

EXISTENCE OF GLOBAL WEAK SOLUTIONS FOR NAVIER-STOKES EQUATIONS WITH LARGE FLUX

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ABSTRACT. Global existence of weak solutions to the Navier-Stokes equation in a cylindrical domain under the slip boundary conditions and with inflow and outflow was proved. To prove the energy estimate, crucial for the proof, we use the Hopf function. This makes us possible to derive such estimate that the inflow and outflow must not vanish as $t \rightarrow \infty$. The proof requires estimates in weighted Sobolev spaces for solutions to the Poisson equation. Finally, the paper is the first step to prove the existence of global regular special solutions to the Navier-Stokes equations with inflow and outflow.

1. INTRODUCTION

We consider viscous incompressible fluid motion in a finite cylinder with large inflow and outflow, assuming boundary slip conditions. Hence, the following initial boundary value problem is examined.

$$\begin{aligned}
 (1.1) \quad & v_t + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v, p) = f && \text{in } \Omega^T = \Omega \times (0, T), \\
 & \operatorname{div} v = 0 && \text{in } \Omega^T, \\
 & v \cdot \bar{n} = 0 && \text{on } S_1^T, \\
 & \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\
 & v \cdot \bar{n} = d && \text{on } S_2^T, \\
 & \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_2^T, \\
 & v|_{t=0} = v(0) && \text{in } \Omega,
 \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is a cylindrical domain, $S = \partial\Omega$, v is the velocity of the fluid motion with $v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}^1$ denotes the pressure, $f = f(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ – the external force field, $x = (x_1, x_2, x_3)$ are the Cartesian coordinates, \bar{n} is the unit outward vector normal to the boundary S and $\bar{\tau}_\alpha$, $\alpha = 1, 2$, are tangent vectors to S and \cdot denotes the scalar product in \mathbb{R}^3 . We define the stress tensor $\mathbb{T}(v, p)$ as

$$\mathbb{T}(v, p) = \nu \mathbb{D}(v) - p \mathbb{I},$$

where ν is the constant viscosity coefficient and \mathbb{I} is the unit matrix. Next, $\gamma > 0$ is the slip coefficient and $\mathbb{D}(v)$ denotes the dilatation tensor of the form

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

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We assume that $\Omega \subset \mathbb{R}^3$ is a cylindrical type domain parallel to the axis x_3 with arbitrary cross section. We set $S = S_1 \cup S_2$ where S_1 is the part of the boundary which is parallel to the axis x_3 and S_2 is perpendicular to x_3 . Hence

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) = c_0, -a < x_3 < a\}, \\ S_2(-a) &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_0, x_3 = -a\}, \\ S_2(a) &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_0, x_3 = a\} \end{aligned}$$

where a, c_0 are positive given numbers and $\varphi_0(x_1, x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$.

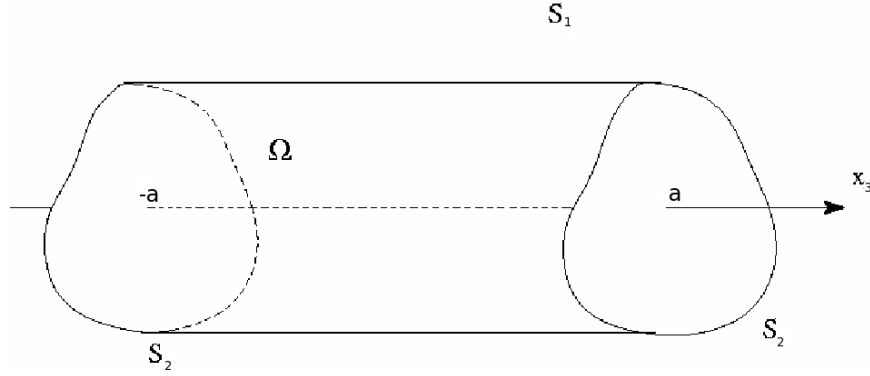


FIGURE 1. Domain Ω .

To describe inflow and outflow we define

$$(1.2) \quad \begin{aligned} d_1 &= -v \cdot \bar{n}|_{S_2(-a)} \\ d_2 &= v \cdot \bar{n}|_{S_2(a)} \end{aligned}$$

with $d_i \geq 0, i = 1, 2$. We infer compatibility conditions

$$(1.3) \quad \int_{S_2(-a)} d_1 dS_2 = \int_{S_2(a)} d_2 dS_2.$$

The aim of this paper is to prove the existence of global weak solutions to problem (1.1) without restrictions on magnitudes of external force f , initial data $v(0)$, inflow d_1 and outflow d_2 . We would like to show the existence of such solutions that the flux does not have to vanish as $t \rightarrow \infty$. The presented in our paper method would allow us to prove the existence of global regular solutions in the cylinder (in the meaning of [RZ3]) which are much more general than in [K1], [K2], [Z] because in these papers the flux must converge to zero sufficiently fast.

We define a space natural for the study of the weak solutions to the Navier-Stokes equations:

$$V_2^0(\Omega^T) = \{u : \|u\|_{V_2^0(\Omega^T)} = \text{ess sup}_{t \in (0, T)} \|u\|_{L_2(\Omega)} + \left(\int_0^T \|\nabla u\|_{L_2(\Omega)}^2 dt \right)^{1/2} < \infty\}.$$

To simplify the notation, we do not distinguish between norms of scalar and vector function and we write

$$\|f\| := \sum_{i=1}^3 \|f_i\| \quad \text{for any } f = (f_1, f_2, f_3).$$

We also use

$$\|d\| := \|d_1\| + \|d_2\|$$

for inflow d_1 and outflow d_2 .

Theorem 1. *Assume the compatibility condition (1.3). Assume that $v(0) \in L_2(\Omega)$, $f \in L_2(0, T; L_{6/5}(\Omega))$, $d_i \in L_\infty(0, T; W_p^{s-1/p}(S_2)) \cap L_2(0, T; W_2^{1/2}(S_2))$, $\frac{3}{p} + \frac{1}{3} \leq s, p > 3$ or $p = 3, s > \frac{4}{3}$, $d_{i,t} \in L_2(0, T; W_{6/5}^{1/6}(S_2))$, $i = 1, 2$. Then there exists a weak solution v to problem (1.1) such that v is weakly continuous with respect to t in $L^2(\Omega)$ norm and v converges to v_0 as $t \rightarrow 0$ strongly in $L^2(\Omega)$ norm. Moreover, $v \in V_2^0(\Omega^T)$, $v \cdot \bar{\tau}_\alpha \in L_2(0, T; L_2(S_1))$, $\alpha = 1, 2$, and v satisfies*

$$\begin{aligned} & \|v\|_{V_2^0(\Omega^t)}^2 + \gamma \sum_{\alpha=1}^2 \int_0^t \|v \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \leq 2 \|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\ & + \varphi \left(\sup_{\tau \leq t} \|d\|_{W_3^{s-1/p}(S_2)} \right) \left(\|d\|_{L_2(0,t;W_2^{1/2}(S_2))}^2 + \|d_t\|_{L_2(0,t;W_{6/5}^{1/6}(S_2))}^2 \right) + \|v(0)\|_{L_2(\Omega)}^2 \end{aligned} \quad (1.4)$$

where φ is a nonlinear positive increasing function of its argument and $t \leq T$.

Theorem 2. *Assume the compatibility condition (1.3). Let $f \in L_2(kT, (k+1)T; L_{6/5}(\Omega))$, $d_i \in L_\infty(R^+; W_p^{s-1/p}(S_2)) \cap L_2(kT, (k+1)T; W_2^{1/2}(S_2))$, where $\frac{3}{p} + \frac{1}{3} \leq s, p > 3$ or $p = 3, s > \frac{4}{3}$, and $d_{i,t} \in L_2(kT, (k+1)T; W_{6/5}^{1/6}(S_2))$, $i = 1, 2$. Let us assume that*

$$\|v(0)\|_{L_2(\Omega)} \leq A$$

for some constant A and

$$\begin{aligned} & 2 \int_{kT}^{(k+1)T} \|f\|_{L_{6/5}(\Omega)}^2 + \varphi \left(\sup_t \|d\|_{W_p^{s-1/p}(S_2)} \right) \int_{kT}^{(k+1)T} \left(\|d\|_{W_2^{1/2}(S_2)}^2 + \|d_t\|_{W_{6/5}^{1/6}(S_2)}^2 \right) \\ & \leq (1 - e^{-\nu T}) A^2 \end{aligned}$$

for $k \in \mathbb{N}_0$, where φ is a nonlinear positive increasing function of its argument. Then there exists a global weak solution v to (1.1) such that

$$v \in V_2^0(\Omega \times (kT, (k+1)T)) \quad \forall k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and

$$\begin{aligned} & \|v\|_{V_2^0(\Omega \times (kT, t))}^2 \leq 2 \int_{kT}^t \|f\|_{L_{6/5}(\Omega)}^2 d\tau + A^2 \\ & + \varphi \left(\sup_\tau \|d\|_{W_p^{s-1/p}(S_2)} \right) \int_{kT}^t \left(\|d\|_{W_2^{1/2}(S_2)}^2 + \|d_t\|_{W_{6/5}^{1/6}(S_2)}^2 \right) d\tau \end{aligned} \quad (1.5)$$

for $t \in (kT, (k+1)T]$.

The main step in this paper is estimate (2.15)- see Lemma 2.2. To derive it, we use the Hopf function (see [L], [G]) and estimates in weighted Sobolev spaces (see [RZ1],[RZ2].) The estimate is such that we can show global estimate (4.4) and prove global existence without assumption of vanishing of the inflow-outflow and

the external force. The paper makes possible to generalize the result from [RZ3] into the inflow-outflow case.

2. ESTIMATES

To show the existence theorem, we need to obtain the energy type estimate and for this purpose, we have to make the Neumann boundary condition (1.1)₅ homogeneous.

To this end, we extend functions corresponding to inflow and outflow so that

$$(2.1) \quad \tilde{d}_i|_{S_2(a_i)} = d_i, \quad i = 1, 2, \quad a_1 = -a, \quad a_2 = a$$

We introduce the function η , see [L].

$$\eta(\sigma; \varepsilon, \rho) = \begin{cases} 1 & 0 \leq \sigma \leq \rho e^{-1/\varepsilon} \equiv r, \\ -\varepsilon \ln \frac{\sigma}{\rho} & r < \sigma \leq \rho, \\ 0 & \rho < \sigma < \infty. \end{cases}$$

We calculate

$$\frac{d\eta}{d\sigma} = \eta'(\sigma; \varepsilon, \rho) = \begin{cases} 0 & 0 < \sigma \leq r, \\ -\frac{\varepsilon}{\sigma} & r < \sigma \leq \rho, \\ 0 & \rho < \sigma < \infty. \end{cases}$$

so that $|\eta'(\sigma; \varepsilon, \rho)| \leq \frac{\varepsilon}{\sigma}$. We define functions η_i on the neighborhood of S_2 (inside Ω):

$$\eta_i = \eta(\sigma_i; \varepsilon, \rho), \quad i = 1, 2,$$

where σ_i denote local coordinates defined on small neighborhood of $S_2(a_i)$:

$$\sigma_1 = a + x_3, \quad \sigma_2 = a - x_3$$

and we set

$$(2.2) \quad \begin{aligned} \alpha &= \sum_{i=1}^2 \tilde{d}_i \eta_i, \\ b &= \alpha \bar{e}_3, \quad \bar{e}_3 = (0, 0, 1). \end{aligned}$$

We construct function u so that

$$(2.3) \quad u = v - b.$$

Therefore,

$$\begin{aligned} \operatorname{div} u &= -\operatorname{div} b = -\alpha_{x_3} \quad \text{in } \Omega, \\ u \cdot \bar{n} &= 0 \quad \text{on } S. \end{aligned}$$

Then, the boundary condition for u is homogeneous. The compatibility condition takes the form

$$\int_{\Omega} \alpha_{,x_3} dx = - \int_{S_2(-a)} \alpha|_{x_3=-a} dS_2 + \int_{S_2(a)} \alpha|_{x_3=a} dS_2 = 0$$

We define function φ as a solution to the Neumann problem

$$(2.4) \quad \begin{aligned} \Delta\varphi &= -\operatorname{div} b \quad \text{in } \Omega, \\ \bar{n} \cdot \nabla\varphi &= 0 \quad \text{on } S, \\ \int_{\Omega} \varphi dx &= 0. \end{aligned}$$

Next, we set

$$(2.5) \quad w = u - \nabla\varphi = v - (b + \nabla\varphi) \equiv v - \delta.$$

Consequently, (w, p) is a solution to the following problem

$$(2.6) \quad \begin{aligned} w_t + w \cdot \nabla w + w \cdot \nabla\delta + \delta \cdot \nabla w - \operatorname{div} \mathbb{T}(w, p) \\ &= f - \delta_t - \delta \cdot \nabla\delta + \nu \operatorname{div} \mathbb{D}(\delta) = F(\delta, t) \quad \text{in } \Omega^T, \\ \operatorname{div} w &= 0 \quad \text{in } \Omega^T, \\ w \cdot \bar{n} &= 0 \quad \text{on } S^T, \\ \nu \bar{n} \cdot \mathbb{D}(w) \cdot \bar{\tau}_\alpha + \gamma w \cdot \bar{\tau}_\alpha \\ &= -\nu \bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha - \gamma \delta \cdot \bar{\tau}_\alpha = B_{1\alpha}(\delta), \quad \alpha = 1, 2, \quad \text{on } S_1^T, \\ \bar{n} \cdot \mathbb{D}(w) \cdot \bar{\tau}_\alpha &= -\bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha = B_{2\alpha}(\delta), \quad \alpha = 1, 2, \quad \text{on } S_2^T, \\ w|_{t=0} &= v(0) - \delta(0) = w(0) \quad \text{in } \Omega, \end{aligned}$$

where $\operatorname{div} \delta = 0$. Moreover, we set

$$\begin{aligned} \bar{n}|_{S_1} &= \frac{(\varphi, x_1, \varphi, x_2, 0)}{\sqrt{\varphi, x_1^2 + \varphi, x_2^2}}, \quad \bar{\tau}_1|_{S_1} = \frac{(-\varphi, x_2, \varphi, x_1, 0)}{\sqrt{\varphi, x_1^2 + \varphi, x_2^2}}, \quad \bar{\tau}_2|_{S_1} = (0, 0, 1) = \bar{e}_3, \\ \bar{n}|_{S_2(-a)} &= -\bar{e}_3, \quad \bar{n}|_{S_2(a)} = \bar{e}_3, \quad \bar{\tau}_1|_{S_2} = \bar{e}_1, \quad \bar{\tau}_2|_{S_2} = \bar{e}_2 \end{aligned}$$

where $\bar{e}_1 = (1, 0, 0)$, $\bar{e}_2 = (0, 1, 0)$.

We define a weak solution to the problem (2.6)

Definition 2.1. *We call w a weak solution to problem (2.6) if for any sufficiently smooth function ψ such that*

$$\operatorname{div} \psi|_{\Omega} = 0, \quad \psi \cdot \bar{n}|_S = 0$$

the integral equality

$$\begin{aligned} \int_{\Omega^T} w_t \cdot \psi dxdt + \int_{\Omega^T} H(w) \cdot \psi dxdt + \nu \int_{\Omega^T} \mathbb{D}(v) \cdot \mathbb{D}(\psi) dxdt + \gamma \sum_{\alpha=1}^2 \int_{S_1^T} w \cdot \bar{\tau}_\alpha \psi \cdot \bar{\tau}_\alpha dS_1 dt \\ - \sum_{\alpha, \sigma=1}^2 \int_{S_\sigma^T} B_{\sigma\alpha} \psi \cdot \bar{\tau}_\alpha dS_\sigma dt = \int_{\Omega^T} F \cdot \psi dxdt \end{aligned}$$

holds, where

$$H(w) = w \cdot \nabla w + w \cdot \nabla\delta + \delta \cdot \nabla w.$$

Lemma 2.2. *Assume the compatibility condition (1.3). Assume that $f \in L_2(0, T; L_{6/5}(\Omega))$, $d_i \in L_\infty(0, T; W_p^{s-1/p}(S_2)) \cap L_2(0, T; W_2^{1/2}(S_2))$, where $\frac{3}{p} + \frac{1}{3} \leq s, p > 3$ or $p = 3, s > \frac{4}{3}$, $d_{i,t} \in L_2(0, T; W_{6/5}^{1/6}(S_2))$, $i = 1, 2$, $w(0) \in L_2(\Omega)$. Then for a weak*

solution to (2.6) holds

$$(2.7) \quad \begin{aligned} & \|w\|_{V_2^0(\Omega^t)}^2 + \gamma \sum_{\alpha=1}^2 \int_0^t \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \leq 2\|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\ & + \varphi(\sup_\tau \|d\|_{W_p^{s-1/p}(S_2)}) \left(\|d\|_{L_2(0,t;W_2^{1/2}(S_2))}^2 + \|d_t\|_{L_2(0,t;W_{6/5}^{1/6}(S_2))}^2 \right) + \|w(0)\|_{L_2(\Omega)}^2 \end{aligned}$$

where $t \leq T$, $d = (d_1, d_2)$ and φ is a nonlinear positive increasing function of its argument.

Proof. We use $\psi = w$ as a test function in a weak solution definition and apply definition of F to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 + \int_\Omega (w \cdot \nabla \delta \cdot w + \delta \cdot \nabla w \cdot w) dx - \int_\Omega \operatorname{div} \mathbb{T}(w + \delta, p) \cdot w dx \\ & = \int_\Omega (f - \delta_t - \delta \cdot \nabla \delta) \cdot w dx \end{aligned}$$

We use boundary conditions (1.1) on S_1 and on S_2 to reformulate the third integral on the l.h.s. of the above inequality as follows

$$\begin{aligned} & \int_\Omega \operatorname{div} \mathbb{T}(w + \delta, p) \cdot w dx = \int_\Omega \operatorname{div} [\nu \mathbb{D}(w + \delta) - p \mathbb{I}] \cdot w dx \\ & = \int_\Omega \operatorname{div} [\nu \mathbb{D}(w + \delta)] \cdot w dx - \int_\Omega p \cdot \nabla w = \int_\Omega D_{ij}(w + \delta) w_{j,x_i} dx \\ & = \int_\Omega D_{ij}(w) w_{j,x_i} dx + \int_\Omega D_{ij}(\delta) w_{j,x_i} dx = \frac{1}{2} \int_\Omega |D_{ij}(w)|^2 dx + \int_\Omega D_{ij}(\delta) w_{j,x_i} dx \end{aligned}$$

Then, we apply Korn inequality to have the estimate

$$(2.8) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 + \nu \|w\|_{H^1(\Omega)}^2 + \gamma \sum_{\alpha=1}^2 \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \\ & \leq - \int_\Omega (w \cdot \nabla \delta \cdot w + \delta \cdot \nabla w \cdot w) dx + c \sum_{\alpha=1}^2 \|\delta \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 + c \|\mathbb{D}(\delta)\|_{L_2(\Omega)}^2 \\ & \quad + \int_\Omega (f - \delta_t - \delta \cdot \nabla \delta) w dx \end{aligned}$$

Next, we focus on the integral

$$\begin{aligned} \int_\Omega \delta \cdot \nabla w \cdot w dx & = \int_\Omega (b + \nabla \varphi) \cdot \nabla w \cdot w dx = \int_\Omega b \cdot \nabla w \cdot w dx + \int_\Omega \nabla \varphi \cdot \nabla w \cdot w dx \\ & = I_1 + I_2 \end{aligned}$$

We can estimate I_1 by Hölder inequality and definition of b

$$\begin{aligned} |I_1| & \leq \|\nabla w\|_{L_2(\Omega)} \|w\|_{L_6(\Omega)} \|b\|_{L_3(\Omega)} \leq c \|w\|_{H^1(\Omega)}^2 \|b\|_{L_3(\tilde{S}_2(\rho))} \\ & \leq c \rho^{1/6} \|w\|_{H^1(\Omega)}^2 \|b\|_{L_6(\tilde{S}_2(\rho))} \leq c \rho^{1/6} \|w\|_{H^1(\Omega)}^2 \|\delta\|_{L_6(\Omega)} \leq c \rho^{1/6} \|w\|_{H^1(\Omega)}^2 \|\tilde{d}\|_{H^1(\Omega)} \end{aligned}$$

where

$$\tilde{S}_2(\rho) = \{x \in \Omega : x_3 \in (-a, -a + \rho) \cup (a - \rho, a)\} = \tilde{S}_2(\rho, a_1) \cup \tilde{S}_2(\rho, a_2).$$

We estimate I_2 as follows

$$(2.9) \quad I_2 = \int_\Omega \nabla \varphi \cdot \nabla w \cdot w dx \leq \|\nabla \varphi\|_{L_3(\Omega)} \|w\|_{L_6(\Omega)} \|\nabla w\|_{L_2(\Omega)}$$

where

$$\begin{aligned} \|\nabla\varphi\|_{L_3(\Omega)} &\leq c\|\nabla\varphi\|_{L_{3,-\mu'}(\Omega)} \leq c\|\nabla_{x_3}\nabla\varphi\|_{L_{3,1-\mu'}(\Omega)} \leq c\|\varphi\|_{L_{3,1-\mu'}^2(\Omega)} \\ &\leq c\|\operatorname{div} b\|_{L_{3,1-\mu'}(\Omega)} \end{aligned}$$

and we denote

$$\|u\|_{L_{p,\mu}^k(\Omega)} = \left(\sum_{|\alpha|=k} \int |D_x^\alpha u|^{p \min_{i=1,2}(\operatorname{dist}(x, S_2(a_i)))^{p\mu}} dx \right)^{1/p}, \mu \in \mathbb{R}, p \in (1, \infty).$$

To estimate the last norm, we have used the result of [RZ2] on Poisson equation in weighted Sobolev spaces and choose $\frac{2}{3} \leq 1 - \mu' \leq 1$. With $\mu = 1 - \mu'$ we have

$$\begin{aligned} c\|\operatorname{div} b\|_{L_{3,\mu}(\Omega)} &\leq c\varepsilon \left(\sum_{i=1}^2 \int_{\tilde{S}_2(a_i)} |\tilde{d}_i|^3 \frac{\sigma_i^{3\mu}}{\sigma_i^3} dx \right)^{1/3} + \left(\sum_{i=1}^2 \int_{\tilde{S}_2(a_i)} |\tilde{d}_{i,x_3}|^3 |\rho(x)|^{3\mu} dx \right)^{1/3} \\ &\leq c \sum_{i=1}^2 \varepsilon \left(\sup_{x_3} \int_{S_2(a_i)} |\tilde{d}_i|^3 dx' \int_r^\rho \frac{\sigma_i^{3\mu}}{\sigma_i^3} d\sigma_i \right)^{1/3} + \sum_{i=1}^2 \left(\sup_{x_3} \int_{S_2(a_i)} |\tilde{d}_{i,x_3}|^3 dx' \int_0^\rho \sigma_i^{3\mu} d\sigma_i \right)^{1/3} \\ &\leq c\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + c\rho^{\mu+1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \end{aligned}$$

where $\sigma_i = \operatorname{dist}\{S_2(a_i), x\}$, $x \in S_2(a_i, \rho)$. We note, that the last bound holds for $\mu > \frac{2}{3}$ since for $\mu = \frac{2}{3}$ the r.h.s. takes the form

$$c \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + c\rho \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)},$$

which can not be made small for large \tilde{d} . Then,

$$I_2 \leq c \left[\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \right] \|w\|_{H^1(\Omega)}^2$$

Next, we consider the term

$$\int_{\Omega} (w \cdot \nabla \delta \cdot w) dx = \int_{\Omega} (w \cdot \nabla b \cdot w) dx + \int_{\Omega} (w \cdot \nabla \nabla \varphi \cdot w) dx = I_3 + I_4.$$

For I_4 , we have

$$\begin{aligned} |I_4| &\leq \left| \int_{\Omega} \operatorname{div} (w \cdot \nabla \varphi \cdot w) dx - \int_{\Omega} (w \cdot \nabla w \cdot \nabla \varphi) dx \right| \\ &\leq \int_S |\tilde{n} \cdot \nabla \varphi \cdot w^2| dS + \int_{\Omega} |\nabla \varphi \cdot (w \cdot \nabla w)| dx \leq \int_{\Omega} |\nabla \varphi \cdot (w \cdot \nabla w)| dx \end{aligned}$$

so I_4 can be treated in the same way as I_2 and therefore

$$(2.10) \quad |I_4| \leq c \left[\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \right] \|w\|_{H^1(\Omega)}^2.$$

On the other hand, using $b = \alpha \bar{e}_3 = \sum_{i=1}^2 \tilde{d}_i \eta_i \bar{e}_3$, we find the bound for I_3

$$\begin{aligned}
|I_3| &\leq \left| \sum_{i=1}^2 \int_{\tilde{S}_2(\rho, a_i)} w \cdot \nabla(\tilde{d}_i \eta_i) w_3 dx \right| \\
&\leq \left| \sum_{i=1}^2 \int_{\tilde{S}_2(\rho, a_i)} (w \cdot \nabla \tilde{d}_i \eta_i w_3 + w \cdot \nabla \eta_i \tilde{d}_i w_3) dx \right| \\
&\leq \sum_{i=1}^2 \left(\int_{\tilde{S}_2(\rho, a_i)} |w \cdot \nabla \tilde{d}_i \eta_i| |w_3| dx + \int_{\tilde{S}_2(\rho, a_i)} \varepsilon \left| \frac{w_3}{\sigma_i} w_3 \tilde{d}_i \right| d\sigma_i dx_1 dx_2 \right) \\
&\leq c \sum_{i=1}^2 \|w\|_{L_6(\tilde{S}_2(\rho, a_i))} \|w_3\|_{L_3(\tilde{S}_2(\rho, a_i))} \|\nabla \tilde{d}_i\|_{L_2(\tilde{S}_2(\rho, a_i))} \\
&\quad + c\varepsilon \sum_{i=1}^2 \|w_3\|_{L_6(\tilde{S}_2(\rho, a_i))} \|\tilde{d}_i\|_{L_3(\tilde{S}_2(\rho, a_i))} \left(\int_{\tilde{S}_2(\rho, a_i)} dx_1 dx_2 \int_r^\rho d\sigma_i \left| \frac{w_3}{\sigma_i} \right|^2 \right)^{1/2} \\
&\leq c\rho^{1/6} \sum_{i=1}^2 \|w\|_{L_6(\tilde{S}_2(\rho, a_i))}^2 \|\nabla \tilde{d}_i\|_{L_2(\tilde{S}_2(\rho, a_i))} \\
&\quad + c\varepsilon \sum_{i=1}^2 \|w\|_{L_6(\tilde{S}_2(\rho, a_i))} \|\nabla w_3\|_{L_2(\tilde{S}_2(\rho, a_i))} \|\tilde{d}_i\|_{L_3(\tilde{S}_2(\rho, a_i))} \\
&\leq c(\rho^{1/6} + \varepsilon) \|w\|_{H^1(\Omega)}^2 \|\tilde{d}\|_{W_3^1(\Omega)}.
\end{aligned}$$

Thus, we can summarize estimates for $I_1 - I_4$ to conclude that nonlinear term in (2.8) is bounded by

$$\begin{aligned}
&\left| \int_{\Omega} (w \cdot \nabla \delta \cdot w + \delta \cdot \nabla w \cdot w) dx \right| \\
(2.11) \quad &\leq c \|w\|_{H^1(\Omega)}^2 \left(\varepsilon \rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \right. \\
&\quad \left. + (\rho^{1/6} + \varepsilon) \|\tilde{d}\|_{W_3^1(\Omega)} + \rho^{1/6} \|\tilde{d}\|_{H^1(\Omega)} \right).
\end{aligned}$$

Next, we examine the second term on the r.h.s. of (2.8):

$$\begin{aligned}
\sum_{\alpha=1}^2 \|\delta \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 &\leq \sum_{\alpha=1}^2 (\|b \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 + \|\nabla \varphi \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2) \\
&\leq \|\alpha\|_{L_2(S_1)}^2 + c \|\nabla \varphi\|_{W_{3/2}^1(\Omega)}^2 \\
&\leq \sum_{i=1}^2 \|d_i\|_{L_2(S_1)}^2 + c \|\operatorname{div} b\|_{L_{3/2}(\Omega)}^2 \\
&\leq c \|\tilde{d}\|_{W_{3/2}^1(\Omega)}^2 + c \sum_{i=1}^2 \|\nabla(\tilde{d}_i \eta_i)\|_{L_{3/2}(\Omega)}^2 \\
&\leq c \|\tilde{d}\|_{W_{3/2}^1(\Omega)}^2 + c \sum_{i=1}^2 \left(\|\nabla \tilde{d}_i \eta_i\|_{L_{3/2}(\Omega)}^2 + \|\tilde{d}_i \nabla \eta_i\|_{L_{3/2}(\Omega)}^2 \right) \\
&\leq c \|\tilde{d}\|_{W_{3/2}^1(\Omega)}^2 + c \sum_{i=1}^2 \|\tilde{d}_i \nabla \eta_i\|_{L_{3/2}(\Omega)}^2.
\end{aligned}$$

The last expression we calculate in details:

$$\begin{aligned}
\sum_{i=1}^2 \|\tilde{d}_i \nabla \eta_i\|_{L_{3/2}(\Omega)}^2 &\leq \varepsilon^2 \left[\left(\int_{-a+r}^{-a+\rho} dx_3 \int_{S_2(a_1)} dx' \left| \frac{\tilde{d}_1}{a+x_3} \right|^{3/2} \right)^{4/3} \right. \\
&\quad \left. + \left(\int_{a-\rho}^{a-r} dx_3 \int_{S_2(a_2)} dx' \left| \frac{\tilde{d}_2}{a-x_3} \right|^{3/2} \right)^{4/3} \right] \\
&\leq \varepsilon^2 \left[\sup_{x_3} \|\tilde{d}_1\|_{L_{3/2}(S_2(a_1))}^2 \left(\int_{-a+r}^{-a+\rho} \left| \frac{1}{a+x_3} \right|^{3/2} dx_3 \right)^{4/3} \right. \\
&\quad \left. + \sup_{x_3} \|\tilde{d}_2\|_{L_{3/2}(S_2(a_2))}^2 \left(\int_{a-\rho}^{a-r} \left| \frac{1}{a-x_3} \right|^{3/2} dx_3 \right)^{4/3} \right] \\
&\leq c \varepsilon^2 \sup_{x_3} \|\tilde{d}\|_{L_{3/2}(S_2)}^2 \left(\int_r^\rho \frac{dy}{y^{3/2}} \right)^{4/3} \leq c \varepsilon^2 \sup_{x_3} \|\tilde{d}\|_{L_{3/2}(S_2)}^2 \left[\frac{1}{r^{1/2}} - \frac{1}{\rho^{1/2}} \right]^{4/3} \\
&\leq c \varepsilon^2 \sup_{x_3} \|\tilde{d}\|_{L_{3/2}(S_2)}^2 \frac{1}{\rho^{2/3}} [e^{1/2\varepsilon} - 1]^{4/3} \leq c \frac{\varepsilon^2}{\rho^{2/3}} e^{2/3\varepsilon} \sup_{x_3} \|\tilde{d}\|_{L_{3/2}(S_2)}^2.
\end{aligned}$$

Combining inequalities above, we infer

$$\sum_{\alpha=1}^2 \|\delta \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \leq c \|\tilde{d}\|_{W_{3/2}^1(\Omega)}^2 + c \frac{\varepsilon^2}{\rho^{2/3}} e^{2/3\varepsilon} \sup_{x_3} \|\tilde{d}\|_{L_{3/2}(S_2)}^2$$

We estimate also the term

$$\begin{aligned}
\|\mathbb{D}(\delta)\|_{L_2(\Omega)}^2 &\leq \|\mathbb{D}(b)\|_{L_2(\Omega)}^2 + \|\mathbb{D}(\nabla\varphi)\|_{L_2(\Omega)}^2 \\
&\leq \sum_{i=1}^2 \left(\|\nabla\tilde{d}_i\eta_i\|_{L_2(\Omega)}^2 + \|\tilde{d}_i\nabla\eta_i\|_{L_2(\Omega)}^2 \right) + \|\nabla^2\varphi\|_{L_2(\Omega)}^2 \\
&\leq \sum_{i=1}^2 \left(\|\nabla\tilde{d}_i\eta_i\|_{L_2(\Omega)}^2 + \|\tilde{d}_i\nabla\eta_i\|_{L_2(\Omega)}^2 \right) + \|\operatorname{div} b\|_{L_2(\Omega)}^2 \\
&\leq c \sum_{i=1}^2 \left(\|\nabla\tilde{d}_i\eta_i\|_{L_2(\Omega)}^2 + \|\tilde{d}_i\nabla\eta_i\|_{L_2(\Omega)}^2 \right) \\
&\leq c \sum_{i=1}^2 \|\tilde{d}_i\|_{W_2^1(\Omega)}^2 + \varepsilon^2 c \int_{-a+r}^{-a+\rho} dx_3 \int_{S_2(a_1)} dx' \left| \frac{\tilde{d}_1}{a+x_3} \right|^2 \\
&\quad + \varepsilon^2 \int_{a-\rho}^{a-r} dx_3 \int_{S_2(a_2)} dx' \left| \frac{\tilde{d}_2}{a-x_3} \right|^2 \\
&\leq c \sum_{i=1}^2 \left(\|\tilde{d}_i\|_{W_2^1(\Omega)}^2 + \varepsilon^2 \sup_{x_3} \|\tilde{d}_i\|_{L_2(S_2)}^2 \int_r^\rho \frac{dy}{y^2} \right) \\
&\leq c \sum_{i=1}^2 \left[\|\tilde{d}_i\|_{W_2^1(\Omega)}^2 + \varepsilon^2 \sup_{x_3} \|\tilde{d}_i\|_{L_2(S_2)}^2 \left(\frac{1}{r} - \frac{1}{\rho} \right) \right] \\
&\leq c \sum_{i=1}^2 \left[\|\tilde{d}_i\|_{W_2^1(\Omega)}^2 + \varepsilon^2 \sup_{x_3} \|\tilde{d}_i\|_{L_2(S_2)}^2 \frac{1}{\rho} (e^{1/\varepsilon} - 1) \right] \\
&\leq c \sum_{i=1}^2 \left[\|\tilde{d}_i\|_{W_2^1(\Omega)}^2 + \frac{\varepsilon^2}{\rho} e^{1/\varepsilon} \sup_{x_3} \|\tilde{d}_i\|_{L_2(S_2)}^2 \right].
\end{aligned}$$

Analyzing the last integral on the r.h.s. of (2.8) we have

$$\begin{aligned}
\int_{\Omega} (f - \delta_t - \delta \cdot \nabla\delta) w dx &\leq \varepsilon_1 \|w\|_{L_6(\Omega)}^2 + c(1/\varepsilon_1) (\|f\|_{L_{6/5}(\Omega)}^2 + \|\delta_t\|_{L_{6/5}(\Omega)}^2) \\
&\quad + \left| \int_{\Omega} \delta \cdot \nabla\delta \cdot w dx \right|
\end{aligned}$$

We estimate $\|\delta_t\|_{L_{6/5}(\Omega)}$ as follows

$$\begin{aligned}
\|\delta_t\|_{L_{6/5}(\Omega)} &= \|b_t + \nabla\varphi_t\|_{L_{6/5}(\Omega)} \leq \|\tilde{d}_t\|_{L_{6/5}(\Omega)} + \|\operatorname{div} b_t\|_{L_{6/5}(\Omega)} \\
&\leq \|\tilde{d}_t\|_{L_{6/5}(\Omega)} + \|\nabla\tilde{d}_t\|_{L_{6/5}(\Omega)} + \|\tilde{d}_t\nabla\eta\|_{L_{6/5}(\Omega)} \\
&\leq \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)} + \varepsilon \sup_{x_3} \|\tilde{d}_t\|_{L_{6/5}(S_2)} \left(\int_r^\rho \frac{dx_3}{x_3^{6/5}} \right)^{5/6} \\
&\leq \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)} + \varepsilon \frac{1}{\rho^{1/6}} e^{1/6\varepsilon} \sup_{x_3} \|\tilde{d}_t\|_{L_{6/5}(S_2)}
\end{aligned}$$

since

$$\left(\int_r^\rho \frac{dx_3}{x_3^{6/5}} \right)^{5/6} = \left(\frac{1}{r^{1/5}} - \frac{1}{\rho^{1/5}} \right)^{5/6} = \frac{1}{\rho^{1/6}} \left(e^{1/5\varepsilon} - 1 \right)^{5/6}$$

Finally, we examine

$$\begin{aligned} \left| \int_\Omega \delta \cdot \nabla \delta \cdot w dx \right| &\leq \|\nabla \delta\|_{L_2(\Omega)} \|w\|_{L_6(\Omega)} \|\delta\|_{L_3(\Omega)} \leq \varepsilon_2 \|w\|_{L_6(\Omega)}^2 + c(1/\varepsilon_2) \|\delta\|_{W_2^1(\Omega)}^4 \\ &\leq \varepsilon_2 \|w\|_{L_6(\Omega)}^2 + c(1/\varepsilon_2) \left(\|\tilde{d}\|_{W_2^1(\Omega)}^4 + \frac{\varepsilon^4}{\rho^2} e^{2/\varepsilon} \sup_{x_3} \|\tilde{d}\|_{L_2(S_2)}^4 \right) \end{aligned}$$

We summarize above estimates to rewrite (2.8) as follows

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 + \nu \|w\|_{H^1(\Omega)}^2 + \gamma \sum_{\alpha=1}^2 \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \\ &\leq \|w\|_{H^1(\Omega)}^2 \left[\varepsilon \rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \right. \\ &\quad \left. + (\rho^{1/6} + \varepsilon) \|\tilde{d}\|_{W_3^1(\Omega)} + \rho^{1/6} \|\tilde{d}\|_{H^1(\Omega)} + \varepsilon_1 + \varepsilon_2 \right] \\ (2.12) \quad &+ \|f\|_{L_{6/5}(\Omega)}^2 + \|\tilde{d}\|_{L_2(\Omega)}^2 + \|\tilde{d}\|_{W_2^1(\Omega)}^4 + \|\tilde{d}\|_{W_2^1(\Omega)}^2 \\ &+ \|\nabla \tilde{d}\|_{L_{6/5}(\Omega)}^2 + \|\nabla \tilde{d}\|_{L_2(\Omega)}^4 + \|\tilde{d}\|_{W_{3/2}^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \\ &+ \frac{\varepsilon^2}{\rho} e^{1/\varepsilon} \sup_{x_3} \|\tilde{d}\|_{L_2(S_2)}^2 + \frac{\varepsilon^4}{\rho^2} e^{2/\varepsilon} \sup_{x_3} \|\tilde{d}\|_{L_2(S_2)}^4 \\ &+ \frac{\varepsilon^2}{\rho^{2/3}} e^{2/3\varepsilon} \sup_{x_3} \|\tilde{d}\|_{L_{3/2}(S_2)}^2 + \frac{\varepsilon^2}{\rho^{1/3}} e^{1/3\varepsilon} \sup_{x_3} \|\tilde{d}_t\|_{L_{6/5}(S_2)}^2 \end{aligned}$$

We apply Sobolev anisotropic imbedding (see [BIN], Ch.3, Section 10) to estimate $\sup_{x_3} \|\tilde{d}\|_{L_3(S_2)}$ and $\sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)}$ with some W_p^s norm and calculate

$$\begin{aligned} 2 \left(\frac{1}{p} - \frac{1}{3} \right) \frac{1}{s} + \frac{1}{p} \cdot \frac{1}{s} + \frac{1}{s} &\leq 1 \quad \text{for } p > 3 \\ 2 \left(\frac{1}{p} - \frac{1}{3} \right) \frac{1}{s} + \frac{1}{p} \cdot \frac{1}{s} + \frac{1}{s} &< 1 \quad \text{for } p = 3 \end{aligned}$$

Then,

$$(2.13) \quad \frac{3}{p} + \frac{1}{3} \leq s \quad \text{for } p > 3 \quad \text{or } p = 3, s > \frac{4}{3}.$$

We set $\mu > \frac{2}{3}$, then since $\rho < 1$, we observe that $\rho^{\mu+1/3} \leq \rho^{1/6}$. Then

$$\begin{aligned} \varepsilon \rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\nabla \tilde{d}\|_{L_3(S_2)} + (\rho^{1/6} + \varepsilon) \|\tilde{d}\|_{W_3^1(\Omega)} + \rho^{1/6} \|\tilde{d}\|_{H^1(\Omega)} \\ \leq (\varepsilon \rho^{\mu-2/3} + \rho^{\mu+1/3} + 2\rho^{1/6} + \varepsilon) \|\tilde{d}\|_{W_p^s(\Omega)} \leq (2\varepsilon + 3\rho^{1/6}) \|\tilde{d}\|_{W_p^s(\Omega)} \end{aligned}$$

We put

$$(2.14) \quad \begin{aligned} \varepsilon &= \frac{\nu}{15 \|\tilde{d}\|_{W_p^s(\Omega)}}, \\ \rho^{1/6} &= \frac{\nu}{15 \|\tilde{d}\|_{W_p^s(\Omega)}}, \\ \varepsilon_1 + \varepsilon_2 &= \frac{\nu}{6}, \end{aligned}$$

with p, s satisfying (2.13). Therefore,

$$\begin{aligned} &\varepsilon \rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\nabla \tilde{d}\|_{L_3(S_2)} \\ &+ (\rho^{1/6} + \varepsilon) \|\tilde{d}\|_{W_3^1(\Omega)} + \rho^{1/6} \|\tilde{d}\|_{H^1(\Omega)} + \varepsilon_1 + \varepsilon_2 \leq \frac{\nu}{2} \end{aligned}$$

and formula (2.12) assumes the form

$$\begin{aligned} \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 &+ \nu \|w\|_{H^1(\Omega)}^2 + \gamma \sum_{\alpha=1}^2 \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \\ &\leq 2 \|f\|_{L_{6/5}(\Omega)}^2 + \varphi(\|\tilde{d}\|_{W_2^1(\Omega)}) (\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2) \\ &\quad + \varphi(\|\tilde{d}\|_{W_p^s(\Omega)}) \left(\sup_{x_3} \|\tilde{d}\|_{L_2(S_2)}^2 + \sup_{x_3} \|\tilde{d}_t\|_{L_{6/5}(S_2)}^2 \right) \end{aligned}$$

where φ is a nonlinear positive increasing function of its argument. We use Sobolev imbedding

$$\begin{aligned} \sup_{x_3} \|\tilde{d}\|_{L_2(S_2)} &\leq c \|\tilde{d}\|_{W_2^1(\Omega)}, \\ \sup_{x_3} \|\tilde{d}_t\|_{L_{6/5}(S_2)} &\leq c \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)} \end{aligned}$$

and hence

$$(2.15) \quad \begin{aligned} \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 &+ \nu \|w\|_{H^1(\Omega)}^2 + \gamma \sum_{\alpha=1}^2 \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \\ &\leq 2 \|f\|_{L_{6/5}(\Omega)}^2 + \varphi(\|\tilde{d}\|_{W_p^s(\Omega)}) \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) \end{aligned}$$

Integrating (2.15) with respect to time we obtain

$$(2.16) \quad \begin{aligned} \|w\|_{V_2^0(\Omega^t)}^2 &+ \gamma \sum_{\alpha=1}^2 \int_0^t \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 dt \leq 2 \|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\ &+ \varphi(\sup_\tau \|\tilde{d}\|_{W_p^s(\Omega)}) \left(\|\tilde{d}\|_{L_2(0,t;W_2^1(\Omega))}^2 + \|\tilde{d}_t\|_{L_2(0,t;W_{6/5}^1(\Omega))}^2 \right) + \|w(0)\|_{L_2(\Omega)}^2, \end{aligned}$$

where $\frac{3}{p} + \frac{1}{3} \leq s, p > 3$ or $p = 3, s > \frac{4}{3}$.

□

3. WEAK SOLUTIONS TO (2.6)

In this section, we use the Galerkin method to prove the existence of weak solutions to the problem (2.6). We follow ideas from [L], chapter 6, section 7.

Namely, we introduce the sequence of approximating functions w_N given as

$$w^N(x, t) = \sum_{k=1}^N C_{kN}(t) a^k(x),$$

where $\{a^k\}_{k=1}^\infty$ is the system of orthogonal functions in $L^2(\Omega) \cap J_2^0(\Omega)$. Here, $J_2^0(\Omega) = \{f \in H^1(\Omega) : \operatorname{div} f = 0\}$ and $\{a^k\}_{k=1}^\infty$ is the fundamental system in $H^1(\Omega)$ with $\sup_{x \in \Omega} |a^k(x)| < \infty, \sup_{x \in \partial\Omega} |a^k(x)| < \infty$. The coefficients $C_{kN}(0)$ are defined by

$$C_{kN}|_{t=0} = (w_0, a_k), \quad k = 1, \dots, N,$$

and the function w^N satisfy the following system with test functions a^k :

$$\left\{ \int_{\Omega} \left(\frac{1}{2} \frac{d}{dt} w^N a^k + w^N \cdot \nabla w^N a^k + \delta \cdot \nabla w^N \cdot a^k + w^N \cdot \nabla \delta \cdot a^k + \nu \mathbb{D}(w^N) \mathbb{D}(a^k) \right) dx \right. \\ \left. + \gamma \int_{S_1} w^N \cdot \bar{\tau}_j a^k \bar{\tau}_j dS_1 \right\} = \left(\sum_{j, \sigma=1}^2 \int_{S_\sigma} B_{\sigma j} a^k \cdot \bar{\tau}_j dS_\sigma + \int_{\Omega} F \cdot a^k dx \right)$$

for $k = 1, \dots, N$. Then, w^N would be the weak solution to (2.6).

With $(f, g) = \int_{\Omega} f g dx$ and $(f, g)_S = \int_S f g dS$ this can be rewritten as:

$$\left\{ (w_t^N, a^k) + (w^N \cdot \nabla w^N, a^k) + (\delta \cdot \nabla w^N, a^k) + (w^N \cdot \nabla \delta, a^k) \right. \\ \left. + \nu (\mathbb{D}(w^N), \mathbb{D}(a^k)) + \gamma (w^N \cdot \bar{\tau}_j, a^k \cdot \bar{\tau}_j)_{S_1} \right\} = \\ \left[\sum_{\sigma, j=1}^2 (B_{\sigma j}, a^k \cdot \bar{\tau}_j)_{S_\sigma} + (F, a^k) \right], \quad k = 1, \dots, N.$$

Thus,

$$(3.1) \quad \left(\frac{d}{dt} w^N, a^k \right) + (w^N \cdot \nabla w^N, a^k) + (\delta \cdot \nabla w^N, a^k) + (w^N \cdot \nabla \delta, a^k) \\ + \nu (\mathbb{D}(w^N), \mathbb{D}(a^k)) + \gamma (w^N \cdot \bar{\tau}_j, a^k \cdot \bar{\tau}_j)_{S_1} \\ = \sum_{j, \sigma=1}^2 (B_{\sigma j}, a^k \cdot \bar{\tau}_j)_{S_\sigma} + (F, a^k), \quad k = 1, \dots, N.$$

The above equations are in fact a system of ordinary differential equations for the functions $C_{kN}(t)$. The properties of the sequence a^k imply

$$\|w^N(x, t)\|_{2, \Omega}^2 = \sum_{k=1}^N C_{kN}^2(t).$$

On the other hand, we can obtain the a priori bounds for the approximative solutions w^N of the same form as (2.16):

$$(3.2) \quad \|w^N\|_{V_2^0(\Omega^T)}^2 = \sup_{0 \leq t \leq T} \|w^N\|_{2, \Omega}^2 + \int_0^T \|\nabla w^N\|_{2, \Omega}^2 dt \\ \leq \int_0^T \|f\|_{L_{6/5}(\Omega)}^2 + \varphi \left(\sup_{0 \leq t \leq T} \|\tilde{d}\|_{W_p^s(\Omega)} \right) \int_0^T \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) dt \\ + \|w^N(0)\|_{L_2(\Omega)}^2 \leq C,$$

where $\frac{3}{p} + \frac{1}{3} \leq s, p > 3$ or $p = 3, s > \frac{4}{3}$. Therefore, $\sup_{0 \leq t \leq T} |C_{kN}(t)|$ is bounded on $[0, T]$ and w^N are well defined for all times t .

Let us define now $\psi_{N,k} \equiv (w^N(x, t), a^k(x))$. This sequence is uniformly bounded by (3.2). We can also show that it is equicontinuous. Namely, we integrate (3.1) with respect to t from t to $t + \Delta t$ to obtain

$$\begin{aligned}
|\psi_{N,k}(t + \Delta t) - \psi_{N,k}(t)| &\leq \sup_{x \in \Omega} |a^k(x)| \int_t^{t+\Delta t} (|w^N \cdot \nabla w^N|_{2,\Omega} + |\delta \cdot \nabla w^N|_{2,\Omega} \\
&+ |w^N \cdot \nabla \delta|_{2,\Omega} + |F|_{2,\Omega}) dt + \nu |\nabla a^k|_{2,\Omega} \int_t^{t+\Delta t} |\nabla w^N|_{2,\Omega} dt \\
&+ \gamma \sup_{x \in S} |a^k(x)| \int_t^{t+\Delta t} \left(|w^N \cdot \bar{\tau}_j|_{2,S_1} + \sum_{j,\sigma=1}^2 |B_{\sigma j}|_{2,S_\sigma} \right) dt \\
&\leq \sup_{x \in \Omega} |a^k(x)| \sqrt{\Delta t} \left(\sup_{x \in \Omega} |w^N|_{2,\Omega} (|\nabla w^N|_{2,\Omega^T} + |\nabla \delta|_{2,\Omega^T}) + \sup_{x \in \Omega} |\delta|_{2,\Omega} |\nabla w^N|_{2,\Omega^T} \right) \\
&+ \sup_{x \in \Omega} |a^k(x)| \int_t^{t+\Delta t} |F|_{2,\Omega} dt + \nu |\nabla a^k|_{2,\Omega} \sqrt{\Delta t} |\nabla w^N|_{2,\Omega^T} \\
&+ \gamma \sup_{x \in S} |a^k(x)| \left(\sqrt{\Delta t} |\nabla w^N|_{2,\Omega^T} + \int_t^{t+\Delta t} \sum_{j=1}^2 |B_j|_{2,S} \right) dt \\
&\leq C(k) \left(\sqrt{\Delta t} + \int_t^{t+\Delta t} (|F|_{2,\Omega} + \sum_{j=1}^2 |B_j|_{2,S}) dt \right).
\end{aligned}$$

We can see that for given k and $N \geq k$ the r.h.s. tends to zero as $\Delta t \rightarrow 0$ uniformly in N . Thus, it is possible to choose a subsequence N_m such that $\psi_{N_m,k}$ converges with $m \rightarrow \infty$ uniformly to some continuous function ψ_k for any given k . Since the limit function w is defined as

$$w(x, t) = \sum_{k=1}^{\infty} \psi_k(t) a^k(x),$$

then we conclude that $(w^{N_m} - w, \psi(x))$ tends to zero as $m \rightarrow \infty$ uniformly with respect to $t \in [0, T]$ for any $\psi \in J_2^0(\Omega)$ and $w(x, t)$ is continuous in t in weak topology. Moreover, estimates (3.2) remain true for the limit function w .

We will show that $\{w^{N_m}\}$ converges strongly in $L^2(\Omega^T)$. To this end, we need to apply the following version of the Friedrichs lemma: for any $\varepsilon > 0$, there exists such N_ε that for any $u \in W_2^1(\Omega)$ the following inequality holds:

$$\|u\|_{2,\Omega}^2 \leq \sum_{k=1}^{N_\varepsilon} (u, a^k) + \varepsilon \|\nabla u\|_{2,\Omega}^2.$$

This in terms of $u = w^{N_m} - w^{N_l}$ reads

$$\|w^{N_m} - w^{N_l}\|_{2,\Omega^T}^2 \leq \sum_{k=1}^{N_\varepsilon} \int_0^T (w^{N_m} - w^{N_l}, a^k) dt + \varepsilon \|\nabla w^{N_m} - \nabla w^{N_l}\|_{2,\Omega^T}^2.$$

By (3.2), we have

$$\|\nabla w^{N_m} - \nabla w^{N_l}\|_{2,\Omega^T}^2 \leq 2C^2$$

for some constant C . The first integral on the r.h.s. for given number N_ε can be arbitrarily small if only m and l are sufficiently large, so it tends to zero as $m, l \rightarrow \infty$. Therefore, $\{w^{N_m}\}$ converges strongly in $L^2(\Omega^T)$.

We summarize the above convergence properties of the sequence $\{w^{N_m}\}$:

- (i) $\{w^{N_m}\} \rightarrow w$ strongly in $L^2(\Omega^T)$ for some w ,
- (ii) $\{w^{N_m}\} \rightarrow w$ weakly in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$,
- (iii) $\nabla\{w^{N_m}\} \rightarrow \nabla w$ weakly in $L^2(\Omega^T)$.

With given $\Phi^k = \sum_{j=1}^k d_j(t) a^j(x)$, the sequence $\{w^{N_m}\}$ satisfy the identities:

$$\begin{aligned} & \int_{\Omega} \left(\frac{d}{dt} w^{N_m} \Phi^k + (w^{N_m} \cdot \nabla w^{N_m} + \delta \cdot \nabla w^{N_m} + w^{N_m} \cdot \nabla \delta) \Phi^k + \nu \mathbb{D}(w^{N_m}) \mathbb{D}(\Phi^k) \right) dx \\ & + \gamma \int_{S_1} w^{N_m} \cdot \bar{\tau}_j \Phi^k \cdot \bar{\tau}_j dS_0 = \sum_{\sigma, j=1}^2 \int_{S_\sigma} B_{\sigma j} \Phi^k \cdot \bar{\tau}_j dS_\sigma + \int_{\Omega} F \Phi^k dx. \end{aligned}$$

Then, we can pass to the limit with $m \rightarrow \infty$ to obtain the identity for w . Conditions $\operatorname{div} w^N = 0, w^N \cdot \bar{n}|_{S^T} = 0$ stay true for the limit function w as well.

It remains to consider the limit $\lim_{t \rightarrow 0} w(x, t)$. We note, that w^{N_m} satisfy the relation (2.8) (if we use the test function w^{N_m}). This yields

$$|w^{N_m}|_{2, \Omega} \leq |w_0|_{2, \Omega} + \int_0^t (|F|_{2, \Omega} + |B|_{2, S}) dt.$$

In the limit $m \rightarrow \infty$ we obtain

$$|w|_{2, \Omega} \leq |w_0|_{2, \Omega} + \int_0^t (|F|_{2, \Omega} + |B|_{2, S}) dt$$

which implies

$$\overline{\lim}_{t \rightarrow 0} |w|_{2, \Omega} \leq |w_0|_{2, \Omega}.$$

On the other hand, since w^{N_m} tends to w as $m \rightarrow \infty$, we have $|w^{N_m} - w_0|_{2, \Omega} \rightarrow 0$. Therefore, $|w^{N_m} - w_0| \rightarrow 0$ weakly in $L^2(\Omega)$ as $t \rightarrow 0$ and

$$|w_0|_{2, \Omega} \leq \underline{\lim}_{t \rightarrow 0} |w|_{2, \Omega}.$$

We conclude that the limit $\lim_{t \rightarrow 0} |w|_{2, \Omega}$ exists and is equal to $|w_0|_{2, \Omega}$ where the convergence is strong - in the norm $L^2(\Omega)$.

Consequently, we have proved the following result.

Lemma 3.3. *Let the assumptions of Lemma 2.2 be satisfied. Then there exists a weak solution w to problem (2.6) such that w is weakly continuous with respect to t in $L^2(\Omega)$ norm and w converges to w_0 as $t \rightarrow 0$ strongly in $L^2(\Omega)$ norm.*

Since $v = w - \delta$ we conclude the analogous existence result for v formulated in Theorem 1.

4. GLOBAL SOLUTIONS TO (2.6)

To obtain a global estimate we write (2.15) in the form

$$\frac{d}{dt} \|w\|_{L^2(\Omega)}^2 + \nu \|w\|_{L^2(\Omega)}^2 \leq 2 \|f\|_{L_{6/5}(\Omega)}^2 + \varphi(\|\tilde{d}\|_{W_p^s(\Omega)}) \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right),$$

where $\frac{3}{p} + \frac{1}{3} \leq s, p > 3$ or $p = 3, s > \frac{4}{3}$. Hence

$$\frac{d}{dt} \left(\|w\|_{L^2(\Omega)}^2 e^{\nu t} \right) \leq 2 \|f\|_{L_{6/5}(\Omega)}^2 e^{\nu t} + \varphi(\|\tilde{d}\|_{W_p^s(\Omega)}) \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) e^{\nu t}$$

Integrating with respect to time from t_1 to t_2 yields

$$\begin{aligned} \|w(t_2)\|_{L_2(\Omega)}^2 e^{\nu t_2} &\leq 2 \int_{t_1}^{t_2} \|f\|_{L_{6/5}(\Omega)}^2 e^{\nu t} dt + \|w(t_1)\|_{L_2(\Omega)}^2 e^{\nu t_1} \\ &\quad + \varphi(\sup_t \|\tilde{d}\|_{W_p^s(\Omega)}) \int_{t_1}^{t_2} \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) e^{\nu t} dt. \end{aligned}$$

Thus,

$$\begin{aligned} \|w(t_2)\|_{L_2(\Omega)}^2 &\leq 2e^{-\nu t_2} \int_{t_1}^{t_2} \|f\|_{L_{6/5}(\Omega)}^2 e^{\nu t} dt + \|w(t_1)\|_{L_2(\Omega)}^2 e^{-\nu(t_2-t_1)} \\ &\quad + \varphi(\sup_t \|\tilde{d}\|_{W_p^s(\Omega)}) e^{-\nu t_2} \int_{t_1}^{t_2} \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) e^{\nu t} dt \end{aligned}$$

and this implies

$$\begin{aligned} (4.1) \quad \|w(t_2)\|_{L_2(\Omega)}^2 &\leq 2 \int_{t_1}^{t_2} \|f\|_{L_{6/5}(\Omega)}^2 dt + \|w(t_1)\|_{L_2(\Omega)}^2 e^{-\nu(t_2-t_1)} \\ &\quad + \varphi(\sup_t \|\tilde{d}\|_{W_p^s(\Omega)}) \int_{t_1}^{t_2} \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) dt \end{aligned}$$

Setting $t_1 = 0$ and $t_2 = t \in R_+$ we obtain the global estimate

$$\begin{aligned} (4.2) \quad \|w(t)\|_{L_2(\Omega)}^2 &\leq 2 \int_0^t \|f\|_{L_{6/5}(\Omega)}^2 d\tau + \|w(0)\|_{L_2(\Omega)}^2 e^{-\nu t} \\ &\quad + \varphi(\sup_t \|\tilde{d}\|_{W_p^s(\Omega)}) \int_0^t \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) d\tau \end{aligned}$$

Let $k \in \mathbb{N}$. Integrating (2.15) with respect to time from kT to $t \in (kT, (k+1)T]$ we get

$$\begin{aligned} (4.3) \quad |w|_{V_2^0(\Omega \times (kT, t))}^2 &\leq 2 \int_{kT}^t \|f\|_{L_{6/5}(\Omega)}^2 d\tau + \|w(kT)\|_{L_2(\Omega)}^2 \\ &\quad + \varphi(\sup_\tau \|\tilde{d}\|_{W_p^s(\Omega)}) \int_{kT}^t \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) d\tau \end{aligned}$$

Therefore,

$$\begin{aligned} (4.4) \quad |v|_{V_2^0(\Omega \times (kT, t))}^2 &\leq 2 \int_{kT}^t \|f\|_{L_{6/5}(\Omega)}^2 d\tau + \|v(kT)\|_{L_2(\Omega)}^2 \\ &\quad + \varphi(\sup_\tau \|\tilde{d}\|_{W_p^s(\Omega)}) \int_{kT}^t \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) d\tau \end{aligned}$$

We have also

$$\begin{aligned} (4.5) \quad \|v(T)\|_{L_2(\Omega)}^2 &\leq 2 \int_0^t \|f\|_{L_{6/5}(\Omega)}^2 d\tau + \|v(0)\|_{L_2(\Omega)}^2 e^{-\nu T} \\ &\quad + \varphi(\sup_t \|\tilde{d}\|_{W_p^s(\Omega)}) \int_0^t \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) d\tau \end{aligned}$$

We set $\mu_1 = e^{-\nu T}$. Let us assume that

$$\|v(0)\|_{L_2(\Omega)} \leq A$$

for some constant A and

$$2 \int_0^t \|f\|_{L_{6/5}(\Omega)}^2 d\tau + \varphi(\sup_t \|\tilde{d}\|_{W_p^s(\Omega)}) \int_0^t \left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2 \right) d\tau \leq (1 - e^{-\nu T}) A^2$$

Thus,

$$\|v(T)\|_{L_2(\Omega)} \leq A$$

so we can control the initial condition for the next time step. This can be repeated for intervals $(kT, (k+1)T)$. Then by (4.4) we can prove global existence of weak solution such that

$$v \in V_2^0(\Omega \times (kT, (k+1)T)) \quad \forall k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

so we conclude Theorem 2.

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