Small deviations for a family of smooth Gaussian processes

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Abstract

We study the small deviation probabilities of a family of very smooth self-similar Gaussian processes. The canonical process from the family has the same scaling properties as standard Brownian motion and plays an important role in the study of zeros of random polynomials.

Our estimates are based on several approaches. The precise result for the L_2-norm is obtained by an appropriate modification of the entropy method, discovered in Kuelbs and Li (1992) and developed further in Li and Linde (1999), Gao (2004), and Aurzada et al. (2009), as well as on classical results about the entropy of classes of operators. In the sup-norm case we use a combination of several methods and obtain also the correct rate. The upper bound can be derived by purely probabilistic arguments, while for the lower bound we again apply the entropy connection.

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1 Introduction

The small deviation problem for a stochastic process $X = (X(t))_{t\geq 0}$ – also called small ball or small value problem – consists in determining the probability

$$-\log \mathbb{P}\left(||X|| \leq \varepsilon\right), \qquad \text{as } \varepsilon \to 0,$$

where $||.||$ is for example the norm in some $L_p[0,1]$ or in $C[0,1]$. Small deviation probabilities play a fundamental role in many problems in probability and analysis, see the lecture notes [17] for details. This is why there has been a lot of interest in small deviation problems in recent years, cf. the survey [19] and the literature compilation [21]. There are many connections to other questions such as the law of the iterated logarithm of Chung type, strong limit laws in statistics, metric entropy properties of linear operators, quantization, and several other approximation quantities for stochastic processes. For Gaussian processes, a reasonable amount of theory has been developed up to date, see e.g. [19].

The aim of this paper is to study a family of very smooth self-similar Gaussian processes and their respective small deviation probabilities. The canonical process from the family has the same scaling properties as Brownian motion and plays an important role in the study of zeros of random polynomials, see Li and Shao [20]. The question of its small deviation rate was posed at the AIM Workshop "Small ball inequalities in analysis, probability, and irregularities of distribution" (Palo Alto, December 2008). During the workshop, a couple of methods for obtaining small deviation estimates were discussed. It is the purpose of this paper to use the mentioned family to illustrate some different methods.

To be more precise, we focus on the family of centered Gaussian processes $X_{\alpha,\beta}(t)$, $t > 0$, defined by the covariance function

$$K(t,s) = \mathbb{E}\, X_{\alpha,\beta}(t) X_{\alpha,\beta}(s) = \frac{2^{2\beta+1}(ts)^{\alpha}}{(t+s)^{2\beta+1}} \tag{1.1}$$

for $\alpha > 0$ and $\beta > -1/2$. Note that for $\alpha > \beta + 1/2$, we can define $X_{\alpha,\beta}(0) = 0$. It is also easy to check that $X_{\alpha,\beta}$ is an $(\alpha - \beta - 1/2)$-self-similar process, i.e. $(X_{\alpha,\beta}(ct))$ has the same law as $(c^{\alpha-\beta-1/2} X_{\alpha,\beta}(t))$. In particular, $X_{\alpha,\beta}$ has the same scaling property as Brownian motion for $\alpha - \beta = 1$. A useful stochastic integral representation is

$$X_{\alpha,\beta}(t) = \sqrt{\frac{2^{2\beta+1}}{\Gamma(2\beta+1)}}\, t^{\alpha} \int_0^{\infty} x^{\beta} e^{-xt}\, \mathrm{d}B(x), \qquad t > 0, \tag{1.2}$$

where B is a standard Brownian motion.

If $\beta = 0$, it is easy to see, using integration by parts, that $X_{\alpha,0}$ has the same law as the process

$$\tilde{X}_{\alpha,0}(t) := \sqrt{2}\, t^{1+\alpha} \int_0^{\infty} e^{-xt} B(x)\, \mathrm{d}x, \qquad t > 0.$$

We also introduce, for $\alpha = 1$, $\beta = 0$, the canonical process

$$X(t) := X_{1,0}(t) = \sqrt{2}\, t \int_0^{\infty} e^{-xt}\, \mathrm{d}B(x), \qquad t \geq 0\,.$$

This paper is structured as follows. In Section 2, an approach for obtaining the small deviation probabilities via the metric entropy of a related linear operator is discussed. We formulate the main results in Section 2.1. In Section 3, we prove further properties of the family of processes defined above and use them to discuss an upper bound for the small deviation rate via purely probabilistic arguments. Section 4 is devoted to the determinant method, which was introduced at the mentioned AIM Workshop for the first time. Finally, a relation to the metric entropy of certain function classes is described and utilized in Section 5.

Let us fix some notation. We write $f \preceq g$ or $g \succeq f$ if $\limsup f/g < \infty$, and the asymptotic equivalence $f \asymp g$ means that we have both $f \preceq g$ and $g \preceq f$. Moreover, we write $f \lesssim g$ or $g \gtrsim f$, if $\limsup f/g \leq 1$. Finally, the strong equivalence $f \sim g$ means that $\lim f/g = 1$.

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2 Small deviations under L_2- and sup-norm

2.1 Results

We can clarify the small deviation order for the whole class of smooth processes introduced above for the L_2-norm and the sup-norm.

For the L_2-norm, we obtain the following result.

Theorem 1 *Let $\alpha > \beta > -1/2$. Let $X_{\alpha,\beta}$ be the process defined by (1.1). Then*

$$-\log \mathbb{P}\left(\int_0^1 |X_{\alpha,\beta}(t)|^2 \, \mathrm{d}t \leq \varepsilon^2\right) \sim \kappa_{\alpha,\beta} |\log \varepsilon|^3,$$

where the constant is given by

$$\kappa_{\alpha,\beta} := \frac{1}{3(\alpha-\beta)\pi^2}. \tag{2.3}$$

For the sup-norm, we obtain the following theorem under optimal assumptions on the parameters α and β. However, the result is less precise with respect to the small deviation constant compared to the L_2-norm result.

Theorem 2 *Let $\alpha > \beta + 1/2 > 0$. Let $X_{\alpha,\beta}$ be the process defined by (1.1) with $X_{\alpha,\beta}(0) = 0$. Then, with some constant $\tilde{\kappa}_{\alpha,\beta} > 0$, we have*

$$\tilde{\kappa}_{\alpha,\beta} |\log \varepsilon|^3 \lesssim -\log \mathbb{P}\left(\sup_{t\in[0,1]} |X_{\alpha,\beta}(t)| \leq \varepsilon\right) \lesssim \kappa_{\alpha-1/2,\beta} |\log \varepsilon|^3, \tag{2.4}$$

where $\kappa_{\alpha,\beta}$ was defined in (2.3), and $\tilde{\kappa}_{\alpha,\beta} \to \infty$ when $\alpha - \beta \to 1/2$.

The proof of these two theorems uses the connection between small deviations of Gaussian processes and the entropy numbers of a linear operator generating the process, cf. [14], [18], [2]. In fact, due to Corollaries 2.2 and 2.4 in [2], Theorem 1 and Theorem 2 are equivalent to Theorem 3 and Theorem 5, respectively, given in the next sections. Other interesting small deviation estimates for smooth Gaussian processes can be found in [2] and [15].

In order to state and use the connection to the entropy numbers, let us first define the entropy numbers. For a linear operator $u : E \to F$ between Banach spaces E and F and $n \in \mathbb{N}$ the entropy numbers are defined as follows

$$e_n(u : E \to F) := \inf \left\{ \varepsilon > 0 \, : \, \exists f_1, \ldots, f_{2^{n-1}} \in F \text{ s.t. } u(B_E) \subseteq \bigcup_{k=1}^{2^{n-1}} (f_k + B_F) \right\} ,$$

where B_E and B_F denote the closed unit balls in E and F, respectively. For elementary properties and further information see e.g. [6]. Since u is compact if and only if $\lim_{n\to\infty} e_n(u) = 0$, the decay rate of the entropy numbers is a measure for the "degree of compactness" of u.

It turns out that there is a close relation between the small deviation problem for a Gaussian process X attaining values in E and the entropy numbers of a compact operator $u : L_2(S) \to E$ related to X via

$$\mathbb{E}\, e^{i\langle X,h\rangle} = \exp\left(-\frac{1}{2} \, \|u'(h)\|^2_{L_2(S)} \right), \qquad h \in E', \tag{2.5}$$

where $u' : E' \to L_2(S)$ is the dual operator and $(S, \mathcal{S}, \lambda)$ is some measure space. Here, $(E, \|.\|)$ is some Banach space.

It can be checked easily that, up to an unimportant multiplicative constant, the process $X_{\alpha,\beta}$ defined in (1.1) is related – via (2.5) – to the operator

$$(uf)(t) = t^\alpha \int_0^\infty x^\beta e^{-xt} f(x) \, \mathrm{d}x, \qquad f \in L_2[0, \infty). \tag{2.6}$$

2.2 Entropy arguments in the L_2-case

Note that the process defined in (1.1) can be considered with values in the Banach spaces $E = L_2[0, 1]$ for $\alpha > \beta > -1/2$. In this L_2-setting, we obtain the precise behavior of the entropy numbers of the operator related to our process on the exponential scale.

Theorem 3 *Let $\alpha > \beta > -1/2$. Let $u : L_2[0, \infty) \to L_2[0, 1]$ be the operator given by (2.6). Then*

$$-\log e_n(u) \sim d_{\alpha,\beta} \, n^{1/3},$$

where

$$d_{\alpha,\beta} := (3(\alpha - \beta)\pi^2 \log 2)^{1/3}. \tag{2.7}$$

Proof. First note that the operator $uu^* : L_2[0, 1] \to L_2[0, 1]$, where u^* denotes the adjoint operator of u, is given by

$$(uu^*g)(t) = \Gamma(2\beta + 1) \, t^\alpha \int_0^1 \frac{x^\alpha}{(t + x)^{2\beta+1}} \, g(x) \, \mathrm{d}x \, .$$

The exact asymptotic behavior of its singular numbers $s_n(uu^*)$ was found by Laptev [16]. He showed that

$$-\log s_n(uu^*) \sim 2\pi(\alpha-\beta)^{1/2} n^{1/2}\,.$$

Similar operators were studied in [23], [12], and [3]. Since $s_n(uu^*) = s_n(u)^2$, this implies

$$-\log s_n(u) \sim \pi(\alpha-\beta)^{1/2} n^{1/2}. \tag{2.8}$$

Since we are in the Hilbert space setting, we can return to the entropy numbers using a result of Gordon, König, and Schütt (Proposition 1.7 in [11]), which states that the entropy numbers of a diagonal operator D_σ with weight sequence (σ_n) can be estimated as follows:

$$\sup_{n\geq 1}\left(2^{-k/n}(\sigma_1\cdots\sigma_n)^{1/n}\right) \leq e_{k+1}(D_\sigma) \leq 6\sup_{n\geq 1}\left(2^{-k/n}(\sigma_1\cdots\sigma_n)^{1/n}\right).$$

In our case,

$$\log e_k(u) \sim \sup_{n\geq 1}\left(-\frac{k}{n}\log 2 + \frac{1}{n}\sum_{i=1}^{n}\log s_i(u)\right).$$

Using the asymptotics of $(s_n(u))$, we get

$$-\log e_k(u) \sim \inf_{n\geq 1}\left(\frac{k}{n}\log 2 + \frac{\pi(\alpha-\beta)^{1/2}}{n}\sum_{i=1}^{n} i^{1/2}\right).$$

Therefore, integrating we obtain

$$-\log e_k(u) \sim \inf_{n\geq 1}\left(\frac{k}{n}\log 2 + \frac{\pi(\alpha-\beta)^{1/2}}{n}\cdot\frac{2}{3}\,n^{3/2}\right).$$

The optimal n is of order

$$n \sim \left(\frac{3\log 2\cdot k}{\pi(\alpha-\beta)^{1/2}}\right)^{2/3}.$$

Inserting this into the formula, we obtain the assertion

$$-\log e_k(u) \sim (3(\alpha-\beta)\pi^2\log 2\cdot k)^{1/3}\,.$$

□

Remark: Since we deal with L_2-norms, one can use e.g. Theorem 4.1 in [5] to derive Theorem 1 directly from (2.8). In the same vein, we can use the well known Karhunen-Loeve expansion and the asymptotics of the associated eigenvalues based on Widom [23] to prove Theorem 1. In fact, the work of Laptev [16] on singular numbers $s_n(uu^*)$ is based on [23].

2.3 Entropy arguments in the sup-norm case

Before we state our entropy result in the sup-norm case for the concrete operators we are interested in, we give a general theorem which might be of independent interest. Roughly speaking, it provides a technique for deriving upper entropy estimates for an operator u from a Banach space H into $C[0,1]$ based on entropy estimates of $u : H \to L_2[0,1]$, i.e. the same operator, but considered as operator into a larger target space. The additional information we need for this argument is that u should map H even into a smaller space than $C[0,1]$, namely into a Hölder space $C_\lambda[0,1]$ for some $0 < \lambda \le 1$. This space consists of all functions $f \in C[0,1]$ such

$$\|f\|_{C_\lambda[0,1]} := \sup_{0\le s<t\le 1} \frac{|f(t)-f(s)|}{|t-s|^\lambda} + \sup_{0\le t\le 1} |f(t)| < \infty\,.$$

Moreover, $C_\lambda[0,1]$ is a Banach space under the norm $\|.\|_{C_\lambda[0,1]}$.

Theorem 4 *Let $0 < \lambda \le 1$, let H be a Banach space and $u : H \to C_\lambda[0,1]$ be an operator of norm $A_\lambda > 0$. Then we have for large enough $k \in \mathbb{N}$*

$$e_k(u : H \to C[0,1]) \le (1 + A_\lambda)\, e_k(u : H \to L_2[0,1])^{\frac{\lambda}{\lambda+1/2}}\,.$$

Proof. Clearly, since $C_\lambda[0,1]$ is compactly embedded in $L_2[0,1]$, the operator $u : H \to L_2[0,1]$ is compact, whence its entropy numbers tend to zero.

For $0 < \delta \le 1$ and $t \in [0,1]$ we consider the interval $I_\delta(t) := [0,1] \cap [t-\delta, t+\delta]$ and define for $g \in L_2[0,1]$, the local averaging operator

$$(P_\delta g)(t) := \frac{1}{|I_\delta(t)|} \int_{I_\delta(t)} g(x)\ \mathrm{d}x\,.$$

One can easily verify that the averaging operators P_δ map $L_2[0,1]$ in $C[0,1]$.

Step 1: We need the following simple norm estimates:

$$\|P_\delta : L_2[0,1] \to C[0,1]\| \le \delta^{-1/2}; \tag{2.9}$$

$$\|\mathrm{id} - P_\delta : C_\lambda[0,1] \to C[0,1]\| \le \delta^\lambda \tag{2.10}$$

For all $t \in [0,1]$ and $g \in L_2[0,1]$ we have, by the Cauchy-Schwarz inequality,

$$|(P_\delta g)(t)| \le \frac{1}{|I_\delta(t)|} \int_{I_\delta(t)} |g(x)|\ \mathrm{d}x \le \frac{1}{|I_\delta(t)|}\, |I_\delta(t)|^{1/2} \|g\|_{L_2[0,1]} \le \delta^{-1/2} \|g\|_{L_2[0,1]}\,,$$

which implies (2.9).

Now let $t \in [0,1]$ and $g \in C_\lambda[0,1]$ with $\|g\|_{C_\lambda[0,1]} \le 1$. Then

$$|g(t) - P_\delta g(t)| = \frac{1}{|I_\delta(t)|} \left| \int_{I_\delta(t)} (g(t) - g(x))\ \mathrm{d}x \right| \le \frac{1}{|I_\delta(t)|} \int_{I_\delta(t)} |g(t) - g(x)|\, dx \le \delta^\lambda\,,$$

since $|g(t) - g(x)| \le |t-x|^\lambda \|g\|_{C_\lambda[0,1]} \le \delta^\lambda$ for every $x \in I_\delta(t)$ and $\frac{1}{|I_\delta(t)|}\int_{I_\delta(t)}\ \mathrm{d}x = 1$. To finish the proof of (2.10), we take the supremum over all $t \in [0,1]$ and $g \in C_\lambda[0,1]$ with $\|g\|_{C_\lambda[0,1]} \le 1$.

Step 2: Using elementary properties of entropy numbers (cf. [6]) and the above norm estimates (2.9) and (2.10), we obtain now for all $k \in \mathbb{N}$ and $0 < \delta < 1$

$$\begin{aligned} e_k(u : H \to C[0,1]) &\leq e_k(P_\delta u : H \to C[0,1]) + \| u - P_\delta u : H \to C[0,1] \| \\ &\leq e_k(u : H \to L_2[0,1]) \cdot \| P_\delta : L_2[0,1] \to C[0,1] \| \\ &\quad + \| u : H \to C_\lambda[0,1] \| \cdot \| \mathrm{id} - P_\delta : C_\lambda[0,1] \to C[0,1] \| \\ &\leq e_k(u : H \to L_2[0,1]) \cdot \delta^{-1/2} + A_\lambda \cdot \delta^\lambda . \end{aligned}$$

Finally, for k large enough, we choose the parameter δ such that

$$e_k(u : H \to L_2[0,1]) \cdot \delta^{-1/2} = \delta^\lambda ,$$

i.e. $\delta = e_k(u : H \to L_2[0,1])^{\frac{1}{\lambda+1/2}}$. With this choice of δ we obtain the desired inequality

$$e_k(u : H \to C[0,1]) \leq (1 + A_\lambda)\delta^\lambda = (1 + A_\lambda)\, e_k(u : H \to L_2[0,1])^{\frac{\lambda}{\lambda+1/2}} .$$

□

Now we apply the general Theorem 4 to the special operators we are interested in. In the sup-norm case we get the following result under the optimal assumption on the parameters $\alpha - \beta > 1/2$, which is when the process defined in (1.1) is almost surely in $E = C[0,1]$.

Theorem 5 *Let $\alpha > \beta + 1/2 > 0$. Let $u : L_2[0,\infty) \to C[0,1]$ be the operator given by (2.6) Then*

$$\tilde{d}_{\alpha,\beta}\, n^{1/3} \lesssim -\log e_n(u) \lesssim d_{\alpha-1/2,\beta}\, n^{1/3},$$

where $d_{\alpha,\beta}$ is defined in (2.7) and

$$\tilde{d}_{\alpha,\beta} := \frac{\min(\alpha - \beta - 1/2, 1/2)}{1/2 + \min(\alpha - \beta - 1/2, 1/2)} \cdot d_{\alpha,\beta}.$$

Note, in particular, that if $\alpha - \beta \to 1/2$, then

$$\tilde{d}_{\alpha,\beta} \to 0 \qquad \text{and} \qquad d_{\alpha-1/2,\beta} \to 0.$$

Proof of the upper bound for the entropy numbers in Theorem 5. *Step 1:* First we show that u maps $L_2[0,\infty)$ into the Hölder space $C_\lambda[0,1]$, where $\lambda := \min(\alpha-\beta-1/2\,,\,1/2)$, i.e. we show $\| u : L_2[0,\infty) \to C_\lambda[0,1] \| =: A_\lambda < \infty$. Let $f \in L_2[0,\infty)$, let $0 \leq s < t \leq 1$, and set $h := t - s$. We consider

$$|(uf)(t) - (uf)(s)| \leq \int_0^\infty |t^\alpha x^\beta e^{-xt} - s^\alpha x^\beta e^{-xs}| \cdot |f(x)| \,\mathrm{d}x.$$

Using the Cauchy-Schwartz inequality, this can be estimated by $A^{1/2} \, \|f\|_{L_2[0,\infty)}$, with

$$A := \int_0^\infty |t^\alpha x^\beta e^{-xt} - s^\alpha x^\beta e^{-xs}|^2 \,\mathrm{d}x.$$

We have to show that $A \le Ch^{2\lambda}$ for some constant C independent of t, s, since then one can take the supremum over all $0 \le s < t \le 1$ and finally over all f.

In order to see that $A \le Ch^{2\lambda}$, define

$$g(y) := \int_0^\infty x^{2\beta} e^{-2yx}\,\mathrm{d}x = y^{-2\beta-1} 2^{-2\beta-1} \Gamma(2\beta+1)$$

and note that

$$\begin{aligned} A &= \int_0^\infty x^{2\beta}(t^{2\alpha}e^{-2xt} - 2(ts)^\alpha e^{-x(t+s)} + s^{2\alpha}e^{-2xs})\,\mathrm{d}x \\ &= t^{2\alpha} g(t) - 2(ts)^\alpha g((t+s)/2) + s^{2\alpha} g(s)\,. \end{aligned}$$

Therefore, using the notation $\gamma := \alpha - \beta - 1/2$, we have

$$\begin{aligned} \frac{2^{2\beta+1}A}{\Gamma(2\beta+1)} &= t^{2\alpha-2\beta-1} - 2(ts)^\alpha \left(\frac{t+s}{2}\right)^{-2\beta-1} + s^{2\alpha-2\beta-1} \\ &= \left[t^{2\gamma} - 2\left(\frac{t+s}{2}\right)^{2\gamma} + s^{2\gamma}\right] + 2\left[\left(\frac{t+s}{2}\right)^{2\gamma} - (ts)^\alpha \left(\frac{t+s}{2}\right)^{-2\beta-1}\right]. \end{aligned} \tag{2.11}$$

If $\gamma < 1/2$, the first term is negative by the concavity of the function $x \mapsto x^{2\gamma}$; if $\gamma \ge 1/2$, the first term is no larger than γh. In either case, we have

$$\frac{2^{2\beta}A}{\Gamma(2\beta+1)} \le \frac{\gamma h}{2} + \left(\frac{t+s}{2}\right)^{-2\beta-1}\left[\left(\frac{t+s}{2}\right)^{2\alpha} - (ts)^\alpha\right].$$

Set $a := st$ and $b := \left(\frac{t+s}{2}\right)^2$ and note that $a \le b$. By the mean value theorem, the last expression becomes, with some $\xi \in (a, b)$,

$$\frac{2^{2\beta}A}{\Gamma(2\beta+1)} \le \frac{\gamma h}{2} + \left(\frac{t+s}{2}\right)^{-2\beta-1}\left[(b-a)\alpha\xi^{\alpha-1}\right]. \tag{2.12}$$

Now we have all crucial estimates to conclude:

Case 1: $h \ge s$. In this case we have $t = s + h \le 2h$ and $s \le h$, and therefore (2.11) implies

$$\frac{2^{2\beta+1}A}{\Gamma(2\beta+1)} \le t^{2\gamma} + s^{2\gamma} \le (2h)^{2\gamma} + h^{2\gamma} = Ch^{2\gamma}.$$

Case 2: $h \le s$. In this case we have $s < t = s + h \le 2s$, implying $s/2 \le (t+s)/2 \le 3s/2$. Therefore $\xi^{\alpha-1} \le Cs^{2(\alpha-1)}$, independently on whether $\alpha \ge 1$ or $\alpha \le 1$. Further, $b - a = \frac{1}{4}(t^2 + 2ts + s^2 - 4ts) = (t-s)^2/4 = h^2/4$. Thus, estimating in (2.12), we get

$$\frac{2^{2\beta}A}{\Gamma(2\beta+1)} \le \frac{\gamma h}{2} + Cs^{-2\beta-1}h^2 s^{2(\alpha-1)} = \frac{\gamma h}{2} + C\left(\frac{h}{s}\right)^{2-2\gamma} h^{2\gamma} \le C' h^{\min(2\gamma,1)},$$

as required.

Step 2: Let $\varepsilon > 0$. Then, by Theorem 3, for large enough k,

$$e_k(u : L_2[0,\infty) \to L_2[0,1]) \le e^{-(d_{\alpha,\beta}-\varepsilon)k^{1/3}}.$$

Now Theorem 4 implies that

$$e_k(u : L_2[0,\infty) \to C[0,1]) \le (1 + A_\lambda) \exp\left(-\frac{\lambda}{\lambda+1/2}(d_{\alpha,\beta}-\varepsilon)k^{1/3}\right),$$

and consequently

$$-\log e_k(u : L_2[0,\infty) \to C[0,1]) \gtrsim \frac{\lambda}{\lambda+1/2}(d_{\alpha,\beta}-\varepsilon)k^{1/3}.$$

In other words, we have

$$\liminf_{k\to\infty} \frac{-\log e_k(u : L_2[0,\infty) \to C[0,1])}{k^{1/3}} \ge \frac{\lambda}{\lambda+1/2}(d_{\alpha,\beta}-\varepsilon).$$

Letting $\varepsilon \to 0$, we arrive at the desired upper bound for the entropy numbers. □

Proof of the lower bound for the entropy numbers in Theorem 5. Obviously, $e_k(u : L_2[0,\infty) \to C[0,1]) \ge e_k(u : L_2[0,\infty) \to L_2[0,1])$. However, we can even gain a bit concerning the constant. For this purpose, let us stress the dependence on α and β in the definition (2.6) by denoting the operator $u_{\alpha,\beta}$. Further, for some fixed $\varepsilon > 0$, we let $v : C[0,1] \to L_2[0,1]$ denote the multiplication operator $(vf)(t) = t^{-1/2+\varepsilon} f(t)$. Note that $v : C[0,1] \to L_2[0,1]$ is bounded. Then one can observe that $vu_{\alpha,\beta} = u_{\alpha-1/2+\varepsilon}$. Therefore,

$$e_k(u_{\alpha-1/2+\varepsilon} : L_2[0,\infty) \to L_2[0,1]) \le e_k(u_{\alpha,\beta} : L_2[0,\infty) \to C[0,1]) \cdot \|v : C[0,1] \to L_2[0,1]\|.$$

Using the L_2 estimate from Theorem 3 for the left-hand side, this shows

$$-\log e_k(u_{\alpha,\beta} : L_2[0,\infty) \to C[0,1]) \lesssim d_{\alpha-1/2+\varepsilon,\beta} k^{1/3},$$

or in other words,

$$\limsup_{k\to\infty} \frac{-\log e_k(u_{\alpha,\beta} : L_2[0,\infty) \to C[0,1])}{k^{1/3}} \le d_{\alpha-1/2+\varepsilon,\beta}.$$

This holds for all $\varepsilon > 0$. Letting ε tend to zero yields the lower bound for the entropy numbers, since the constant $d_{\alpha,\beta}$ (defined in (2.7)) is continuous in the parameters. □

3 Properties of the processes and probabilistic upper bound

In this section, we prove several properties of the processes $X_{\alpha,\beta}$. These properties are used later on and they can also be of independent interest. Additionally, we show how they provide a probabilistic argument for the upper bound of the small deviation probability under the sup-norm. The approach only seems to work for the canonical process $X(t)$ and is of independent interest. It is instructive to compare this with the standard Brownian motion, which has the same scaling property but completely different path behavior and thus small deviation probabilities.

First we prove the following proposition on some basic properties of the processes defined above.

Proposition 1 *Fix $t > 0$ and let $\widehat{X}$ be an independent copy of $X = X_{1,0}$. Then we have*

(a) Let $\gamma > \alpha - (2\beta + 1)$. Then, in distribution,

$$\{s^\gamma X_{\alpha,\beta}(1/s), s \geq 0\} = \{X_{2\beta+1+\gamma-\alpha,\beta}(s), s \geq 0\}.$$

In particular, the process $(s^\gamma X_{\alpha,\beta}(1/s))$ coincides in distribution with $X_{\alpha,\beta}$ if $\gamma = 2\alpha - 2\beta - 1$.

(b) In distribution,

$$\{X(s), s \geq 0\} = \left\{\frac{t-s}{t+s}X(s) + \frac{2s}{t+s}\widehat{X}(t), s \geq 0\right\}.$$

(c) In distribution,

$$\{X(t) - X(s), s \geq 0\} = \left\{\frac{t-s}{t+s}(X(s) + \widehat{X}(t)), s \geq 0\right\}.$$

(d) The process

$$Y_t(s) = X(s) - \frac{2s}{s+t}X(t), \quad s \geq 0$$

is independent of $X(t)$.

(e) In distribution,

$$\left\{Y_t(s) = X(s) - \frac{2s}{s+t}X(t), s \geq 0\right\} = \left\{\frac{t-s}{t+s}X(s), s \geq 0\right\}.$$

Proof. To see (a), one just has to calculate the covariance structure: For $s, t \geq 0$,

$$\begin{aligned}\mathbb{E}\left[s^\gamma X_{\alpha,\beta}(1/s)t^\gamma X_{\alpha,\beta}(1/t)\right] &= (st)^\gamma \frac{2^{2\beta+1}(ts)^{-\alpha}}{(t^{-1}+s^{-1})^{2\beta+1}} = \frac{2^{2\beta+1}(ts)^{2\beta+1+\gamma-\alpha}}{(t+s)^{2\beta+1}} \\ &= \mathbb{E}\left[X_{2\beta+1+\gamma-\alpha,\beta}(s)X_{2\beta+1+\gamma-\alpha,\beta}(t)\right].\end{aligned}$$

The property in (b) can be proved in the same way. Now, all other properties easily follow from (b). $\square$

Now we use (d) and (e) from the above proposition to give an alternative proof of the upper bound of the small deviation probability in (2.4) for $\alpha = 1$ and $\beta = 0$.

Proof of the upper bound of the small deviation probability in (2.4). Let $0 < t_1 < t_2 < \cdots < t_n \le 1$. Then

$$\begin{aligned}
\mathbb{P}(\max_{1\le i\le n} |X(t_i)| \le \varepsilon) &= \mathbb{P}(|X(t_n)| \le \varepsilon, \max_{1\le i\le n-1} |X(t_i) - \frac{2t_i}{t_i+t_n}X(t_n) + \frac{2t_i}{t_i+t_n}X(t_n)| \le \varepsilon) \\
&\le \mathbb{P}(|X(t_n)| \le \varepsilon, \max_{1\le i\le n-1} |X(t_i) - \frac{2t_i}{t_i+t_n}X(t_n)| \le \varepsilon) \\
&= \mathbb{P}(|X(t_n)| \le \varepsilon) \cdot \mathbb{P}(\max_{1\le i\le n-1} |X(t_i) - \frac{2t_i}{t_i+t_n}X(t_n)| \le \varepsilon) \\
&= \mathbb{P}(|X(t_n)| \le \varepsilon) \cdot \mathbb{P}\left(|X(t_i)| \le \frac{t_n+t_i}{t_n-t_i}\varepsilon,\ 1 \le i \le n-1\right) \\
&\le \prod_{i=1}^{n} \mathbb{P}(|X(t_i)| \le a_i\varepsilon), \qquad (3.13)
\end{aligned}$$

where the first inequality follows from Anderson's inequality and (d) in Proposition 1, the third step is due to (d) in Proposition 1, and the last equality is by property (e) in Proposition 1 and the last inequality follows from iteration. Here $a_n := 1$ and

$$a_i := \prod_{j=i+1}^{n} \frac{t_j+t_i}{t_j-t_i},\ i = 1, ..., n-1.$$

Now let $t_i := (1+\delta)^i \varepsilon^{1/2}$ for $1 \le i \le n$, where $\delta := 24/|\log\varepsilon|$ and n is the integer part of $4^{-1}|\log\varepsilon|/\log(1+\delta)$. Then it is clear that $\varepsilon^{1/2} \le t_i \le t_n \le 2\varepsilon^{1/4}$ for small ε and that $n \sim |\log\varepsilon|^2/96$. Further, it follows from Lemma 1 below that

$$a_i = \prod_{j=i+1}^{n} \frac{(1+\delta)^{j-i}+1}{(1+\delta)^{j-i}-1} \le \prod_{i=1}^{\infty} \frac{(1+\delta)^{i}+1}{(1+\delta)^{i}-1} \le e^{6/\delta}, \qquad \text{for all } i = 1, \dots, n.$$

Therefore, by the self-similarity of X,

$$\mathbb{P}(|X(t_i)| \le a_i\varepsilon) \le \mathbb{P}(|Z| \le \varepsilon e^{6/\delta}/\sqrt{t_i}) \le \mathbb{P}(|Z| \le \varepsilon^{1/4}e^{6/\delta}) = \mathbb{P}(|Z| \le \varepsilon^{1/2})$$

where we used the relation $\delta = 24/|\log\varepsilon|$ and Z is the standard normal random variable. Together with the early estimate (3.13), we obtain

$$\begin{aligned}
\log\mathbb{P}(\max_{1\le i\le n} |X(t_i)| \le \varepsilon) &\le \sum_{i=1}^{n} \log\mathbb{P}(|X(t_i)| \le a_i\varepsilon) \\
&\le n\log\mathbb{P}(|Z| \le \varepsilon^{1/2}) \\
&\le -cn \cdot |\log\varepsilon| \le -c'|\log\varepsilon|^3
\end{aligned}$$

which implies the upper bound for the small deviation probability in (2.4).

□

Lemma 1 *For all $\delta > 0$ we have*

$$\prod_{i=1}^{\infty} \frac{(1+\delta)^i + 1}{(1+\delta)^i - 1} \leq \exp\left(\frac{\pi^2}{4\delta}\right).$$

Proof. We let $a := 1 + \delta > 1$ and observe that

$$\sum_{i=1}^{\infty} \log \frac{a^i + 1}{a^i - 1} = \sum_{i=1}^{\infty} \int_{-1}^{1} \frac{1}{a^i + z} \, \mathrm{d}z = \int_{-1}^{1} \sum_{i=1}^{\infty} \frac{1}{a^i + z} \, \mathrm{d}z. \tag{3.14}$$

Using twice the formula for the geometric sum $(1+x)^{-1} = \sum_{n=0}^{\infty} (-x)^n$, $|x| < 1$, we have

$$\sum_{i=1}^{\infty} \frac{1}{a^i + z} = \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{a^i} \cdot \left(\frac{-z}{a^i}\right)^n = \sum_{n=0}^{\infty} \frac{(-z)^n}{a^{n+1} - 1}.$$

Therefore, the expression in (3.14) equals

$$\int_{-1}^{1} \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1} - 1} z^n \, \mathrm{d}z = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1} - 1} \int_{-1}^{1} z^n \, \mathrm{d}z = \sum_{n=0}^{\infty} \frac{2}{(2n+1)(a^{2n+1} - 1)}.$$

Note that all changes of summations and integrations are easy to justify. Now, since $a > 1$ we have $\frac{a^{2n+1}-1}{a-1} = \sum_{j=0}^{2n} a^j \geq 2n + 1$. Thus,

$$\sum_{i=1}^{\infty} \log \frac{a^i + 1}{a^i - 1} = \sum_{n=0}^{\infty} \frac{2}{(2n+1)(a^{2n+1} - 1)} \leq \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2 (a-1)} = \frac{2}{a-1} \frac{\pi^2}{8},$$

which shows the assertion of the lemma. □

4 The determinant method

The main point of this section is to illustrate the method of determinants, which was first presented in the AIM workshop by the fourth author as a possible tool to attack the well known Brownian sheet problem, see AIM web page for more details. This method seems very promising for providing a general tool to obtain upper bounds for small deviation problems. Note also that the determinant method is used in [9] for a closely related problem. Here we only work out the estimate for $X(t) = X_{1,0}(t)$ since precise determinant evaluations is lacking for covariance matrix associated with $X_{\alpha,\beta}(t)$.

Proof of the upper bound of the small deviation probability in (2.4). We start with the time reversed Gaussian process (cf. Proposition 1 (b)):

$$Y(t) = X(1/t), \quad t > 0,$$

with covariance

$$\mathbb{E}\, Y(t)Y(s) = 2/(s+t).$$

We will prove the relation

$$\log \mathbb{P}(\sup_{0<t\leq 1} |X(t)| \leq \varepsilon) = \log \mathbb{P}(\sup_{t\geq 1} |Y(t)| \leq \varepsilon) \leq -\frac{1}{4}\,|\log \varepsilon|^3,$$

for ε small enough. Note that by scaling, $\mathbb{P}(\sup_{t>0} |X(t)| \leq \varepsilon) = 0$.

The determinant method starts with the following simple observation. For any sequence of numbers $(\delta_i)_{i=1}^n$ with $\delta_i \geq 1$, consider the covariance matrix

$$\Sigma = (\mathbb{E}\, Y(\delta_i)Y(\delta_j))_{1\leq i,j\leq n} = \left(\frac{2}{\delta_i+\delta_j}\right)_{1\leq i,j\leq n}.$$

Then one has

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\geq 1} |Y(t)| \leq \varepsilon\right) &\leq \mathbb{P}\left(\max_{1\leq i\leq n} |Y(\delta_i)| \leq \varepsilon\right) \\
&= (2\pi)^{-n/2}(\det \Sigma)^{-1/2} \int_{[-\varepsilon,+\varepsilon]^n} \exp\left(-\langle y, \Sigma^{-1} y\rangle\right)\, \mathrm{d}y \\
&\leq (2\pi)^{-n/2}(\det \Sigma)^{-1/2}(2\varepsilon)^n \leq \varepsilon^n (\det \Sigma)^{-1/2}.
\end{aligned} \tag{4.15}$$

Now the main difficulty is posed by the evaluation or estimation of the determinant and judicious choices of partition points. Using Cauchy's determinant identity, see e.g. [13], we know

$$\begin{aligned}
\det \Sigma &= 2^n \frac{\prod_{1\leq i<j\leq n}(\delta_j-\delta_i)^2}{\prod_{1\leq i,j\leq n}(\delta_j+\delta_i)} \\
&= \prod_{i=1}^n \delta_i^{-1} \cdot \prod_{1\leq i<j\leq n} \left(\frac{\delta_j-\delta_i}{\delta_j+\delta_i}\right)^2.
\end{aligned}$$

We will use a geometric sequence $\delta_j := a^j$, $j = 1,\ldots,n$, where $a > 1$ is chosen later (depending on n) and n is chosen later (depending on ε). We get that

$$\begin{aligned}
\det \Sigma &= a^{-\sum_{i=1}^n i} \cdot \left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{a^j-a^i}{a^j+a^i}\right)^2 \\
&= a^{-n(n+1)/2} \cdot \left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{a^{j-i}-1}{a^{j-i}+1}\right)^2 \\
&\geq a^{-n^2} \cdot \left(\prod_{i=1}^{n-1}\prod_{j=1}^{\infty} \frac{a^j-1}{a^j+1}\right)^2 \\
&\geq e^{-n^2 \log a} \cdot \exp\left(2(n-1)\,\frac{\pi^2}{4(a-1)}\right) \\
&\geq \exp\left(-n^2(a-1) + \frac{2n}{a-1}\right),
\end{aligned}$$

where we used Lemma 1 in the second inequality. We can optimize the choice of a by setting

$$a := 1 + 2/\sqrt{n},$$

which gives

$$\det \Sigma \geq e^{-n^{3/2}}.$$

By (4.15), this yields

$$\mathbb{P}\left(\sup_{t\geq 1} |Y(t)| \leq \varepsilon\right) \leq e^{-n|\log \varepsilon| + n^{3/2}/2}.$$

This estimate is optimized for n being the integer part of $|\log \varepsilon|^2$, which shows the upper bound of the probability in (2.4). $\square$

5 Relation to the entropy of function classes

In this section, we relate the small deviation problem for $X_{\alpha,\beta}$ under the sup-norm to another small deviation problem, which in turn is related to a metric entropy problem of a certain function class. The function class related to the canonical case $\alpha = 1$, $\beta = 0$ was studied in Theorem 1.2 in [9].

Let us define the process

$$S(t) := t^{\alpha'} \int_0^1 x^\beta e^{-xt} \,\mathrm{d}B(x), \qquad t \geq 1,$$

where $\alpha' := 2\beta + 1 - \alpha$ and the natural restrictions are $\alpha > \beta + 1/2 > 0$. Note that exactly under these restrictions, S is bounded on $[1, \infty]$. Our main theorem concerning S is as follows.

Theorem 6 *Let $\alpha > \beta + 1/2 > 0$. Then*

$$-\log \mathbb{P}\left(\sup_{t\geq 1} |S(t)| \leq \varepsilon\right) \asymp |\log \varepsilon|^3. \tag{5.16}$$

Using the technique in [9], one finds that the associated convex hull for the process $S(t), t \geq 1$ is the function class $\mathcal{F}$ consisting of all the functions f on $[0, 1]$ corresponding to the kernel $K(t, x) = t^{\alpha'} x^\beta e^{-tx}$. More precisely, $\mathcal{F}$ can be expressed as

$$\mathcal{F} := \left\{ f \;:\; f(x) = x^\beta \int_1^\infty t^{\alpha'} e^{-tx} \mu(\,\mathrm{d}t) \;:\; \|\mu\|_{TV} \leq 1 \right\}.$$

Under the $L_2[0, 1]$ norm $\|f\|_2 = (\int_0^1 f^2(x)dx)^{1/2}$, the class $\mathcal{F}$ is compact and its metric entropy is denoted by $\log N(\varepsilon, \mathcal{F}, \|\cdot\|_2)$ where $N(\varepsilon, \mathcal{F}, \|\cdot\|_2)$ is the minimum number of ε-radii balls in the norm $\|\cdot\|_2$ to cover the class $\mathcal{F}$. Thus as discussed in detail in [9] via the connection between the small ball probability and the metric entropy, we obtain the following statement for the function class $\mathcal{F}$ associated with S:

Corollary 1 *For the class $\mathcal{F}$ defined above, and $\alpha' = 2\beta + 1 - \alpha$ with $\alpha > \beta + 1/2 > 0$,*

$$\log N(\varepsilon, \mathcal{F}, \|\cdot\|_2) \asymp |\log \varepsilon|^3. \tag{5.17}$$

Our original proof of the lower bounds for the estimates of the probability in (5.16) and (2.4) follows from a covering estimates of the upper bound in (5.17) which is lengthy and unpleasant. The current approach for this part, which is turned around, is based on the simple and soft arguments summarized in Theorem 5. However, the argument that the upper bound of the metric entropy implies the lower bound of small ball probability, as discussed in Section 2, is the same, like many other instances we know before. The key point is that it seems easier to find an upper bound of the metric entropy via analytic tools than a lower bound of the small ball probability via probabilistic tools, even though they are equivalent. It would be interesting to find a probabilistic proof for the probability lower bound in (5.16) or (2.4) for all parameters in the range $\alpha > \beta + 1/2 > 0$.

Proof of Theorem 6. We recall the time inversion of the processes $X_{\alpha,\beta}$ from Proposition 1 (a): $(X_{\alpha,\beta}(1/s))$ has the same law as $X_{\alpha',\beta}$. For simplicity, set $\rho := \sqrt{\frac{\Gamma(2\beta+1)}{2^{2\beta+1}}}$. Theorem 2 yields that

$$-|\log \varepsilon/\rho|^3 \asymp \log \mathbb{P}\left(\sup_{t\le 1} |\rho X_{\alpha,\beta}(t)| \le \varepsilon\right) = \log \mathbb{P}\left(\sup_{t\ge 1} |\rho X_{\alpha',\beta}(t)| \le \varepsilon\right).$$

Clearly, by Andersons inequality and the integral representation (1.2), the last expression is smaller than

$$\log \mathbb{P}\left(\sup_{t\ge 1} |S(t)| \le \varepsilon\right),$$

which already shows the lower bound of the small deviation probability of S in (5.16).

To see the opposite bound, note that

$$\rho X_{\alpha',\beta}(t) = t^{\alpha'} \int_0^{2/\varepsilon} x^\beta e^{-xt}\, \mathrm{d}B(x) + t^{\alpha'} \int_{2/\varepsilon}^{\infty} x^\beta e^{-xt}\, \mathrm{d}B(x) =: V(t) + U(t)$$

and thus

$$e^{-c|\log \varepsilon|^3} \ge \mathbb{P}\left(\sup_{t\ge 1} |\rho X_{\alpha',\beta}(t)| \le \varepsilon\right) \ge \mathbb{P}\left(\sup_{t\ge 1} |V(t)| \le \varepsilon/2\right) \cdot \mathbb{P}\left(\sup_{t\ge 1} |U(t)| \le \varepsilon/2\right). \tag{5.18}$$

Since

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\ge 1} |V(t)| \le \varepsilon/2\right) &= \mathbb{P}\left(\sup_{t\ge 1} \left|\int_0^{2/\varepsilon} t^{\alpha'} x^\beta e^{-tx}\, \mathrm{d}B(x)\right| \le \varepsilon/2\right) \\
&= \mathbb{P}\left(\sup_{t\ge 1} \left|(\varepsilon/2)^{\alpha'} \int_0^1 (2t/\varepsilon)^{\alpha'} (2x/\varepsilon)^\beta e^{-2xt/\varepsilon} (\varepsilon/2)^{-1/2}\, \mathrm{d}B(x)\right| \le \varepsilon/2\right) \\
&= \mathbb{P}\left(\sup_{t\ge 2/\varepsilon} \left|\int_0^1 t^{\alpha'} x^\beta e^{-xt}\, \mathrm{d}B(x)\right| \le (\varepsilon/2)^{1+1/2+\beta-\alpha'}\right) \\
&\ge \mathbb{P}(\sup_{t\ge 1} |S(t)| \le (\varepsilon/2)^{1+1/2+\beta-\alpha'})
\end{aligned}$$

and $1+1/2+\beta-\alpha'>0$, it is sufficient to show that the second term in (5.18) is bounded from below by a constant. This can be seen as follows: Note that the finite dimensional distributions of U are the same as of the following process

$$\begin{aligned} & t^{\alpha'} e^{-t/\varepsilon} \int_{1/\varepsilon}^{\infty} (x+1/\varepsilon)^{\beta} e^{-tx}\, \mathrm{d}B(x) \\ = & \; t^{\alpha'} \varepsilon^{-\beta} e^{-t/\varepsilon} \int_{1/\varepsilon}^{\infty} (\varepsilon x+1)^{\beta} e^{-tx}\, \mathrm{d}B(x) \\ = & \; (t/\varepsilon)^{\alpha'} \varepsilon^{\alpha'-\beta-1/2} e^{-t/\varepsilon} \int_{1}^{\infty} (u+1)^{\beta} e^{-tu/\varepsilon}\, \mathrm{d}B(u). \end{aligned}$$

Therefore, estimating $e^{t/\varepsilon} \geq e^{1/\varepsilon}$ in the first step, we obtain

$$\begin{aligned} \mathbb{P}(\sup_{t\geq 1} |U(t)| \leq \varepsilon/2) \;\geq\; & \mathbb{P}\left(\sup_{t\geq 1} \left|(t/\varepsilon)^{\alpha'} \int_1^{\infty} (u+1)^{\beta} e^{-tu/\varepsilon}\, \mathrm{d}B(u)\right| \leq \varepsilon^{1+\beta+1/2-\alpha'} e^{1/\varepsilon}/2\right) \\ = \; & \mathbb{P}\left(\sup_{s\geq 1/\varepsilon} \left|s^{\alpha'} \int_1^{\infty} (u+1)^{\beta} e^{-su}\, \mathrm{d}B(u)\right| \leq \varepsilon^{1+\beta+1/2-\alpha'} e^{1/\varepsilon}/2\right) \\ \geq \; & \mathbb{P}\left(\sup_{s\geq 1} \left|s^{\alpha'} \int_1^{\infty} (u+1)^{\beta} e^{-su}\, \mathrm{d}B(u)\right| \leq 1\right), \end{aligned}$$

for small ε because $\varepsilon^{1+\beta+1/2-\alpha'} e^{1/\varepsilon} \to \infty$ as $\varepsilon \to 0^+$. Note that the Gaussian process

$$Z(s) = s^{\alpha'} \int_1^{\infty} (u+1)^{\beta} e^{-su}\, \mathrm{d}B(u)$$

is sample bounded on $[1,\infty)$ under the assumption $\alpha-\beta>1/2$. Indeed,

$$\begin{aligned} \mathbb{E}\,|Z(t)-Z(s)|^2 &= \int_1^{\infty} (u+1)^{2\beta} (t^{\alpha'} e^{-tu} - s^{\alpha'} e^{-su})^2\, \mathrm{d}u \\ &\leq C \int_0^{\infty} u^{2\beta} (t^{\alpha'} e^{-tu} - s^{\alpha'} e^{-su})^2\, \mathrm{d}u \\ &= C'\mathbb{E}\,|X_{\alpha',\beta}(t) - X_{\alpha',\beta}(s)|^2 \\ &= C'\mathbb{E}\,|X_{\alpha,\beta}(1/t) - X_{\alpha,\beta}(1/s)|^2. \end{aligned}$$

Now, Theorem 2 implies that when $\alpha-\beta>1/2$, $Z(t)$ is sample bounded on $[1,\infty)$. Therefore,

$$\mathbb{P}\left(\sup_{s\geq 1} \left|s^{\alpha'} \int_1^{\infty} (u+1)^{\beta} e^{-su}\, \mathrm{d}B(u)\right| \leq 1\right)$$

is a positive constant, as required. □

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