The Realizable Extension Problem and the Weighted Graph $(K_{3,3}, l)$

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To my parents John and Colette

Abstract. This note outlines the realizable extension problem for weighted graphs and provides results of a detailed analysis of this problem for the weighted graph $(K_{3,3}, l)$. This analysis is then utilized to provide a result relating to the connectedness of the moduli space of planar realizations of $(K_{3,3}, l)$. The note culminates with two examples which show that in general, realizability and connectedness results relating to the moduli spaces of weighted cycles which are contained in a larger weighted graph cannot be extended to similar results regarding the moduli space of the larger weighted graph.

Mathematics Subject Classification (2000). Primary 55R80; Secondary 51-XX.

Keywords. Moduli spaces, realizations, weighted graphs, configuration space.

1. Introduction

Given a graph with preassigned edge lengths then a common problem is to determine if this weighted graph can be realized in $\mathbb{E}^2$. A *graph* G is a pair (V_G, E_G) where V_G, known as the *vertex set* of G, is a finite set, and E_G, known as the *edge set* of G, is a multiset whose elements are elements of $[V_G]^2$, the set of 2-element subsets of V_G. Each edge $\{i, j\}$ is denoted ij in the sequel. In this note, graphs can have parallel edges but not loops. For further detail regarding graph theory, see [5]. A length function on a graph G is a function $l : E_G \to \mathbb{R}^{\geq 0}$. A weighted graph is a pair (G, l) where G is a graph and l is a length function on E_G. Given a weighted graph (G, l), then the *configuration space* $C(G, l)$ of (G, l) is defined as

$$C(G, l) = \{p : V_G \to \mathbb{E}^2 \ \mid \ d(p(u), p(v)) = l(uv) \text{ for all } uv \in E_G\}$$

Each p contained in $C(G, l)$ is called a *realization* of (G, l) and if there exists a realization of (G, l), then the weighted graph (G, l) is said to be *realizable*. Note that in the sequel, and particularly in figures, given a realization p then $p_{|v_i}$ is denoted p_i. Given a graph G with vertex set V_G then the group $\mathbb{E}^+(2)$ of *orientation preserving isometries* of $\mathbb{E}^2$ acts on $C(G,l)$ by

$$(\mathbf{g}.p)(v) = \mathbf{g}.(p(v)) \text{ for all } v \in V_G$$

Given a weighted graph (G,l) and the configuration space $C(G,l)$, then the *moduli space* $M(G,l)$ of (G,l) is the quotient space

$$M(G,l) = C(G,l)/\mathbb{E}^+(2)$$

Elements of a moduli space $M(G,l)$ are equivalence classes and so are usually denoted by $[p]$, however, whenever no confusion can arise, by a slight abuse of notation, the elements of $M(G,l)$ are simply denoted p in the sequel.

A subspace of a configuration space which is utilized in the sequel is now described. Given a weighted graph (G,l), the vertices a and b in V_G such that $ab \in E_G$ and that $l(ab) > 0$, then define

$$C_{a,b}(G,l) = \{p \in C(G,l) \mid p(a) = (0,0) \text{ and } p(b) = (l(ab),0)\}$$

Note that $C_{a,b}(G,l)$ and $C_{b,a}(G,l)$ are different as sets but are homeomorphic topological spaces. Observe that given a weighted graph (G,l) then the space $C_{a,b}(G,l)$ is homeomorphic to the moduli space $M(G,l)$.

The *realizability problem* for a weighted graph is the problem of establishing whether or not there exists a realization of (G,l) and, in general, this problem is hard. Note that this problem is sometimes referred to as the *molecule problem* and for further details on this see [1] and [8]. One of the simplest weighted graphs for which the realizability problem has been solved is (K^4,k), where K^4 is the complete graph on four vertices and this solution is now briefly outlined. Consider (K^4,k), with vertex set $V_{K^4} = \{v_1,v_2,v_3,v_4\}$ and edge set $E_{K^4} = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$. It is assumed throughout this section that the lengths assigned by k are denoted as follows $k(v_1v_2) = a$, $k(v_2v_4) = b$, $k(v_3v_4) = c$, $k(v_1v_3) = d$, $k(v_2v_3) = \alpha$ and $k(v_1v_4) = \beta$. This notation is illustrated in Fig. 1.

It is well known, see [4] for instance, that (K^4,k) is realizable if and only if all cyclic permutations of the four inequalities $a \leq b+\beta$, $b \leq c+\alpha$, $c \leq d+\beta$ and

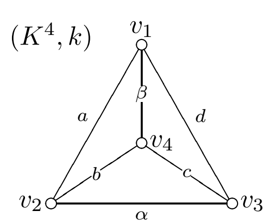

FIGURE 1. The weighted graph (K^4,k)

$d \leq a+\alpha$ are satisfied and equation 1.1 holds. Note that the determinant contained in equation 1.1 is known as the *Cayley-Menger determinant.*

$$\det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & d^2 & \beta^2 \\ 1 & a^2 & 0 & \alpha^2 & b^2 \\ 1 & d^2 & \alpha^2 & 0 & c^2 \\ 1 & \beta^2 & b^2 & c^2 & 0 \end{pmatrix} = 0 \qquad (1.1)$$

The fact that realizability conditions exist for the weighted graph (K^4, k) appears to be something of a rarity as there does not appear to exist in the literature general realizability conditions, analogous to the (K^4, k) case for other (non-trivial) weighted graphs. However, one recent development to this end, is a result contained in [10] (and will appear in [2]) which gives realizability conditions for weighted graphs where the graph is contained in the class of series-parallel graphs.

At this point the focus switches from the *realizability problem* to the following, more tractable, *realizable extension problem.* Given a realizable weighted graph (H, h) where $H \subset G$, then what conditions must an extension of h, denoted l, satisfy so that (G, l) is realizable. Observe that as every graph has a spanning tree (or spanning forest if the graph is not connected) then it is possible to state the following elementary existence result for such extensions.

Lemma 1.1. *Given a graph G and a realizable weighted graph (H, h) where $H \subset G$, then it is possible to find an extension of h, denoted l, such that (G, l) is realizable.*

2. The Realizable Extension Problem for $(K_{3,3}, l)$

The realizable extension problem is now examined in the case of the weighted graph $(K_{3,3}, l)$. The reason for choosing $(K_{3,3}, l)$ is that this graph is essentially the simplest graph for which the realizable extension problem is non-trivial. With the exception of K^4, for which the realizable extension problem is essentially trivial, all graphs smaller than $K_{3,3}$ are series-parallel and so the realizability problem and hence, the realizable extension problem, can be solved using the results of [10].

Consider the weighted complete bi-partite graph $(K_{3,3}, l)$, where $V_{K_{3,3}} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E_{K_{3,3}} = \{v_6v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_1v_4, v_2v_5, v_3v_6\}$. It is assumed throughout this section that the lengths assigned by $l : E_{K_{3,3}} \to \mathbb{R}^{\geq 0}$ are denoted $l(v_1v_6) = a$, $l(v_1v_2) = b$, $l(v_2v_3) = c$, $l(v_3v_4) = d$, $l(v_4v_5) = e$, $l(v_5v_6) = f$, $l(v_1v_4) = \alpha$, $l(v_3v_6) = \beta$ and $l(v_2v_5) = \gamma$. The values $a, b, ..., \gamma$ are not assumed to be fixed at this stage. This notation is illustrated in Fig. 2.

Consider also the four specific subgraphs of $K_{3,3}$ which are defined as $G_3 = (V_{K_{3,3}}, E_{K_{3,3}} \setminus v_2v_5)$, $G_2 = (V_{K_{3,3}}, E_{G_3} \setminus v_3v_6)$, $G_1 = (V_{K_{3,3}}, E_{G_2} \setminus v_1v_4)$ and $G_0 = (V_{K_{3,3}}, E_{G_1} \setminus v_5v_6)$ which is a path. The former three of the aforementioned

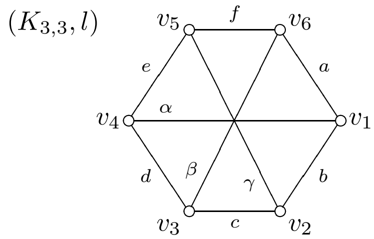

FIGURE 2. The weighted graph $(K_{3,3}, l)$

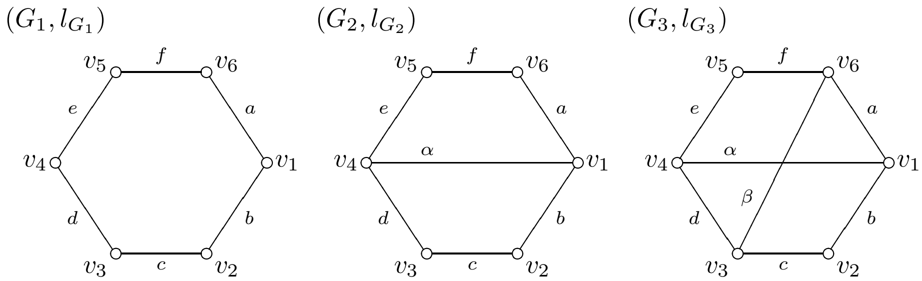

FIGURE 3. The weighted graphs (G_1, l_{G_1}), (G_2, l_{G_2}) and (G_3, l_{G_3})

subgraphs of $K_{3,3}$ are shown in Fig. 3.

Assuming that l_{G_0} is given, thus fixing the edge lengths a, b, c, d and e, then determining conditions which the extensions $l_{G_1}, l_{G_2}, l_{G_3}$ and l must satisfy so that (G_1, l_{G_1}), (G_2, l_{G_2}), (G_3, l_{G_3}) and $(K_{3,3}, l)$, respectively, are realizable, is the focus of the remainder of this section.

Lemma 2.1. *Given a weighted graph (G_0, l_{G_0}), as above, then (G_1, l_{G_1}) is realizable if and only if l_{G_1} assigns a value for f such that*

$$f \in [\max\{0, 2.\max\{a, b, c, d, e\} - (a + b + c + d + e)\}, a + b + c + d + e]$$

Proof. As G_0 is a path then (G_0, l_{G_0}) is always realizable. The graph $G_1 = (V_{G_0}, E_{G_0} \cup v_5v_6)$ is a cycle, and so (G_1, l_{G_1}) is realizable if and only if the inequality $f \leq a+b+c+d+e$, and all of the other five cyclic permutations of this inequality, are satisfied. Choosing $f \in [\max\{0, 2.\max\{a, b, c, d, e\} - (a+b+c+d+e)\}, a+b+c+d+e]$ ensures all six inequalities are satisfied. □

Lemma 2.2. *Given a realizable weighted graph (G_1, l_{G_1}), as above, and letting $\mu_1 = 2.\max\{a, e, f\}$ and $\mu_2 = 2.\max\{b, c, d\}$, then (G_2, l_{G_2}) is realizable if and only if l_{G_2} assigns a value for α such that*

$$\alpha \in [\max\{0, \mu_1 - (a + e + f)\}, a + e + f] \cap [\max\{0, \mu_2 - (b + c + d)\}, b + c + d]$$

Proof. Consider the paths P^1 and P^2 contained in G_1 with respective edge sets $E_{P^1} = \{v_4v_5, v_5v_6, v_6v_1\}$ and $E_{P^2} = \{v_1v_2, v_2v_3, v_3v_4\}$. Consider also the cycles C^1 and C^2 contained in G_2 with respective edge sets $E_{C^1} = E_{P^1} \cup v_1v_4$ and $E_{C^2} = E_{P^2} \cup v_1v_4$. Clearly (G_2, l_{G_2}) is realizable if and only if both (C^1, l_{C^1}) and (C^2, l_{C^2}) are realizable *and* both l_{C^1} and l_{C^2} assign the same (permissible) value of α to the edge v_1v_4. It now follows from Lemma 2.1 that $\alpha \in [\max\{0, \mu_1 - (a + e + f)\}, a + e + f] \cap [\max\{0, \mu_2 - (b + c + d)\}, b + c + d]$ where $\mu_1 = 2.\max\{a, e, f\}$ and $\mu_2 = 2.\max\{b, c, d\}$. □

Before considering the weighted graph (G_3, l_{G_3}) the concept of a *workspace* is introduced. For more details regarding workspaces see [3], [11] or [13], where the concept first appears. Given a weighted graph (G, l), then the *workspace* of a vertex v with respect to the graph G, the length function l and an edge $ab \in E_G$ where $l(ab) > 0$, is defined as the image of the map $M(G, l) \to M(H, l_{|H})$ i.e.

$$W_{G,l,ab}(v) = im(M(G, l) \to M(H, l_{|H}))$$

where $H = (\{a, b, v\}, \{ab\})$ and $l_{|H}$ is the restriction of l induced by $H \subset G$.

Note that the moduli space $M(H, l_{|H})$ is in fact a copy of $\mathbb{E}^2$. It is possible to construct an explicit homeomorphism φ_a as follows. For each $[p] \in M(H, l_{|H})$, let q be the unique realization in $C(H, l_{|H})$ that satisfies $q(a) = (0, 0)$, $q(b) = (l(ab), 0)$, and $[q] = [p]$ in $M(H, l_{|H})$. It is now possible to define $\varphi_a([p]) = q(v)$. It is clear that $\varphi_a : M(H, l_{|H}) \to \mathbb{E}^2$ is a homeomorphism. In the sequel, the map φ_a is used to identify the workspace of a vertex with a particular subset of $\mathbb{E}^2$.

Lemma 2.3. *Given a realizable weighted graph (G_2, l_{G_2}), as above, then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\beta = l_{G_3}(v_3v_6)$ can be chosen so that (G_3, l_{G_3}) is realizable is an interval or the disjoint union of two intervals.*

Proof. Given a weighted graph (C, l) where C is a cycle such that $ij, jk \in E_C$, then it is well known, see [3], that the image of $\varphi_i|_{W_{C,l,ij}(k)}$ has one of three types; a circle S with centre $(l(ij), 0)$ and radius $l(jk)$, a contractible subset of S or two disjoint contractible subsets of S. All three of these subsets of S are also symmetric about the x-axis i.e. $w \in im(\varphi_i|_{W_{C,l,ij}(k)}) \iff \rho_x(w) \in im(\varphi_i|_{W_{C,l,ij}(k)})$ where ρ_x is the reflection in the x-axis. Returning to the (G_3, l_{G_3}) case at hand. Consider the circle S_1 with centre $(l_{G_2}(v_1v_4), 0)$ and radius $l_{G_2}(v_1v_6)$ and the circle S_2 with centre $(0, 0)$ and radius $l_{G_2}(v_3v_4)$. Observe that the images of $\varphi_{v_4}|_{W_{G_2,l_{G_2},v_4v_1}(v_3)}$ and $\varphi_{v_4}|_{W_{G_2,l_{G_2},v_4v_1}(v_6)}$ are subsets of circles S_1 and S_2, respectively, and these images are denoted $W(v_3)$ and $W(v_6)$, respectively, for the rest of this proof. The structure of the set $X = \{d(w, w') \mid w \in W(v_3) \text{ and } w' \in W(v_6)\}$ is now determined.

Consider the value $m = \min\{d(w, w') \mid w \in W(v_3) \text{ and } w' \in W(v_6)\}$ and the value $N = \max\{d(w, w') \mid w \in W(v_3) \text{ and } w' \in W(v_6)\}$. A brief consideration of subsets of two circles (centred on the x-axis) which are symmetric about the

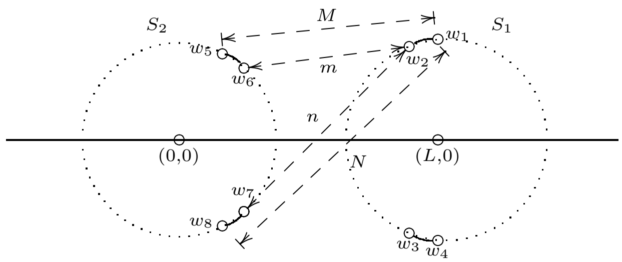

Figure 4. The subset of $\mathbb{R}^{\geq 0}$ from which the value of β can be chosen so that (G_3, l_{G_3}) is realizable can be the disjoint union $[m, M] \sqcup [n, N]$

x-axis leads to the conclusion that there is only one case where $X \neq [m, N]$. This case is a special case of the instance where $W(v_3)$ and $W(v_6)$ are themselves two disjoint contractible subsets of S_1 and S_2 respectively. In order to describe this special case denote by $W(v_3)^+$ the component of $W(v_3)$ contained in the upper half-plane and denote by $W(v_3)^-$ the component of $W(v_3)$ contained in the lower half-plane. The components $W(v_6)^+$ and $W(v_6)^-$ of $W(v_6)$ are defined similarly. Now, consider the value $M = \max\{d(w, w') \mid w \in W(v_3)^+ \text{ and } w' \in W(v_6)^+\}$ and the value $n = \min\{d(w, w') \mid w \in W(v_3)^- \text{ and } w' \in W(v_6)^+\}$. The aforementioned special case occurs whenever $n > M$ and so the subset of $\mathbb{R}^{\geq 0}$ from which the value of β can be chosen so that (G_3, l_{G_3}) is realizable is the disjoint union of two intervals $[m, M] \sqcup [n, N]$. Consider Fig. 4 and note that the subsets $W(v_3) = W(v_3)^+ \sqcup W(v_3)^- = [w_1, w_2] \sqcup [w_3, w_4]$ and $W(v_6) = W(v_6)^+ \sqcup W(v_6)^- = [w_5, w_6] \sqcup [w_7, w_8]$ of the circles S_1 and S_2, respectively, and let $L = l_{G_2}(v_1 v_4)$.

Hence, the subset of $\mathbb{R}^{\geq 0}$ from which the value of β can be chosen so that (G_3, l_{G_3}) is realizable is either an interval $[m, N]$ or the disjoint union of two intervals $[m, M] \sqcup [n, N]$, where m, M, n and N are defined as above. □

Lemma 2.4. *Given a realizable weighted graph (G_3, l_{G_3}), as above, then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\gamma = l(v_2 v_5)$ can be chosen so that $(K_{3,3}, l)$ is realizable is an interval or the disjoint union of two, three or four intervals.*

Proof. Consider a weighted graph (H, h), where $V_H = \{u_1, u_2, u_3, u_4, u_5\}$ and $E_H = \{u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_2u_3, u_3u_4, u_4u_5\}$ as shown in Fig. 5. Observe that $M(H, h)$ is homeomorphic to $C_{u_4,u_1}(H, h)$. If (H, h) is realizable, then for every $q \in C_{u_4,u_1}(H, h)$ there exists a $\rho q \in C_{u_4,u_1}(H, h)$ whose image is a reflection of the image of q in the x-axis. Observe that $C_{u_4,u_1}(H, h)$ can have at most 2^3 connected components. Further motivation of this statement is provided in Fig. 5. The images of the realizations p, q, r and s of (H, h) are shown, and there also exists four corresponding realizations $\rho p, \rho q, \rho r$ and ρs of (H, h) in $C_{u_4,u_1}(H, h)$ which

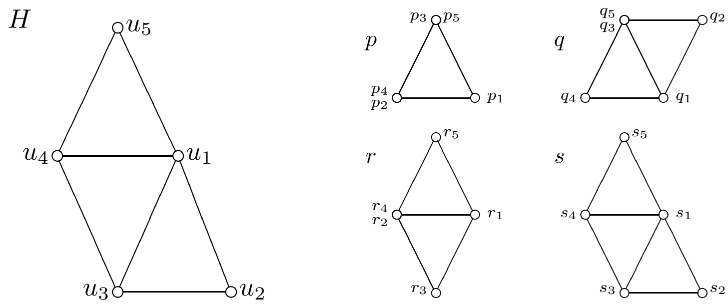

FIGURE 5. The graph H and the images of the realizations p, q, r and s of (H, h) given an equilateral length function h

are not shown. Clearly it is possible to chose length functions l_{G_3} and h so that $b := l_{G_3}(v_1v_2) = h(u_1u_2), c := l_{G_3}(v_2v_3) = h(u_2u_3), d := l_{G_3}(v_3v_4) = h(u_3u_4), e := l_{G_3}(v_4v_5) = h(u_4u_5), f := l_{G_3}(v_5v_1) = h(u_5u_1), \alpha := l_{G_3}(v_1v_4) = h(u_1u_4)$ and $\beta := l_{G_3}(v_1v_3) = h(u_1u_3)$ and that the edge length $a := l_{G_3}(v_1v_6)$ is assigned an arbitrarily small length $\epsilon \ll 1$ by l_{G_3}. Consider the image of the realization $p \in C_{u_4,u_1}(G_3, l_{G_3})$ which is contained in Fig. 6. Note that the incidence structure of the larger node, labeled $\frac{p_6}{p_1}$, is shown in the detailed (*blown-up*) section contained in the circle on the right-hand-side of Fig. 6.

As a result of such length functions, then each connected component of the moduli space $M(G_3, l_{G_3})$ is a circle whereas each connected component of the moduli space $M(H, h)$ is a point. The salient point here is that the connected components

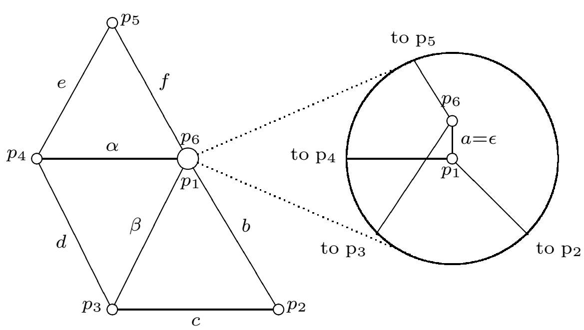

FIGURE 6. The image of a realization p of the weighted graph (G_3, l_{G_3}) where l_{G_3} assigns the edge v_1v_6 the length ϵ such that $l_{G_3}(v_1v_6) = a = \epsilon \ll 1$

of $M(H,h)$ and $M(G_3, l_{G_3})$ are in a one-to-one correspondence.

Observe that if $C_{u_4,u_1}(G_3, l_{G_3})$ has 8 connected components then these eight components must occur in pairs such that for any realization p contained in one component of $C_{u_4,u_1}(G_3, l_{G_3})$ there exists a realization, denoted ρp, in another component of $C_{u_4,u_1}(G_3, l_{G_3})$ such that the image of ρp is a reflection of the image of p in the x-axis. As a reflection is an isometry, then the distance $d(p_2, p_5)$ must be equal to the distance $d(\rho p(v_2), \rho p(v_5))$, for each $p \in C_{u_4,u_1}(G_3, l_{G_3})$. Hence, the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\gamma = l(v_2 v_5)$ can be chosen so that $(K_{3,3}, l)$ is realizable, can have at most four connected components. Examples 1, 2, 3 and 4 are, respectively, occurrences of the subset of $\mathbb{R}^{\geq 0}$ from which the value of γ can be chosen so that $(K_{3,3}, l)$ is realizable being one, two, three and four, disjoint intervals. This completes the proof. □

Example 1. Suppose that $a = b = c = d = e = f = \alpha = \beta = 1$ then the subset of $\mathbb{R}^{\geq 0}$ from which the value of γ can be chosen so that $(K_{3,3}, l)$ is realizable is the set $\{1\}$. See Figure 7.

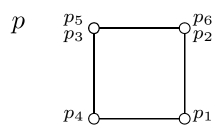

Figure 7. The subset of $\mathbb{R}^{\geq 0}$ from which the value of γ can be chosen so that $(K_{3,3}, l)$ is realizable is a single point

In Examples 2, 3 and 4 the larger nodes labeled $\frac{p_6}{p_1}$, $\frac{q_6}{q_1}$, $\frac{r_6}{r_1}$ and $\frac{s_6}{s_1}$ are each analogous to the larger node contained in Fig. 6 i.e. they possess the same incidence structure. It should also be noted that, in the interest of brevity, the images of the four realizations $\rho p, \rho q, \rho r$ and ρs of $(K_{3,3}, l)$, (whose images are the reflections in the x-axis of the images of p, q, r and s, respectively) are omitted from Fig. 8, Fig. 9 and Fig. 10.

Example 2. Suppose that $a = \epsilon \ll b = c = d = e = \alpha = 1$ and $f = \beta = \sqrt{2}$ then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\gamma = l(v_2 v_5)$ can be chosen so that $(K_{3,3}, l)$ is realizable is $[1 - \delta_1, 1 + \delta_1] \sqcup [\sqrt{5} - \delta_2, \sqrt{5} - \delta_2]$ where $\delta_1, \delta_2 \ll 1$. See Figure 8.

Example 3. Suppose that $a = \epsilon \ll b = c = \frac{\sqrt{5}}{2}, d = \alpha = 1, e = f = \frac{\sqrt{5}}{4}$ and $\beta = \sqrt{2}$ then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\gamma = l(v_2 v_5)$ can be chosen so that $(K_{3,3}, l)$ is realizable is $[\frac{\sqrt{5}}{4} - \delta_1, \frac{\sqrt{5}}{4} + \delta_1] \sqcup [\frac{\sqrt{13}}{2} - \delta_2, \frac{\sqrt{13}}{2} + \delta_2] \sqcup [\frac{\sqrt{29}}{2} - \delta_3, \frac{\sqrt{29}}{2} + \delta_3]$ where $\delta_1, \delta_2, \delta_3 \ll 1$. See Figure 9.

Example 4. Suppose that $a = \epsilon \ll b = \sqrt{17}, c = \sqrt{29}, d = \sqrt{5}, \alpha = 3, e = \sqrt{13}, f = \sqrt{10}$ and $\beta = 5\sqrt{2}$ then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\gamma = l(v_2 v_5)$ can

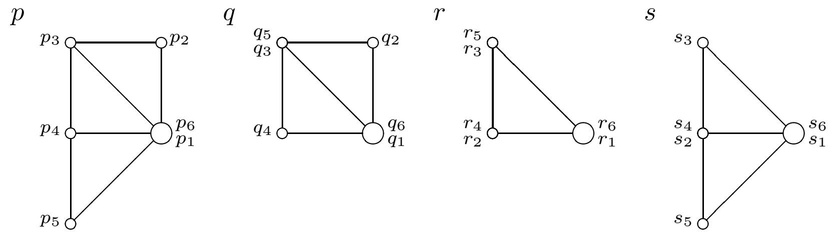

FIGURE 8. The subset of $\mathbb{R}^{\geq 0}$ from which the value of γ can be chosen so that $(K_{3,3}, l)$ is realizable is two disjoint intervals

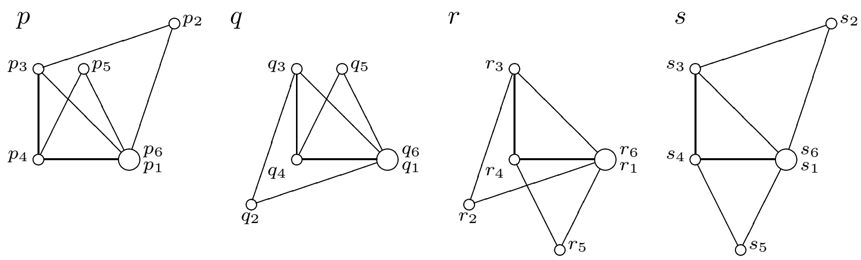

FIGURE 9. The subset of $\mathbb{R}^{\geq 0}$ from which the value of γ can be chosen so that $(K_{3,3}, l)$ is realizable is three disjoint intervals

be chosen so that $(K_{3,3}, l)$ is realizable is $[\ell_1 - \delta_1, \ell_1 + \delta_1] \sqcup [\ell_2 - \delta_2, \ell_2 + \delta_2] \sqcup [\ell_3 - \delta_3, \ell_3 + \delta_3] \sqcup [\ell_4 - \delta_4, \ell_4 + \delta_4]$ where $\delta_1, \delta_2, \delta_3, \delta_4 \ll 1$ and $\ell_1, \ell_2, \ell_3, \ell_4$ are all distinct and $d(\ell_i, \ell_j) > \delta_i + \delta_j$ for all distinct 2-element subsets $\{i, j\}$ contained in $\{1, 2, 3, 4\}$. See Figure 10.

This analysis of the $(K_{3,3}, l)$ case is now distilled into Theorem 2.5.

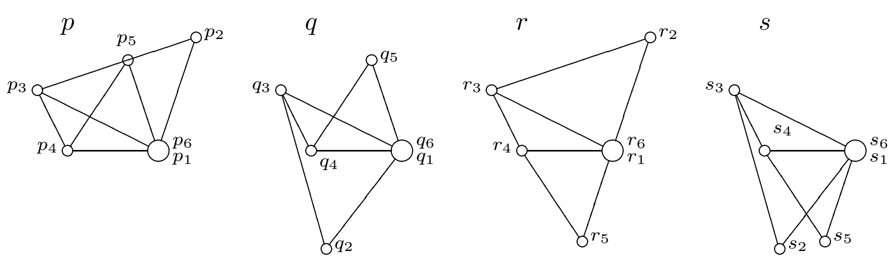

FIGURE 10. The subset of $\mathbb{R}^{\geq 0}$ from which the value of γ can be chosen so that $(K_{3,3}, l)$ is realizable is four disjoint intervals

Theorem 2.5. *Given the weighted graph (G_0, l_{G_0}), then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $f = l_{G_1}(v_5v_6)$ can be chosen so that (G_1, l_{G_1}) is realizable is an interval; having chosen l_{G_1} and hence fixed f, then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\alpha = l_{G_2}(v_1v_4)$ can be chosen so that (G_2, l_{G_2}) is realizable is an interval; having chosen l_{G_2} and hence fixed α, then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\beta = l_{G_3}(v_3v_6)$ can be chosen so that (G_3, l_{G_3}) is realizable is either an interval or the disjoint union of two intervals; finally, having chosen l_{G_3} and hence fixed β, then the subset of $\mathbb{R}^{\geq 0}$ from which the value of $\gamma = l(v_2v_5)$ can be chosen so that $(K_{3,3}, l)$ is realizable is an interval or the disjoint union of two, three or four intervals.*

3. The Moduli Space $M(K_{3,3}, l)$

A nice corollary of the analysis of previous section is that it is possible to establish a result relating to the connectedness of the moduli space $M(K_{3,3}, l)$.

Lemma 3.1. *Given the weighted graph $(K_{3,3}, l)$ then the moduli space $M(K_{3,3}, l)$ can only have one, two, four, six or eight connected components.*

Proof. As outlined above in relation to the weighted graph (H, h) where H contains three 3-cycles, the moduli space $M(K_{3,3}, l)$ can contain at most eight connected components. If $(K_{3,3}, l)$ is not realizable then the moduli space $M(K_{3,3}, l)$ is the empty set and has, by definition, a single component. It is now required to show that the moduli space $M(K_{3,3}, l)$ cannot contain three, five or seven connected components. In a similar fashion to the proof of Lemma 2.3, let the image of $\varphi_{v_4}|_{W_{K_{3,3},l,v_1v_4}(v_i)}$ be denoted $W(v_i)$ for $i \in \{2, 3, 5, 6\}$. Clearly if a $W(v_i)$ is connected then this does not imply that the corresponding moduli space is connected. However, if all $W(v_i)$, for $i \in \{2, 3, 5, 6\}$, are connected then the corresponding moduli space must be connected. It follows that when $M(K_{3,3}, l)$ is disconnected, there must exist at least one $W(v_i)$, for $i \in \{2, 3, 5, 6\}$, which is disconnected.

Each $W(v_i)$, for $i \in \{2, 3, 5, 6\}$, is symmetric about the x-axis i.e. $w \in W(v_i) \iff \rho_x(w) \in W(v_i)$, where ρ_x is the reflection in the x-axis. This means that if $M(K_{3,3}, l)$ is disconnected then there exists some disconnected $W(v_i)$, for $i \in \{2, 3, 5, 6\}$, such that the images of the realizations contained in the fibres of $\pi : M(K_{3,3}, l) \to W(v_i)$ over $W(v_i)^+ \subset W(v_i)$ are all reflections in the x-axis of the images of realizations contained in the fibres of π over $W(v_i)^- \subset W(v_i)$. Hence, if $M(K_{3,3}, l)$ is disconnected then the connected components of $M(K_{3,3}, l)$ must occur in pairs where the images of realizations contained in these components differ by a reflection in the x-axis. As $M(K_{3,3}, l)$ can have at most eight connected components, and as the empty set has one connected component, then $M(K_{3,3}, l)$ cannot contain three, five or seven connected components.

In Example 1 the moduli space $M(K_{3,3}, l)$ is homeomorphic to the moduli space of an equilateral 4-cycle (C, l_C). The moduli space $M(C, l_C)$ is well known to be a connected space, see [12], hence the moduli space $M(K_{3,3}, l)$ can have one (non-empty) component. In Example 2 there are three realizations q, r and s in which the length of γ is contained in the interval $[1-\delta_1, 1+\delta_1]$, with $\delta_1 \ll 1$, such that there does not exist a continuous deformation between any two of the realizations q, r and s. This means that such a choice of γ results in the moduli space $M(K_{3,3}, l)$ containing six connected components. In the same example, choosing γ to be contained in the interval $[\sqrt{5}-\delta_2, \sqrt{5}+\delta_2]$, where $\delta_2 \ll 1$, results in the moduli space $M(K_{3,3}, l)$ containing two connected components. In Example 3 there does not exist a continuous deformation between realizations p and r. The length of γ is contained in the interval $[\frac{\sqrt{5}}{4}-\delta_3, \frac{\sqrt{5}}{4}+\delta_3]$, where $\delta_3 \ll 1$, and results in the moduli space $M(K_{3,3}, l)$ containing four connected components. Finally, Example 5, below, illustrates that there exists a scenario where it is possible to choose a value for γ which results in the moduli space $M(K_{3,3}, l)$ containing eight connected components. This completes the proof. □

Example 5. Suppose that $a = \epsilon \ll b = f = \alpha = \beta = 1$ and $c = d = e = \sqrt{2}$ then the subset of $\mathbb{R}^{\geq 0}$ from which the value of γ can be chosen so that $(K_{3,3}, l)$ is realizable is the interval $[\sqrt{2}-\delta, \sqrt{2}+\delta]$ where $\delta \ll 1$. Observe that the moduli space $M(K_{3,3}, l)$ has eight connected components. Recall that $M(K_{3,3}, l)$ is homeomorphic to $C_{v_4,v_1}(K_{3,3}, l)$. The images of realizations p, q, r and s which are each contained in distinct connected components of $C_{v_4,v_1}(K_{3,3}, l)$ are shown in Fig. 11. Again, the larger nodes labeled $\frac{p_6}{p_1}$, $\frac{q_6}{q_1}$, $\frac{r_6}{r_1}$ and $\frac{s_6}{s_1}$ are analogous to the larger node contained in Fig. 6. The reflection ρ_x in the x-axis applied to the images of each of the realizations p, q, r and s results in the images of the four realizations $\rho p, \rho q, \rho r$ and ρs which are each contained in one of the remaining four distinct connected components of $C_{v_4,v_1}(K_{3,3}, l)$.

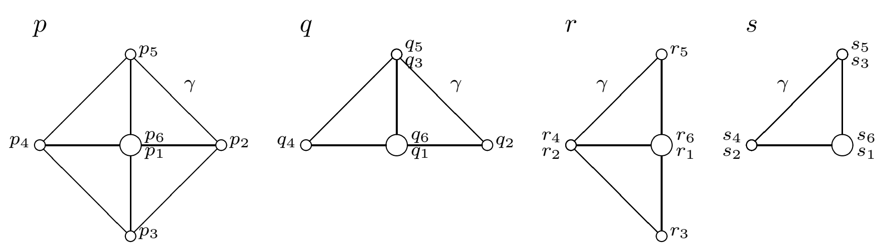

FIGURE 11. $M(K_{3,3}, l)$ can have eight connected components

4. Moduli Spaces of Weighted Cyclic Subgraphs

The moduli space of a weighted cycle is a well understood object, see for example [6],[7], [11] and [12]. It may seem reasonable therefore that whenever a weighted graph (G, l) contains weighted cycles that by determining realizability and/or connectedness results for certain weighted cyclic subgraphs of (G, l) then these results may be extended to realizability and/or connectedness results relating to the weighted graph (G, l). This section contains two examples which show that properties of a moduli space $M(C, l_{|C})$ are not necessarily possessed by the moduli space of $M(G, l)$ where C is a cyclic subgraph of G.

4.1. Realizability

This section contains an example which shows that even though all weighted cyclic subgraphs of a given (G, l) are realizable, the weighted graph (G, l) may not itself be realizable.

Example 6. Consider the graph (G, l) where $V_G = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E_G = \{v_1v_4, v_1v_5, v_1v_6, v_2v_4, v_2v_5, v_2v_7, v_3v_4, v_3v_6, v_3v_7\}$ and l assigns the lengths $l(v_1v_4) = 2$, $l(v_1v_5) = l(v_1v_6) = 4$, $l(v_2v_4) = l(v_3v_4) = \sqrt{13}$, $l(v_2v_5) = l(v_3v_6) = 1$, $l(v_3v_7) = \frac{7}{2}$ and $l(v_2v_7) = \frac{1}{2}$. Observe that such a length assignment results in the situation where the weighted graph (G, l) is not realizable and so $M(G, l)$ is empty. Further justification of the fact that $M(G, l)$ is empty can be found in Fig. 12 which contains the weighted graph (G, l) and two "attempted realizations" of (G, l) which are labeled "p" and "q".

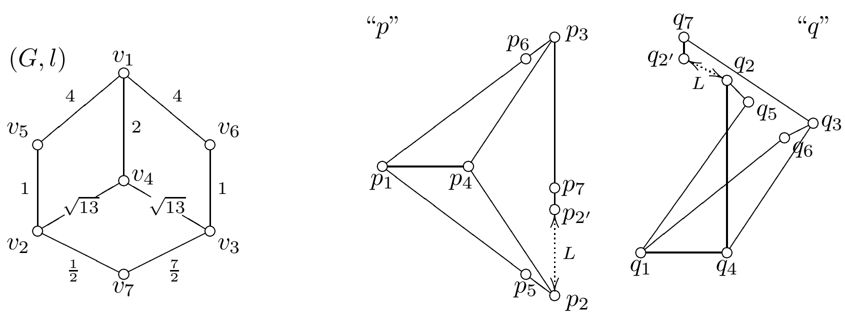

FIGURE 12. The weighted graph (G, l) and two "attempted realizations" of (G, l) which are labeled "p" and "q"

Observe that G contains seven cyclic subgraphs and that all seven of these weighted cyclic subgraphs contained in (G, l) are realizable. However, as the value of $L = d(p_2, p_{2'})$ is always strictly positive in any "attempted realization" of (G, l), for example "p" and "q" in Fig. 12, then (G, l) is not realizable.

4.2. Connectedness

This section contains an example which shows that even though the moduli spaces of some weighted cyclic subgraphs of a given (G, l) are not connected, the moduli space $M(G, l)$ may itself be connected.

Example 7. Consider the graph (H, h) where $V_H = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E_H = \{v_1v_4, v_1v_5, v_1v_6, v_2v_4, v_2v_5, v_2v_7, v_3v_4, v_3v_6, v_3v_7\}$ and h assigns the lengths $h(v_1v_4) = h(v_2v_4) = h(v_3v_4) = \sqrt{3}$ and $h(v_1v_5) = h(v_1v_6) = h(v_2v_5) = h(v_2v_7) = h(v_3v_6) = h(v_3v_7) = \frac{3}{2}$ as illustrated in Fig. 13.

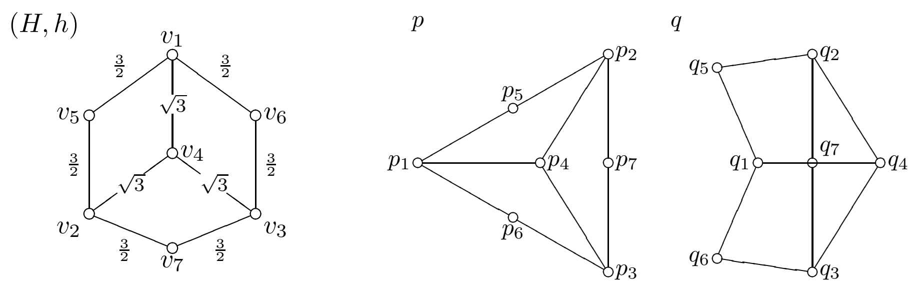

FIGURE 13. The weighted graph (H, h) and the images of two realizations p and q of (H, h) (the image of ρp is the reflection of the image of p in the line containing p_1 and p_4)

Note that there exists a realization ρp in the moduli space $M(H, h)$ whose image is a reflection of the image of p in the half-line containing the images p_1 and p_4. Given $p, \rho p, q \in M(H, h)$, whose images are illustrated in Fig. 13, then there does not exist a path $\alpha_1 : [0, 1] \to M(H, h)$ such that $\alpha_1(0) = p$ and $\alpha_1(1) = \rho p$, a path $\alpha_2 : [0, 1] \to M(H, h)$ such that $\alpha_2(0) = p$ and $\alpha_2(1) = q$,or a path $\alpha_3 : [0, 1] \to M(H, h)$ such that $\alpha_3(0) = q$ and $\alpha_3(1) = \rho p$. However, observe that there does exist a path $\beta : [0, 1] \to M(H, h)$ such that $\beta(0) = q$ and $\beta(1) = \rho q$, where ρq is the realization whose image is the reflection of the image of q in the line containing q_1 and q_4. It follows that the moduli space $M(H, h)$ has three connected components.

Given (H, h) as per Fig. 13, then consider the weighted graph (G, l) where G has vertex set $V_G = V_H \cup \{v_8\}$, edge set $E_G = E_H \cup \{v_2v_8, v_3v_8\}$ and l is an extension of h which also assigns the lengths $l(v_2v_8) = l(v_3v_8) = \sqrt{2}$.

Consider now the inclusion map $\iota : M(G, l) \to M(H, h)$. Observe that ι is not surjective as neither p nor ρp, as per Fig. 13, are mapped onto by ι. Note that the components of $M(H, l)$ which contain the realizations p and ρp each contain just a single point. Note also that there exists a path $\gamma : [0, 1] \to M(G, l)$ such that $\gamma(0) = r$ and $\gamma(1) = \rho r$ where ρr is the realization of (G, l) whose image is the reflection of the image of r in the line containing r_1 and r_4. It follows from these notes that $M(G, l)$ is connected.

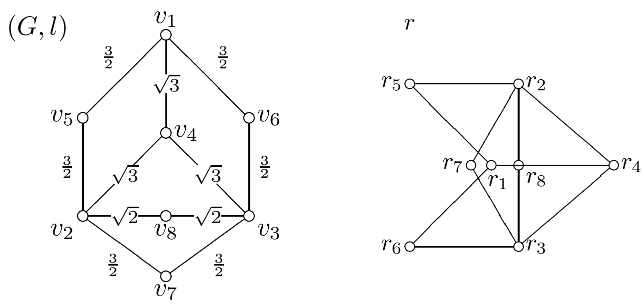

FIGURE 14. The weighted graph (G, l) and the image of a realization r of the weighted graph (G, l)

Acknowledgments

The author would like to thank Dr. James Cruickshank who acted as Ph.D advisor during the writing of [10], in which this note constitutes a chapter.

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