive integers, and the complex manifolds $\mathcal{V}_j := \{x \in U : H_j(x) = 0\}$ intersect transversely at c. Then there exist an open neighborhood U' of c, functions G_J holomorphic (and possibly zero) on U', and positive integers $b_j \leq a_j$ such that

$$\frac{G}{H_1^{a_1}\cdots H_r^{a_r}} = \sum_{J\in\mathcal{J}} \frac{G_J}{\prod_{j\in J} H_j^{b_j}},$$

where $\mathcal{J}$ is the set of all size d subsets of $\{1, \dots, r\}$.

Lemma 5.2 [AY83, Theorem 17.6], for step 2b. Suppose that $\underline{G}$ and $H_1, \dots, H_r$ are holomorphic functions in a neighborhood of U of c, $r \leq d$, and the complex manifolds $\mathcal{V}_j := \{x \in U : H_j(x) = 0\}$ intersect transversely at c. Then there exist an open neighborhood U' of c and functions $\underline{G}_J$ holomorphic (and possibly zero) on U' such that the holomorphic form

$$\frac{\underline{G}(x)dx}{\prod_{j=1}^r H_j^{b_j}(x)},$$

where $dx = dx_1 \wedge \cdots \wedge dx_d$ and the b_j are positive integers, is de Rham cohomologous in $U' \setminus (\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_r)$ to the holomorphic form

$$\sum_{J\in\mathcal{J}} \frac{\underline{G}_J(x)dx}{\prod_{j\in J} H_j(x)},$$

where $\mathcal{J}$ is the set of all subsets of $\{1, \dots, r\}$. In particular, the integrals of the two forms above over a polycircle in $U' \setminus (\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_r)$ are equal.

Remark 5.3. When applying Lemma 5.2 in step 2b, $\underline{G}(x)$ will be of the form $G(x)/x^{n\alpha+1}$ where $G(x)$ does not contain n and each $b_j \leq a_j$. Thus upon inspection of the constructive proof of Lemma 5.2, the cohomologous form will have n-degree at most $\sum_{j=1}^r (b_j - 1)$.

In particular, if $r \geq d$ and the other assumptions of Theorem 3.6 hold, then we can combine Lemmas 5.1 and 5.2 and Theorem 3.6 to conclude that $c^{n\alpha} F_{n\alpha}$ will be asymptotic with exponentially decaying error term to a polynomial of degree at most $\sum_{j=1}^r a_j - r$, as is also shown in [PW04, Theorem 3.6].

Remark 5.4. In case c is finitely minimal, for each point x of $\mathcal{V} \cap C(c)$ we simply find an open set around x and apply the general procedure above. After that we sum the resulting asymptotic expansions over the finitely many x.

6. Examples

Let us apply the formulas and procedures of Sections 3 and 5 to a few combinatorial examples, that is, to functions with all nonnegative Maclaurin coefficients. We wrote a Sage package called `mgf.sage` to do this. Its source code and the worksheets for the examples below are available at `http://www.cs.auckland.ac.nz/~raichev/research.html`. A detailed description of `mgf.sage` will appear in another article.

We focus on combinatorial examples $F(x)$, because for any $\alpha \in \mathbb{N}_+^d$ there is a minimal point in $\mathcal{V} \cap \mathbb{R}_+^d$ that determines the asymptotics for $F_{n\alpha}$ ([PW08, Theorem 3.16]).

Since there is no known computable procedure to factor an arbitrary polynomial H in the analytic local ring of germs of holomorphic functions about c, we choose examples where H is a polynomial whose local factorization in the algebraic local ring about c equals its factorization in the analytic local ring, that is, H is a polynomial whose irreducible factors in $\mathbb{C}[x]$ are all smooth at c.

Example 6.1 ($r < d$, NO REPEATED FACTORS). Consider the trivariate rational function

$$F(x, y, z) = \frac{1}{(1 - x(1 + y))(1 - zx^2(1 + 2y))}$$

in a sufficiently large neighborhood Ω of the origin; cf [PW08, Example 4.10]. Its coefficients F_ν are all nonnegative, and its denominator $H(x, y, z)$ factors over $\mathbb{C}[x, y, z]$ into irreducible terms $H_1(x, y, z) = 1 - x(1 + y)$ and $H_2(x, y, z) = 1 - zx^2(1 + 2y)$, both of which are globally smooth.

The set of non-smooth/singular points of $\mathcal{V} = \{(x, y, z) \in \Omega : H(x, y, z) = 0\}$ is $\mathcal{V}' = \{(x, y, z) \in \Omega : H(x, y, z) = \nabla H(x, y, z) = 0\} = \{(1/(a + 1), a, (a + 1)^2/(2a + 1)) : a \in \mathbb{C} \setminus \{-1\}\}$, which consists entirely of multiple points of order $r = 2$. A simple check shows that the points $(1/(a + 1), a, (a + 1)^2/(2a + 1))$ for $a > 0$ are strictly minimal.

The critical cone for each such point is the conical hull of the vectors $\gamma_1 = (1, a/(a + 1), 0)$ and $\gamma_2 = (1, a/(2a + 1), 1/2)$.

For instance, $c = (1/2, 1, 4/3)$ controls asymptotics for all α in the conical hull of the vectors $\gamma_1(c) = (1, 1/2, 0)$ and $\gamma_2(c) = (1, 1/3, 1/2)$. For instance, $\alpha = (8, 3, 3)$ is in this critical cone, and applying Theorem 3.4 we get

$$F_{n\alpha} = 108^n \left[\frac{\sqrt{3}}{\sqrt{7\pi}} \left(n^{-1/2} - \frac{1231}{24696} n^{-3/2} \right) + O(n^{-5/2}) \right]$$

as $n \to \infty$.

Comparing this approximation with the actual values of $F_{n\alpha}$ for small n, we get the following table.

n	1	2	4	8
$F_{n\alpha}/108^n$	0.3518518519	0.2548010974	0.1823964231	0.1297748629
$\frac{\sqrt{3}}{\sqrt{7\pi}} n^{-1/2}$	0.3693487820	0.2611690282	0.1846743909	0.1305845142
$\frac{\sqrt{3}}{\sqrt{7\pi}} \left(n^{-1/2} - \frac{1231}{24696} n^{-3/2}\right)$	0.3509381749	0.2546598957	0.1823730650	0.1297708726
one-term relative error	-0.1823730650	-0.02499177148	-0.01248910347	-0.006238891584
two-term relative error	0.002596766210	0.0005541644108	0.0001280622701	0.00003074786527

Example 6.2 ($r < d$, NO REPEATED FACTORS). Consider the trivariate rational function

$$F(x, y, z) = \frac{16}{(4 - 2x - y - z)(4 - x - 2y - z)}$$

in a sufficiently large neighborhood Ω of the origin; cf [PW04, Example 3.10]. Its coefficients F_ν are all nonnegative, and its denominator $H(x, y, z)$ factors over $\mathbb{C}[x, y, z]$ into irreducible terms $H_1(x, y, z) = 4 - 2x - y - z$ and $H_2(x, y, z) = 4 - x - 2y - z$, both of which are globally smooth.

The set of non-smooth points of $\mathcal{V} = \{(x, y, z) \in \Omega : H(x, y, z) = 0\}$ is $\mathcal{V}' = \{(x, y, z) \in \Omega : H(x, y, z) = \nabla H(x, y, z) = 0\} = \{(1 - a, 1 - a, 1 + 3a : a \in \mathbb{C}\}$, which contains a line segment $\{(1 - a, 1 - a, 1 + 3a : -1/3 < a < 1\}$ of multiple points of order $r = 2$. The multiple point $c = (1, 1, 1)$ is strictly minimal and its critical cone is the conical hull of the vectors $\gamma_1(c) = (2, 1, 1)$ and $\gamma_2(c) = (1, 2, 1)$.

For instance, $\alpha = (3, 3, 2)$ is in the critical cone and applying Theorem 3.4 we get

$$F_{n\alpha} = \frac{1}{\sqrt{3\pi}} \left(4n^{-1/2} - \frac{25}{72} n^{-3/2} \right) + O(n^{-5/2}),$$

as $n \to \infty$.

Comparing this approximation with the actual values of $F_{n\alpha}$ for small n, we get the following table.

n	1	2	4	8	16
$F_{n\alpha}$	0.7849731445	0.7005249476	0.5847732654	0.4485547669	0.3237528587
$\frac{4}{\sqrt{3\pi}} n^{-1/2}$	1.302940032	0.9213177320	0.6514700159	0.4606588663	0.3257350080
$\frac{1}{\sqrt{3\pi}} \left(4n^{-1/2} - \frac{25}{72} n^{-3/2}\right)$	1.189837599	0.8813299832	0.6373322118	0.4556603976	0.3239677825
one-term relative error	-0.6598530041	-0.3151819006	-0.1140557451	-0.02698466340	-0.006122414820
two-term relative error	-0.5157685423	-0.2580993528	-0.2580993528	-0.01584116640	-0.0006638514355

Example 6.3 ($r < d$, REPEATED FACTORS). Consider the trivariate rational function

$$F(x, y, z) = \frac{16}{(4 - 2x - y - z)^2 (4 - x - 2y - z)}$$

in a sufficiently large neighborhood Ω of the origin. Its coefficients F_ν are all nonnegative, and its denominator $H(x, y, z) = (4 - 2x - y - z)^2 (4 - x - 2y - z)$ is shown factored over $\mathbb{C}[x, y, z]$. Since H contains repeated factors, we first reduce

$$\frac{F(x, y, z)\, dx \wedge dy \wedge dz}{x^{\alpha_1 n+1} y^{\alpha_2 n+1} z^{\alpha_3 n+1}},$$

the differential form of the Cauchy integral of F, to a de Rham cohomologous form with no repeated factors, namely

$$\frac{[16(2\alpha_3 y - \alpha_2 z)n + 16(2y - z)]/(yz)\, dx \wedge dy \wedge dz}{(4 - 2x - y - z)(4 - x - 2y - z) x^{\alpha_1 n+1} y^{\alpha_2 n+1} z^{\alpha_3 n+1}},$$

which determines the asymptotics of $F_{n\alpha}$. The constructive procedure in the proof of Lemma 5.2 to find such a cohomologous form is implemented in `mgf.sage`.

The singular variety $\mathcal{V}$ of this new form is the same as in the previous example and so the singularity analysis is the same.

Taking $\alpha = (3, 3, 2)$ again, for instance, and applying Theorem 3.4 we get

$$F_{n\alpha} = \frac{1}{\sqrt{3\pi}} \left(4n^{1/2} + \frac{47}{72} n^{-1/2} \right) + O(n^{-3/2}),$$

as $n \to \infty$.

It is a coincidence that the leading coefficient above is the same as the leading coefficient in the previous example without repeated factors. Using the denominator $(4 - 2x - y - z)^3 (4 - x - 2y - z)$ instead, for instance, gives a different leading coefficient.

Comparing our approximation with the actual values of $F_{n\alpha}$ for small n, we get the following table.

n	1	2	4	8	16
$F_{n\alpha}$	0.9812164307	1.576181132	2.485286378	3.700576827	5.260983954
$\frac{4}{\sqrt{3\pi}}n^{1/2}$	1.302940032	1.842635464	2.605880063	3.685270927	5.211760127
$\frac{1}{\sqrt{3\pi}}\left(4n^{1/2}+\frac{47}{72}n^{-1/2}\right)$	1.515572607	1.992989400	2.712196350	3.760447895	5.264918270
one-term relative error	-0.3278824031	-0.1690505784	-0.04852305395	0.004136084917	0.009356391776
two-term relative error	-0.5445854345	-0.2644418586	-0.09130133815	-0.01617884746	-0.0007478289298

Notice that in this case the two-term approximation to $F_{n\alpha}$ is not an improvement over the one-term approximation until somewhere between $n = 8$ and $n = 16$. The question of how many terms of a divergent asymptotic series expansion to use for a given argument to obtain the best approximation/least error is called the question of 'optimal truncation' or 'optimal approximation'. See [PK01], for instance, for more details.

Example 6.4 ($r \geq d$ WITH NO REPEATED FACTORS). Consider the bivariate function

$$F(x, y) = \frac{9\exp(x + y)}{(3 - 2x - y)(3 - x - 2y)}$$

in a sufficiently large neighborhood Ω of the origin; cf [PW08, Example 4.12].

Its coefficients F_ν are all nonnegative, and its denominator $H(x, y)$ factors over $\mathbb{C}[x, y]$ into irreducible terms $H_1(x, y) = 3 - 2x - y$ and $H_2(x, y) = 3 - x - 2y$, both of which are globally smooth.

The set of non-smooth points of $\mathcal{V} = \{(x, y) \in \Omega : H(x, y) = 0\}$ is $\mathcal{V}' = \{(x, y) \in \Omega : H(x, y) = \nabla H(x, y) = 0\}$, which consists of the multiple point $c = (1, 1)$ of order $r = 2$. The point c is strictly minimal and its critical cone is the conical hull of the vectors $\gamma_1(c) = (2, 1)$ and $\gamma_2(c) = (1/2, 1)$.

By Theorem 3.6, for any α in this critical cone we get

$$F_{n\alpha} = 3\mathrm{e}^2 + O(\epsilon^n),$$

as $n \to \infty$, where $\epsilon \in (0, 1)$.

Example 6.5 ($r \geq d$ WITH REPEATED FACTORS). Consider the bivariate function

$$F(x, y) = \frac{9\exp(x + y)}{(3 - 2x - y)^2(3 - x - 2y)^2},$$

which is a variation of the function of the previous example.

Since the denominator of F contains repeated factors, we first reduce

$$\frac{F(x, y)\, dx \wedge dy}{x^{\alpha_1 n+1} y^{\alpha_2 n+1}},$$

the differential form of the Cauchy integral of F, to a de Rham cohomologous form with no repeated factors which `mgf.sage` computes.

Reusing the analysis of the previous example and applying Theorem 3.6 to any α in conical hull of the vectors $\gamma_1(c) = (2, 1)$ and $\gamma_2(c) = (1/2, 1)$ we get

$$F_{n\alpha} = -3\mathrm{e}^2(2\alpha_1 - \alpha_2)(\alpha_1 - 2\alpha_2)n^2 - 6\mathrm{e}^2\,(\alpha_1 + \alpha_2)n - 12\mathrm{e}^2 + O(\epsilon^n),$$

as $n \to \infty$, where $\epsilon \in (0, 1)$.

7. Remaining Proofs

Proof of Lemma 4.2. Let $f_n(w,y) = -y^{\alpha_d n-1}\check{G}(w,y^{-1})$. Then for $r \geq 2$,

$$\begin{aligned}
R_n(w) &= \sum_{j=1}^{r} \lim_{y\to h_j(w)^{-1}} y^{-\alpha_d n-1}(y-h_j(w)^{-1})F(w,y) \\
&= \sum_{j=1}^{r} \lim_{y\to h_j(w)^{-1}} -y^{-\alpha_d n}h_j(w)^{-1}(y^{-1}-h_j(w))\frac{\check{G}(w,y)}{\prod_{k=1}^{r}(y^{-1}-h_k(w))} \\
&= \sum_{j=1}^{r} \frac{f_n(w,h_j(w))}{\prod_{k\neq j}(h_j(w)-h_k(w))} \\
&= \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{r-2}} \left(\frac{\partial}{\partial y}\right)^{r-1} f_n(w,(1-\sigma_1)h_1+(\sigma_1-\sigma_2)h_2+\cdots \\
&\quad (\sigma_{r-2}-\sigma_{r-1})h_{r-1}+\sigma_{r-1}h_r)\, d\sigma_{r-1} \\
&\quad \text{(by [DL93, Chapter 4, Section 7, equations (7.7) and (7.12)])} \\
&= \int_\Delta \left(\frac{\partial}{\partial y}\right)^{r-1} (-1)^{r-1} f_n(w, s_1h_1+\cdots+s_{r-1}h_{r-1}+(1-\sum_{j=1}^{r-1})h_r)\, ds \\
&\quad \text{(by the change of variables } (s_1,\ldots,s_{r-1}) = (1-\sigma_1,\sigma_1-\sigma_2,\ldots,\sigma_{r-2}-\sigma_{r-1})\text{)},
\end{aligned}$$

as desired.

Notice that the $(-1)^{r-1}$ cancels with the $(-1)^{r-1}$ in the definition of f_n.

For $r=1$, we have $R_n(w) = \lim_{y\to h_0(w)^{-1}} y^{-\alpha_d n-1}(y-h_0(w)^{-1})F(w,y) = f_n(w,h(w))$. ∎

Proof of Lemma 4.3. First, for $r \geq 2$,

$$\begin{aligned}
&\left(\frac{\partial}{\partial y}\right)^{r-1} (-1)^{r-1} f(w,y) \\
&= \left(\frac{\partial}{\partial y}\right)^{r-1} (-1)^{r} y^{\alpha_d n-1}\check{G}(w,y^{-1}) \\
&= -\sum_{j=0}^{r-1} \binom{r-1}{j} \left(\frac{\partial}{\partial y}\right)^{r-1-j} y^{\alpha_d n-1}(-1)^{r-1} \left(\frac{\partial}{\partial y}\right)^{j} \check{G}(w,y^{-1}) \\
&= -\sum_{j=0}^{r-1} \binom{r-1}{j} (\alpha_d n-1)\underline{{}^{r-1-j}} y^{\alpha_d n-r+j}(-1)^{r-1} \left(\frac{\partial}{\partial y}\right)^{j} \check{G}(w,y^{-1}) \\
&= -\sum_{j=0}^{r-1} P_j(n) y^{-\alpha_d n} A_j(w,y).
\end{aligned}$$

Thus

$$
\begin{aligned}
&c^{n\alpha}(2\pi\mathrm{i})^{1-d}\int_X \frac{-R(w)}{w^{n\hat{\alpha}+1}}dw\\
=&c^{n\alpha}(2\pi\mathrm{i})^{1-d}\int_X \frac{1}{w^{n\hat{\alpha}+1}}\int_\Delta \left(\frac{\partial}{\partial y}\right)^{r-1}(-1)^{r-1}f(w,y)\Big|_{y=h(w,s)}ds\,dw\\
&\text{(by Lemma 4.2)}\\
=&c^{n\alpha}(2\pi\mathrm{i})^{1-d}\sum_{j=0}^{r-1}P_j(n)\int_X \frac{1}{w^{n\hat{\alpha}+1}}\int_\Delta h(w,s)^{\alpha_d n}A_j(w,h(w,s))ds\,dw\\
=&(2\pi\mathrm{i})^{1-d}\sum_{j=0}^{r-1}P_j(n)\int_X\int_\Delta \frac{\widehat{c}^{n\widehat{\alpha}}}{w^{n\widehat{\alpha}}}A_j(w,h(w,s))(c_d h(w,s))^{\alpha_d n}ds\frac{dw}{\prod_{m=1}^{d-1}w_m}\\
=&(2\pi)^{1-d}\sum_{j=0}^{r-1}P_j(n)\int_{\widetilde{X}}\int_\Delta \prod_{m=1}^{d-1}\exp(-\mathrm{i}\alpha_m n t_m)\widetilde{A}_j(t,s)(c_d\widetilde{h}(t,s))^{\alpha_d n}ds\,dt\\
&\text{(via the change of variables } w=e(t))\\
=&(2\pi)^{1-d}\sum_{j=0}^{r-1}P_j(n)\int_{\widetilde{X}}\int_\Delta \widetilde{A}_j(t,s)\,\exp(-\alpha_d n\widetilde{\Phi}(t,s))ds\,dt,
\end{aligned}
$$

which with Lemma 4.1 proves the stated formula for $c^{n\alpha}F_{n\alpha}$.

The formula for the case $r=1$ follows similarly. □

Proof of Lemma 4.5. First $\widetilde{\Phi}(0,s)=0$ and

$$\Re\widetilde{\Phi}(t,s) = -\log|c_d\widetilde{h}(t,s)| \geq -\log\sum_{j=1}^r s_j|c_d h_j(e(t))| > 0$$

for $t \neq 0$, because the sum is convex and $|h_j(w)^{-1}| > |c_d|$ for $w \neq \widehat{c}$ since c is strictly minimal.

Now, by the implicit function theorem, $\partial_k h_j(w) = h_j(w)^2\partial_k H_j(w,1/h_j(w))/\partial_d H_j(w,1/h_j(w))$ for $k<d$, $j \leq r$, and $w \in W$. Thus for all $k<d$ we have

$$
\begin{aligned}
\partial_k\widetilde{\Phi}(\theta^*) =& -\mathrm{i}\frac{c_k\exp(\mathrm{i}t_k)}{h(e(t),s)}\sum_{j=1}^r s_j^*\frac{\partial_k h_j(e(t))^2 H_j(e(t),1/h_j(e(t))}{\partial_d H_j(e(t),1/h_j(e(t)))} + \mathrm{i}\frac{\alpha_k}{\alpha_d}\Big|_{\theta^*}\\
=& -\mathrm{i}\sum_{j=1}^r s_j^*\frac{c_k\partial_k H_j(c)}{c_d\partial_d H_j(c)} + \mathrm{i}\frac{\alpha_k}{\alpha_d}\\
=&0,
\end{aligned}
$$

since c is critical for α. Also $\partial_k\widetilde{\Phi}(\theta^*) = 0$ for $d \leq k \leq r+d-2$ since $\widetilde{\Phi}(0,s)$ is constant. Thus $\widetilde{\Phi}'(\theta^*) = 0$. Now $\det\widetilde{\Phi}''(\theta^*) \neq 0$, since c is nondegenerate for α. So there is a neighborhood of θ^* in which θ^* is the only zero of $\widetilde{\Phi}'$. Thus, shrinking $\widetilde{X}\times\Delta$ if needed, θ^* is the unique stationary point of $\widetilde{\Phi}$. □

References

[AY83] I. A. Aĭzenberg and A. P. Yuzhakov, *Integral representations and residues in multidimensional complex analysis*, Translations of Mathematical Monographs, vol. 58, American Mathematical Society, Providence, RI, 1983, Translated from the Russian by H. H. McFaden, Translation edited by Lev J. Leifman. MR MR735793 (85a:32006) [dJP00] Theo de Jong and Gerhard Pfister, *Local analytic geometry*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2000, Basic theory and applications. MR MR1760953 (2001c:32001)

[DL93] Ronald A. DeVore and George G. Lorentz, *Constructive approximation*, Grundlehren der Mathematischen Wissenschaften, vol. 303, Springer-Verlag, Berlin, 1993. MR MR1261635 (95f:41001)

[GKP94] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete mathematics*, second ed., Addison-Wesley Publishing Company, Reading, MA, 1994, A foundation for computer science. MR 1397498 (97d:68003)

[Hör83] Lars Hörmander, *The analysis of linear partial differential operators. I*, Grundlehren der Mathematischen Wissenschaften, vol. 256, Springer-Verlag, Berlin, 1983, Distribution theory and Fourier analysis. MR MR717035 (85g:35002a)

[Lla06] Manuel Lladser, *Uniform formulae for coefficients of meromorphic functions in two variables. I*, SIAM J. Discrete Math. **20** (2006), no. 4, 811–828 (electronic). MR MR2272232 (2007i:32001)

[Pem00] Robin Pemantle, *Generating functions with high-order poles are nearly polynomial*, Mathematics and computer science (Versailles, 2000), Trends Math., Birkhäuser, Basel, 2000, pp. 305–321. MR MR1798308 (2002k:05011)

[PK01] R. B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes integrals*, Encyclopedia of Mathematics and its Applications, vol. 85, Cambridge University Press, Cambridge, 2001. MR 1854469 (2002h:33001)

[PW02] Robin Pemantle and Mark C. Wilson, *Asymptotics of multivariate sequences. I. Smooth points of the singular variety*, J. Combin. Theory Ser. A **97** (2002), no. 1, 129–161. MR MR1879131 (2003a:05015)

[PW04] ———, *Asymptotics of multivariate sequences. II. Multiple points of the singular variety*, Combin. Probab. Comput. **13** (2004), no. 4-5, 735–761. MR MR2095981 (2005i:05008)

[PW08] ———, *Twenty combinatorial examples of asymptotics derived from multivariate generating functions*, SIAM Rev. **50** (2008), no. 2, 199–272. MR MR2403050

[RW08] Alexander Raichev and Mark C. Wilson, *Asymptotics of coefficients of multivariate generating functions: improvements for smooth points*, Electron. J. Combin. **15** (2008), no. 1, Research Paper 89, 17. MR MR2426152

[Tay02] Joseph L. Taylor, *Several complex variables with connections to algebraic geometry and Lie groups*, Graduate Studies in Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2002. MR MR1900941 (2004b:32001)

Department of Computer Science, University of Auckland, Private Bag 92019, Auckland, New Zealand
E-mail address: raichev@cs.auckland.ac.nz

Department of Computer Science, University of Auckland, Private Bag 92019, Auckland, New Zealand
E-mail address: mcw@cs.auckland.ac.nz