MAÑÉ'S CONJECTURES IN CODIMENSION ONE

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Abstract. We prove Mañé's conjectures ([Mn96]) in the context of codimension one Aubry-Mather theory.

1. Introduction

We study variational problems on tori in the spirit of [Mo86]. The objects we are interested in are maps u from $\mathbb{R}^n$ to $\mathbb{R}$ which minimize globally the integral

$$\int_{\mathbb{R}^n} L(x,u,\nabla(u))dx \tag{1}$$

where the cost function L is called the Lagrangian of the problem. This theory is also known as codimension one Aubry-Mather theory, because it generalizes the classical Aubry-Mather theory of twist maps. It runs parallel to the "dimension one" theory of Bangert [Ba90], Mather [Mr91], Mañé [Mn96] and Fathi [F].

We begin by recalling the hypotheses on the Lagrangian. Let $L(x_1,\dots,x_n,u,p_1,\dots,p_n)$ be a Lagrangian such that

- (H1) : $L \in C^{l,\gamma}(\mathbb{R}^{2n+1})$, $l \geq 2$, $\gamma > 0$.
- (H2) : L has period 1 in $x_1,\dots,x_n,u$.
- (H3) : There is $\delta > 0$ such that
$$\delta I \leq \frac{\partial^2 L}{\partial p_i \partial p_j} \leq \frac{1}{\delta} I$$
where I denotes the identity matrix on $\mathbb{R}^n$.
- (H4) : There is $C > 0$ such that
$$\left|\frac{\partial^2 L}{\partial p \partial x}\right| + \left|\frac{\partial^2 L}{\partial p \partial u}\right| \leq C(1+|p|)$$
$$\left|\frac{\partial^2 L}{\partial x \partial x}\right| + \left|\frac{\partial^2 L}{\partial u \partial x}\right| + \left|\frac{\partial^2 L}{\partial u \partial u}\right| \leq C(1+|p|^2).$$

The main example we have in mind is a Lagrangian of the form

$$L(x,u,\nabla u) = \frac{1}{2}|\nabla u(x)|^2 + f(x,u)$$

where $f \in C^{l,\gamma}(\mathbb{R}^{n+1})$ is $\mathbb{Z}^{n+1}$-periodic. Observe that for any Lagrangian L satisfying Hypothesis (H1-4) and for any $\mathbb{Z}^{n+1}$-periodic $f \in C^{l,\gamma}(\mathbb{R}^{n+1})$, $L+f$ is again a Lagrangian satisfying Hypothesis (H1-4). Adding a function to the Lagrangian is also called perturbing the Lagrangian by a potential. In this paper, after Mañé ([Mn95]), the phrase "for a generic Lagrangian

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L, Property P holds" means "for any Lagrangian L, there exists a residual subset $\mathcal{O}(L)$ of the set of potentials, such that for any f in $\mathcal{O}(L)$, Property P holds for $L+f$".

Since the integral (1) is infinite in general, we must explain what we mean by minimizing in (1). We say that $u \in W^{1,2}_{loc}(\mathbb{R}^n)$ is a minimizer for L if

$$\int_{\mathbb{R}^n} [L(x, u+\phi, \nabla(u+\phi)) - L(x, u, \nabla u)]dx \geq 0 \qquad \forall \phi \in C_0^\infty(\mathbb{R}^n). \tag{2}$$

Since L is periodic, if u is a minimizer and $(k,j) \in \mathbb{Z}^n \times \mathbb{Z}$, then $u(x+k)+j$ is a minimizer, too; we say that u is non self intersecting if

$$\forall (k,j) \in \mathbb{Z}^n \times \mathbb{Z}, \quad \text{either} \quad u(x+k)+j > u(x) \ \forall x$$

$$\text{or} \quad u(x+k)+j < u(x) \ \forall x \qquad \text{or} \quad u(x+k)+j = u(x) \ \forall x. \tag{3}$$

In [Mo86], it is proven that non self-intersecting minimizers lie within finite distance of some hyperplane:

Theorem 1.1 ([Mo86])**.** *Let $u \in W^{1,2}_{loc}(\mathbb{R}^n)$ be minimal and non self intersecting; then there exists $\rho \in \mathbb{R}^n$ and a constant $C_L(||\rho||) > 0$, depending only on L and $||\rho||$, such that*

$$||u - u(0) - \rho \cdot x||_{C^{l,\gamma}(\mathbb{R}^n)} \leq C_L(||\rho||).$$

In particular, any minimizer $u \in W^{1,2}_{loc}(\mathbb{R}^n)$ is actually as regular as the Lagrangian. The vector ρ is called the rotation vector, or the slope, of u; an important fact is that there are minimal, non self intersecting solutions of any rotation vector.

Theorem 1.2 ([Mo86])**.** *For any $\rho \in \mathbb{R}^n$, there is a minimal, non self intersecting solution of slope ρ.*

Definition 1.3. *A minimal, non self intersecting solution of slope ρ, is called a (L,ρ)-minimizer. When $\rho \in \mathbb{Q}^n$, we can consider the subclass of periodic minimizers: we say that a (L,ρ) minimizer u is periodic if $u(x+k)+j = u(x)$ for all $(k,j) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $\rho \cdot k + j = 0$. If u is a (L,ρ)-minimizer, with $\rho \in \mathbb{Q}^n$, then u is either periodic, or asymptotic to some periodic (L,ρ)-minimizer (see* [Ba89]*).*

We want to study uniqueness of (L,ρ)-minimizers; since we saw before that, if u is a (L,ρ)-minimizer, also $u(\cdot + k) + j$ is such, we have to identify u with its integer translations. Even with this identification, the answer is negative, because in [Ba89] it is proven that, if $\rho \notin \mathbb{Q}^n$, or if $\rho \in \mathbb{Q}^n$ and $n \geq 2$, there are always uncountably many (L,ρ)-minimizers. The situation changes if we look at the currents induced by minimizers (see section 3.1 for the precise definitions). Indeed, we are able to prove that, generically, all (L,ρ)-minimizers induce the same current; with the added bonus that, if ρ is irrational, we can drop the "generically."

The problem of uniqueness can be formulated not only for (L,ρ)-minimizers, but also for the dual notion of $(L-c)$-minimizers. We briefly explain what we mean; we recall, that, as proven in [S91], a mean action is defined.

Theorem 1.4 ([S91]). *For any (L,ρ)-minimizer u, the following limit exists and depends only on L and ρ:*

$$\lim_{R\to\infty}\frac{1}{|B(0,R)|}\int_{B(0,R)} L(x,u,\nabla u)dx := \beta(\rho).$$

Moreover, the function β is strictly convex and superlinear.

Since β is strictly convex, its Legendre-Fenchel transform, traditionally denoted by α, is C^1; it is easy to see that $-\alpha(c)$ is the minimum, over all u minimal and non self intersecting, of

$$\lim_{R\to\infty}\frac{1}{|B(0,R)|}\int_{B(0,R)} [L(x,u,\nabla u) - c\cdot\nabla u]dx.$$

Note that for any c in $\mathbb{R}^n$, the Lagrangian $L(x,u,\nabla u)-c\cdot\nabla u$, denoted $L-c$ for short, still satisfies Hypothesis (H1-4). A minimal, non self-intersecting u such that

$$\lim_{R\to\infty}\frac{1}{|B(0,R)|}\int_{B(0,R)} [L(x,u,\nabla u) - c\cdot\nabla u]dx = -\alpha(c)$$

is called a $(L-c)$-minimizer. As for (L,ρ)-minimizers, we may ask about the uniqueness of the $(L-c)$-minimizer for a given c, and similarly the question should be rephrased in terms of currents. One difference between (L,ρ)-minimizers and $(L-c)$-minimizers is that we don't know a priori when an $(L-c)$-minimizer is periodic, so another question we adress is *how large is the set of c for which $(L-c)$-minimizers have a rational slope ?* Note that by Fenchel duality an $(L-c)$-minimizer is an $(L,\alpha'(c))$-minimizer, so the question boils down to *how large is the set of c for which $\alpha'(c)\in\mathbb{Q}^n$?*.

Now we can state our result.

Theorem 1.5. *For a generic Lagrangian satisfying Hypothesis (H1-4),*

- *for every $\rho\in\mathbb{R}^n$, the (L,ρ)-minimizers induce a unique current; if ρ is rational, there is a unique periodic (L,ρ)-minimizer*
- *for every $c\in\mathbb{R}^n$, the $(L-c)$-minimizers induce a unique current*
- *there exists an open dense subset U of $\mathbb{R}^n$ such that for every $c\in U$, $\alpha'(c)\in\mathbb{Q}^n$.*

Our theorem solves, in the affirmative, the codimension one versions of the problems posed by Mañé in [Mn95], [Mn96]. In the "dimension one" theory much less is known. The best result about the first point of the theorem is that of [BC08], which says that for a generic Lagrangian L on a manifold of dimension n, for every homology class ρ, there exists at most $n+1$ (L,ρ)-minimizing currents. The second point of the theorem is trivially false in the dimension one theoretical setting (see Hedlund's example in [Ba90]). To be precise about the third point, recall that the problem originally proposed by Mañé was : *is it true that for a generic Lagrangian L, there exists a dense open subset U of the cohomology of the configuration space such that for any $c\in U$, there exists a unique minimizing measure, and it is supported on a periodic orbit.* This is true, by [O09], when the base manifold is the circle and the Lagrangian depends periodically on time, and by [Mt03] when the base manifold has dimension two and the Lagrangian is autonomous. In the codimension one theory, the notion corresponding to *minimizing measure* is that of *recurrent minimizer*. Thus, in this case Mañé conjecture follows by the first and third points of theorem 1.5.

Thus Mañé's conjectures seem taylor-made for the codimension one case. One possible reason for this is that Mañé had in mind the twist map case, which in some respects is more typical of the codimension one case than it is of the dimension one case.

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2. The derivative of α is rational on a dense set

We define

$$\mathrm{rat}(\rho,1)=\mathrm{Vect}\left((\rho,1)^{\perp}\cap\mathbb{Z}^{n+1}\right)$$

where $\mathrm{Vect}(A)$ denotes the smallest subspace of $\mathbb{R}^{n+1}$ containing the set A.

Let α and β be as in the introduction; we recall that they are dual convex functions; since β is superlinear and strictly convex by theorem 1.4, α is C^1 and superlinear.

We call flat of slope ρ the set

$$D_\rho=\{(c,\alpha(c))\quad:\quad\alpha'(c)=\rho\}.$$

We shall need the following result of Senn [S95]; it says that the linear space generated by the flat of slope ρ is contained in $\mathrm{rat}(\rho,1)$. If $A\subset\mathbb{R}^p$, let $L(A)$ be the linear space generated by the differences $a-b$ with $a,b\in A$. Clearly, if $0\in A$, then $L(A)=\mathrm{Vect}(A)$.

Theorem 2.1 ([S95])**.** *Let L be a Lagrangian on $\mathbb{R}^{2n+1}$ satisfying Hypothesis (H1-4), and let*

$$D_\rho=\{(c,\alpha(c))\quad:\quad\alpha'(c)=\rho\}.$$

Then

$$L(D_\rho)=\mathrm{rat}(\rho,1)$$

unless the recurrent (L,ρ)-minimizers (i. e. the periodic ones when ρ is rational, and the functions u^α defined in lemma 5.1 below when ρ is irrational) foliate $\mathbb{T}^{n+1}$, in which case $L(D_\rho)=\{0\}$.

Proposition 2.2. *Let L be a Lagrangian on $\mathbb{R}^{2n+1}$ satisfying Hypothesis (H1-4). Then the set $\{c\in\mathbb{R}^n\colon\ \alpha'(c)\in\mathbb{Q}^n\}$ is dense in $\mathbb{R}^n$.*

Proof. Let U be any open subset of $\mathbb{R}^n$.

First case : there exists c in U such that the flat $D_{\alpha'(c)}$ of α containing $(c,\alpha(c))$ is reduced to a point. Then, by the convexity of α,

$$\forall d\in\mathbb{R}^n\setminus\{c\},\ \left\langle\alpha'(c)-\alpha'(d),c-d\right\rangle>0.$$

Let B be a closed ball centered at c and contained in U. By Theorem 1.4, α' is continuous. Hence, by Lemma A.1, $\alpha'(U)$ contains a neighborhood of $\alpha'(c)$; thus there exists $d\in U$ such that $\alpha'(d)\in\mathbb{Q}^n$.

Second case : any $c\in U$ is contained in a non-trivial face of α, that is to say, for any $c\in U$, the face $D_{\alpha'(c)}$ of α is not reduced to a point. Then by Theorem 2.1, for any $c\in U$, the vector space $L(D_{\alpha'(c)})$ generated by $D_{\alpha'(c)}$ is $\mathrm{rat}(\alpha'(c),1)$, which is a rational subspace of $\mathbb{R}^{n+1}$: it is generated, practically by definition, by integer vectors. There are only countably many rational subspaces of $\mathbb{R}^{n+1}$, so by Baire's theorem (a countable union of nowhere dense subsets of a complete metric space is nowhere dense) there exists an open subset U_1 of U, and a rational subspace N_1 of $\mathbb{R}^{n+1}$, such that for any $c \in U_1$,

$$N_1 = L(D_{\alpha'(c)}) = \mathrm{rat}(\alpha'(c), 1).$$

Let M_1 be the canonical projection to $\mathbb{R}^n$ of N_1. If M_1 has dimension n, then $\alpha'(c)$ is rational; therefore, we shall suppose that M_1 is a proper subspace of $\mathbb{R}^n$.

Observation : First let us observe that, if $c \in U_1$, then $(c, \alpha(c))$ lies in the relative interior of $D_{\alpha'(c)}$. Indeed, let us take $c \in U_1$, and a convex neighborhood V of 0 in M_1, such that $c + V \subset U_1$. Let us denote by $\tilde{\alpha}$ the map α restricted to $c + V$; then $\tilde{\alpha}$ is affine and convex. The convexity is trivial, to prove that $\tilde{\alpha}$ is affine, we recall one fact from convex analysis: $\tilde{\alpha}$ is affine on $c + V$ if and only if, for any $d \in c + V$, the flat of $\tilde{\alpha}$ containing $(d, \alpha(d))$ has maximal dimension. In our case, $c+V$ is an open set of $c+M_1$, and maximal dimension means the dimension of M_1. Now, $D_{\tilde{\alpha}'(d)}$ is simply $D_{\alpha'(d)}$ intersected with $(c + V) \times \mathbb{R}$; our assumptions on $D_{\alpha'(d)}$ and V yield that $L(D_{\tilde{\alpha}'(d)}) = \mathrm{rat}(\alpha'(c), 1)$, and $\mathrm{rat}(\alpha'(c), 1)$ has the same dimension as M_1.

This proves that $\tilde{\alpha}$ is affine on the set $c + V$, and that $c + V$ is open in $c + M_1$; in other words, $(c, \alpha(c))$ lies in the relative interior of $D_{\alpha'(c)}$.

From this we now deduce that for any $c \in U_1$, the map α restricted to $(c + M_1^{\perp}) \cap U_1$, which we denote α_c for simplicity, is strictly convex at c, that is, c is not contained in any non-trivial face of α_c. Indeed, [Mt], Lemma A.2 says that, if some $c \in U_1$ is contained in a non-trivial face of α_c and the observation above holds, then c is contained in a face D of α such that $L(D)$ properly contains N_1; but this contradicts the fact that the flat at c generates N_1.

So for any $c \in U_1$, the map α_c is strictly convex at c. Therefore, by the same argument as in the first case, for any $c \in U_1$, there exists $d \in (c + M_1^{\perp}) \cap U_1$, such that $\alpha_c'(d) \in M_1^{\perp} \cap \mathbb{Q}^n$.

Observe that $M_1^{\perp} \cap \mathbb{Q}^n \neq \{0\}$. To show this, we note that $M_1^{\perp} \neq \{0\}$, because we are supposing that M_1 is proper; moreover, $M_1^{\perp}$, being the orthogonal of a rational subspace of $\mathbb{R}^n$, is itself a rational subspace of $\mathbb{R}^n$.

Now $\alpha'(d)$ is the sum of $\alpha_c'(d)$ and the derivative at d of the restriction of α to $(c+M_1)\cap U_1$, which is the orthogonal projection of $\alpha'(d)$ to M_1. The latter lies in $\mathbb{Q}^n \cap M_1$ by Lemma B.2 in the appendix. Therefore $\alpha_c'(d) \in \mathbb{Q}^n$. $\square$

3. Currents and recurrent minimizers

We define the current induced by a minimal u. For $p \in \mathbb{R}^n$, we denote by $\omega(x, u) \cdot (p, 1)$ the n-form ω applied to the n-vector

$$\begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 1, & \dots, & 0 \\ \dots, & \dots, & \dots, & \dots \\ p_1, & p_2, & \dots, & p_n \end{pmatrix}.$$

Let ω be a smooth n-form on the torus and let u be a (L,ρ)-minimizer; let $B(0,R)$ be the ball of radius R centered at the origin in $\mathbb{R}^n$, and let $|B(0,R)|$ be its Euclidean n-dimensional volume. It can be proven that the following limit exists

$$\lim_{R\to\infty}\frac{1}{|B(0,R)|}\int_{B(0,R)}\omega(x,u(x))\cdot(\nabla u(x),1)dx. \tag{4}$$

We define $T_u(\omega)$ to be the limit above. It is proven in [Be09] that T_u is a n-current of finite mass, and that $\partial T_u = 0$. This means the following: let us denote by $\Omega_n^{(0)}$ the set of continuous n-foms on $\mathbb{T}^{n+1}$, equipped with the sup norm; then T_u is a linear, continuous operator on $\Omega_n^{(0)}$ and $T_u(\mathrm{d}\eta) = 0$ for every $(n-1)$-form η of class C^1. In particular, we can restrict T_u to the subspace of closed forms and quotient on the exact forms; what we obtain is a linear operator $slope_{T_u}$ from $H^n(\mathbb{T}^{n+1})$, the n-th real cohomology group of $\mathbb{T}^{n+1}$, to $\mathbb{R}$. Thus, $slope_{T_u}$ belongs to the dual of $H^n(\mathbb{T}^{n+1})$, which identifies with the n-th homology group $H_n(\mathbb{T}^{n+1})$. On $H^n(\mathbb{T}^{n+1})$ we introduce, as a basis, the equivalence classes of the differential forms

$$\mathrm{d}\hat{x}_i :\ = (-1)^{n-i+1}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{i-1}\wedge\mathrm{d}x_{i+1}\wedge\cdots\wedge\mathrm{d}x_{n+1}\quad 1\le i\le n$$

$$\mathrm{d}\hat{x}_{n+1} = \mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n.$$

On $H_n(\mathbb{T}^{n+1})$ we introduce the basis e_i dual to $\mathrm{d}\hat{x}_i$. It is easy to see that, with this choice of the basis, if u is (L,ρ)-minimal, then $slope_{T_u} = (\rho,1)$.

Given a current T of finite mass on $\mathbb{T}^{n+1}$, we can define a signed measure μ_T on $\mathbb{T}^{n+1}$ by the formula

$$T(f\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n) = \int_{\mathbb{T}^{n+1}} f d\mu_T \tag{5}$$

for any function f continuous on the torus.

We note that, by (4) and (5), if $T = T_u$, the measure μ_T is defined by

$$\int_{\mathbb{T}^{n+1}} f(x,x_{n+1})d\mu_T = \lim_{R\to\infty}\frac{1}{|B(0,R)|}\int_{B(0,R)} f(x,u(x))dx. \tag{6}$$

From the formula above, it is immediate that μ_T is a probability measure.

The following lemma will be useful along the way.

Lemma 3.1. *For every u in M_ρ, for every (z,z_{n+1}) in $\mathbb{Z}^n\times\mathbb{Z}$, denoting $v(x) := u(x+z)+z_{n+1}$, we have $T_u = T_v$.*

Proof. Take

- u in M_ρ
- a smooth n-form ω on $\mathbb{T}^{n+1}$
- (z,z_{n+1}) in $\mathbb{Z}^n\times\mathbb{Z}$.

We have

$$\begin{aligned} T_u(\omega) &= \lim_{R\to\infty} \frac{1}{|B(0,R)|}\int_{B(0,R)} \omega(x,u(x))\cdot(\nabla u(x),1)dx \\ &= \lim_{R\to\infty} \frac{1}{|B(0,R)|}\int_{B(-z,R)} \omega(x,u(x+z)+z_{n+1})\cdot(\nabla u(x+z),1)dx \\ &= \lim_{R\to\infty} \frac{1}{|B(0,R)|}\int_{B(0,R)} \omega(x,u(x+z)+z_{n+1})\cdot(\nabla u(x+z),1)dx \\ &= T_v(\omega). \end{aligned}$$

The second equality comes from the change of variables $x \mapsto x+z$ and the fact that ω is $\mathbb{Z}^{n+1}$-periodic, the third one from the fact that ω is bounded and

$$\lim_{R\to\infty} \frac{|B(0,R)\setminus B(-z,R)|}{|B(0,R)|} = 0.$$

□

3.1. **Action of a current.** This action has been defined for dimension 1 currents in [BB07]; as shown in [Be09], the same definition applies to codimension 1 currents. We are not going to recall this definition here, we only recall some facts; the first one is that this definition extends the notion of mean action we gave in the introduction.

Indeed, the following holds: if u is a (L,ρ) minimizer, then the mean action of T_u, say $MA(L,T_u)$, is given by

$$MA(L,T_u) = \lim_{R\to\infty} \frac{1}{|B(0,R)|}\int_{B(0,R)} L(x,u,\nabla u)dx = \beta(\rho).$$

3.2. **Rational rotation numbers.** Let $\rho \in \mathbb{Q}^n$ and let $r>0$; we define

$$\Gamma\colon = \{k\in\mathbb{Z}^n \quad : \quad k\cdot\rho\in\mathbb{Z}\}. \tag{7}$$

It is easy to see that Γ is a subgroup of $\mathbb{Z}^n$; actually, it is the projection of $\mathrm{rat}(\rho,1)\cap(\mathbb{Z}^n\times\mathbb{Z})$ to $\mathbb{Z}^n$. Since ρ is rational, Γ contains a basis of $\mathbb{R}^n$; in particular, the action of Γ on $\mathbb{R}^n$ admits a bounded, measurable fundamental domain D.

We define a set $J_r(\rho)$ which will come handy in the next section. The set $J_r(\rho)$ is the set of all functions $u\colon\mathbb{R}^n\to\mathbb{R}$ satisfying the three points below:

- $u\in C^{l,\gamma}(\mathbb{R}^n)\subset W^{1,2}_{loc}(\mathbb{R}^n)$
- the $C^{l,\gamma}(\mathbb{R}^n)$-norm of the map $x\mapsto u(x)-u(0)-\rho\cdot x$ is smaller than r.
- $u(x+k)+j=u(x)$ whenever $(k,j)\in\mathbb{Z}^n\times\mathbb{Z}\cap(\rho,1)^{\perp}$.

Then u induces a current T_u by (4). The mean action of T_u is given, as expected, by

$$MA(L,T_u) = \frac{1}{|D|}\int_D L(x,u,\nabla u)dx. \tag{8}$$

We shall need a theorem, due to Moser, which says that, when ρ is rational and r is larger than the constant $C_L(||\rho||)$ of theorem 1.1, there are minimizers in the class $J_r(\rho)$. This is in sharp contrast to the dimension one case, where in general there are no periodic minimizers.

Theorem 3.2 ([Mo86]). *Let $\rho \in \mathbb{Q}^n$, let Γ be defined as in (7) and let D be a fundamental domain. We set*

$$W = \{u \in W^{1,2}_{loc}(\mathbb{R}^n) \quad : \quad u(x+k) - u(x) - \rho \cdot k \equiv 0 \quad \forall k \in \Gamma\}.$$

Then

$$\beta(\rho) = \inf\{\frac{1}{|D|}\int_D L(x,u,\nabla u)dx \quad : \quad u \in W\}.$$

Moreover, the inf *is a minimum and the functions $u \in W$ on which the minimum is attained are (L,ρ) minimizers.*

Conversely, if u is a (L,ρ) minimizer such that $u(x+k) - u(x) - \rho \cdot k \equiv 0$ for any $k \in \Gamma$, then

$$\beta(\rho) = \frac{1}{|D|}\int_D L(x,u,\nabla u)dx.$$

4. Generic uniqueness of periodic minimizers, and of the minimizing currents with rational slope

Here we prove (Proposition 4.1) that given a rational rotation number ρ, for a generic Lagrangian L, there is a unique periodic minimizer with rotation number ρ. Then we prove (Lemma 4.4) that for such a Lagrangian, all (L,ρ)-minimizers, including the non-periodic ones, induce the same current.

Proposition 4.1. *Let*

- *L be a Lagrangian on $\mathbb{R}^{2n+1}$ satisfying Hypothesis (H1-4).*
- *ρ be a vector in $\mathbb{Q}^n$.*

Then there exists a residual subset $\mathcal{O}(L,\rho)$ of $C^\infty(\mathbb{T}^{n+1})$ such that for any $f \in \mathcal{O}(L,\rho)$, there is only one periodic $(L-f,\rho)$-minimizer. Moreover, all $(L-f,\rho)$-minimizers induce the same current.

Proof. We will see how this result follows from [BC08]. This paper considers the following situation:

$$\begin{array}{cccc}
 & H_r & & \\
 & \cap & & \\
E\times & F & & \\
 & \downarrow \pi & \searrow & \\
E\times & G & \xrightarrow{\alpha} & \mathbb{R} \\
 & \cup & & \\
 & K & &
\end{array}$$

where E, F, G are topological vector spaces, π is a linear continuous map between F and G and α is a bilinear coupling. The hypotheses are the following ones.

- The bilinear pairing α is continuous.
- K is a compact and convex set, separated by E; the latter means that, if η and ν are two different points of K, then there exists $u \in E$ such that $\alpha(u, \eta - \nu) \neq 0$.
- E is a Frechet space.
- H_r is compact, convex, and $\pi(H_r) \subset K$.

We define H_r^* as the set of all the affine, continuous functions on H_r; for L in H_r^*, we denote by $MIN_{H_r}(L)$ the set of minima of L over H_r.

Under these hypotheses, theorem 5 of [BC08] holds:

Theorem 4.2 ([BC08]). *For any finite dimensional affine subspace B of H_r^*, there exists a residual subset $\mathcal{O}(B) \subset E$ such that, for all $f \in \mathcal{O}(B)$ and $L \in B$, we have that $\pi(MIN_{H_r}(L-f))$ is contained in an affine subspace of G, whose dimension is not larger than the dimension of B.*

We want to apply this theorem to our situation. To do this, we let

- E be the Fréchet space $C^{l,\gamma}(\mathbb{T}^{n+1})$
- F be the space of closed n-currents of finite mass on $\mathbb{T}^{n+1}$. This space is the dual of the space $\Omega_n^{(0)}$ of continuous n-forms on $\mathbb{T}^{n+1}$, equipped with the sup norm.
- G be the dual space of $C^0(\mathbb{T}^{n+1})$, i.e. the space of Borel signed measures on $\mathbb{T}^{n+1}$
- $\pi\colon F \longrightarrow G$ be the continuous linear map $T \longrightarrow \mu_T$ defined as in (5).
- α be the continuous bilinear pairing between E and G defined by integration
- $K \subset G$ be the metrizable, compact, convex set of Borel probability measures on $\mathbb{T}^{n+1}$. Observe that K is separated by E.

The definition of H_r is a bit trickier. We let $J_r(\rho)$ be the set of functions defined in section 3.2.

Define $\tilde{H}_r$ to be the set of currents of the form T_u, with u in $J_r(\rho)$, and let H_r be the weak* closure of the convex hull of $\tilde{H}_r$. Then H_r is contained in a ball in F, so by the Banach-Alaoglu Theorem it is compact with respect to the weak* topology. It is also metrizable because the space $\Omega_n^0(\mathbb{T}^{n+1})$ of continuous n-forms on $\mathbb{T}^{n+1}$, equipped with the sup norm, is separable.

We saw in formula (6) that π brings any $T_u \in \tilde{H}_r$ to a probability measure, i. e. to an element of K; taking convex combinations, the same is true for H_r.

We show in lemma 4.3 below that $MA(L,\cdot) \in H_r^*$, i. e. it is an affine, continuous functional on H_r.

Now we can apply Theorem 4.2. For us, B will be a singleton, i. e. $B = \{MA(L,\cdot)\}$. By theorem 4.2 there exists a residual subset $\mathcal{O}_r(L)$ of E such that for any $f \in \mathcal{O}_r(L)$, $\pi(MIN_{H_r}(L-f))$ is reduced to a point. Clearly, if r is smaller than the constant $C_{L-f}(||\rho||)$ of theorem 1.1, the minima in H_r may not correspond to any $(L-f,\rho)$-minimizer. That's why we consider

$$\mathcal{O}(L) = \bigcap_{r\in\mathbb{N}} \mathcal{O}_r(L).$$

We get that $\mathcal{O}(L)$ is a residual set too and, if $f \in \mathcal{O}(L)$, then $\pi(MIN_{H_r}(L-f))$ is reduced to a point for any r.

We show how this implies the thesis. Let us suppose by contradiction that there are two different periodic (L,ρ)-minimizers, say u and v. We prove below that, if $r > C_{L-f}(||\rho||)$, then T_u and T_v are minimal in H_r. We recall from [Mo86] that the graphs of the two periodic minimals u and v are disjoint; this implies by (6) that μ_{T_u} and μ_{T_v} are different. In other words, $\pi(MIN_{H_r}(L-f))$ contains at least two elements, while we have just proven that it is reduced to a point; this contradiction proves the thesis.

Now we show that, if u is a $(L-f,\rho)$-minimizer, then T_u minimizes $MA(L-f,\cdot)$ in H_r for r large enough. To show this, it suffices to show that the minimum of $MA(L-f,\cdot)$ on the currents of H_r coincides with the minimum of $MA(L-f,\cdot)$ on the currents T_u, where u is a periodic minimizer. By (8) and theorem 3.2, the latter minimum is $\beta_{L-f}(\rho)$, where by β_{L-f} we denote the β-function of the Lagrangian $L-f$. Thus, it suffices to show that the minimum of $MA(L-f,\cdot)$ on the currents of H_r coincides with $\beta_{L-f}(\rho)$. That's what we do next.

We begin by noting the following: let $r > C_{L-f}(||\rho||)$ and let u be (L,ρ)-minimal; theorem 3.2 yields the first equality below, formula (8) the second one:

$$\beta_{L-f}(\rho) = \frac{1}{|D|}\int_D (L-f)(x,u,\nabla u)dx = MA(L-f,T_u).$$

Since $T_u \in H_r$, we get that

$$\beta_{L-f}(\rho) \geq \min_{T\in H_r} MA(L-f,T).$$

To show the opposite inequality, we recall that $MA(L-f,\cdot)$ is affine, continuous and H_r is the closed, convex hull of the currents T_u, with $u \in J_r(\rho)$; thus, it suffices to prove that $MA(L-f,T_u) \geq \beta_{L-f}(\rho)$ for any $u \in J_r(\rho)$. But this follows immediately from theorem 3.2.

□

Lemma 4.3. *Let L satisfy hypotheses H1)-H4) of the introduction. Then the function $T \longmapsto MA(L,T)$ is affine and continuous on H_r.*

Proof. We refer the reader to [BB07] for the proof that $MA(L,\cdot)$ is affine; we prove that it is continuous on H_r.

Let $T \in H_r$ and let the measure μ_T be defined as in (5); then [BB07] implies that there is a multi-vector field $X \in L^1(\mu_T)$ such that $T = X \wedge \mu_T$ and $X = (X_1, X_2, \ldots, X_n, 1)$ in the coordinates introduced above; moreover,

$$MA(L,T) = \int_{\mathbb{T}^n\times\mathbb{T}} L(x,u,X_1(x,u),\ldots,X_n(x,u))d\mu_T(x,u).$$

Let γ_T be the push-forward of the measure μ_T by the map

$$(x,u) \longmapsto (x,u,X_1(x,u),\ldots,X_n(x,u)).$$

By the formula above, we have that

$$MA(L,T) = \int_{\mathbb{T}^n\times\mathbb{T}\times\mathbb{R}^n} L(x,u,p)d\gamma_T(x,u,p).$$

It is easy to see the following: if T_k is a sequence in H_r, then it converges weak* to T if and only if the measures γ_{T_k} converge weak* to γ_T. We also note that, by definition, the support of γ_T with $T \in H_r$ is contained in $\mathbb{T}^n \times \mathbb{T} \times B(0,r)$; since on this set L is bounded, we get that the linear function

$$\gamma_T \longmapsto \int_{\mathbb{T}^n\times\mathbb{T}\times\mathbb{R}^n} L(x,u,p)d\gamma_T(x,u,p)$$

is continuous; by the aforesaid this implies that also $MA(L,\cdot)$ is continuous.

□

We have shown that, generically, there is only one periodic minimizer, i. e. the first part of proposition 4.1; by the next lemma, this implies that all the (L,ρ)-minimizers induce the same current, i. e. the second part of proposition 4.1.

Lemma 4.4. *Let $\rho\in\mathbb{Q}^n$, and let us suppose that there is a unique periodic (L,ρ)-minimizer u; let us call T_u the current it induces. Let v be any (L,ρ)-minimizer; then $T_u=T_v$.*

Proof. By lemma 3.1 and integer translation, we can suppose that $u(0)\le v(0)<u(0)+1$. By [Ba89], there is a vector $\gamma\in\mathbb{R}^n$ such that

$$\lim_{t\to-\infty}||v-u||_{C^1(\{x\quad:\quad x\cdot\gamma<t\})}=0$$

$$\text{and}\quad \lim_{t\to+\infty}||v-u-1||_{C^1(\{x\quad:\quad x\cdot\gamma>t\})}=0. \tag{9}$$

Let $\omega\in\Omega_n^{(0)}$; by the formula above, we can fix $t>0$ so large that

$$|\omega(x,v)\cdot(\nabla v,1)-\omega(x,u)\cdot(\nabla u,1)|<\epsilon\quad\text{if}\quad|x\cdot\gamma|>t.$$

For this t,

$$|T_v(\omega)-T_u(\omega)|\le\lim_{R\to\infty}\frac{1}{|B(0,R)|}\int_{B(0,R)}|\omega(x,v)\cdot(\nabla v,1)-\omega(x,u)\cdot(\nabla u,1)|dx=$$

$$\lim_{R\to\infty}\frac{1}{|B(0,R)|}\int_{B(0,R)\cap\{x\quad:\quad|x\cdot\gamma|>t\}}|\omega(x,v)\cdot(\nabla v,1)-\omega(x,u)\cdot(\nabla u,1)|dx\le\epsilon.$$

Since ϵ is arbitrary, the last formula implies the thesis.

□

5. Uniqueness of the minimizing current within a given homology class: irrational case

Lemma 5.1. *Let*

- *L be a Lagrangian on $\mathbb{R}^{2n+1}$ satisfying Hypothesis (H1-4).*
- *ρ be a vector in $\rho\in\mathbb{R}^n\setminus\mathbb{Q}^n$*
- *u_1,u_2 be minimizers in M_ρ.*

Then $T_{u_1}=T_{u_2}$.

Proof. For $l=1,2$, we define

$$u_l^{\alpha^-}(x)=\sup\{u_l(x+k)+j\quad:\quad\rho\cdot k+j<\alpha\}$$

and

$$u_l^{\alpha^+}(x)=\inf\{u_l(x+k)+j\quad:\quad\rho\cdot k+j>\alpha\}.$$

We recall a few results of Bangert's on the properties of $u_l^{\alpha^\pm}$. We set

$$\Gamma=\{(k,j)\in\mathbb{Z}^n\times\mathbb{Z}\quad:\quad k\cdot\rho+j=0\}.$$

1) It is proven in proposition 5.6 of [Ba89] that $u_l^{\alpha^+}=u_l^{\alpha^-}$ save for at most countably many α, for which $u_l^{\alpha^+}>u_l^{\alpha^-}$. We call u_l^α their common value, defined for α outside a countable set.
2) By the same proposition, $u_l^{\alpha^\pm}$ is Γ-periodic; it follows from the definition that $u_l^{\alpha^-}\le u_l^{\alpha^+}$. Let M be the projection of $\mathrm{rat}(\rho,1)\subset\mathbb{R}^n\times\mathbb{R}$ on $\mathbb{R}^n$; corollary 4.6 of [Ba89] implies that, for all $\epsilon>0$ we can find $C>0$ such that

$$u_l^{\alpha^+}(x+z)-u_l^{\alpha^-}(x+z)\le\epsilon\quad\text{if}\quad x\in M,z\in M^\perp,\quad||z||\ge C.$$

3) By [Ba87], there is $a \in \mathbb{R}$ such that $u_1^\alpha = u_2^{\alpha+a}$.
4) Setting $\alpha_1 = 0$, $\alpha_2 = -a$, we have by the last point and the definition of u_l^α that, for $l = 1, 2$,

$$u_1^{\alpha_l^-} \le u_l \le u_1^{\alpha_l^+}.$$

In the formula above, there are only two possibilities: either there are two equality signs, or there are two strict inequalities.

We define $\tilde{G}$ as the closure of

$$\cup_{\alpha\in\mathbb{R}}\{(x, u_1^{\alpha^\pm}(x)) \quad : \quad x \in \mathbb{R}^n\}$$

and we call G the projection of $\tilde{G}$ on $\mathbb{T}^n \times \mathbb{T}$.
Observation 1. We assert that for $l = 1, 2$, $supp(T_{u_l}) \subset G$. Indeed, let ω be a continuous n-form compactly supported on $(\mathbb{T}^n \times \mathbb{T}) \setminus G$; we shall show that $T_{u_i}(\omega) = 0$. For starters, ω induces a periodic form $\tilde{\omega}$ on $\mathbb{R}^n \times \mathbb{R}$. Since ω is compactly supported on $\mathbb{T}^n \times \mathbb{T}$, the distance between the support of ω and G is positive, which implies that the distance between the support of $\tilde{\omega}$ and $\tilde{G}$ is positive. By point 2) above, this implies that, for C large enough,

$$supp(\tilde{\omega}) \cap \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \quad : \quad u_1^{\alpha_l^-}(x) < x_{n+1} < u_1^{\alpha_l^+}(x)\} \subset M \times (B(0, C) \cap M^\perp).$$

In particular, if s is the dimension of M, we get that

$$|\{x \in B(0, R) \quad : \quad (x, u_l(x)) \in supp(\tilde{\omega})\}| \le C_1 R^s.$$

Since ρ is irrational, $s < n$, and thus

$$|T_{u_l}(\omega)| = \lim_{R\to\infty} \frac{1}{|B(0, R)|} \left| \int_{B(0,R)} \tilde{\omega}(x, u_l(x)) \cdot (\nabla u_l, 1)(x) dx \right| \le \lim_{R\to\infty} \frac{C_2 R^s}{|B(0, R)|} = 0$$

because $s < n$, since ρ is irrational.
Observation 2. We assert that Lemma 5.1 follows if we prove that $\mu_{T_{u_1}} = \mu_{T_{u_2}}$. Indeed, let

$$X_l \colon \tilde{G} \cup \{(x, u_l(x)) \quad : \quad x \in \mathbb{R}^n\} \to \Lambda_n(\mathbb{R}^{n+1})$$

be defined by

$$X_l(x, u_1^{\alpha\pm}(x)) = (\nabla u_1^{\alpha\pm}(x), 1), \qquad X_l(x, u_l(x)) = (\nabla u_l(x), 1).$$

In the formula above, we have written the coordinates of X_l with respect to the basis $\{e_i\}$ of $\Lambda_n(\mathbb{R}^{n+1})$ which is dual to the basis $d\hat{x}_i$ of $\Lambda^n(\mathbb{R}^{n+1})$ we defined in section 3.

By point 3) above, we have that $X_1 = X_2$ on G; we call X their common value on this set. Now T_{u_l} is supported on G, where X is defined; clearly, the observation follows if we prove that $T_{u_l} = X \wedge \mu_{T_{u_l}}$.

To show this, we recall that, by (6), $\mu_{T_{u_l}}$ is the weak* limit of the measures $\mu_{l,R}$ on $\mathbb{T}^n \times \mathbb{T}$ defined by

$$\int_{\mathbb{T}^n\times\mathbb{T}} f(x, x_{n+1}) d\mu_{l,R}(x, x_{n+1}) = \frac{1}{|B(0, R)|} \int_{B(0,R)} f(x, u_l(x)) dx$$

for all continuous functions f. Now, it follows from theorem 4.5 of [Mo86] that X_l is Lipschitz on the union of G and the graph of u_l; if $\tilde{X}_l$ is a Lipschitz extension of X_l to $\mathbb{T}^n \times \mathbb{T}$, we get that

$$\begin{aligned} T_{u_l}(\omega) &= \lim_{R\to\infty} \frac{1}{|B(0,R)|} \int_{B(0,R)} \omega(x, u_l(x)) \cdot (\nabla u_l(x), 1) dx \\ &= \lim_{R\to\infty} \int_{\mathbb{T}^n \times \mathbb{T}} \omega(x, x_{n+1}) \cdot \tilde{X}_l(x, x_{n+1}) d\mu_{l,R}(x, x_{n+1}) \\ &= \int_{\mathbb{T}^n \times \mathbb{T}} \omega(x, x_{n+1}) \cdot \tilde{X}_l(x, x_{n+1}) d\mu_{T_{u_l}}(x, x_{n+1}) \\ &= \int_{\mathbb{T}^n \times \mathbb{T}} \omega(x, x_{n+1}) \cdot X(x, x_{n+1}) d\mu_{T_{u_l}}(x, x_{n+1}) \end{aligned}$$

where the first equality is the definition of T_{u_l}; the second one follows from the definition of $\mu_{l,R}$ and the fact that $\tilde{X}_l = (\nabla u_l, 1)$ on the graph of u_l. The third equality follows since $\omega \cdot \tilde{X}_l$ is a continuous function on $\mathbb{T}^n \times \mathbb{T}$ and $\mu_{l,R}$ converges weakly. We note that, by (5), if T_{u_l} is supported on G, the measure $\mu_{T_{u_l}}$ is supported on G too; since on this set $\tilde{X}_1 = \tilde{X}_2 = X$, the last equality follows.

The formula above implies that $T_{u_l} = X \wedge \mu_{T_{u_l}}$.

Observation 3. We define the map

$$\tilde{\Phi} \colon \tilde{G} \to \mathbb{R}^{n+1}, \qquad \tilde{\Phi}(x, u_1^{\alpha^\pm}(x)) = (x, \rho \cdot x + \alpha).$$

We recall from [Mo86] that this map quotients to a map $\Phi \colon G \to \mathbb{T}^{n+1}$. We call P the canonical projection $\mathbb{T}^n \times \mathbb{T} \longrightarrow \mathbb{T}^n$, i. e. $P(x, x_{n+1}) = x$. We shall prove the following three facts.

- $P_\sharp(\mu_{T_{u_l}})$ and $(P \circ \Phi)_\sharp(\mu_{T_{u_l}})$ are the Lebesgue measure
- the measures $\mu_{T_{u_l}}$ on G are invariant by the map

$$\psi_k \colon G \to G, \qquad \psi_k \colon (x, u_1^\alpha(x)) \to (x, u_1^\alpha(x + k))$$

- the measures $\Phi_\sharp(\mu_{T_{u_l}})$ on $\mathbb{T}^n \times \mathbb{T}$ are invariant by the map

$$\tilde{\psi}_k \colon \mathbb{T}^n \times \mathbb{T} \to \mathbb{T}^n \times \mathbb{T}, \quad \tilde{\psi}_k(x, x_{n+1}) = (x, x_{n+1} + k \cdot \rho).$$

For the first statement, we note that, if $f \colon \mathbb{T}^n \to \mathbb{R}$ is continuous, then

$$\begin{aligned} \int_{\mathbb{T}^n} f dP_\sharp(\mu_{T_{u_l}}) &= \int_{\mathbb{T}^n \times \mathbb{T}} f(x) d\mu_{T_{u_l}}(x, x_{n+1}) \\ = \lim_{R\to\infty} \frac{1}{|B(0,R)|} \int_{B(0,R)} f(x) dx &= \int_{\mathbb{T}^n} f(x) dx \end{aligned}$$

where the second equality comes from (6) and the last one from the periodicity of f. This proves that $P_\sharp(\mu_{T_{u_l}})$ is Lebesgue. The statement for $(P \circ \Phi)_\sharp(\mu_{T_{u_l}})$ follows as above, noting that $P \circ \Phi(x, u^{\alpha^\pm}(x)) = x$.

The second statement follows from Lemma 3.1 and (5).

To prove the third statement, we note that $u_1^{\alpha + k\cdot\rho} = u_1^\alpha(x + k)$; we can rewrite this fact as

$$\Phi \circ \psi_k(x, u^{\alpha^\pm}(x)) = \tilde{\psi}_k \circ \Phi(x, u^{\alpha^\pm}(x)).$$

This implies the first equality below, while the second one follows from the previous point.

$$(\tilde{\psi}_k)_\sharp \Phi_\sharp(\mu_{T_{u_l}}) = \Phi_\sharp(\psi_k)_\sharp(\mu_{T_{u_l}}) = \Phi_\sharp(\mu_{T_{u_l}}).$$

The last formula proves the invariance of $\Phi_\sharp(\mu_{T_{u_l}})$.

Observation 4. We assert that $\Phi_\sharp(\mu_{T_{u_l}})$ is the Lebesgue measure. We prove this using observation 3 and the Fourier transform; we set

$$m_{k,j} = \int_{\mathbb{T}^n \times \mathbb{T}} e^{-2\pi i[k\cdot x + jx_{n+1}]} d\Phi_\sharp(\mu_{T_{u_l}})(x, x_{n+1}).$$

We choose $\tilde{k}$ such that $\rho \cdot \tilde{k} \notin \mathbb{Q}$; invariance under $\tilde{\psi}_{\tilde{k}}$ implies the second equality below

$$\begin{aligned}
m_{k,j} &= \int_{\mathbb{T}^n \times \mathbb{T}} e^{-2\pi i[k\cdot x + jx_{n+1}]} d\Phi_\sharp(\mu_{T_{u_l}})(x, x_{n+1}) \\
&= \int_{\mathbb{T}^n \times \mathbb{T}} e^{-2\pi i[k\cdot x + jx_{n+1}]} d(\tilde{\psi}_{\tilde{k}})_\sharp \Phi_\sharp(\mu_{T_{u_l}})(x, x_{n+1}) \\
&= \int_{\mathbb{T}^n \times \mathbb{T}} e^{-2\pi i[k\cdot x + j(x_{n+1} + \tilde{k}\cdot\rho)]} d\Phi_\sharp(\mu_{T_{u_l}})(x, x_{n+1}) = e^{-2\pi i j(\tilde{k}\cdot\rho)} m_{k,j}.
\end{aligned}$$

Since $\rho \cdot \tilde{k}$ is irrational, we deduce that $m_{k,j} = 0$ unless $j = 0$. Since the marginal of $\Phi_\sharp(\mu_{T_{u_l}})$ on $\mathbb{T}^n$ is the Lebesgue measure by the first point of observation 3, we see that $m_{k,0} = 0$ unless $k = 0$; in other words, $\Phi_\sharp(\mu_{T_{u_l}})$ has the same Fourier transform as the Lebesgue measure, which implies that it is the Lebesgue measure.

End of the proof. We prove that $\mu_{T_{u_1}} = \mu_{T_{u_2}}$; by observation 2, this implies the thesis.

If Φ were injective, observation 4 would imply that $\mu_{T_{u_1}}$ and $\mu_{T_{u_2}}$ coincide. Indeed,

$$\begin{aligned}
\mu_{T_{u_1}}(A) &= \mu_{T_{u_1}}(\Phi^{-1}(\Phi(A))) = \Phi_\sharp \mu_{T_{\mu_1}}(\Phi(A)) = \mathcal{L}^{n+1}(\Phi(A)) \\
&= \Phi_\sharp \mu_{T_{\mu_2}}(\Phi(A)) = \mu_{T_{u_2}}(\Phi^{-1}(\Phi(A))) = \mu_{T_{u_2}}(A)
\end{aligned}$$

where the third and fourth equalities come from observation 4. The same argument would apply if we could prove that the set on which Φ is not injective is negligible for $\mu_{T_{u_1}}$ and $\mu_{T_{u_2}}$. But we saw in point 4) at the beginning of the proof that the set on which Φ is two to one is exactly the union of the boundaries of the gaps of G, which have the form

$$\{(x, u_1^{\alpha^\pm}(x)) \quad : \quad x \in \mathbb{R}^n\}$$

projected to the torus. By countable additivity, it suffices to prove that each piece

$$\{(x, u_1^{\alpha^\pm}(x)) \quad : \quad x \in [0,1]^n\}$$

has measure zero. But the measures $\mu_{T_{u_i}}$ are invariant by the action of ψ_k; thus, if one of the sets above had positive measure, the measure of the whole torus would be infinite, a contradiction.

$\square$

6. Proof of the main theorem

Theorem 6.1. *Let L be a Lagrangian on $\mathbb{R}^{2n+1}$ satisfying Hypothesis (H1-4). Then there exists a residual subset $\mathcal{O}(L)$ of $C^\infty(\mathbb{T}^{n+1})$ such that for any $f \in \mathcal{O}(L)$,*

- *for any $\rho \in \mathbb{R}^n$, all the $(L-f,\rho)$-minimizers induce the same current; if $\rho \in \mathbb{Q}^n$, there is a unique periodic (L,ρ)-minimizer*
- *for any $c \in \mathbb{R}^n$, all the $(L-f-c)$-minimizers induce the same current*
- *there exists an open dense subset $U(L,f)$ of $\mathbb{R}^n$, such that for any $c \in U(L,f)$, we have $\rho := \alpha'_{L-f}(c) \in \mathbb{Q}^n$.*

Proof. **First statement.** Set

$$\mathcal{O}(L) := \bigcap_{\rho\in\mathbb{Q}^n} \mathcal{O}(L,\rho)$$

where $\mathcal{O}(L,\rho)$ comes from Proposition 4.1. Then $\mathcal{O}(L)$ is residual in $C^{l,\gamma}(\mathbb{T}^{n+1})$. Take $f \in \mathcal{O}(L)$. We remark that, if $\rho \in \mathbb{R}^n$ is irrational, then by Lemma 5.1, all the $(L-f, \alpha'_{L-f}(c))$-minimizers induce the same current. On the other hand, if $\rho \in \mathbb{Q}^n$, then by the definition of $\mathcal{O}(L)$ there is only one periodic $(L-f,\rho)$- minimizer; moreover, by lemma 4.4, the (L,ρ)-minimizers induce a unique current. This proves the first part of the theorem.

Second statement. Take $f \in \mathcal{O}(L)$ and $c \in H^n(\mathbb{T}^{n+1})$. Then, if $\alpha'_{L-f}(c) \notin \mathbb{Q}^n$, we know by Lemma 5.1 that there exists a unique $(L-f, \alpha'_{L-f}(c))$-minimizing current, hence there exists a unique $(L-f-c)$-minimizing current. If $\alpha'_{L-f}(c) \in \mathbb{Q}^n$, by the definition of $\mathcal{O}(L)$, all the $(L-f, \alpha'_{L-f}(c))$-minimizers induce the same current, hence there exists a unique $(L-f-c)$-minimizing current. This proves the second part of the theorem.

Third statement. By the first statement above and Theorem 2.1, if $f \in \mathcal{O}(L)$ and $\rho \in \mathbb{Q}^n$, the dimension of $D_\rho(L-f)$ is n. Now $D_\rho(L-f) \subset \mathbb{R}^{n+1}$; let $\mathrm{P}\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection to the first n coordinates, and let $\mathrm{int}(D_\rho(L-f))$ denote the interior of $D_\rho(L-f)$. Since the dimension of $D_\rho(L-f)$ is n, and this set is not vertical, we easily get that $\mathrm{P}\{[\mathrm{int}(D_\rho(L-f))]\}$ is an open set.

Set

$$U(L,f) := \bigcup_{\rho\in\mathbb{Q}^n} \mathrm{P}\{[\mathrm{int}(D_\rho(L-f))]\},$$

then $U(L,f)$ is open in $\mathbb{R}^n$, and it is dense in $\mathbb{R}^n$ by Proposition 2.2. Besides, if $\rho \in \mathbb{Q}^n$ and $c \in \mathrm{P}\{[\mathrm{int}(D_\rho(L-f))]\}$, then by Proposition 4.1 there is a unique periodic $(L-f-c)$-minimizer with slope $\alpha'_{L-f}(c)$. ☐

Appendix A. A bit of topology

We denote by

- $B(0,r)$ the closed ball in $\mathbb{R}^n$ of radius r, centered at the origin
- $\langle .,. \rangle$ the canonical inner product in R^n.

Lemma A.1. *Let f be a continuous map from $\mathbb{R}^n$ to itself, such that for any x in $\mathbb{R}^n \setminus \{0\}$, we have $\langle x, f(x)\rangle > 0$. Then for any neighborhood U of zero in $\mathbb{R}^n$, $f(U)$ contains a neighborhood of zero.*

Proof. By modifying f outside some neighborhood of zero, we may assume that $\|f(x)\|$ goes to infinity when $\|x\|$ goes to infinity. Thus, setting $\tilde{f}(\infty) := \infty$, f extends to a continuous self-map $\tilde{f}$ of $\mathbb{R}^n \cup \{\infty\}$, the one-point compactification of $\mathbb{R}^n$. We identify $\mathbb{R}^n \cup \{\infty\}$ with $\mathbb{S}^n$ by the stereographic projection, i. e. by the map

$$\psi \colon \{(x,z) \quad : \quad x \in \mathbb{R}^n, z \in \mathbb{R}, \quad |x|^2 + z^2 = 1\} \to \mathbb{R}^n$$

defined by

$$\psi(x,z) = \frac{1}{1-z}x.$$

We consider the continuous map $F \colon \mathbb{S}^n \to \mathbb{S}^n$ given by $F(x,z) = \psi^{-1} \circ \tilde{f} \circ \psi(x,z)$. Next, we observe that $\langle \psi(x,z), \psi(-x,-z)\rangle < 0$ save when $x = 0$; in other words, if two points on S^n are diametrically opposite, then the internal product of their ψ-images is negative. In particular, if (x,z) and $F(x,z)$ were diametrically opposite, then we would have that $\langle \psi(x,z), \tilde{f}(\psi(x,z))\rangle < 0$; but this is excluded by our hypotheses on f. Since (x,z) and $F(x,z)$ are never diametrically opposite, F is homotopic to the identity by the shortest geodesic homotopy. Therefore F has degree one, hence it is onto; as a consequence, $\tilde{f}$ is onto too.

Now we want to show that for any $\delta > 0$, there exists $\epsilon > 0$ such that $B(0,\epsilon) \subset f(B(0,\delta))$. Assume not. Then for any positive integer k, there exists x_k, such that $\|x_k\| \leq \frac{1}{k}$ and x_k is not in $f(B(0,\delta))$. Since f is onto, there exists y_k, such that $\|y_k\| \geq \delta$ and $f(y_k) = x_k$. Take a limit point y of y_k in the n-dimensional sphere. Since $||y_k|| \geq \delta$, we have $y \neq 0$. But, since f is continuous, we get $f(y) = 0$, a contradiction with $\langle y, f(y)\rangle > 0$. □

Appendix B. A bit of linear algebra

We say an affine subspace of $\mathbb{R}^n$ is rational if it is defined by equations of the form $\langle c_i, h\rangle = \tau_i$, $i = 1, \ldots s$, where c_i, $i = 1, \ldots s$, are integer vectors, and $\tau_i \in \mathbb{Z}$, $i = 1, \ldots s$. The intersection of two rational affine subspaces is a rational affine subspace, so given $\rho \in \mathbb{R}^n$, there exists a smallest rational affine subspace containing ρ. We denote it $A(\rho)$.

With an abuse of notation, we shall set

$$A(\rho)^\perp = \mathrm{Vect}(c_1, \ldots c_s).$$

In other words, $A(\rho)^\perp$ is the vector space orthogonal to $L(A(\rho))$, i. e. to the smallest space containing the differences $a - b$ with $a, b \in A(\rho)$.

We define $\mathrm{rat}(\rho, 1)$ as the subspace of $\mathbb{R}^n$ generated by $\mathbb{Z}^n \times \mathbb{Z} \cap (\rho, 1)^\perp$; we also define $M(\rho)$ as the projection on $\mathbb{R}^n$ of $\mathrm{rat}(\rho, 1)$. Recall from [Mt09] (Proposition 18) that the irrationality $I_{\mathbb{Z}}(\rho)$ of ρ is the dimension of $A(\rho)$. The next lemma implies that

$$I_{\mathbb{Z}}(\rho) = n - \dim M(\rho) = n - \dim \mathrm{rat}(\rho, 1).$$

Lemma B.1. *For any $\rho \in \mathbb{R}^n$, $M(\rho)$ is the vector subspace $A(\rho)^\perp$ of $\mathbb{R}^n$ orthogonal to $A(\rho)$.*

Proof. Assume $A(\rho)$ is defined by the equations $c_i \cdot v = \tau_i$, with $c_i \in \mathbb{Z}^n$, $\tau_i \in \mathbb{Z}$, $i = 1, \dots k$. Then $A(\rho)^\perp = \mathrm{Vect}(c_1, \dots c_k)$. Recall from [Be09] that $M(\rho)$ is generated by the vectors $k \in \mathbb{Z}^n$ such that $\rho \cdot k \in \mathbb{Z}$. Thus $c_i \in M(\rho)$, $i = 1, \dots k$, whence $A(\rho)^\perp \subset M(\rho)$.

On the other hand, if $k \in \mathbb{Z}^n$ is such that $\rho \cdot k \in \mathbb{Z}$, the equations $c_i \cdot v = \tau_i$, $i = 1, \dots s$ together with $k \cdot v = \rho \cdot k$ define a rational affine subspace B of $\mathbb{R}^n$, containing ρ, and contained in $A(\rho)$, so by the definition of $A(\rho)$, $B = A(\rho)$. Therefore $k \in \mathrm{Vect}(c_1, \dots c_s)$. Since $A(\rho)^\perp = \mathrm{Vect}(c_1, \dots c_s)$, we conclude that $A(\rho)^\perp \supset M(\rho)$.

□

Lemma B.2. *For any $\rho \in \mathbb{R}^n$, $\rho = \rho_1 + \rho_2$, where $\rho_1 \in M(\rho) \cap \mathbb{Q}^n$, $\rho_2 \in M(\rho)^\perp$, and ρ_2 is completely irrational in $M(\rho)^\perp$, that is, ρ_2 is not contained in any proper rational affine subspace of $M(\rho)^\perp$.*

Proof. Let P denote the orthogonal projection on $M(\rho)$; we begin to prove that $P(\rho)$ is rational. Let $w_1, \dots w_k$ be vectors in $\mathbb{Z}^n$ which form a basis of $M(\rho)$; by the definition of $M(\rho)$ we get that $\rho \cdot w_i \in \mathbb{Z}$. Since $P(\rho) \cdot w_i = \rho \cdot w_i$, we get that $P(\rho) \cdot w_i \in \mathbb{Z}$; if we set

$$P(\rho) = a_1 w_1 + \dots a_k w_k, \quad a = (a_1, \dots, a_k), \quad b = (P(\rho) \cdot w_1, \dots, P(\rho) \cdot w_k)$$

and we define W to be the matrix of the internal products $w_i \cdot w_j$, we see that $Wa = b$, i. e. $a = W^{-1} b$. Since W and b have integer entries, this implies that a is rational, which in turn implies that $P(\rho)$ is rational.

Now set $\rho_2 = \rho - P(\rho)$, we have $\rho_2 \in M(\rho)^\perp$. Assume ρ_2 lies in a rational affine subspace B contained in $M(\rho)^\perp$. Then $P(\rho) + B$ is a rational affine subspace of $\mathbb{R}^n$, and it contains ρ, so it contains $A(\rho)$. On the other hand, the dimension of $P(\rho) + B$ is at most $\dim M(\rho)^\perp = \dim A(\rho)$, so $B = M(\rho)^\perp$. Thus ρ_2 is completely irrational in $M(\rho)^\perp$. □

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