EXCITED BROWNIAN MOTIONS AS LIMITS OF EXCITED RANDOM WALKS

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ABSTRACT. We obtain the convergence in law of a sequence of excited (also called cookies) random walks toward an excited Brownian motion. This last process is a continuous semi-martingale whose drift is a function, say φ, of its local time. It was introduced by Norris, Rogers and Williams as a simplified version of Brownian polymers, and then recently further studied by the authors. To get our results we need to renormalize together the sequence of cookies, the time and the space in a convenient way. The proof follows a general approach already taken by Tóth and his coauthors in multiple occasions, which goes through Ray-Knight type results. Namely we first prove, when φ is bounded and sufficiently regular, that the convergence holds at the level of the local time processes. This is done via a careful study of the transition kernel of an auxiliary Markov chain which describes the local time at a given level. Then we prove a tightness result and deduce the convergence at the level of the full processes.

1. INTRODUCTION

1.1. **General overview.** Self-interacting random processes play a prominent role in the probability theory and in statistical physic. One fascinating aspect is that behind an apparent simplicity, they can be extremely hard to analyze rigorously. Just to mention one striking example, it is still not known whether once reinforced random walks on a ladder are recurrent in general (see however [Sel] and [Ver] for a partial answer and the surveys [MR] and [Pem] for other problems on reinforced processes). A major difficulty in these models is that we loose the Markovian property and in particular the usual dichotomy between recurrence and transience can be broken. A famous example where this happens is for vertex reinforced random walks on $\mathbb{Z}$: it is now a well known result in the field, first conjectured and partially proved by Pemantle and Volkov [PemV], that almost surely these processes eventually get stuck on five sites [Tar]. For analogous results concerning self-attracting diffusions, see [CLJ], [HRo] and [R].

Beside this very basic, yet fundamental, problem of recurrence, a question of particular interest is to understand the connections between the various discrete and continuous models. In particular an important challenging conjecture is that self-avoiding random walks on $\mathbb{Z}^2$ converge, after renormalization, toward the $SLE_{8/3}$ (see [LSW] for a discussion on this and [DCS] for some recent progress). There are in fact not many examples where invariance principles or central limit theorems were fully established. But for instance it was proved that random walks perturbed at extrema converge after the usual renormalization toward a perturbed Brownian motion (see e.g. [Dav] and [W]).

In this paper we are interested in the class of so-called excited random processes, which are among the most elementary examples of self-interacting processes. By

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this we mean that the interaction with the past trajectory is as localized as possible: the evolution of these processes at any time only depend on their local time at their present position. A discrete version was introduced relatively recently by Benjamini and Wilson [BW] and a generalization, called multi-excited or cookie random walks, was then further studied by Zerner [Z] and many other authors (see in particular [MPRV] and references therein). Closely related models were also considered in [ABK], [BKS], [K] and [KRS]. Dolgopyat [D] observed that in dimension 1, in the recurrent regime, and after the usual renormalization, multi-excited random walks also converge toward a perturbed Brownian motion (we will give a more precise statement later). However, as we will see below, the latter are not, in some sense, the most natural continuous versions of excited processes. Somewhat more natural ones were introduced two decades ago by Norris, Rogers and Williams [NRW2], in connection with the excluded volume problem [NRW1], and as a simplified model for Brownian polymers. They were later called excited Brownian motions by the authors [RS].

The aim of this paper is to show that excited Brownian motions can be approached in law by multi-excited random walks in the Skorokhod space, i.e. in the sense of the full process. For this we need to use a nonstandard renormalization, namely we need to scale together and appropriately the sequence of cookies, which govern the drift of the walk, the space and the time. But let us give more details, starting with some definitions:

A multi-excited or cookie random walk $(X_\varepsilon(n), n \geq 0)$ is associated to a sequence

$$\varepsilon := (\varepsilon_i, i \geq 1) \in [-1, 1]^{\mathbb{N}},$$

of cookies in the following way: set

$$p_{\varepsilon,i} := \frac{1}{2}(1 + \varepsilon_i),$$

for all $i \geq 1$, and let $(\mathcal{F}_{\varepsilon,n}, n \geq 0)$ be the filtration generated by X_ε. Then $X_\varepsilon(0) := 0$ and for all $n \geq 0$,

$$\mathbb{P}[X_\varepsilon(n+1) - X_\varepsilon(n) = 1 \mid \mathcal{F}_{\varepsilon,n}] = 1 - \mathbb{P}[X_\varepsilon(n+1) - X_\varepsilon(n) = -1 \mid \mathcal{F}_{\varepsilon,n}] = p_{\varepsilon,i},$$

if $\#\{j \leq n : X_\varepsilon(j) = X_\varepsilon(n)\} = i$. We notice that the case of random cookies has also been studied in the past, for instance by Zerner [Z], but here we consider only deterministic ε.

On the other hand excited Brownian motions are solutions of a stochastic differential equation of the type:

$$dY_t = dB_t + \varphi(L_t^{Y_t}) \, dt,$$

where B is a Brownian motion, $L_\cdot$ is the local time process of Y and $\varphi : \mathbb{R} \to \mathbb{R}$ is some measurable (bounded) function.

So at an heuristic level the discrete and the continuous models are very similar: the drift is a function of the local time at the present position. But the analogy can be pushed beyond this simple observation. In particular criteria for recurrence and nonzero speed in both models (see respectively [KZ] and [RS]) are entirely similar (see below). Our results here give now a concrete link. We first prove that when φ is bounded and sufficiently regular, the local time process of X_ε, conveniently renormalized, converges to the one of Y, exactly in the same spirit

as in Tóth's papers on self-interacting random walks (see [T2]). Then we obtain a tightness result and deduce a convergence in the Skorokhod space at the level of the processes (see Theorem 1.4 below). For proving the convergence of the local time processes we use a standard criterion of Ethier and Kurtz [EK] on approximation of diffusions. To show that we can apply it here we introduce some auxiliary Markov chains describing the local time on each level and we make a careful analysis of the transition kernels of these Markov chains (see Section 2).

1.2. **Description of the results.** For $a \in \mathbb{Z}$ and $v \in \mathbb{N}$, let

$$\tau_{\varepsilon,a}(v) := \inf \left\{ j \ : \ \#\{i \le j \ : \ X_\varepsilon(i) = a \text{ and } X_\varepsilon(i+1) = a-1\} = v+1 \right\}.$$

We assume here that X_ε is recurrent, in which case all these stopping times are a.s. finite. A criterion when $\varepsilon_i \ge 0$ for all i or when $\varepsilon_i = 0$ for i large enough is given in [Z] and [KZ] (namely in these cases, X_ε is recurrent if, and only if, $\sum_i \varepsilon_i \in [-1,1]$).
   Then we consider the process $(S_{\varepsilon,a,v}(k), k \in \mathbb{Z})$ defined by

$$S_{\varepsilon,a,v}(k) = \#\{j \le \tau_{\varepsilon,a}(v) - 1 \ : \ X_\varepsilon(j) = k \text{ and } X_\varepsilon(j+1) = k-1\}.$$

In particular $S_{\varepsilon,a,v}(a) = v$. Assume now that φ is bounded and let $\varepsilon_n = (\varepsilon_i(n), i \ge 1)$ be defined by

$$\varepsilon_i(n) := \frac{1}{2n}\varphi\left(\frac{i}{2n}\right) \quad \text{for all } n \ge 1 \text{ and all } i \ge 1.$$

Since φ is bounded, if n is large enough then $\varepsilon_n \in [-1,1]^{\mathbb{N}}$ and X_{ε_n} is well defined. Then for $a \in \mathbb{R}$ and $v \ge 0$, set

$$\Lambda_{a,v}^{(n)}(x) := \frac{1}{n} S_{\varepsilon_n,[2na],[nv]}([2nx]) \quad \text{for all } x \in \mathbb{R}.$$

We give now the analogous definitions in the continuous setting. First for $a \in \mathbb{R}$, let

$$\tau_a(v) := \inf\{t > 0 \ : \ L_t^a > v\} \quad \text{for all } v \ge 0,$$

be the right continuous inverse of the local time of Y at level a. To simplify we assume also that Y is recurrent. This is equivalent (see [RS, Theorem 1.1]) to requiring $C_1^+ = C_1^- = +\infty$, where

$$C_1^\pm := \int_0^\infty \exp\left[\mp \int_0^x \frac{dl}{l} \int_0^l \varphi(u)\, du\right]\, dx.$$

In particular when φ is nonnegative or compactly supported this is equivalent to $\int_0^\infty \varphi(\ell)\, d\ell \in [-1,1]$. In this case $\tau_a(v)$ is a.s. finite for any a and v. Then set

$$\Lambda_{a,v}(x) := L_{\tau_a(v)}^x,$$

for all a, v, and x. The Ray-Knight theorem describes the law of $(\Lambda_{a,v}(x), x \in \mathbb{R})$ (a proof is given in [NRW2] when $v = 0$, but it applies as well for $v > 0$) and we recall this result now. To fix ideas we assume that $a \le 0$. An analogue result holds for $a \ge 0$. So first we have $\Lambda_{a,v}(a) = v$. Next on $[a,0]$, $\Lambda_{a,v}$ is solution of the stochastic differential equation:

$$(1) \qquad d\Lambda_{a,v}(x) = 2\sqrt{\Lambda_{a,v}(x)}\, dB_x + 2(1 + h(\Lambda_{a,v}(x)))\, dx,$$

where B is a Brownian motion and

$$h(z) := \int_0^z \varphi(\ell)\, d\ell \quad \text{for all } z \ge 0.$$

On $[0, +\infty)$, it is solution of

$$dΛ_{a,v}(x) = 2\sqrt{Λ_{a,v}(x)}\, dB_x + 2h\left(Λ_{a,v}(x)\right)\, dx, \tag{2}$$

up to the first time, say $w^+_{a,v}$, when it hits 0, and then is absorbed in 0 (i.e. $Λ_{a,v}(x) = 0$ for $x \geq w^+_{a,v}$). Similarly $(Λ_{a,v}(a - x), x \geq 0)$ is solution of (2) (with a drift $-2h$ instead of $2h$) up to the first time, say $w^-_{a,v}$, when it hits 0, and then is absorbed in 0.

We denote by $\mathbb{D}(\mathbb{R})$ the Skorokhod space, i.e. the space of càdlàg functions on $\mathbb{R}$, which is a separable metric space (see for instance Section 12 in [Bil]).

Our first result is the following theorem:

Theorem 1.1. *Assume that φ is bounded, Lipschitz and such that $(\ell \mapsto \ell\varphi(\ell))$ is also Lipschitz. Assume further that for n large enough, X_{ε_n} is recurrent and that Y is recurrent. Then for any $a \in \mathbb{R}$ and $v \geq 0$,*

$$(Λ^{(n)}_{a,v}(x), x \in \mathbb{R}) \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (Λ_{a,v}(x), x \in \mathbb{R}),$$

in the Skorokhod space $\mathbb{D}(\mathbb{R})$.

As announced above, this theorem gives the convergence of a sequence of excited random walks toward the excited Brownian motion (associated to φ) at the level of the local times. We will later extend this result in two directions. First in a non homogeneous setting, i.e. when φ is allowed to depend also on the space variable. We refer the reader to Subsection 2.7 for more details. Then we will prove a multidimensional version (see Theorem 5.1): given any finite set I and (a_i, v_i), $i \in I$, the sequence $(Λ^{(n)}_{a_i,v_i}, i \in I)$ converges in law toward $(Λ_{a_i,v_i}, i \in I)$.

A consequence of Theorem 1.1 is the following

Corollary 1.2. *For any $a \in \mathbb{R}$ and $v \geq 0$, the random variable $\tau_{\varepsilon_n,[2na]}([nv])/(4n^2)$ converges in law toward $\tau_a(v)$, when $n \to \infty$.*

For $u \in \mathbb{R}$, denote by θ_u some geometric random variable with parameter $1 - e^{-u}$, independent of X_{ε_n} and Y. Denote also by γ_u some independent exponential random variable with parameter u. Then as in [T1], we can deduce from the previous results the

Corollary 1.3. *For any $\lambda \geq 0$, $X_{\varepsilon_n}(\theta_{\lambda/(4n^2)})/(2n)$ converges in law toward Y_{γ_λ}, when $n \to \infty$.*

Finally we get the following:

Theorem 1.4. *For $t \geq 0$, set $X^{(n)}(t) := X_{\varepsilon_n}([4n^2 t])/(2n)$. Then under the hypotheses of Theorem 1.1,*

$$(X^{(n)}(t), t \geq 0) \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (Y(t), t \geq 0).$$

To obtain this result we need to prove the tightness of the sequence $X_{\varepsilon_n}([4n^2 \cdot])/(2n)$, $n \geq 1$. This is done by using a coupling between different branching processes, similar as what we use for proving Corollary 1.2. The convergence of finite-dimensional distributions follows from the multi-dimensional extension of Theorem 1.1 mentioned above.

As for Theorem 1.1 an extension of this result to the non homogeneous setting can be proved (see Remark 5.4 at the end of the paper).

Let us mention now a related result of Dolgopyat [D]. He proved a functional central limit theorem for excited random walks when ε is fixed, and in the recurrent regime; more precisely when $\varepsilon_i \geq 0$ for all i and $\alpha := \sum_i \varepsilon_i < 1$. In this case the limiting process is a perturbed Brownian motion, i.e. the process defined by

$$X_t = B_t + \alpha \left(\sup_{s \leq t} X_s - \inf_{s \leq t} X_s \right) \quad \text{for all } t \geq 0,$$

with B a Brownian motion.

We will prove Theorem 1.1 in Section 2, Corollaries 1.2 and 1.3 respectively in Section 3 and 4, and Theorem 1.4 in Section 5.

2. Proof of Theorem 1.1

To fix ideas we assume that $a \leq 0$. The case $a \geq 0$ is similar. Moreover we only prove the convergence of $\Lambda_{a,v}^{(n)}$ on the time interval $[a, \infty)$, since the proofs of the convergence on $(-\infty, a]$ and on $[0, +\infty)$ are the same.

2.1. A criterion of Ethier and Kurtz.
It is now a standard fact and not difficult to check (see however [BaS] or [KZ] for more details) that for all $a \in \mathbb{N}^-$ and $v \in \mathbb{N}$, the sequence $(S_{\varepsilon,a,v}(a), \ldots, S_{\varepsilon,a,v}(0))$ has the same law as $(V_{\varepsilon,v}(0), \ldots, V_{\varepsilon,v}(-a))$, where $(V_{\varepsilon,v}(k), k \geq 0)$ is some Markov chain starting from v, independent of a. Similarly $(S_{\varepsilon,a,v}(k), k \geq 0)$ has the same law as some other Markov chain $(\widetilde{V}_{\varepsilon,w}(k), k \geq 0)$ starting from $w = S_{\varepsilon,a,v}(0)$, and $(S_{\varepsilon,a,v}(a - k), k \geq 1)$ has the same law as $(\widetilde{V}_{-\varepsilon,v+1}(k), k \geq 1)$, where by definition $(-\varepsilon)_i = -\varepsilon_i$ for all $i \geq 1$.

The laws of these Markov chains will be described in Subsection 2.2 in terms of another Markov chain W_ε, see in particular (7) and (8).

In the following, in order to lighten the presentation we will forget about the dependence in the starting point (which does not play any serious role here) in the notation for V_ε and $\widetilde{V}_\varepsilon$. Thus V_ε and V_{ε_n} should be understood respectively as $V_{\varepsilon,v}$ and $V_{\varepsilon_n,[nv]}$, where the v will be clear from the context, and similarly for $\widetilde{V}_\varepsilon$ and $\widetilde{V}_{\varepsilon_n}$.

Now we first prove the convergence of $\Lambda_{a,v}^{(n)}$ on $[a, 0]$. The proofs of the convergence on $[0, +\infty)$ and on the full interval $[a, +\infty)$ are similar and will be explained in Subsection 2.6.

So on $[a, 0]$, $\Lambda_{a,v}^{(n)}$ can be decomposed as a sum of a martingale part $M_{a,v}^{(n)}$ and a drift part $B_{a,v}^{(n)}$:

$$(3) \qquad \Lambda_{a,v}^{(n)}(x) = \frac{[nv]}{n} + M_{a,v}^{(n)}(x) + B_{a,v}^{(n)}(x) \quad \text{for all } x \in [a, 0],$$

with the following equalities in law:

$$(4) \qquad M_{a,v}^{(n)}(x) = \frac{1}{n} \sum_{k=1}^{[2nx]-[2na]} \left\{ V_{\varepsilon_n}(k) - \mathbb{E}[V_{\varepsilon_n}(k) \mid V_{\varepsilon_n}(k-1)] \right\},$$

and

$$(5) \qquad B_{a,v}^{(n)}(x) = \frac{1}{n} \sum_{k=1}^{[2nx]-[2na]} \left\{ \mathbb{E}[V_{\varepsilon_n}(k) \mid V_{\varepsilon_n}(k-1)] - V_{\varepsilon_n}(k-1) \right\}.$$

Let also $A_{a,v}^{(n)}$ be the previsible compensator of $(M_{a,v}^{(n)})^2$ given by

$$(6) \quad A_{a,v}^{(n)}(x) = \frac{1}{n^2} \sum_{k=1}^{[2nx]-[2na]} \left\{ \mathbb{E}[V_{\varepsilon_n}(k)^2 \mid V_{\varepsilon_n}(k-1)] - \mathbb{E}[V_{\varepsilon_n}(k) \mid V_{\varepsilon_n}(k-1)]^2 \right\},$$

for all $x \in [a, 0]$.

We will deduce the convergence of $\Lambda_{a,v}^{(n)}$ from a criterion of Ethier and Kurtz [EK], namely Theorem 4.1 p.354. According to this result the convergence on $[a, 0]$ in Theorem 1.1 follows from Propositions 2.1 and 2.2 below. In addition we need to verify that the martingale problem associated to the operator $2\lambda d^2/(d\lambda)^2 + 2(1 + h(\lambda))d/d\lambda$ is well posed. This follows from Theorem 2.3 p.372 in [EK] (with the notation of [EK] take $r_0 = 0$ and $r_1 = +\infty$).

Proposition 2.1. *Let $R > 0$ be given. Set $\tau_n^R := \inf\{x \geq a \; : \; \Lambda_{a,v}^{(n)}(x) \geq R\}$. Then for $a \leq x \leq 0 \wedge \tau_n^R$,*

$$B_{a,v}^{(n)}(x) = 2 \int_a^x (1 + h(\Lambda_{a,v}^{(n)}(y))) \, dy + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

where the $\mathcal{O}(n^{-1/2})$ is deterministic and only depends on a and R.

Proposition 2.2. *Let $R > 0$ be given. Then for $a \leq x \leq 0 \wedge \tau_n^R$,*

$$A_{a,v}^{(n)}(x) = 4 \int_a^x \Lambda_{a,v}^{(n)}(y) \, dy + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

where the $\mathcal{O}(n^{-1/2})$ is deterministic and only depends on a and R.

These propositions will be proved in the Subsections 2.2–2.5.

2.2. **An auxiliary Markov chain.** Let ε and $v \geq 0$ be given. We express here (see in particular (7) and (8) below) the laws of $V_\varepsilon = V_{\varepsilon,v}$ and $\widetilde{V}_\varepsilon = \widetilde{V}_{\varepsilon,v}$ in terms of the law of another Markov chain W_ε. A similar representation already appeared in Tóth's paper [T1] on "true" self-avoiding walks. So let us first define $(s_{\varepsilon,i}, i \geq 0)$ by $s_{\varepsilon,0} = 0$ and for $i \geq 1$,

$$s_{\varepsilon,i} := \sum_{j=1}^{i} 1_{\{U_j \geq p_{\varepsilon,j}\}},$$

where $(U_j, j \geq 1)$ is a sequence of i.i.d random variables with uniform distribution in $[0, 1]$. This $s_{\varepsilon,i}$ is equal in law to the number of times the excited random walk jumps from level k to $k - 1$, for some arbitrary $k \in \mathbb{Z}$, after i visits at this level k. For $m \geq 0$, set

$$W_\varepsilon(m) := \inf\{i \geq 0 \; : \; s_{\varepsilon,i} = m\}.$$

Then $W_\varepsilon(m)$ is equal in law to the number of visits to level k before the m-th jump from k to $k - 1$. Moreover $(W_\varepsilon(m), m \geq 0)$ is a Markov chain on $\mathbb{N}$ starting from 0 and with transition operator Q_ε defined for any function f by

$$Q_\varepsilon f(r) = \sum_{\ell \geq 1} f(r + \ell) 2^{-\ell} (1 + \varepsilon_{r+1}) \ldots (1 + \varepsilon_{r+\ell-1})(1 - \varepsilon_{r+\ell}),$$

for all $r \in \mathbb{N}$. Furthermore it is immediate that the law of $V_\varepsilon(k+1)$ conditionally to $\{V_\varepsilon(k) = m\}$ is equal to the law of $W_\varepsilon(m+1) - (m+1)$:

$$(7) \qquad \mathcal{L}(V_\varepsilon(k+1) \mid V_\varepsilon(k) = m) = \mathcal{L}(W_\varepsilon(m+1) - (m+1)).$$

Similarly the law of $\widetilde{V}_\varepsilon(k+1)$ conditionally to $\{\widetilde{V}_\varepsilon(k) = m\}$ is equal to the law of $W_\varepsilon(m) - m$:

$$(8) \qquad \mathcal{L}(\widetilde{V}_\varepsilon(k+1) \mid \widetilde{V}_\varepsilon(k) = m) = \mathcal{L}(W_\varepsilon(m) - m).$$

By convention we denote by Q_0 the transition operator associated to the sequence $(\varepsilon_i, i \geq 1)$, where $\varepsilon_i = 0$ for all i. In other words

$$Q_0 f(r) = \mathbb{E}[f(r + \xi)] \quad \text{for all } r \in \mathbb{N},$$

where ξ is a geometric random variable with parameter $1/2$, i.e. $\mathbb{P}(\xi = \ell) = 2^{-\ell}$, for all $\ell \geq 1$. Note that $\mathbb{E}(\xi) = 2$ and $\mathbb{V}(\xi) = 2$. In particular, if u is defined by $u(r) = r$ for all $r \in \mathbb{N}$, then for all $m \geq 1$,

$$Q_0^m u(0) = \mathbb{E}[\xi_1 + \cdots + \xi_m] = 2m,$$

where $\xi_1, \ldots, \xi_m$ are i.i.d. geometric random variables with parameter $1/2$. Note also that for all $m \geq 1$, $\mathbb{E}[W_\varepsilon(m)] = Q_\varepsilon^m u(0)$. Thus (7) shows that

$$(9) \qquad \mathbb{E}[V_\varepsilon(k) \mid V_\varepsilon(k-1)] - V_\varepsilon(k-1) = Q_\epsilon^{V_\varepsilon(k-1)+1} u(0) - Q_0^{V_\varepsilon(k-1)+1} u(0) + 1,$$

for all $k \geq 1$. So in view of (5) and (9), our strategy for proving Proposition 2.1 will be to estimate terms of the form $Q_\varepsilon^m u(0) - Q_0^m u(0)$. Note that since $x < \tau_n^R$ by hypothesis, we can restrict us to the case when $m \leq Rn + 1$. Likewise
$$(10)$$
$$\mathbb{E}[V_\varepsilon^2(k) \mid V_\varepsilon(k-1)] - \mathbb{E}[V_\varepsilon(k) \mid V_\varepsilon(k-1)]^2 = Q_\varepsilon^{V_\varepsilon(k-1)+1} u^2(0) - (Q_\varepsilon^{V_\varepsilon(k-1)+1} u(0))^2,$$

for all $k \geq 1$. So in view of (6) and (10) we will have also to estimate terms of the form $Q_\varepsilon^m u^2(0) - (Q_\varepsilon^m u(0))^2$, for proving Proposition 2.2.

2.3. **Some elementary properties of the operators Q_ε and Q_0.** For $f : \mathbb{N} \to \mathbb{R}$, we set

$$|f|_\infty = \sup_{r \in \mathbb{N}} |f(r)| \quad \text{and} \quad \mathrm{Lip}(f) = \sup_{r \neq r'} \frac{|f(r) - f(r')|}{|r - r'|}.$$

If $\varepsilon \in [-1, 1]^\mathbb{N}$ is given we set

$$|\varepsilon|_\infty := \sup_{i \geq 1} |\varepsilon_i| \quad \text{and} \quad \mathrm{Lip}(\varepsilon) = \sup_{i \neq j} \frac{|\varepsilon_i - \varepsilon_j|}{|i - j|}.$$

Throughout this section we will always assume that $|\varepsilon|_\infty \leq 1/2$. Let $R_\varepsilon := Q_\varepsilon - Q_0$. Observe that $Q_\varepsilon 1 = Q_0 1 = 1$, where 1 is the constant function on $\mathbb{N}$. In particular $R_\varepsilon 1 = 0$. Observe also that for any function f, $|Q_\varepsilon f|_\infty \leq |f|_\infty$, and

$$(11) \qquad |Q_\varepsilon f - f|_\infty \leq C \mathrm{Lip}(f),$$

where $C = \sum_{\ell \geq 1} \ell (4/3)^{-\ell}$. As a corollary we get the

Lemma 2.3. *There exists a constant $C > 0$ such that for all $|\varepsilon|_\infty \leq 1/2$, all $j \geq 0$ and all functions f,*

$$|Q_\varepsilon^j f - f|_\infty \leq Cj \mathrm{Lip}(f).$$

Proof. Write

$$Q_\varepsilon^j f - f = \sum_{i=1}^j Q_\varepsilon^{i-1}(Q_\varepsilon f - f),$$

and then use (11) for each term of the sum.   □

Set for all $r \geq 0$ and $\ell \geq 1$,

$$\widetilde{\varepsilon}_{r,\ell} := -\varepsilon_{r+\ell} + \sum_{i=1}^{\ell-1} \varepsilon_{r+i},$$

and define $\widetilde{R}_\varepsilon$ by

$$\widetilde{R}_\varepsilon f(r) = \sum_{\ell \geq 1} f(r+\ell) 2^{-\ell} \widetilde{\varepsilon}_{r,\ell}.$$

The next result is immediate.

Lemma 2.4. *For any function f and any r,*

$$\widetilde{R}_\varepsilon f(r) = -\sum_{\ell \geq 1} \varepsilon_{r+\ell} \left(f(r+\ell) 2^{-\ell} - \sum_{i=\ell+1}^{\infty} f(r+i) 2^{-i} \right).$$

In particular $\widetilde{R}_\varepsilon 1 = 0$ since $\sum_{i=\ell+1}^{\infty} 2^{-i} = 2^{-\ell}$. We also get the following

Lemma 2.5. *There exists a constant $C > 0$ such that for all $|\varepsilon|_\infty \leq 1/2$ and all f,*

$$|\widetilde{R}_\varepsilon f|_\infty \leq C Lip(f) \times |\varepsilon|_\infty.$$

Proof. By using that $f(r)\widetilde{R}_\varepsilon 1(r) = 0$ for all r, we get

$$\widetilde{R}_\varepsilon f(r) = -\sum_{\ell \geq 1} \varepsilon_{r+l} \left((f(r+l) - f(r)) 2^{-\ell} - \sum_{i=\ell+1}^{\infty} (f(r+i) - f(r)) 2^{-i} \right).$$

Thus

$$|\widetilde{R}_\varepsilon f|_\infty \leq \mathrm{Lip}(f) \times |\varepsilon|_\infty \times \sum_{\ell \geq 1} \left(\ell 2^{-\ell} + \sum_{i=\ell+1}^{\infty} i 2^{-i} \right).$$

The lemma follows.   □

Next we have

Lemma 2.6. *There exists a constant $C > 0$ such that for all $|\varepsilon|_\infty \leq 1/2$ and all f,*

$$|R_\varepsilon f - \widetilde{R}_\varepsilon f|_\infty \leq C Lip(f) \times |\varepsilon|_\infty^2,$$

Proof. Recall that

$$(R_\varepsilon - \widetilde{R}_\varepsilon) f(r) = \sum_{\ell \geq 1} f(r+\ell) 2^{-\ell} (\varepsilon_{r,\ell} - \widetilde{\varepsilon}_{r,\ell}),$$

where for all r and ℓ,

$$\varepsilon_{r,\ell} := (1 + \varepsilon_{r+1}) \ldots (1 + \varepsilon_{r+\ell-1})(1 - \varepsilon_{r+\ell}) - 1.$$

Since $f(r)R_\varepsilon 1(r) = f(r)\widetilde{R}_\varepsilon 1(r) = 0$ for all r, we get

$$(R_\varepsilon - \widetilde{R}_\varepsilon) f(r) = \sum_{\ell \geq 1} (f(r+\ell) - f(r)) 2^{-\ell} (\varepsilon_{r,\ell} - \widetilde{\varepsilon}_{r,\ell}).$$

But for any r, $\varepsilon_{r,0} = \widetilde{\varepsilon}_{r,0}$, and for $\ell \geq 1$,

$$
\begin{aligned}
|\varepsilon_{r,\ell} - \widetilde{\varepsilon}_{r,\ell}| &\leq (1 + |\varepsilon|_\infty)^\ell - 1 - \ell|\varepsilon|_\infty. \\
&\leq \ell^2(1 + |\varepsilon|_\infty)^{\ell-2}|\varepsilon|_\infty^2 \\
&\leq \ell^2(3/2)^{\ell-2}|\varepsilon|_\infty^2.
\end{aligned}
$$

(12)

The lemma follows. $\qquad\square$

We will need also the following

Lemma 2.7. *We have $Lip(Q_0^i f) \leq Lip(f)$, for all $i \geq 0$, and*

$$
|Q_0^i f(r) - f(r + 2i)| \leq \sqrt{2} Lip(f)\sqrt{i} \quad \text{for all } r.
$$

Proof. Just recall that for all i and r, $Q_0^i f(r) = \mathbb{E}[f(r + \xi_1 + \cdots + \xi_i)]$, where $\xi_1, \ldots, \xi_i$ are i.i.d. geometric random variables with parameter $1/2$. The first claim of the lemma follows. Next write

$$
\begin{aligned}
|Q_0^i f(r) - f(r + 2i)| &\leq \mathbb{E}[|f(r + \xi_1 + \cdots + \xi_i) - f(r + 2i)|] \\
&\leq Lip(f)\mathbb{E}[|\xi_1 + \cdots + \xi_i - 2i|] \\
&\leq \sqrt{2}Lip(f)\sqrt{i},
\end{aligned}
$$

by using Cauchy-Schwarz inequality and the fact that $\mathbb{E}(\xi_i) = 2$ and $\mathbb{V}(\xi_i) = 2$, for all i. $\qquad\square$

Lemma 2.8. *There exists a constant $C > 0$ such that for all $|\varepsilon|_\infty \leq 1/2$, all $i \geq 0$ and all f,*

$$
|(Q_\varepsilon^i - Q_0^i)f|_\infty \leq Ci Lip(f) \times |\varepsilon|_\infty.
$$

Proof. First write

$$
Q_\varepsilon^i - Q_0^i = \sum_{j=1}^{i} Q_\varepsilon^{i-j} R_\varepsilon Q_0^{j-1}.
$$

Then by using that $|Q_\varepsilon^j f|_\infty \leq |f|_\infty$, for all j, we get

$$
\begin{aligned}
|(Q_\varepsilon^i - Q_0^i)f|_\infty &\leq \sum_{j=0}^{i-1} |R_\varepsilon Q_0^j f|_\infty \\
&\leq \sum_{j=0}^{i-1} |\widetilde{R}_\varepsilon Q_0^j f|_\infty + \sum_{j=0}^{i-1} |(R_\varepsilon - \widetilde{R}_\varepsilon)Q_0^j f|_\infty \\
&\leq C \sum_{j=0}^{i-1} Lip(Q_0^j f)|\varepsilon|_\infty \quad \text{(by using Lemma 2.5 and 2.6).}
\end{aligned}
$$

We conclude the proof of the lemma by using that $Lip(Q_0^j f) \leq Lip(f)$ for all j. $\qquad\square$

Recall now that u is defined by $u(r) = r$ for all $r \in \mathbb{N}$.

Lemma 2.9. *We have $Lip(\widetilde{R}_\varepsilon u) \leq Lip(\varepsilon)$.*

Proof. By using Lemma 2.4 and the fact that $\widetilde{R}_\varepsilon 1 = 0$, we get for all r,

$$
\widetilde{R}_\varepsilon u(r) = r\widetilde{R}_\varepsilon 1(r) + \widetilde{R}_{\varepsilon(r)} u(0) = \widetilde{R}_{\varepsilon(r)} u(0),
$$

where $\varepsilon(r)$ is defined by $(\varepsilon(r))_i = \varepsilon_{r+i}$ for all i. The result of the lemma follows immediately by using for instance Lemma 2.4. $\qquad\square$

Our last result in this subsection is the following

Lemma 2.10. *There exists a constant $C > 0$ such that for all $m \geq 0$ and all $|\varepsilon|_\infty \leq 1/2$,*

$$|Q_\varepsilon^m u - Q_0^m u - \sum_{i=1}^m Q_0^{m-i} \widetilde{R}_\varepsilon u|_\infty \leq C(m|\varepsilon|_\infty^2 + m^2 |\varepsilon|_\infty Lip(\varepsilon)).$$

Proof. First observe that for all $j \geq 0$, $Q_0^j u = u + 2j$. Since $\widetilde{R}_\varepsilon$ is linear and $\widetilde{R}_\varepsilon 1 = 0$, we get $\widetilde{R}_\varepsilon Q_0^{i-1} u = \widetilde{R}_\varepsilon u$ for all $i \geq 1$. Thus

$$\sum_{i=1}^m Q_0^{m-i} \widetilde{R}_\varepsilon u = \sum_{i=1}^m Q_0^{m-i} \widetilde{R}_\varepsilon Q_0^{i-1} u.$$

Next we have

$$Q_\varepsilon^m u - Q_0^m u - \sum_{i=1}^m Q_0^{m-i} \widetilde{R}_\varepsilon Q_0^{i-1} u \quad = \quad \sum_{i=1}^m Q_0^{m-i}(R_\varepsilon - \widetilde{R}_\varepsilon) Q_0^{i-1} u$$

$$+ \sum_{i=1}^m (Q_\varepsilon^{m-i} - Q_0^{m-i}) \widetilde{R}_\varepsilon Q_0^{i-1} u.$$

By using Lemma 2.6 and the fact that $\mathrm{Lip}(Q_0^i u) \leq \mathrm{Lip}(u) = 1$ for all i, we get

$$\sum_{i=1}^m |Q_0^{m-i}(R_\varepsilon - \widetilde{R}_\varepsilon) Q_0^{i-1} u|_\infty \quad \leq \quad C \sum_{i=1}^m \mathrm{Lip}(Q_0^{i-1} u)|\varepsilon|_\infty^2$$

$$\leq \quad Cm|\varepsilon|_\infty^2.$$

Then by using Lemma 2.8 we obtain

$$\sum_{i=1}^m |(Q_\varepsilon^{m-i} - Q_0^{m-i}) \widetilde{R}_\varepsilon Q_0^{i-1} u|_\infty \leq C|\varepsilon|_\infty \sum_{i=1}^m (m-i) \mathrm{Lip}(\widetilde{R}_\varepsilon u).$$

We conclude the proof of the lemma by using Lemma 2.9. $\square$

2.4. **Proof of Proposition 2.1.** Recall that $\varepsilon_n = (\varepsilon_i(n), i \geq 1)$, with $\varepsilon_i(n) = \varphi(i/2n)/(2n)$. Since φ is bounded, we can always assume by taking large enough n if necessary, that $|\varepsilon_n|_\infty \leq 1/2$. Note also that $\mathrm{Lip}(\varepsilon_n) = \mathcal{O}(1/n^2)$. Assume now that $m = \mathcal{O}(n)$. Then Lemma 2.10 shows that

$$Q_{\varepsilon_n}^m u - Q_0^m u = \sum_{i=0}^{m-1} Q_0^i \widetilde{R}_{\varepsilon_n} u + \mathcal{O}\left(\frac{1}{n}\right).$$

Next write

$$\sum_{i=0}^{m-1} Q_0^i \widetilde{R}_{\varepsilon_n} u(0) = \sum_{i=0}^{m-1} \widetilde{R}_{\varepsilon_n} u(2i) + \sum_{i=0}^{m-1} (Q_0^i \widetilde{R}_{\varepsilon_n} u(0) - \widetilde{R}_{\varepsilon_n} u(2i)).$$

By using Lemma 2.7 we get

$$\sum_{i=0}^{m-1} |Q_0^i \widetilde{R}_{\varepsilon_n} u(0) - \widetilde{R}_{\varepsilon_n} u(2i)| \quad \leq \quad \sqrt{2} \sum_{i=0}^{m-1} \mathrm{Lip}(\widetilde{R}_{\varepsilon_n} u)\sqrt{i}$$

$$\leq \quad Cm^{3/2} \mathrm{Lip}(\varepsilon_n)$$

$$\leq \quad \frac{C}{\sqrt{n}}.$$

On the other hand, set

(13) $$a_\ell := -\ell 2^{-\ell} + \sum_{j=\ell+1}^\infty j2^{-j} = 2^{-\ell+1}.$$

Then by using Lemma 2.4 we get

$$\begin{aligned}
\sum_{i=0}^{m-1} \widetilde{R}_{\varepsilon_n} u(2i) &= \sum_{i=0}^{m-1} \sum_{\ell=1}^\infty a_\ell(\varepsilon_n)_{2i+\ell} \\
&= \sum_{\ell=1}^\infty 2^{-\ell+1} \times \frac{1}{2n} \sum_{i=0}^{m-1} \varphi\left(\frac{2i+\ell}{2n}\right).
\end{aligned}$$

But $\sum_{\ell=1}^\infty 2^{-\ell+1} = 2$, and since φ is Lipschitz and bounded

$$\frac{1}{n} \sum_{i=0}^{m-1} \varphi\left(\frac{2i+\ell}{2n}\right) = \int_0^{m/n} \varphi(s)\,ds + \mathcal{O}\left(\frac{\ell}{n}\right).$$

Thus putting the pieces together we get

$$Q_{\varepsilon_n}^m u(0) - Q_0^m u(0) = h\left(\frac{m}{n}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Finally (9) gives for $a \le x \le 0 \wedge \tau_n^R$,

$$\begin{aligned}
B_{a,v}^{(n)}(x) &= \frac{1}{n} \sum_{k=1}^{[2nx]-[2na]} \left(\mathbb{E}[V_{\varepsilon_n}(k) \mid V_{\varepsilon_n}(k-1)] - V_{\varepsilon_n}(k-1)\right) \\
&= \frac{1}{n} \sum_{k=1}^{[2nx]-[2na]} \left(1 + Q_{\varepsilon_n}^{V_{\varepsilon_n}(k-1)+1} u(0) - Q_0^{V_{\varepsilon_n}(k-1)+1} u(0)\right) \\
&= \frac{1}{n} \sum_{k=1}^{[2nx]-[2na]} \left\{1 + h\left(\Lambda_{a,v}^{(n)}\left(a + \frac{k-1}{2n}\right)\right)\right\} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \\
&= 2\int_a^x \left\{1 + h(\Lambda_{a,v}^{(n)}(y))\right\} dy + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

which proves Proposition 2.1.   □

2.5. **Proof of Proposition 2.2.** We assume throughout this subsection that $m = \mathcal{O}(n)$. Then on the one hand by using Lemma 2.10, we get

$$Q_{\varepsilon_n}^m u(0) = 2m + \sum_{i=1}^m Q_0^{m-i}\widetilde{R}_{\varepsilon_n} u(0) + \mathcal{O}\left(\frac{1}{n}\right).$$

Moreover Lemma 2.5 shows that $|Q_0^{m-i}\widetilde{R}_{\varepsilon_n} u(0)| = \mathcal{O}(1/n)$ uniformly in i. Thus

$$(Q_{\varepsilon_n}^m u(0))^2 = 4m^2 + 4m \sum_{i=1}^m Q_0^{m-i}\widetilde{R}_{\varepsilon_n} u(0) + \mathcal{O}(1).$$

On the other hand we have for all ε,

$$Q_\varepsilon^m u^2 = Q_0^m u^2 + \sum_{i=1}^m Q_\varepsilon^{m-i} R_\varepsilon Q_0^{i-1} u^2.$$

A variance calculus shows that

$$Q_0^{i-1}u^2 = u^2 + 4(i-1)u + 4(i-1)^2 + 2(i-1),$$

which implies that for all ε,

$$R_\varepsilon Q_0^{i-1}u^2 = R_\varepsilon u^2 + 4(i-1)R_\varepsilon u,$$

since $R_\varepsilon 1 = 0$. Thus

$$Q_\varepsilon^m u^2(0) = 4m^2 + 2m + E_{\varepsilon,m} + F_{\varepsilon,m},$$

where

$$E_{\varepsilon,m} = \sum_{i=1}^{m} Q_\varepsilon^{m-i} R_\varepsilon u^2(0),$$

and

$$F_{\varepsilon,m} = 4\sum_{i=1}^{m} (i-1)Q_\varepsilon^{m-i} R_\varepsilon u(0).$$

We now prove the following

Lemma 2.11. *We have*

$$(14) \qquad E_{\varepsilon_n,m} = \sum_{i=1}^{m} Q_0^{m-i} \widetilde{R}_{\varepsilon_n} u^2(0) + \mathcal{O}(1),$$

and

$$(15) \qquad F_{\varepsilon_n,m} = 4\sum_{i=1}^{m} (i-1)Q_0^{m-i} \widetilde{R}_{\varepsilon_n} u(0) + \mathcal{O}(1).$$

Proof. We have

$$(16) \qquad (R_\varepsilon - \widetilde{R}_\varepsilon)u^2(r) = \sum_{\ell \geq 1} (2r\ell + \ell^2)2^{-\ell}(\varepsilon_{r,\ell} - \widetilde{\varepsilon}_{r,\ell}) \quad \text{for all } r.$$

Thus, by using (12), we see that there exists a constant $C > 0$ such that

$$(17) \qquad |(R_{\varepsilon_n} - \widetilde{R}_{\varepsilon_n})u^2(r)| \leq C|\varepsilon_n|_\infty^2(x+1) \leq C\frac{r+1}{n^2} \quad \text{for all } r.$$

By using Lemma 2.3 (applied to $f(x) = x+1$), we see that there exists $C > 0$ such that

$$\begin{aligned}
|E_{\varepsilon_n,m} - \sum_{i=1}^{m} Q_{\varepsilon_n}^{m-i} \widetilde{R}_{\varepsilon_n} u^2(0)| &\leq \frac{C}{n^2}\sum_{i=1}^{m}(1+m-i) \\
&\leq C\frac{m^2}{n^2} = \mathcal{O}(1).
\end{aligned}$$

Next by using Lemma 2.8, we get for all ε,

$$(18) \qquad |(Q_\varepsilon^{m-i} - Q_0^{m-i})\widetilde{R}_\varepsilon u^2|_\infty \leq C(m-i)|\varepsilon|_\infty \text{Lip}(\widetilde{R}_\varepsilon u^2).$$

Recall the formula for a_ℓ given in (13) and let

$$b_\ell := -\ell^2 2^{-\ell} + \sum_{i=\ell+1}^{\infty} i^2 2^{-i}.$$

Then Lemma 2.4 shows that

$$
\begin{aligned}
\widetilde{R}_{\varepsilon_n} u^2(r) &= 2r \sum_{\ell \geq 1} a_\ell(\varepsilon_n)_{r+\ell} + \sum_{\ell \geq 1} b_\ell(\varepsilon_n)_{r+\ell} \\
&= 2 \sum_{\ell \geq 1} a_\ell \frac{r+\ell}{n} \varphi\left(\frac{r+\ell}{n}\right) + \sum_{\ell \geq 1} (b_\ell - 2\ell a_\ell)(\varepsilon_n)_{r+\ell}.
\end{aligned}
$$

We deduce that if $(s \mapsto s\varphi(s))$ is Lipschitz, then

$$
\text{(19)} \qquad\qquad \text{Lip}(\widetilde{R}_{\varepsilon_n} u^2) = \mathcal{O}\left(\frac{1}{n}\right).
$$

Thus (18) implies that

$$
|(Q_{\varepsilon_n}^{m-i} - Q_0^{m-i})\widetilde{R}_{\varepsilon_n} u^2|_\infty = \mathcal{O}\left(\frac{1}{n}\right).
$$

This proves (14). Now Lemma 2.6, 2.8 and 2.9 show that

$$
\begin{aligned}
|Q_{\varepsilon_n}^{m-i} R_{\varepsilon_n} u - Q_0^{m-i}\widetilde{R}_{\varepsilon_n} u|_\infty &\leq |Q_{\varepsilon_n}^{m-i}(R_{\varepsilon_n} u - \widetilde{R}_{\varepsilon_n})u|_\infty + |(Q_{\varepsilon_n}^{m-i} - Q_0^{m-i})\widetilde{R}_{\varepsilon_n} u|_\infty \\
&= \mathcal{O}\left(|\varepsilon_n|_\infty^2 + (m-i)|\varepsilon_n|_\infty \text{Lip}(\varepsilon_n)\right) = \mathcal{O}\left(\frac{1}{n^2}\right).
\end{aligned}
$$

This proves (15) and finishes the proof of the lemma. $\qquad\square$

We can now write

$$
\begin{aligned}
Q_{\varepsilon_n}^m u^2(0) - (Q_{\varepsilon_n}^m u(0))^2 &= 2m + \sum_{j=0}^{m-1} Q_0^j \widetilde{R}_{\varepsilon_n} u^2(0) \\
&\quad + 4 \sum_{j=0}^{m-1} (m-j-1)Q_0^j \widetilde{R}_{\varepsilon_n} u(0) \\
&\quad - 4m \sum_{j=0}^{m-1} Q_0^j \widetilde{R}_{\varepsilon_n} u(0) + \mathcal{O}(1).
\end{aligned}
$$

By using Lemma 2.7 and 2.9 and (19), we get for $j \leq m-1$,

$$
Q_0^j \widetilde{R}_{\varepsilon_n} u^2(0) = \widetilde{R}_{\varepsilon_n} u^2(2j) + \mathcal{O}(n^{-1/2}),
$$

and

$$
Q_0^j \widetilde{R}_{\varepsilon_n} u(0) = \widetilde{R}_{\varepsilon_n} u(2j) + \mathcal{O}(n^{-3/2}).
$$

Therefore

$$
\begin{aligned}
Q_{\varepsilon_n}^m u^2(0) - (Q_{\varepsilon_n}^m u(0))^2 &= 2m + \sum_{j=0}^{m-1} \widetilde{R}_{\varepsilon_n} u^2(2j) \\
&\quad - 4 \sum_{j=0}^{m-1} (j+1)\widetilde{R}_{\varepsilon_n} u(2j) + \mathcal{O}(n^{1/2}).
\end{aligned}
$$

Lemma 2.4 shows that

$$
\widetilde{R}_{\varepsilon_n} u^2(2j) = 4j \sum_{\ell \geq 1} a_\ell(\varepsilon_n)_{2j+\ell} + \mathcal{O}\left(\frac{1}{n}\right),
$$

and

$$\widetilde{R}_{\varepsilon_n} u(2j) = \sum_{\ell \geq 1} a_\ell(\varepsilon_n)_{2j+\ell}.$$

Thus

$$Q_{\varepsilon_n}^m u^2(0) - (Q_{\varepsilon_n}^m u(0))^2 = 2m + \mathcal{O}(n^{1/2}).$$

Then (10) shows that for $a \leq x \leq 0 \wedge \tau_n^R$,

$$
\begin{aligned}
A_{a,v}^{(n)}(x) &= \frac{1}{n^2} \sum_{k=1}^{[2nx]-[2na]} \left(Q_{\varepsilon_n}^{V_{\varepsilon_n}(k-1)+1} u^2(0) - (Q_{\varepsilon_n}^{V_{\varepsilon_n}(k-1)+1} u(0))^2 \right) \\
&= \frac{2}{n} \sum_{k=1}^{[2nx]-[2na]} \Lambda_{a,v}^{(n)} \left(a + \frac{k-1}{2n} \right) + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right) \\
&= 4 \int_a^x \Lambda_{a,v}^{(n)}(y) \, dy + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right).
\end{aligned}
$$

This finishes the proof of Proposition 2.2. $\qquad\square$

2.6. **Proof of the convergence on $[0, +\infty)$.** The proof of the convergence of $\Lambda_{a,v}^{(n)}$ on $[0, +\infty)$ is essentially the same as the proof on $[a, 0]$. Namely we can define $\widetilde{M}_{a,v}^{(n)}$, $\widetilde{B}_{a,v}^{(n)}$ and $\widetilde{A}_{a,v}^{(n)}$, respectively as in (4), (5) and (6) with $\widetilde{V}$ everywhere instead of V. Let also

(20)
$$
\begin{cases}
w_{a,v}^{(n,-)} := \frac{1}{2n} \sup\{k \leq 0 \ : \ S_{\varepsilon_n,[2na],[nv]}(k) = 0\} \\
w_{a,v}^{(n,+)} := \frac{1}{2n} \inf\{k \geq a \ : \ S_{\varepsilon_n,[2na],[nv]}(k) = 0\}.
\end{cases}
$$

Then

$$\Lambda_{a,v}^{(n)}(x) = \Lambda_{a,v}^{(n)}(0) + \widetilde{M}_{a,v}^{(n)}(x) + \widetilde{B}_{a,v}^{(n)}(x) \quad \text{for all } x \in [0, w_{a,v}^{(n,+)}).$$

Moreover (8) shows that

$$\mathbb{E}[\widetilde{V}_\varepsilon(k) \mid \widetilde{V}_\varepsilon(k-1)] - \widetilde{V}_\varepsilon(k-1) = Q_\epsilon^{\widetilde{V}_\varepsilon(k-1)} u(0) - Q_0^{\widetilde{V}_\varepsilon(k-1)} u(0),$$

and

$$\mathbb{E}[\widetilde{V}_\varepsilon(k)^2 \mid \widetilde{V}_\varepsilon(k-1)] - \mathbb{E}[\widetilde{V}_\varepsilon(k) \mid \widetilde{V}_\varepsilon(k-1)]^2 = Q_\varepsilon^{\widetilde{V}_\varepsilon(k-1)} u^2(0) - (Q_\varepsilon^{\widetilde{V}_\varepsilon(k-1)} u(0))^2,$$

for all $k \geq 1$. Then by following the proofs given in the previous subsections we get the analogues of Proposition 2.1 and 2.2:

Proposition 2.12. *Let $R > 0$ and $T > 0$ be given. Then for $0 \leq x \leq T \wedge \tau_n^R \wedge w_{a,v}^{(n,+)}$,*

$$\widetilde{B}_{a,v}^{(n)}(x) = 2 \int_0^x h(\Lambda_{a,v}^{(n)}(y)) \, dy + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right),$$

where the $\mathcal{O}(n^{-1/2})$ is deterministic and only depends on a, T and R.

Proposition 2.13. *Let $R > 0$ and $T > 0$ be given. Then for $0 \leq x \leq T \wedge \tau_n^R \wedge w_{a,v}^{(n,+)}$,*

$$\widetilde{A}_{a,v}^{(n)}(x) = 4 \int_0^x \Lambda_{a,v}^{(n)}(y) \, dy + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right),$$

where the $\mathcal{O}(n^{-1/2})$ is deterministic and only depends on a, T and R.

So according again to the criterion of Ethier and Kurtz (Theorem 4.1 p.354 in [EK]), we deduce the convergence in law of $\Lambda_{a,v}^{(n)}$ on $[0, +\infty)$.

Actually one can deduce the convergence on $[a, +\infty)$ as well. For this we just need to observe that the criterion of Ethier and Kurtz applies in the same way for non homogeneous operators. For reader's convenience let us recall the main steps of its proof. First Propositions 2.1, 2.2, 2.12 and 2.13 imply the tightness of the sequence $(\Lambda_{a,v}^{(n)}, n \geq 1)$ on $[a, +\infty)$. Next Itô Formula shows that any limit of a subsequence is a solution of the non-homogeneous martingale problem (see the definition in [EK] p.221) associated to the operator

$$G_x f(\lambda) := \begin{cases} 2\lambda f''(\lambda) + 2(1 + h(\lambda))f'(\lambda) & \text{if } x \in [a, 0] \\ 2\lambda f''(\lambda) + 2h(\lambda)f'(\lambda) & \text{if } x \in [0, +\infty). \end{cases}$$

Then Theorem 2.3 p.372 in [EK] (with their notation replace t by x, x by λ and take $r_0 = 0$ and $r_1 = +\infty$) shows that this martingale problem is well posed (in particular it has a unique solution). This proves the desired convergence on $[a, +\infty)$. Since the proof of the convergence on $(-\infty, a]$ is the same as on $[0, +\infty)$, this concludes the proof of Theorem 1.1. $\qquad\square$

2.7. **Extension to the non homogeneous setting.** We give here an extension of Theorem 1.1 when φ is allowed to be space dependent. Apart from its own interest, we will use this extension in the proof of Theorem 1.4.

We now define non homogeneous cookies random walks. If

$$\varepsilon = (\varepsilon_{i,x}, i \geq 1, x \in \mathbb{Z}),$$

is given, we set

$$p_{\varepsilon,i,x} := \frac{1}{2}(1 + \varepsilon_{i,x}),$$

for all i and x. Then X_ε is defined by

$$\mathbb{P}[X_\varepsilon(n+1) - X_\varepsilon(n) = 1 \mid \mathcal{F}_{\varepsilon,n}] = 1 - \mathbb{P}[X_\varepsilon(n+1) - X_\varepsilon(n) = -1 \mid \mathcal{F}_{\varepsilon,n}] = p_{\varepsilon,i,x},$$

if $\#\{j \leq n : X_\varepsilon(j) = X_\varepsilon(n)\} = i$ and $X_\varepsilon(n) = x$. Similarly non homogeneous excited Brownian motions are defined by

$$dY_t = dB_t + \varphi(Y_t, L_t^{Y_t}) \, dt,$$

for some bounded and measurable φ. Such generalized version of excited BM was already studied in [NRW2] and [RS]. In particular Ray-Knight results were obtained in this context and a sufficient condition for recurrence is given in [RS] (see below). Now let φ be fixed. Assume that for each $n \geq 1$, a function $\varphi_n : \mathbb{Z} \times [0, \infty) \to \mathbb{R}$ is given, and assume that

(21) $(\varphi_n([2nx], \ell), x \in \mathbb{R}, \ell \geq 0) \to (\varphi(x, \ell), x \in \mathbb{R}, \ell \geq 0)$ when $n \to \infty$,

uniformly in ℓ and in the Skorokhod space $\mathbb{D}(\mathbb{R})$ in the x variable. Assume moreover that $\sup_{k,\ell} |\varphi_n(k, \ell)| \leq 2n$ for n large enough and define $\varepsilon_n = (\varepsilon_{i,x}(n), i \geq 1, x \in \mathbb{Z})$ by

$$\varepsilon_{i,x}(n) = \frac{1}{2n}\varphi_n\left(x, \frac{i}{2n}\right),$$

for all $i \geq 1$ and $x \in \mathbb{Z}$. Finally define $\Lambda_{a,v}^{(n)}$ and $\Lambda_{a,v}$ as in the homogeneous setting (see the introduction). We can state now the following natural extension of Theorem 1.1:

Theorem 2.14. *Assume that φ is bounded, Lipschitz in both variables, and that for each x, $(\ell \mapsto \ell\varphi(x,\ell))$ is also Lipschitz. Assume further that for n large enough, X_{ε_n} is recurrent and that Y is recurrent. Assume eventually that (21) holds. Then for any $a \in \mathbb{R}$ and $v \geq 0$,*

$$(\Lambda_{a,v}^{(n)}(x), x \in \mathbb{R}) \underset{n \to \infty}{\overset{\mathcal{L}}{\Longrightarrow}} (\Lambda_{a,v}(x), x \in \mathbb{R}).$$

The proof of this result is exactly the same as the proof of Theorem 1.1. Note in particular that if φ is Lipschitz in both variables, then the convergence in (21) is actually uniform in x on any compact. Note also that as at the end of the previous subsection, we need to use here a non homogeneous version of Ethier–Kurtz's result (Theorem 4.1 p.354 in [EK]). This time we just have to verify that the martingale problem associated to the operator

$$G_x f(\lambda) := \begin{cases} 2\lambda f''(\lambda) + 2(1 + h(x,\lambda))f'(\lambda) & \text{if } x \in [a, 0] \\ 2\lambda f''(\lambda) + 2h(x,\lambda)f'(\lambda) & \text{if } x \in [0, +\infty), \end{cases}$$

is well posed, where $h(x,\lambda) = \int_0^\lambda \varphi(x,\mu)\,d\mu$, for any x and λ. But again this follows from Theorem 2.3 p.372 in [EK].

In particular the above theorem applies to the following situation, which we will use in the proof of Theorem 5.1. Assume that $\varphi : [0,\infty) \to \mathbb{R}$ satisfies the hypotheses of Theorem 1.1. Let us give for each $n \geq 1$, a function $(\lambda(n,x), x \in \mathbb{Z})$. Set $\lambda_n := \lambda(n, [2n\cdot])$. Assume that λ_n converges to some other function λ in $\mathbb{D}(\mathbb{R})$, when $n \to \infty$. For any λ, set

$$\varphi_\lambda(x,\ell) := \varphi(\lambda(x) + \ell) \quad \text{for all } x \in \mathbb{R} \text{ and } \ell \geq 0.$$

Set also

$$\varphi_n(x,\ell) := \varphi(\lambda(n,x) + \ell) \quad \text{for all } x \in \mathbb{Z} \text{ and } \ell \geq 0.$$

Then $\lambda \mapsto \varphi_\lambda$ is continuous on $\mathbb{D}(\mathbb{R})$ and thus $\varphi_{\lambda_n} = \varphi_n([2n\cdot], \cdot)$ converges to φ_λ as in (21). Let now $\varepsilon_{n,\lambda_n} = (\varepsilon_{i,x}(n,\lambda_n), i \geq 1, x \in \mathbb{Z})$ be defined by

$$\varepsilon_{i,x}(n,\lambda_n) := \frac{1}{2n}\varphi_n\left(x, \frac{i}{2n}\right).$$

Let $\Lambda^{(n,\lambda_n)}$ and $\Lambda^{(\lambda)}$ be the processes associated to $\varepsilon_{n,\lambda_n}$ and φ_λ as in the introduction. The following is an immediate application of Theorem 2.14:

Corollary 2.15. *Assume that the hypotheses of Theorem 2.14 are satisfied. Then with the above notation, for any a and v,*

$$\left(\Lambda_{a,v}^{(n,\lambda_n)}(x), x \in \mathbb{R}\right) \underset{n \to \infty}{\overset{\mathcal{L}}{\Longrightarrow}} \left(\Lambda_{a,v}^{(\lambda)}(x), x \in \mathbb{R}\right).$$

To finish this subsection, we recall some sufficient condition for recurrence of X_ε and Y proved respectively in [Z, Corollary 7] and [RS, Corollary 5.6] in the non homogeneous case. We notice that it applies only when for all i and x, $\varepsilon_{i,x}$, respectively φ, is nonnegative. We only state the result in the continuous setting, the result for X_ε being analogous. So if for $x \in \mathbb{R}$,

$$\delta^x(\varphi) := \int_0^\infty \varphi(x,\ell)\,d\ell,$$

then Y is recurrent as soon as

$$\liminf_{z \to +\infty} \frac{1}{z}\int_0^z \delta^x(\varphi)\,dx < 1.$$

3. Proof of Corollary 1.2

Note that

$$\tau_{\varepsilon_n,[2na]}([nv]) = [2na] + 2\sum_{k\in\mathbb{Z}} S_{\varepsilon_n,[2na],[nv]}(k).$$

Thus

$$\frac{\tau_{\varepsilon_n,[2na]}([nv])}{4n^2} = \int_{\mathbb{R}} \Lambda_{a,v}^{(n)}(y)\, dy + o(1).$$

On the other hand, the occupation times formula ([RY] p.224) gives

$$\tau_a(v) = \int_{\mathbb{R}} \Lambda_{a,v}(y)\, dy.$$

Now Theorem 1.1 shows that for any fixed $A > 0$, the following convergence in law holds:

$$\int_{-A}^{A} \Lambda_{a,v}^{(n)}(y)\, dy \quad \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} \quad \int_{-A}^{A} \Lambda_{a,v}(y)\, dy.$$

So Corollary 1.2 follows from the following lemma (recall that $w_{a,v}^{(n,\pm)}$ is defined in (20)):

Lemma 3.1. *Let $\epsilon > 0$ be given. Then there exists $A > 0$, such that*

$$\mathbb{P}\left[|w_{a,v}^{(n,\pm)}| \geq A\right] \leq \epsilon,$$

for all n large enough.

Proof. We prove the result for $w_{a,v}^{(n,+)}$. The proof for $w_{a,v}^{(n,-)}$ is the same. First observe that $w_{a,v}^+$ is nonnegative and a.s. finite: it is equal to $\sup\{Y_t \ : \ t \leq \tau_a(v)\}$ and $\tau_a(v)$ is a.s. finite since Y is recurrent. So for any $\epsilon > 0$, there exists $A > a$ such that

$$\mathbb{P}[w_{a,v}^+ \geq A] \leq \epsilon.$$

Moreover by using Theorem 1.1 and Skorokhod's representation Theorem, it is possible to define $\Lambda_{a,v}^{(n)}$ and $\Lambda_{a,v}$ on the same probability space, such that for any $\eta > 0$,

$$\mathbb{P}\left[\sup_{0\leq x\leq A} |\Lambda_{a,v}^{(n)}(x) - \Lambda_{a,v}(x)| \geq \eta\right] \leq \epsilon,$$

for n large enough. Thus

(22) $$\mathbb{P}[T_n(\eta) \geq A] \leq 2\epsilon,$$

where

$$T_n(\eta) = \inf\{x > 0 \ : \ \Lambda_{a,v}^{(n)}(x) \leq \eta\}.$$

Recall now that on $[0, +\infty)$, $\Lambda_{a,v}^{(n)}(\cdot)$ is equal in law to $\widetilde{V}_{\varepsilon_n}([2n\cdot])/n$ (see the beginning of Section 2.1). But since $|\varepsilon_n|_\infty = \mathcal{O}(1/n)$, $(\widetilde{V}_{\varepsilon_n}(k), k \geq 0)$ is stochastically dominated by a Galton-Watson process $(W_n(k), k \geq 0)$ with offspring distribution a geometrical law with parameter $1 - p_n = 1/2 - c/n$, for some constant $c > 0$. Moreover, when $W_n(0) = 1$, the probability for W_n to extinct before time $[nA]$ can be computed explicitly. If $f_k^{(n)}(\cdot)$ is the generating function of $W_n(k)$, then this

probability is equal to $f_{[nA]}^{(n)}(0)$. An expression for $f_k^{(n)}(0)$ is given for instance in [AN] p.6-7:

$$f_k^{(n)}(0) = 1 - m_n^k \frac{1 - s_n}{m_n^k - s_n} \quad \text{for all } k \geq 1,$$

where

$$m_n = \frac{p_n}{1 - p_n} = 1 + \frac{4c}{n} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

and

$$s_n = \{1 - m_n(1 - p_n)\}/p_n = 1 - \frac{4c}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

It follows that $f_{[nA]}^{(n)}(0) = 1 - c'/n + \mathcal{O}(1/n^2)$, with $c' = 4c/(1 - e^{-4cA}) > 0$. Now the law of W_n starting from $[\eta n]$ is equal to the law of the sum of $[\eta n]$ independent copies of W_n starting from 1. Thus if $W_n(0) = [\eta n]$, the probability for W_n to extinct before time $[nA]$ is $f_{[nA]}^{(n)}(0)^{[\eta n]}$. If η is small enough and n large enough, this probability is larger than $(1 - \epsilon)$. By using now that $\widetilde{V}_{\varepsilon_n}$ is stochastically dominated by W_n, (22) and the strong Markov property, we get

$$\mathbb{P}\left[w_{a,v}^{(n,+)} \geq 2A\right] \quad \leq \quad \mathbb{P}\left[w_{a,v}^{(n,+)} \geq 2A \text{ and } T_n(\eta) \leq A\right] + \mathbb{P}\left[T_n(\eta) \geq A\right]$$
$$\leq \quad \mathbb{P}\left[W_n([nA]) > 0 \mid W_n(0) = [\eta n]\right] + 2\epsilon \leq 3\epsilon.$$

This concludes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

4. Proof of Corollary 1.3

First note that the law of Y_{γ_λ} has for density the function $a \mapsto \lambda \mathbb{E}[L_{\gamma_\lambda}^a]$. Indeed, for any bounded and measurable function ϕ,

$$\mathbb{E}[\phi(Y_{\gamma_\lambda})] = \mathbb{E}\left[\int_0^\infty \phi(Y_s)\lambda e^{-\lambda s}\, ds\right]$$
$$= \mathbb{E}\left[\int_{0 < s < t} \lambda^2 \phi(Y_s) e^{-\lambda t}\, ds\, dt\right]$$
$$= \mathbb{E}\left[\int_{\mathbb{R}} \int_0^\infty \lambda^2 \phi(a) L_t^a e^{-\lambda t}\, dt\, da\right]$$
$$= \int_{\mathbb{R}} \mathbb{E}[\phi(a)\lambda L_{\gamma_\lambda}^a]\, da,$$

where in the third equality we have used the occupation times formula (see Corollary (1.6) p.224 in [RY]).

We now follow the argument given by Tóth in [T2]. First observe that if

$$\widetilde{\tau}_{\varepsilon,a}(v) := \inf\left\{j \; : \; \#\{i \leq j \; : \; X_\varepsilon(i) = a \text{ and } X_\varepsilon(i+1) = a+1\} = v+1\right\},$$

then exactly as we proved Corollary 1.2, we can show that $\widetilde{\tau}_{\varepsilon_n,[2na]}([nv])/(4n^2)$ converges in law toward $\tau_a(v)$ for any $a \in \mathbb{R}$ and $v \geq 0$. Next observe that for any $a \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\mathbb{P}[X_{\varepsilon_n}(k) = a] = \sum_{v \in \mathbb{N}} \left\{\mathbb{P}[\tau_{\varepsilon_n,a}(v) = k] + \mathbb{P}[\widetilde{\tau}_{\varepsilon_n,a}(v) = k]\right\}.$$

Thus for any $a \in \mathbb{R}$,

$$
\begin{aligned}
2n\mathbb{P}\left(X_{\varepsilon_n}(\theta_{\lambda/(4n^2)}) = [2na]\right) &= 2n(1 - e^{-\lambda/(4n^2)})\sum_{k \geq 0} e^{-k\lambda/(4n^2)} \\
&\quad \times \sum_{v \in \mathbb{N}}\left\{\mathbb{P}[\tau_{\varepsilon_n,[2na]}(v) = k] + \mathbb{P}[\widetilde{\tau}_{\varepsilon_n,[2na]}(v) = k]\right\} \\
&\sim \frac{\lambda}{2n}\sum_{v \in \mathbb{N}}\left\{\mathbb{E}\left[e^{-\lambda\frac{\tau_{\varepsilon_n,[2na]}(v)}{4n^2}}\right] + \mathbb{E}\left[e^{-\lambda\frac{\widetilde{\tau}_{\varepsilon_n,[2na]}(v)}{4n^2}}\right]\right\},
\end{aligned}
$$

since $2n(1 - e^{-\lambda/(4n^2)}) \sim \lambda/(2n)$. Note now that

$$
\frac{1}{n}\sum_{v \in \mathbb{N}}\mathbb{E}\left[e^{-\lambda\frac{\tau_{\varepsilon_n,[2na]}(v)}{4n^2}}\right] = \int_0^\infty \mathbb{E}\left[e^{-\lambda\frac{\tau_{\varepsilon_n,[2na]}([nv])}{4n^2}}\right]dv,
$$

and that for any $v \in \mathbb{R}^+$, Corollary 1.2 implies

$$
\mathbb{E}\left[e^{-\lambda\frac{\tau_{\varepsilon_n,[2na]}([nv])}{4n^2}}\right] \to \mathbb{E}\left[e^{-\lambda\tau_a(v)}\right],
$$

when $n \to \infty$. The same remark applies with $\widetilde{\tau}$ instead of τ. Thus by application of Fatou's lemma, for every $a \in \mathbb{R}$,

$$
(23) \qquad \liminf_{n \to \infty}(2n)\mathbb{P}\left(X_{\varepsilon_n}(\theta_{\lambda/(4n^2)}) = [2na]\right) \geq \lambda\int_0^\infty \mathbb{E}\left[e^{-\lambda\tau_a(v)}\right]dv.
$$

But notice that for every $a \in \mathbb{R}$ and $v \geq 0$,

$$
\begin{aligned}
\mathbb{E}\left[e^{-\lambda\tau_a(v)}\right] &= \lambda\int_0^\infty e^{-\lambda s}\mathbb{P}[\tau_a(v) \leq s]\,ds \\
&= \lambda\int_0^\infty e^{-\lambda s}\mathbb{P}[L_s^a \geq v]\,ds \\
&= \mathbb{P}[L_{\gamma_\lambda}^a \geq v].
\end{aligned}
$$

Therefore

$$
(24) \qquad \lambda\int_{\mathbb{R}}\int_0^\infty \mathbb{E}\left[e^{-\lambda\tau_a(v)}\right]dv\,da = \lambda\int_{\mathbb{R}}\mathbb{E}[L_{\gamma_\lambda}^a]\,da = \lambda\mathbb{E}[\gamma_\lambda] = 1.
$$

On the other hand for any n,

$$
(25) \qquad \int_{\mathbb{R}}(2n)\mathbb{P}\left(X_{\varepsilon_n}(\theta_{\lambda/(4n^2)}) = [2na]\right)da = 1.
$$

It follows now from (23) (24) and (25) that for almost every $a \in \mathbb{R}$,

$$
\lim_{n \to \infty}(2n)\mathbb{P}\left(X_{\varepsilon_n}(\theta_{\lambda/(4n^2)}) = [2na]\right) = \lambda\mathbb{E}[L_{\gamma_\lambda}^a].
$$

The corollary is then a consequence of Sheffé's lemma. $\qquad \square$

5. Proof of Theorem 1.4

Tightness: We first need to show that the sequence $(X^{(n)}(\cdot), n \geq 1)$ is tight. All we have to prove (see e.g. Lemma (1.7) p.516 in [RY]) is that for each $T > 0$, $\alpha > 0$ and $\eta > 0$, there are n_0 and $\kappa > 0$, such that for $n \geq n_0$,

$$
(26) \qquad \mathbb{P}\left[\sup_{t \leq s \leq t+\kappa}|X^{(n)}(s) - X^{(n)}(t)| \geq \eta\right] \leq \alpha\kappa \quad \text{for all } t \leq T.
$$

We first prove the above inequality for $t = 0$. For this it suffices to find $\kappa > 0$ such that

$$(27) \qquad \mathbb{P}\left[\tau_{\varepsilon_n,[2n\eta]}(0) \leq 4n^2\kappa\right] \leq \alpha\kappa,$$

for n large enough ($\eta > 0$ and α being arbitrary and fixed), since the analogous result for $\eta < 0$ is similar (use the same proof with the process $-X^{(n)}$ instead of $X^{(n)}$). But Corollary 1.2 shows that as n tends to ∞, the left hand side in (27) converges toward $\mathbb{P}[\tau_\eta(0) \leq \kappa]$. Since φ is bounded, the excited Brownian motion Y is stochastically dominated by a Brownian motion with some constant drift. Thus this last term in turn is bounded by the analogous quantity for the Brownian motion with drift. But it is well known that for such process $\mathbb{P}[\tau_\eta(0) \leq \kappa]$ is a $o(\kappa)$ (see for instance Proposition (3.7) p.105 in [RY]). This proves (27). To obtain (26) it suffices to observe that after time t, $X^{(n)}$ is equal in law to a renormalized non-homogeneous cookie random walk starting from $X^{(n)}(t)$ and evolving in a shifted cookie environment (see also (30) below). So we can apply the same proof and we obtain the same result with the same constants everywhere. This finishes to prove the tightness of $(X^{(n)}(\cdot), n \geq 0)$. It remains to prove the convergence of the finite-dimensional distributions. For reader's convenience we first present a proof of the convergence of one-dimensional distributions.

Convergence of one-dimensional distributions: Let $(W_t, t \geq 0)$ be some limit in law of $(X^{(n_k)}(t), t \geq 0)$, for a subsequence $(n_k, k \geq 0)$. Then for any bounded and measurable function ϕ,

$$\mathbb{E}\left[\phi\left(\frac{X_{\varepsilon_{n_k}}(\theta_{\lambda/(4n_k^2)})}{2n_k}\right)\right] \quad \sim_{k\to\infty} \quad \lambda\int_0^\infty e^{-\lambda t}\mathbb{E}\left[\phi(X^{(n_k)}(t))\right]\,dt$$

$$\to_{k\to\infty} \quad \lambda\int_0^\infty e^{-\lambda t}\mathbb{E}[\phi(W_t)]\,dt$$

$$= \quad \mathbb{E}[\phi(W_{\gamma_\lambda})].$$

On the other hand Corollary 1.3 shows that the term on the left hand side converges toward $\mathbb{E}[\phi(Y_{\gamma_\lambda})]$. Since this holds for any λ and any ϕ, we deduce that W_t and Y_t have the same law for every $t \geq 0$ (see [F, Theorem 1a p.432]). This proves the convergence of one-dimensional distributions.

Convergence of finite-dimensional distributions: We first need to show that for any finite set I, a_i, $v_i \geq 0$, $i \in I$,

$$(28) \qquad \left(\frac{\tau_{\varepsilon_n,[2na_i]}([nv_i])}{4n^2}, \; i \in I\right) \xRightarrow[n\to\infty]{\mathcal{L}} (\tau_{a_i}(v_i), \; i \in I).$$

For this it is sufficient to prove (see the proof of Corollary 1.2) that:

Theorem 5.1. *Under the hypotheses of Theorem 1.1, for any finite set I, any $a_i \in \mathbb{R}$ and $v_i \geq 0$, $i \in I$,*

$$\left(\Lambda_{a_i,v_i}^{(n)}, \; i \in I\right) \xRightarrow[n\to\infty]{\mathcal{L}} \left(\Lambda_{a_i,v_i}, \; i \in I\right).$$

Proof. Note that when the cardinality of I equals 1 the result is given by Theorem 1.1. The general case can then be proved by induction on the cardinality of I. To simplify the notation we only make the proof of the induction step when the cardinality of I equals 2, but it would work similarly in general. So let a, a', v and

v' be given. All we have to prove is that for any continuous and bounded functions H and $\widetilde{H}$,

$$(29) \qquad \mathbb{E}\left[H\left(\Lambda^{(n)}_{a',v'}\right)\widetilde{H}\left(\Lambda^{(n)}_{a,v}\right)\right] \to \mathbb{E}\left[H\left(\Lambda_{a',v'}\right)\widetilde{H}\left(\Lambda_{a,v}\right)\right],$$

when $n \to \infty$. Consider the events

$$\mathcal{A}_n := \left\{\Lambda^{(n)}_{a,v}(a') < v'\right\},$$

for $n \geq 1$, and

$$\mathcal{A} := \left\{\Lambda_{a,v}(a') < v'\right\}.$$

Observe that conditionally to $\Lambda^{(n)}_{a,v}$ and on the set $\mathcal{A}_n$ we have the equality in law:

$$(30) \qquad \Lambda^{(n)}_{a',v'} - \Lambda^{(n)}_{a,v} = \Lambda^{(n,\Lambda^{(n)}_{a,v}(a+\cdot))}_{a'-a,v'-\Lambda^{(n)}_{a,v}(a')},$$

with the notation of Corollary 2.15. This identity is straightforward. Maybe less immediate is the analogous equality in the continuous setting, so we state it as a lemma:

Lemma 5.2. *Let a, a', v and v' be given. Conditionally to $\Lambda_{a,v}$ and on $\mathcal{A}$, we have the equality in law:*

$$(31) \qquad \Lambda_{a',v'} - \Lambda_{a,v} = \Lambda^{(\Lambda_{a,v}(a+\cdot))}_{a'-a,v'-\Lambda_{a,v}(a')}.$$

Proof. One just has to observe (see also (2) in [RS]) that conditionally to $\Lambda_{a,v}$ and on $\mathcal{A}$, the law of $(Y_{t+\tau_a(v)}, t \geq 0)$ is equal to the law of an excited BM starting from a and associated to the nonhomogeneous function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(x,\ell) = \varphi(\Lambda_{a,v}(x) + \ell).$$

The lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

It follows from (30) that for any continuous and bounded H,

$$\mathbb{E}\left[H\left(\Lambda^{(n)}_{a,v} + (\Lambda^{(n)}_{a',v'} - \Lambda^{(n)}_{a,v})\right) \Big| \Lambda^{(n)}_{a,v}\right] 1_{\mathcal{A}_n} = \mathbb{H}_n\left(\Lambda^{(n)}_{a,v}\right) 1_{\mathcal{A}_n},$$

where

$$\mathbb{H}_n(\lambda) := \mathbb{E}\left[H\left(\lambda + \Lambda^{(n,\lambda(a+\cdot))}_{a'-a,v'-\lambda(a')}\right)\right],$$

for any λ in the Skorokhod space $\mathbb{D}(\mathbb{R})$ such that $\lambda(a') \leq v'$. Define similarly $\mathbb{H}$ by

$$\mathbb{H}(\lambda) := \mathbb{E}\left[H\left(\lambda + \Lambda^{(\lambda(a+\cdot))}_{a'-a,v'-\lambda(a')}\right)\right],$$

for any $\lambda \in \mathbb{D}(\mathbb{R})$ such that $\lambda(a') \leq v'$. Now Corollary 2.15 shows that for any sequence of functions λ_n, satisfying $\lambda_n(a') \leq v'$, and converging to some λ (in $\mathbb{D}(\mathbb{R})$), $\mathbb{H}_n(\lambda_n)$ converges to $\mathbb{H}(\lambda)$. We recall now a standard result in analysis:

Lemma 5.3. *Let E be a metric space. Let $(h_n, n \geq 0)$ be some sequence of real functions on E and let h be some other function. If for any sequence $(\lambda_n, n \geq 0)$ converging to some $\lambda \in E$, $h_n(\lambda_n)$ converges toward $h(\lambda)$, then h_n converges to h uniformly on compact subsets.*

By using this result we deduce that $\mathbb{H}_n$ converges to $\mathbb{H}$ uniformly on compact subsets of the space of functions $f \in \mathbb{D}(\mathbb{R})$ such that $f(a') \leq v'$. Moreover by using the Skorokhod's representation theorem (see Theorem 6.7 in [Bil]), we can assume that $\Lambda^{(n)}_{a,v}$ converges almost surely toward $\Lambda_{a,v}$. This implies in particular

that $1_{\mathcal{A}_n}$ also converges a.s. to $1_{\mathcal{A}}$. So if H and $\widetilde{H}$ are two continuous and bounded functions, we deduce from the dominated convergence theorem that

$$(32) \qquad \mathbb{E}\left[H\left(\Lambda_{a',v'}^{(n)}\right)\widetilde{H}\left(\Lambda_{a,v}^{(n)}\right)1_{\mathcal{A}_n}\right] \to \mathbb{E}\left[H\left(\Lambda_{a',v'}\right)\widetilde{H}\left(\Lambda_{a,v}\right)1_{\mathcal{A}}\right],$$

when $n \to \infty$. We claim moreover that if $(a', v') \neq (a, v)$, then $\mathbb{P}[\Lambda_{a,v}(a') = v'] = 0$. Indeed the set $\{\Lambda_{a,v}(a') = v'\}$ is included in the set $\{e_{a'}(v') \neq 0\}$, where $e_{a'}(v')$ denotes the excursion of Y out of level a' starting from $\tau_{a'}(v'-)$, and this last event has probability 0 (this is well known to be the case for the Brownian motion, and can be deduced for Y by an absolute continuity argument, see also [RS]). We deduce that $\mathbb{P}[\Lambda_{a,v}^{(n)}(a') = v'] \to 0$ when $n \to +\infty$. Thus the convergence in (32) also holds if we replace $\mathcal{A}_n$ and $\mathcal{A}$ respectively by $\mathcal{A}_n^c$ and $\mathcal{A}^c$. Then (29) follows and this concludes the proof of Theorem 5.1. $\qquad\qquad\square$

Now with (28) in hand, we can mimic the rest of the proof of the convergence of one-dimensional distributions. This concludes the proof of Theorem 1.4. $\qquad\square$

Remark 5.4. Notice that the whole proof works as well in the setup of Theorem 2.14. So, as Theorem 1.1, Theorem 1.4 can be extended in the non homogeneous setting.

Remark 5.5. The notation $\Lambda_{a,v}(x)$ is taken from Tóth and Werner [TW]. We notice by the way that here also the set

$$\Lambda = \{(\Lambda_{a,v}(x), x \geq a)\}_{(a,v)\in\mathbb{R}\times[0,\infty)},$$

forms a family of reflected/absorbed coalescing processes. In [TW] the $\Lambda_{a,v}$'s were moreover independent Brownian motions (reflected or absorbed in 0 depending on the time interval) and therefore Λ was called (in their Section 2.1) a FICRAB (for family of independent coalescing reflected and absorbed Brownian motions). Such family of coalescing Brownian motions seems to have been first studied by Arratia [Arr] and is now better known under the name of Brownian web (see for instance [FINR]). Here the situation is slightly different: first each $\Lambda_{a,v}$ is some diffusion which is not a Brownian motion and before they coalesce two $\Lambda_{a,v}$'s are not independent. For instance if $v < v'$, then $(\Lambda_{a,v}, \Lambda_{a,v'})$ satisfies the following system of stochastic differential equations:

$$(33) \quad \begin{cases} d\Lambda_{a,v}(x) & = \quad 2\sqrt{\Lambda_{a,v}(x)}\,dB_x + 2(1_{\{a\leq x\leq 0\}} + h(\Lambda_{a,v}(x)))\,dx \\ d\Lambda_{a,v'}(x) & = \quad 2\sqrt{\Lambda_{a,v}(x)}\,dB_x + 2\sqrt{\Lambda_{a,v'}(x) - \Lambda_{a,v}(x)}\,d\widetilde{B}_x \\ & \qquad +2(1_{\{a\leq x\leq 0\}} + h(\Lambda_{a,v'}(x)))\,dx, \end{cases}$$

for all $x \in [a, +\infty)$, where B and $\widetilde{B}$ are two independent Brownian motions. This result follows from (31) and the Ray-Knight theorem (see for instance [RS, Theorem 6.1]). Note that we could describe similarly the law of $(\Lambda_{a_i,v_i}, i \in I)$, for any finite set I, and any (a_i, v_i), $i \in I$. In [TW], the family Λ was called a sequence of forward lines and the dual sequence, the sequence of backward lines, was defined by

$$\Lambda^* = \left\{\Lambda_{a,v}^*(\cdot) = (\Lambda_{-a,v}(-x), x \geq a)\right\}_{(a,v)\in\mathbb{R}\times[0,\infty)}.$$

As in [TW] we can define Λ^* here and it is also a deterministic function of Λ:

$$(34) \qquad\qquad\qquad \Lambda_{a,v}^*(x) = \sup\{w \; : \; \Lambda_{-x,w}(-a) < v\},$$

for almost all $x \geq a$ and $v \geq 0$. It is important to observe that

$$(35) \quad (\Lambda_{a,v}(x), x \in \mathbb{R}) \text{ is a function of } ((\Lambda_{a,v}(x), x \geq a), (\Lambda_{-a,v}^*(x)), x \geq -a)).$$

We notice now some other notable differences with the situation in [TW]. First if we denote by Q_h the law of Λ, then the law of Λ^* is Q_{-h}. In particular Λ and Λ^* do not have the same law (in other words Λ is not self-dual), except if $h = 0$. Moreover, for any a (say $a < 0$) and $v \geq 0$, the process $\Lambda_{a,v}$ will almost surely not hit 0 in the time interval $[a, 0]$. The reason is that in the time interval $[0, \tau_a(v)]$ the excited BM will cross each level $x \in [a, 0]$ and strictly increase its local time on these levels (by using the absolute continuity between the laws of a standard BM and the excited BM). Similarly given any $a < a'$, v and v', we have $\Lambda_{a,v}(x) \neq \Lambda^*_{-a',v'}(-x) = \Lambda_{a',v'}(x)$ for all $x \in [a, a']$ almost surely. Let us also notice that couples of processes such as $(\Lambda_{a,v}(x), a \leq x \leq 0)$ and $(\Lambda^*_{0,v'}(x), 0 \leq x \leq -a)$, if $a < 0$, are *conjugate diffusions* (see [T3] for a definition). Similarly $(\Lambda_{0,v}(x), x \geq 0)$ and $(\Lambda^*_{a,v'}(x), x \geq -a)$, if $a < 0$, are also conjugate.

Now we can sketch another proof of Theorem 5.1 which bypass the use of Corollary 2.15 and uses instead these notions of forward and backward lines. The idea is to first prove that

$$(36) \qquad \left\{ (\Lambda^{(n)}_{a_i,v_i}(x), x \geq a_i), i \in I \right\} \underset{n \to \infty}{\overset{\mathcal{L}}{\Longrightarrow}} \left\{ (\Lambda_{a_i,v_i}(x), x \geq a_i), i \in I \right\}.$$

This can be done by using Ethier-Kurtz's result (Theorem 4.1 p.354 in [EK]), (30) and (33). One can next define analogues $\Lambda^{(n)}$ and $\Lambda^{(n),*}$ respectively of Λ and Λ^*, in the discrete setting and it then suffices to use (34) (and its discrete counterpart), (35) and (36) to deduce the desired convergence. Since we already gave another proof, we omit the details here.

References

[ABK] **Amir G., Benjamini I, Kozma G.:** *Excited random walk against a wall*, Probab. Theory and Related Fields 140, (2008), 83–102.

[Arr] **Arratia R. A.:** *Coalescing Brownian motions on the line*, Ph.D. Thesis, University of Wisconsin, Madison, (1979).

[AN] **Athreya K., Ney P.:** *Branching processes*, Die Grundlehren der mathematischen Wissenschaften, Band 196. Springer-Verlag, New York-Heidelberg, (1972), xi+287 pp.

[BaS] **Basdevant A.-L., Singh A.:** *On the speed of a cookie random walk*, Probab. Theory Related Fields 141, (2008), 625–645.

[Bil] **Billingsley P.:** *Convergence of probability measures*, Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, (1999), x+277 pp.

[BKS] **Benjamini I., Kozma G., Schapira Br.:** *A balanced excited random walk*, preprint, arXiv:1009.0741.

[BW] **Benjamini I., Wilson D. B.:** *Excited random walk*, Electron. Comm. Probab. 8 (electronic), (2003), 86–92.

[CLJ] **Cranston M., Le Jan Y.:** *Self-attracting diffusions: Two case studies*, Math. Ann. 303, (1995), 87–93.

[Dav] **Davis B.:** *Weak limits of perturbed Brownian motion and the equation $Y_t = B_t + \alpha \sup\{Y_s : s \leq t\} + \beta \inf\{Y_s : s \leq t\}$*, Ann. Probab. 24, (1996), 2007–2023.

[D] **Dolgopyat D.:** *Central limit theorem for excited random walk in the recurrent regime*, preprint, http://www.math.umd.edu/ dmitry/papers.html

[DCS] **Duminil-Copin H., Smirnov S.:** *The connective constant of the honeycomb lattice equals $\sqrt{2 + \sqrt{2}}$*, arXiv:1007.0575.

[EK] **Ethier N., Kurtz G.:** *Markov processes. Characterization and convergence*, Wiley Series Probab. Math. Stat., New York, (1986), x+534 pp.

[F] **Feller W.:** *An introduction to probability theory and its applications. Vol. II.*, Second edition John Wiley & Sons, Inc., New York-London-Sydney (1971), xxiv+669 pp.

[FINR] **Fontes L. R. G., Isopi M., Newman C. M., Ravishankar K.:** *The Brownian web: characterization and convergence*, Ann. Probab. 32, (2004), 2857–2883.

[HRo] **Herrmann S., Roynette B.:** *Boundedness and convergence of some self-attracting diffusions*, Math. Ann. 325, (2003), 81–96.

[KRS] **Kesten H., Raimond O., Schapira Br.:** *Random walks with occasionally modified transition probabilities*, arXiv:0911.3886.

[K] **Kozma G.:** *Problem session*, in: Oberwolfach report 27/2007, Non-classical interacting random walks. `www.mfo.de`

[KZ] **Kosygina E., Zerner M. P. W.:** *Positively and negatively excited random walks on integers, with branching processes*, Electron. J. Probab. 13, (2008), 1952–1979.

[LSW] **Lawler G., Schramm O., Werner W.:** *On the scaling limit of planar self-avoiding walk*, Fractal geometry and application, A jubilee of Benoit Mandelbrot, Proc. Sympos. Pure Math. 72, Part 2, Amer. Math. Soc., Providence, RI, (2004), 339–364.

[MR] **Merkl F., Rolles S.W.W.:** *Linearly edge-reinforced random walks*, Dynamics & stochastics, 66–77, IMS Lecture Notes Monogr. Ser. 48, Inst. Math. Statist., Beachwood, OH, (2006).

[MPRV] **Menshikov M., Popov S., Ramirez A., Vachkovskaia M.:** *On a general many-dimensional excited random walk*, arXiv:1001.1741.

[NRW1] **Norris J.R., Rogers L.C.G., Williams D.:** *An excluded volume problem for Brownian motion*, Phys. Letters A., 112, (1985), 16–18.

[NRW2] **Norris J.R., Rogers L.C.G., Williams D.:** *Self-avoiding random walk: a Brownian motion model with local time drift*, Probab. Theory and Related Fields 74, (1987), 271–287.

[Pem] **Pemantle R.:** *A survey of random processes with reinforcement*, Probab. Surv. 4 (electronic), (2007), 1–79.

[PemV] **Pemantle R., Volkov S.:** *Vertex-reinforced random walk on Z has finite range*, Ann. Probab. 27, (1999), 1368–1388.

[R] **Raimond O.:** *Self-attracting diffusions: Case of the constant interaction*, Probability Theory and Related Fields 107, (1997), 177–196.

[RS] **Raimond O., Schapira Br.:** *Excited Brownian motion*, arXiv:0810.3538.

[RY] **Revuz D., Yor M.:** *Continuous martingales and Brownian motion*, Springer-Verlag, third ed. (1999).

[Sel] **Sellke T.:** *Recurrence of reinforced random walk on a ladder*, Electron. J. Probab. 11, (2006), 301–310.

[Tar] **Tarrès P.:** *Vertex-reinforced random walk on $\mathbb{Z}$ eventually gets stuck on five points*, Ann. Probab. 32, (2004), 2650–2701.

[T1] **Tóth B.:** *The "true" self-avoiding walk with bond repulsion on $\mathbb{Z}$: limit theorems*, Ann. Probab. 23, (1995), 1523–1556.

[T2] **Tóth B.:** *Self-interacting random motions*, Proceedings of the 3rd European Congress of Mathematics, Barcelona 2000, Birkhauser, (2001), 555–565.

[T3] **Tòth B.:** *Generalized Ray-Knight theory and limit theorems for self-interacting random walks on Z^1*, Ann. Probab. 24, (1996), 1324–1367.

[TW] **Tóth B., Werner W.:** *The true self-repelling motion*, Probab. Theory and Related Fields 111, (1998), 375–452.

[Ver] **Vervoort M.R.:** *Reinforced Random Walks*, in preparation, draft version available on http://staff.science.uva.nl/ vervoort/.

[W] **Werner W.:** *Some remarks on perturbed reflecting Brownian motion*, Sém. Probab. XXIX, LNM 1613, Springer, Berlin, (1995), 37–43.

[Z] **Zerner M. P. W.:** *Multi-excited random walks on integers*, Probab. Theory and Related Fields 133, (2005), 98–122.

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