

# Global existence of solutions for some coupled systems of reaction-diffusion equations

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ABSTRACT. The aim of this work is to study the global existence of solutions for some coupled systems of reaction diffusion which describe the spread within a population of infectious disease. We consider a triangular matrix diffusion and we show that we can prove global existence of classical solutions for the nonlinearities of weakly exponential growth.

## 1. INTRODUCTION

In this work we shall be concerned with a reaction-diffusion system of the form:

$$(1.1) \quad \begin{cases} \frac{du}{dt} - a\Delta u = \Lambda - \lambda(t)f(u, v) - \mu u & \text{in } \mathbb{R}^+ \times \Omega \\ \frac{dv}{dt} - b\Delta u - d\Delta v = \lambda(t)f(u, v) - \mu v & \text{in } \mathbb{R}^+ \times \Omega \end{cases}$$

with boundary conditions

$$(1.2) \quad \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega$$

and the nonnegative continuous and bounded initial data

$$(1.3) \quad u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \quad \text{in } \bar{\Omega}$$

where  $\Omega$  is a bounded domain of class  $C^1$  in  $\mathbb{R}^n$ , with boundary  $\partial\Omega$ ,  $\frac{\partial}{\partial \eta}$  is the outward normal derivative to  $\partial\Omega$ .

The constants  $a, b, d, \Lambda, \mu$  are such that

$$(H.1) \quad a > 0, \quad b > 0, \quad d \geq a + b, \quad \mu > 0, \quad \text{and } \Lambda \geq 0.$$

We assume that  $t \rightarrow \lambda(t)$  is a nonnegative and bounded function in  $C(\mathbb{R}^+)$  with  $0 \leq \lambda(t) \leq \hat{\lambda}$  and nonlinearity  $f$  is a nonnegative continuously differentiable function on  $(0, +\infty) \times (0, +\infty)$  satisfying

$$(H.2) \quad f(0, \eta) = 0 \quad \text{for all } \eta \text{ in } \mathbb{R}^+ \quad \text{and} \quad \lim_{\eta \rightarrow +\infty} \frac{\log(1 + f(\cdot, \eta))}{\eta} = 0.$$

The reaction-diffusion system (1.1) – (1.3) may be viewed as a diffusive epidemic model where  $u$  and  $v$  represent the nondimensional population densities of susceptibles and infectives, respectively. We can consider the system (1.1) – (1.3) as a model describing the spread of an infection disease (such as AIDS for instance)

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within a population assumed to be divided into the susceptible and infective classes as precised (for further motivation see for instance [2, 3] and the references therein).

We note that in the case  $\Lambda = \mu = 0$ , this problem corresponds to the problem of R. H. Martin and much works have been done in the litterature and positive answers have been under different forms for subgrowth nonlinearity (se for instance [1, 9, 5, 7, 8, 11]).

We mentionne that in the case  $\Lambda > 0$ , it is not obvious to obtain global existence of solutions when the nonlinearity are at most polynomial growth and the methods as above can not be applied.

Our main purpose is to study the global existence of solutions of the system (1.1) – (1.3) with nonlinearities of weakly exponential growth.

## 2. Preminilaries results

**2.1. Local existence of solutions.** We denote by  $W^{m,p}(\Omega)$ , the Sobolev space of order  $m \geq 0$  for  $1 \leq p \leq +\infty$  and by  $C(\bar{\Omega})$  the Banach space of continuous functions on  $\bar{\Omega}$ .

We can convert equations (1.1) – (1.3) to an abstract first order system in the Banach space  $X = C(\bar{\Omega}) \times C(\bar{\Omega})$  of the form

$$\begin{cases} \frac{dU(t)}{dt} = \tilde{A}U(t) + F(U(t)) & t > 0, \\ U(0) = U_0 \in X \end{cases}$$

where

$$\tilde{A} : D_\infty(A) \times D_\infty(B) \rightarrow X,$$

with

$$D_\infty(A) = D_\infty(B) = \left\{ z \in W^{2,p}(\Omega) \text{ for all } p > n, \Delta z \in C(\bar{\Omega}), \frac{\partial z}{\partial \eta} = 0 \right\},$$

$$\tilde{A}U = (Au, Bv)$$

and

$$F(U) = (\Lambda - \lambda f(u, v); \lambda f(u, v)).$$

It is clear that from the general theory of semigroup we deduce the existence of an unique local classical solution in some interval  $]0, T^*[$ , where  $T^*$  is the eventual blowing-up time in  $L^\infty(\Omega)$  (see for example Henry [6] or Pazy [12]).

**2.2. Positivity of solutions.** From the nonnegativity of the initial data  $u_0$  on  $\Omega$ , one easily deduce from the maximum applied to the first equation of (1.1) that the component  $u$  remains nonnegative and bounded on  $(0, T^*) \times \Omega$  so that

$$0 \leq u(t, x) \leq K := \max(\|u_0\|_\infty, \frac{\Lambda}{\mu}).$$

In order to obtain the positivity of  $v$  we assume that

$$(H.3) \quad \|u_0\|_\infty \leq \frac{\Lambda}{\mu}.$$

**Lemma 1.** *Let  $(u, v)$  be the solution of (1.1) – (1.3). If the initial data  $v_0$  satisfies the condition*

$$v_0 \geq \frac{b}{d-a} \left( \frac{\Lambda}{\mu} - \|u_0\|_\infty \right)$$

then for all  $(t, x) \in (0, T^*) \times \Omega$  we have

$$v(t, x) \geq \frac{b}{d-a} \left( \frac{\Lambda}{\mu} - u(t, x) \right).$$

PROOF. Let us consider  $w = v - \frac{b}{d-a} \left( \frac{\Lambda}{\mu} - u \right)$  and in this way the system (1.1) – (1.3) may be equivalent to the system

$$(2.1) \quad \begin{cases} \frac{du}{dt} - a\Delta u = \Lambda - \lambda f(u, v) - \mu u \\ \frac{dw}{dt} - d\Delta w = \left(1 - \frac{b}{d-a}\right) \lambda f(u, v) - \mu w \end{cases} \quad \text{in } \mathbb{R}^+ \times \Omega$$

with

$$(2.2) \quad \begin{cases} u(0, \cdot) = u_0(\cdot) \geq 0 \\ w(0, \cdot) = w_0(\cdot) \geq 0 \end{cases} \quad \text{in } \Omega.$$

Using (H.1) and maximum principle to the second equation of the system (2.1) we obtain

$$w(t, x) \geq 0 \quad \forall (t, x) \in (0, T^*) \times \Omega$$

that means

$$(2.3) \quad v(t, x) \geq \frac{b}{d-a} \left( \frac{\Lambda}{\mu} - u(t, x) \right) \geq 0 \quad \text{for all } (t, x) \in (0, T^*) \times \Omega.$$

□

Now we are able to state our main result.

### 3. Statement and proof of the main results

Using the idea of Haraux and Youkana [5], let us consider the fonctionnal

$$J(t) = \int_{\Omega} (1 + \delta(1 + u + u^2)) e^{\varepsilon w} dx$$

where  $\delta$  and  $\varepsilon$  are constants such that

$$(3.1) \quad 0 < \delta \leq \min \left( \frac{\sigma}{2\Lambda(1+2K)}, 2 \left( \frac{2\sqrt{ab}}{a+b} \cdot \frac{1}{1+2K} \right)^2 \right)$$

and

$$(3.2) \quad 0 < \varepsilon \leq \frac{\delta}{1 + \delta(K + K^2)} \min \left( 1, \frac{d-a}{b} \right).$$

Our main results are as follows:

**Theorem 1.** *Let  $(u, v)$  be a solution of (1.1) – (1.3) on  $(0, T^*)$ , then there exist a positive constant  $\gamma$  such that for all  $t \in (0, T^*)$  the fonctionnal*

$$(3.3) \quad J(t) = \int_{\Omega} (1 + \delta(1 + u + u^2)) e^{\varepsilon w} dx$$

satisfies the relation

$$(3.4) \quad \frac{d}{dt} J(t) \leq -J(t) + \gamma.$$

**Corollary 1.** *Under the hypothesis (H.1) – (H.3), the solutions of the parabolic system (1.1) – (1.3) with nonnegative and bounded initial data  $u_0, v_0$  are global and uniformly bounded in  $(0, +\infty) \times \Omega$ .*

PROOF OF THE THEOREM. By simple use of Green's formula and from equations (1.1) – (1.3) we have for all  $t \in (0, T^*)$

$$\frac{d}{dt}J(t) = G + H_1 + H_2 + H_3$$

where

$$\begin{aligned} G = & -\delta \int_{\Omega} [2a + b\varepsilon(1 + 2u)] e^{\varepsilon w} |\nabla u|^2 dx \\ & - \varepsilon \int_{\Omega} [(a + d)\delta(1 + 2u) + b\varepsilon(1 + \delta(u + u^2))] e^{\varepsilon w} \nabla u \nabla v dx \\ & - d\varepsilon^2 \int_{\Omega} [1 + \delta(u + u^2)] e^{\varepsilon w} |\nabla v|^2 dx \end{aligned}$$

and

$$\begin{aligned} H_1 = & \int_{\Omega} \left( \Lambda \frac{\delta(1+2u)}{1+\delta(u+u^2)} - \mu u \frac{\delta(1+2u)}{1+\delta(u+u^2)} - \mu \right) (1 + \delta(u + u^2)) e^{\varepsilon w} dx \\ H_2 = & \int_{\Omega} \left( \varepsilon - \frac{\delta(1+2u)}{1+\delta(u+u^2)} \right) \lambda f(u, v) (1 + \delta(u + u^2)) e^{\varepsilon w} dx \\ H_3 = & \mu \int_{\Omega} (1 - \varepsilon w) e^{\varepsilon w} (1 + \delta(u + u^2)) dx. \end{aligned}$$

We observe that G involves a quadratic form with respect to  $\nabla u$  and  $\nabla v$

$$\begin{aligned} Q = & \delta(2a + b\varepsilon(1 + 2u)) e^{\varepsilon w} |\nabla u|^2 \\ & + \varepsilon [(a + d)\delta(1 + 2u) + b\varepsilon(1 + \delta(u + u^2))] e^{\varepsilon w} \nabla u \nabla v \\ & + d\varepsilon^2 (1 + \delta(u + u^2)) e^{\varepsilon w} |\nabla v|^2 \end{aligned}$$

and the discriminant

$$D = \left[ \varepsilon [(a + d)\delta(1 + 2u) + b\varepsilon(1 + \delta(u + u^2))] \right]^2 - 4[\delta(2a + b\varepsilon(1 + 2u))] [d\varepsilon^2(1 + \delta(u + u^2))],$$

may be nonpositive since the constants  $\delta$  and  $\varepsilon$  satisfies (3.1) and (3.2).

Consequently

$$G \leq 0 \text{ for all } t \text{ in } (0, T^*).$$

Concerning the terms  $H_i$ ,  $i = 1, 2, 3$  we have again from (3.1) – (3.2) where now

$$(3.5) \quad 0 < \delta \leq \frac{\mu}{2\Lambda(1 + 2K)}, \quad 0 < \varepsilon \leq \frac{\delta}{1 + \delta(K + K^2)},$$

one checks that

$$(3.6) \quad \Lambda \frac{\delta(1 + 2u)}{1 + \delta(u + u^2)} - \mu u \frac{\delta(1 + 2u)}{1 + \delta(u + u^2)} - \mu \leq 2\Lambda\delta(1 + 2K) - \mu \leq -\frac{\mu}{2},$$

and

$$(3.7) \quad \varepsilon - \frac{\delta(1 + 2u)}{1 + \delta(u + u^2)} \leq \varepsilon - \frac{\delta}{1 + \delta(K + K^2)} \leq 0,$$

from we deduce

$$H_1 \leq -\frac{\sigma}{2}J(t)$$

and

$$H_2 \leq 0.$$

Now for the terme  $H_3$ , one observe that the function

$$\pi : \eta \rightarrow (1 - \varepsilon\eta) e^{\varepsilon\eta},$$

is bounded on  $\mathbb{R}^+$ . Indeed, one has

$$\frac{d\pi}{d\eta}(\eta) = -\varepsilon^2 \eta e^{\varepsilon\eta} \leq 0,$$

so that  $\pi$  is nonincreasing in  $[0, +\infty[$  and

$$\max_{\eta \geq 0} (1 - \varepsilon\eta) e^{\varepsilon\eta} = 1.$$

In this way one can deduce that there a positive constant  $\gamma$

$$\gamma = \mu (1 + \delta (K + K^2)) |\Omega|$$

such that

$$(3.8) \quad H_3 \leq \gamma.$$

Aditioning  $G$ ,  $H_1$ ,  $H_2$  and  $H_3$  we get

$$(3.9) \quad \frac{d}{dt} J(t) = G + H_1 + H_2 + H_3 \leq -\frac{\mu}{2} L(t) + \gamma.$$

Thus, the proof of theorem is complete.  $\square$

Now we are able to prove the global existence for the solutions of (1.1) – (1.3).

PROOF OF THE COROLLARY. From (H.2) we see that there exist a positive consant  $C$  such that

$$(3.10) \quad 1 + f(., v) \leq C e^{\frac{v}{n}}, \quad \forall v \geq 0.$$

where  $\varepsilon$  is chosen as in (3.2). By virtue of (3.9), it is seen that

$$J(t) \leq C, \quad \forall t \in (0, T^*)$$

and it follows in particular that

$$f(., v) \in L^\infty((0, T^*); L^n(\Omega)).$$

By the regularising effect of the heat equation [6, 4] we conclude that

$$u \in L^\infty((0, T^*); L^\infty(\Omega)).$$

Finally the solutions of the system (1.1)–(1.3) are global and uniformly bounded on  $(0, +\infty) \times \Omega$  and the corollary is completely proved.  $\square$

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