ON MODULAR BALL-QUOTIENT SURFACES WITH KODAIRA DIMENSION ONE

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ABSTRACT. Let $\Gamma \subset \mathbf{PU}(2,1)$ be a lattice which is not co-compact, of finite Bergman-covolume and acting freely on the open unit ball $\mathbf{B} \subset \mathbb{C}^2$. Then the compactification $X = \overline{\Gamma \setminus \mathbf{B}}$ is a smooth projective surface with an elliptic compactification divisor $D = X \setminus (\Gamma \setminus \mathbf{B})$. In this short note we discover a new class of unramified ball-quotients X. We consider ball-quotients X with $kod(X) = h^1(X, \mathcal{O}_X) = 1$. We prove that all minimal surfaces with finite Mordell-Weil group in the class described are pull-backs of the elliptic modular surface which parametrizes triples (E, x, y) of elliptic curves E with 6-torsion points $x, y \in E[6]$ such that $\mathbb{Z}x + \mathbb{Z}y = E[6]$.

1. Introduction

Let the symbol $\mathcal{T}$ denote the class of complex projective smooth surfaces X which contain pairwise disjoint elliptic curves $D_1, ..., D_{h_X}$ such that $U = X \setminus \bigcup D_i$ admits the open unit ball $\mathbf{B} \subset \mathbb{C}^2$ as universal holomorphic covering; as explained in [7], $\mathcal{T}$ forms the 'generic' class of compactified ball-quotient surfaces. There are several motivations to study surfaces in $\mathcal{T}$ without assuming that $\pi_1(U, *)$ with its Poincaré action on $\mathbf{B}$ is an arithmetic lattice of $\mathbf{PU}(2,1)$; we refer to [1] or to the introduction of [7]. Since the discovery of blown-up abelian surfaces in $\mathcal{T}$ by Hirzebruch and Holzapfel some years ago (cf. [2]) there have been no further examples of surfaces of special type in $\mathcal{T}$. In this short note we present a new class of modular surfaces $X \in \mathcal{T}$ with $kod(X) = 1$.

In what follows we only consider complex projective smooth surfaces. Recall that a minimal elliptic surface $\pi : X \longrightarrow C$ with finite Mordell-Weil group $MW(X)$ of sections and fulfilling the identity $rank\, NS(X) = h^{1,1}(X)$ is said to be **extremal**. Particular examples arise in the following way. To each pair of positive integers

$$(m, n) \notin \{(1,1), (1,2), (2,2), (1,3), (1,4), (2,4)\}$$

there exists a modular elliptic surface over $\overline{\mathbb{Q}}$ in the sense of Shioda [10]

$$\pi_n(m) : X_n(m) \longrightarrow C_n(m)$$

such that $\pi_n(m)$ admits no multiple fibers and has a non-constant j-invariant. By [10], $X_n(m)$ is an extremal elliptic surface with the following properties.

- $MW(X_n(m)) = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

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- $C_n(m)$ is the (compactified) curve $\overline{\Gamma_m(n) \setminus \mathbb{H}}$ where $\Gamma_n(m) \subset \mathbf{Sl}_2(\mathbb{Z})$ is the group
$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} mod\, m, b \equiv 0\, mod\, n \right\}.$$
- $C_n(m)$ parametrizes triples $\big((E, e_E), x, y\big)$ of elliptic curves E with neutral element $e_E \in E(\mathbb{C})$ and elements $x \in E[m], y \in E[n]$ such that $|\mathbb{Z}x + \mathbb{Z}y| = mn$.
- All singular fibers of $\pi_n(M)$ are of type I_k in Kodaira's notation; they lie over the cusps of $c \in C_n(m)$. A representant of c in $\mathbb{Q} \cup \{\infty\}$ is stabilized by a matrix $\gamma \in \Gamma$ which is a $\mathbf{Sl}_2(\mathbb{Z})$-conjugate of
$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Here, x and y arise from to the intersection of E with generators of $MW(X_n(m))$. More generally, by [8, Thm. 1.2, Thm. 1.3] each extremal elliptic surface $\pi : X \longrightarrow C$ with non-constant j-invariant, no multiple fibers and $MW(\tilde{X}) = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, (m, n) as above, allows a cartesian diagram (where v is an isogeny)

$$\begin{array}{ccc} X & \xrightarrow{\pi} & C \\ \downarrow & & \downarrow v \\ X_n(m) & \xrightarrow{\pi_n(m)} & C_n(m) \end{array}$$

With this perspective we are able to formulate our main result. We call a complex projective smooth surface X **irregular** if $h^1(X, \mathcal{O}_X) > 1$.

Theorem 1.1. *Let X be an irregular minimal surface in $\mathcal{T}$ with kod$(X) = 1$. If X has a finite Mordell-Weil group then X is an extremal elliptic surface fibered over an elliptic curve C such that:*

(1) *The j-invariant of π induces a cartesian diagram defined over $\overline{\mathbb{Q}}$*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & C \\ \downarrow & & \downarrow \\ X_6(6) & \xrightarrow{\pi_6(6)} & C_6(6) \end{array}$$

(2) *The compactification divisor D of X consists of 36 sections of π, each having self-intersection number $-\chi(X)$. The fibration π admits $2\chi(X)$ singular fibers of type I_6, and each component of an I_6 intersects D in precisely* 6 *points. We have rank $NS(X) = 10\chi(X) + 2$.*

Conversely, $X_6(6)$ is an extremal elliptic and irregular surface in $\mathcal{T}$.

2. Some basic properties of surfaces in $\mathcal{T}$

We cite two results on ball-quotient surfaces needed for the proof of the theorem. The first result is essentially [9, Thm. 3.1] specified to dim $X = 2$ with attention to sign conventions, except the assertion on semi-stability. The latter assertion follows from [5]. Thereby, a reduced effective divisor is called *semi-stable* if it has normal crossings and if every rational smooth prime component intersects the remaining components in more than one point.

Theorem 2.1 (Tian-Yau/Miyaoka-Sakai)**.** *Let X be a smooth projective surface and $D \subset X$ a divisor with normal crossings. Suppose that $K_X + D$ is big and ample modulo D. Then*

$$c_1^2(\Omega_X^1(log\, D)) \leq 3c_2(\Omega_X^1(log\, D)),$$

with equality holding if an only if $X \setminus D$ is an unramified ball quotient $\Gamma \setminus \mathbf{B}$ and D is semi-stable.

There is a canonical exact sequence

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(log\, D) \xrightarrow{res} \mathcal{O}_D \longrightarrow 0$$

where res is the Poincaré residue map. With this one proves that $c_1(\Omega_X^1(log\, D)) = [D] - c_1(X) \in H^2(X, \mathbb{C})$ and $c_2(\Omega_X^1(log\, D)) = c_2(X) - (c_1(X), [D]) + ([D], [D]) \in H^4(X, \mathbb{C})$. Therefore,

$$c_1^2(\Omega_X^1(log\, D)) = (K_X + D)^2.$$

In fact, it is interesting to note that if equality holds in the theorem then D is smooth. Namely, if $\Gamma' \subset \Gamma$ is a neat normal subgroup with finite index in Γ then $\Gamma' \setminus \mathbf{B}$ is compactified by a smooth elliptic divisor, and $\Gamma \setminus \mathbf{B}$ is compactified by a divisor D. As D is the quotient D'/G, $G = \Gamma/\Gamma'$, it is a normal curve. Hence, D is smooth and consists of elliptic curves, for rational curves cannot appear because of semi-stability. The next is proved *verbatim* as [7, Lemma 3.2].

Lemma 2.2. *Let X be in $\mathcal{T}$ with compactification divisor D and consider an irreducible curve $L \subset X$. If L is smooth rational then $|L \cap D| \geq 3$. If L is a smooth elliptic curve then $|L \cap D| \geq 1$.*

3. Proof of the results

General theory asserts that X admits an elliptic fibration $\pi : X \longrightarrow C$ which is the Albanese morphism. As $K_X + D$ is ample modulo D, it follows that a general fiber F has positive self-intersection with D. Thus, a component of D dominates C. Hence, C is an elliptic curve and $h^1(X, \mathcal{O}_X) = 1$. Moreover, after transition to an etale cover $\tilde{C}$ of C and performing a base change, we can achieve that every D_i is a section, as soon as it dominates C ([7, Lemma 3.3]). We will assume this for the time being. Since the curves D_i are pair-wise disjoint, in fact all must be sections.

Claim 3.1. *We have $36\chi(X) = DF \cdot \chi(X) = -D^2$ and $36 = DF$.*

Proof. According to the canonical bundle-formula we have $K_X = \pi^*(\mathfrak{c})$ for a divisor Weil divisor $\mathfrak{c} \in Div(C)$ and $h^0(X, mK_X) = h^0(C, m\mathfrak{c})$. Riemann-Roch on C yields $h^0(X, K_X) = \deg\, \mathfrak{c} > 0$. Adjunction formula implies that

$$D_i^2 = -\deg\, \mathfrak{c} = -h^0(X, K_X) = -\chi(X).$$

Hence, $-D^2 = -\sum D_i^2 = DF\chi(X)$. Furthermore, $12\chi(X) = c_2(X)$ by Noether's formula. So, Thm. 2.1 yields the remaining identities. □

We consider the Mordell-Weil group $MW(X) = MW_{tor}(X)$. It follows that $|MW_{tor}(X)| \geq 36$. We prove the following lemma of general interest.

Lemma 3.2. *Let $\pi : X \longrightarrow C$ be a minimal elliptic surface over an elliptic curve C and assume that $kod(X) \geq 1$ and that each rational curve $L \subset X$ meets at least three sections of π. Suppose moreover that $D = MW_{tor}(X) \geq 33$. Then all singular fibers of π are semi-stable of type I_6, X has $2\chi(X)$ singular fibres and $MW(X) = MW_{tor}(X) = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and the rank of the Neron severi group $NS(X)$ equals $h^{1,1}(X) = 10\chi(X) + 2$.*

Proof. The assertion concerning $MW(X)$ follows directly from [4, (4.8)] (in fact, it is sufficient to assume '≥ 33'). [4, Lemma 1.1] implies then that all singular fibers are of type I_n. If $H_n \subset M(X)$ is the non-trivial isotropy group of a node $x \in I_n$ then $MW_{tor}(X)/H_n$ is cyclic by [4, Lemma 2.2]. Moreover, all nodes from one and the same fiber admit the same isotropy group by [4, Lemma 2.1, (c)], and this isotropy group is non-trivial by [4, Lemma 2.1, (b)] and because a component of I_n meets at least three sections. Thus, always $|H_n| \geq 6$. On the other hand, by [4, p. 251] and [4, Lemma 2.3, (f)], $\sum_{I_n} n = c_2(X)$ and

$$36c_2(X) = |MW_{tor}(X)|c_2(X) = \sum_{I_n} n|H_n|^2.$$

Hence, always $|H_n| = 6$. Let $S \in MW(X)$ be the neutral element. By the proof of [4, Lemma 2.2], H_n consists of precisely those sections meeting the prime component $L \subset I_n$ which contains $S \cap I_n$. However, since we may take any section to be the neutral element of $MW(X)$, for each component $L \subset I_n$ we have $LD = 6$. As $DI_n = 36$, we get $n = 6$. Finally, recalling that $\sum_{I_n} n = c_2(X)$, we find for the number t of singular fibers:

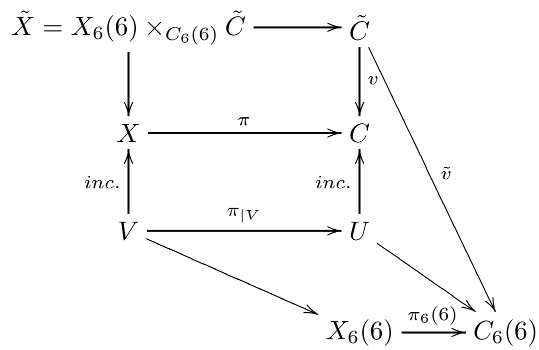
$$t = 2\chi(X) = 2g(C) - 2 + rank\, MW(X) + 2\chi(X).$$

According to [4, Prop. 1.6] this happens precisely when $rank\, NS(X) = h^{1,1}(X)$, and an easy calculation shows that $h^{1,1}(X) = 10\chi(X) + 2$. □

It follows that X is isomorphic to a pull-back $X_6(6) \times_{C_6(6)} C$. However, we remember that in the beginning of the proof we assumed that all curves D_i dominating C are sections. *A priori*, this additional assumption holds only after performing an etale base change. In the final part of the proof we are going to withdraw the additional assumption:
Assume that $\tilde{X} = X_6(6) \times_{C_6(6)} \tilde{C}$ arises from X by a non-trivial base change $v : \tilde{C} \longrightarrow C$. Let $\tilde{D}_i, \tilde{D}_j \in MW(\tilde{X})$ be two generators of $MW(\tilde{X})$ and view $\tilde{X}$ as a parameter space of level structures

$$\tilde{\mathfrak{F}} = \big((\tilde{F}, e_{\tilde{F}} = \tilde{D}_1 \cap \tilde{F}), a_{\tilde{F}} = \tilde{D}_i \cap \tilde{F}, b_{\tilde{F}} = \tilde{D}_j \cap \tilde{F}\big).$$

Choose a smooth fiber $F \subset X$ and let $e_F \subset F \cap D$ be a point in the image of the neutral element $\tilde{D}_1 \cap \tilde{F} \in MW(\tilde{F})$ for some smooth fiber $\tilde{F}$ with neutral element $e_{\tilde{F}} = \tilde{D}_1 \cap \tilde{F}$. Let $x_F, y_F \in F$ be the images of $\tilde{D}_i \cap \tilde{F}$ and $\tilde{D}_j \cap \tilde{F}$ respectively. Consider the unique group structure on F with neutral element e_F. Then $F \cap D = F[6]$ and x_F, y_F generate $F[6]$. Let $U \subset C$ be a connected open neighborhood of $\pi(F)$ with local sections $\sigma, \sigma_1, \sigma_2 : U \longrightarrow D \subset X$ such that $e_F \in \sigma(U), x_F \in \sigma_1(U), y_F \in \sigma_2(U)$. For each $u \in U$ lying over a smooth fiber F_u, we consider the elliptic curve F_u with neutral element $e_{F_u} = \sigma(u)$, so that again $F_u \cap D = F[6]$ with generators $\sigma_1(u), \sigma_2(u)$. Write $V = \pi^{-1}(U)$. We receive an unique commutative modular diagram



We view V as a parameter space of level structures $\mathfrak{F} = \big((F, e_F), x_F, y_F\big)$. Over $\mathfrak{F}$ there lie level structures $\tilde{\mathfrak{F}}$ on $\tilde{X}$ which are easily seen to be isomorphic to $\mathfrak{F}$. By the universal modular property of $X_6(6)$ and the modularity of $\tilde{v}$, all level structures $\tilde{\mathfrak{F}}$ on $\tilde{X}$, which are isomorphic to $\mathfrak{F}$, are mapped to one and the same level structure on $X_6(6)$. It follows that $\tilde{v}$ factors through v. This means that π results from a pull-back of $\pi_6(6)$. As explained in the introduction, π and fulfills (1) and (2) in Thm. 1.1. Conversely, it is known that $X_6(6)$ is fibred over an elliptic curve. It is then clear from the above that $X_6(6)$ satisfies the equality in Thm. 2.1. Thm. 1.1 follows.

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