Noncommutative Biorthogonal Polynomials

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The idea of orthogonal polynomials is well established and has many applications. For any commutative moments $\{S_i\}$, it is known that a sequence of orthogonal polynomials can be generated by the formula

$$p_n = \begin{vmatrix} S_n & \cdots & S_{2n-1} & x^n \\ \vdots & \ddots & \vdots & \vdots \\ S_0 & \cdots & S_{n-1} & 1 \end{vmatrix}.$$

The idea of orthogonal polynomials and this method of generating them has been generalized in two ways to achieve new types of polynomials: noncommutative orthogonal polynomials and biorthogonal polynomials.

In 1994, Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon[2] extended the theory to situations in which the moments do not commute with one another. This paper generates noncommutative polynomials by setting p_n equal to the quasideterminant of a similar matrix. It also shows that the 3-term recurrence relation, which is well-known for commutative orthogonal polynomials, still holds in this case. As a special case, some work has been done for orthogonal polynomials with matrix coefficients.

Second, orthogonal polynomials have been generalized in several ways to biorthogonal polynomials. Bertola, Gekhtman and Szmigielski[1] describe a family of biorthogonal polynomials as a set of two sequences of real polynomials $\{p_n(x)\}$ and $\{q_m(y)\}$ so that $\int\int p_n(x) * q_m(y) * K(x,y) d\alpha(x) d\beta(y) = 0$ when $n \neq m$ for particular K, α and β. In their paper, it is shown that these polynomials may be generated by taking determinants of matrices whose entries are bimoments and, for a specific $K(x,y)$, a 4-term recurrence relation is obtained. In this paper, we define biorthogonal polynomials in a similar way which can be generalized to noncommutative rings. For our purposes, a biorthogonal family is a set consisting of two sequences of polynomials $\{p_n(x)\}$ and $\{q_n(y)\}$, over a division ring R, along with a function $< \cdot, \cdot >$: $R[x] \times R[y] \to R$ so that $< p_n(x), q_m(y) >= 0$ for all $n \neq m$

In this paper, we bring these two different generalizations together to present a completely algebraic definition of noncommutative biorthogonal polynomials. We then go on to obtain recurrence relations for some types of biorthogonal polynomials as a generalization to the 4-term recurrence relations mentioned in [1] and conclude with a broad extension of Favard's theorem.

1 Set-Up and Definitions

Let R be a division ring with center C. We will view R[x] as an R-C bimodule of R and R[y] as a C-R bimodule of R. That is, elements of R[x] will be of the form $\sum a_i x^i$ and elements of R[y] will be of the form $\sum y^j b_j$ so that $xc = cx$ and $yc = cy$ for all $c \in C$. Let $< \cdot, \cdot >: R[x] \times R[y] \to R$ so that

$$< \sum a_i x^i, \sum y^j b_j >= \sum a_i < x^i, y^j > b_j.$$

A system of polynomials $\{p_n\}, \{q_n\}_{n\in\mathbb{N}}$ is biorthogonal with respect to $< \cdot, \cdot >$ if $< p_n(x), q_m(y) >= 0$ for all $n \neq m$.

Let $I_{a,b} =< x^a, y^b >$. The set $\{I_{a,b}\}_{a,b\in\mathbb{N}}$ is called the set of bimoments for $< \cdot, \cdot >$. The bimoments completely define the function $< \cdot, \cdot >$ so we will say that a set of polynomials is biorthogonal with respect to the bimoments.

In keeping wit the notation of [1], we will let I be the matrix of bimoments and write Id for the identity matrix. Note in these cases, and below, all matrices and vectors are infinite, with rows and columns indexed by $\mathbb{Z}_{\geq 0}$.

We extend $< \cdot, \cdot >$ to $R[x]^n \times R[y]$ and to $R[x] \times R[y]^n$ in the following way:

If $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in R[x]^n$ and $g \in R[y], then < B, g >= \begin{bmatrix} < b_1, g > \\ \vdots \\ < b_n, g > \end{bmatrix}$.

Similarly, if $f \in R[x]$ and $D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in R[y]^n$, then $< f, D >= \begin{bmatrix} < f, d_1 > \\ \vdots \\ < f, d_n > \end{bmatrix}$.

If $C \in Mat_{r\times n}(R), B \in R[x]^n$ and $g \in R[y]$, then $< CB, g >= C < B, g >$.

For an (n+1)x(n+1) matrix A, let $A^{i,j}$ denote the nxn matrix formed by removing the ith row and jth column. Then(c.f. [2], def. 2.1) the i,j-quasideterminant of A $|A|_{i,j}$ is

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & \boxed{a_{i,j}} & \cdots & a_{i,n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,j} & \cdots & a_{n+1,n+1} \end{vmatrix}$$

$$= a_{i,j} - \begin{bmatrix} a_{i,1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{i,n+1} \end{bmatrix} \cdot (A^{i,j})^{-1} \cdot \begin{bmatrix} a_{1,j} \\ \cdots \\ a_{i-1,j} \\ a_{i+1,j} \\ \cdots \\ a_{n+1,j} \end{bmatrix}.$$

2 Constructing Biorthogonal Polynomials Using Bimoments

Throughout, we will assume that several quasideterminants exist. This is our only restriction on the set of bimoments.

Theorem: Let $\{I_{a,b} | a, b \in \mathbb{Z}_{\geq 0}\} \subseteq R$. For all $n \in \mathbb{N}$, define

$$p_n(x) = |I|_{1,n+1} = \begin{vmatrix} I_{n,0} & \cdots & I_{n,n-1} & \boxed{x^n} \\ \vdots & \ddots & \vdots & \vdots \\ I_{1,0} & \cdots & I_{1,n-1} & x \\ I_{0,0} & \cdots & I_{0,n-1} & 1 \end{vmatrix}$$

and

$$q_n(y) = \begin{vmatrix} 1 & y & \cdots & \boxed{y^n} \\ I_{n-1,0} & I_{n-1,1} & \cdots & I_{n-1,n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{0,0} & I_{0,1} & \cdots & I_{0,n} \end{vmatrix}.$$

Then $\{p_n\}, \{q_n\}$ is a (monic) biorthogonal system of polynomials with respect to the set of bimoments $\{I_{a,b}\}$.

To prove the theorem we need the following lemma:

Lemma: Let $n \in \mathbb{Z}_{\geq 0}$ and p_n, q_n be as defined as in the proposition. Then $< x^i, q_n >=< p_n, y^i >= 0$ for all $0 \leq i \leq n-1$.

Proof of Lemma:

Let $n \in \mathbb{N}$ and $0 \leq i \leq n-1$. We see that

$$< p_n, y^i >=< x^n - \begin{bmatrix} I_{n,0} & \cdots & I_{n,n-1} \end{bmatrix} \cdot (I^{1,n+1})^{-1} \cdot \begin{bmatrix} x^{n-1} \\ \vdots \\ 1 \end{bmatrix}, y^i >$$

$=$

$$I_{n,i} - \begin{bmatrix} I_{n,0} & \cdots & I_{n,n-1} \end{bmatrix} \cdot (I^{1,n+1})^{-1} \cdot \begin{bmatrix} I_{n-1,i} \\ \vdots \\ I_{0,i} \end{bmatrix}.$$

Applying the definition of quasideterminant, we see that this is

$$\begin{vmatrix} I_{n,0} & \cdots & I_{n,n-1} & \boxed{I_{n,i}} \\ \vdots & \ddots & \vdots & \vdots \\ I_{0,0} & \cdots & I_{0,n-1} & I_{0,i} \end{vmatrix}.$$

Thus, since $0 \leq i \leq n-1$, $< p_n, y^i >$ is the quasideterminant of a matrix whose nth column is equal to its $(i+1)st$ column and hence is 0 (c.f. [3], prop. 1.4.6).

Similarly,

$$< x^i, q_n >= \begin{vmatrix} I_{i,0} & \cdots & \boxed{I_{i,n}} \\ I_{n-1,0} & \cdots & I_{n-1,n} \\ \vdots & \ddots & \vdots \\ I_{0,0} & \cdots & I_{0,n} \end{vmatrix}.$$

Thus, since $0 \leq i \leq n-1$, the top row will be equal to the $(n-i+1)th$ row, again making the quasideterminant 0 (c.f. [3], prop. 1.4.6).

Proof of Proposition:

Let $n, m \in \mathbb{N}$ so that $n \neq m$. Suppose $n < m$. Now $p_n(x) = \sum_{k=0}^{n} a_k x^k$ for some $a_0, \cdots, a_n \in R$. Thus $< p_n, q_m >= \sum_{k=0}^{n} a_k < x^k, q_m >$. For all $0 \leq k \leq n$, $k < m$ so by the lemma, $< x^k, q_m >= 0$. Thus $< p_n, q_m >= 0$. The case for $n > m$ is similar.

3 Connections to Orthogonal Polynomials:

We note here that we can recover the orthogonal polynomials of Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibbon [2] from the construction above. Let R be the free associative algebra on generators $S_0, S_1, \cdots$ with $S_{a+b} = I_{a,b}$ for all $a, b \in \mathbb{N}$. Following the notation of Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibbon let * be the anti-automorphism so $(S_k)^* = S_k$ and $(\sum c_i x^i)^* = \sum (c_i)^* x^i$. A little examination shows that $q_n = p_n^*$. The collection $\{p_n\}$ is orthogonal with respect to the (very similar) inner product $< \cdot, \cdot >_*$ where $< \sum c_i x^i, \sum d_j y^j >_* = \sum c_i S_{i+j} (d_j)^*$. Thus $< p_n, q_m >=< p_n, p_m >_*$.

4 Banded Matrices:

Let $a \leq 0$ and $b \geq 0$. $M_{[a,b]}$ is defined to be $span\{E_{i,j} : a \leq i - j \leq b\}$. We will refer to these matrices as "banded". For example, the set of diagonal matrices is $M_{[0,0]}$. Let $X \in M_{[a,b]}$ and $Y \in M_{[c,d]}$.

Claim: $X + Y \in M_{[min(a,c),max(b,d)]}$ and $XY \in M_{[a+c,b+d]}$.

Proof:

The proof that $X + Y \in M_{[min(a,c),max(b,d)]}$ is trivial. Suppose $[XY]_{u,v} \neq 0$. Then $[X]_{u,w} \neq 0$ and $[Y]_{w,v} \neq 0$ for some w. This implies $a \leq w - u \leq b$ and $c \leq v - w \leq d$. Adding these equations shows that $a + c \leq v - u \leq b + d$. Thus $XY \in M_{[a+c,b+d]}$.

5 Recurrence Relations:

In the commutative case, Bertola, Gekhtman and Szmigielski [1] achieve a 4 term recurrence relation when $I_{a+1,b} + I_{a,b+1} = \alpha_a \beta_b$. This means there is a formula for p_{n+1} in terms of p_n, p_{n-1}, and p_{n-2} and a similar formula for q_{n+1}.This condition corresponds to what they called the "Cauchy kernel". K is called the kernel of a system of biorthogonal polynomials if $< a(x), b(y) > = \int\int a(x)b(y)K(x,y)dxdy$. The Cauchy kernel is $\dfrac{1}{x+y}$. Below, we achieve similar, but longer, recurrences that would correspond to kernels of the form $\dfrac{1}{f(x)+g(y)}$ where f and g are polynomials.

For all $n \in \mathbb{N}$, let

$$p_n = \begin{vmatrix} I_{n,0} & \cdots & \boxed{I_{n,n}} \\ \vdots & \ddots & \vdots \\ I_{0,0} & \cdots & I_{0,n} \end{vmatrix}^{-1} \cdot \begin{vmatrix} I_{n,0} & \cdots & I_{n,n-1} & \boxed{x^n} \\ \vdots & \ddots & \vdots & \vdots \\ I_{0,0} & \cdots & I_{0,n-1} & 1 \end{vmatrix}$$

and let

$$q_n = \begin{vmatrix} 1 & \cdots & \boxed{y^n} \\ I_{n-1,0} & \cdots & I_{n-1,n} \\ \vdots & \ddots & \vdots \\ I_{0,0} & \cdots & I_{0,n} \end{vmatrix}.$$

These are scalar multiples of the orthogonal polynomials constructed earlier in Section 2. Therefore they are still biorthogonal. A quick check will show that we also have that $< p_n, q_n >= 1$ for all $n \in \mathbb{N}$. Thus this system of polynomials is biorthonormal.

Theorem: Let $\{p_k\}, \{q_k\}$ be any biorthonormal polynomials. Suppose there exist polynomials over the center of R $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(y) = \sum_{j=0}^{m} y^j b_j$ so that $\sum_{i=0}^{n} a_i I_{r+i,s} + \sum_{j=0}^{m} I_{r,s+j} b_j = \alpha_r \beta_s$ for all $r, s \in \mathbb{N}$. Then there exist $n+m+2$ term recurrence relations for p_i and q_i. That is, we can express p_{i+1} in terms of $p_i, \cdots, p_{i-n-m-2}$ and q_{i+1} can be expressed likewise. The recurrences we achieve for p_{i+1} and q_{i+1} have polynomial coefficients for $p_i, p_{i-1}, q_i,$ and q_{i-1} and scalar coefficients for all other terms.

Proof:

Let

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Let $p(x)$ and $q(y)$ be column vectors with entries p_k and q_k respectively. Note that for each $k \in \mathbb{Z}_{\geq 0}$, p_k and q_k are polynomials of degree k so for each so the products $p_k f(x)$ and $g(y) q_k$ can be written as a linear combination of $p_{n+k}, \cdots, p_1, p_0$ and $q_{m+k}, \cdots, q_1, q_0$ respectively.

Let X and Y be the infinite scalar matrices so that $p(x)f(x) = Xp(x)$ and $g(y)q^T(y) = q^T(y)Y^T$. Since $< p(x), q^T(y) >= Id$, we know that $< p(x)f(x), q^T(y) >= X$ and $< p(x), g(y)q^T(y) >= Y^T$.

Suppose $p_k(x) = \sum_{i=0}^{k} c_i x^i$ and $q_l(y) = \sum_{i=0}^{l} y^i d_i$. Let $\pi_k = \sum_{i=0}^{k} c_i \alpha_i$ and $\eta_l = \sum_{i=0}^{l} \beta_i d_i$.

$$(X + Y^T)_{k,l} =< p_k(x)f(x), q_l(y) > + < p_k(x), g(y)q_l(y) >=$$

$$\sum_{i,j} c_i < f(x)x^i, y^j > d_j + \sum_{i,j} c_i < x^i, y^j g(y) > d_j = \sum_{i,j} c_i \alpha_i \beta_j d_j = \pi_k \eta_l.$$

If π and η are vectors with entries π_n and η_n respectively, then $X + Y^T = \pi\eta^T = D_\pi(\underline{1})(\underline{1}^T)D_\eta$ where D_π and D_η are diagonal matrices with (i,i) entries π_i and η_i, respectively.

Let $A = (\Lambda - Id)D_\pi^{-1}X$ and $B^T = Y^T D_\eta^{-1}(\Lambda^T - Id)$. Since $\underline{1}$ is a null vector of $\Lambda - Id$, $(\Lambda - Id)D_\pi^{-1}(X + Y^T) = 0$ and $(X + Y^T)D_\eta^{-1}(\Lambda^T - Id) = 0$. Then $A = -(\Lambda - Id)D_\pi^{-1}Y^T$ and $B^T = -XD_\eta^{-1}(\Lambda^T - Id)$.

We claim that A and B are banded matrices. Note that $X \in M_{[-\infty,n]}$ since $X_{i,j} =< p_i(x)f(x), q_j(y) >= 0$ if $i + n < j$ (because the degree $p_i * f(x)$ is less than the degree of q_j) and that $Y^T \in M_{[-m,\infty]}$ since $Y^T_{i,j} =< p_i(x), g(y)q_j(y) >= 0$ if $i > m + j$. Note also that $(\Lambda - I) \in M_{[0,1]}$.

Applying the results we obtained for banded matrices, we see that $A = (\Lambda - Id)D_\pi^{-1}X \in M_{[-\infty,n+1]}$ and that $A = -(\Lambda - Id)D_\pi^{-1}Y^T \in M_{[-m,\infty]}$. Thus $A \in M_{[-m,n+1]}$. Similarly, $B^T \in M_{[-\infty,m+1]}$ and $B^T \in M_{[-n,\infty]}$ so $B^T \in M_{[-n,m+1]}$ and $B \in M_{[-m-1,n]}$.

Recall that $p(x)f(x) = Xp(x)$ and $g(y)q^T(y) = q^T(y)Y^T$. Then $(\Lambda - Id)D_\pi^{-1}p(x)f(x) = (\Lambda - Id)D_\pi^{-1}Xp(x) = Ap(x)$ and $g(y)q^T(y)D_\eta^{-1}(\Lambda^T - Id) = q^T(y)Y^T D_\eta^{-1}(\Lambda^T - Id) = q^T(y)B^T$.

Thus examining the k-1th row of these equations gives the following n+m+2 term recurrence relations, as desired:

$$(\pi_k^{-1}p_k - \pi_{k-1}^{-1}p_{k-1})f(x) = \sum_{i=k-m}^{k+n+1} A_{k-1,i}p_i,$$

$$g(y)(\eta_k^{-1}q_k - \eta_{k-1}^{-1}q_{k-1}) = \sum_{i=k-n}^{k+m+1} B_{k-1,i}q_i.$$

6 Biorthogonal Analogue of Favard's Theorem:

Favards theorem states that if $\{p_n(x)\}$ is a sequence of polynomials which obeys the usual 3-term recurrence relation then there exists an inner product for which these polynomials are orthogonal. Here we show that any set of two sequences of polynomials are biorthogonal with respect to some function, for which we construct the bimoments. It is important to note that no recurrence relation is required here.

Theorem: Let $\{p_n\}, \{q_n\}$ be any set of polynomials over any division ring R so that p_n and q_n are of degree n for all $n \in \mathbb{N}$. For any $\{c_k\}_{k\in\mathbb{Z}_{\geq 0}}$ in R, there exists a unique set of bimoments for which $\{p_n\}, \{q_n\}$ is a biorthogonal system of polynomials and $< p_k, q_k >= c_k$.

Proof:

It is equivalent to show that there is a set of bimoments so that for all $a, b \in \mathbb{N}$, the following conditions hold:

1) If $a < b$ then $< x^a, q_b(y) >= 0$.

2) If $a > b$ then $< p_a(x), y^b >= 0$.

3) If $a = b$, then $< p_a(x), q_b(y) >= c_a$.

We will define $I_{a,b}$ inductively on $a + b$. It is pivotal to note that the equations $< x^a, q_b(y) >= 0$, $< p_a(x), y^b >= 0$, and $< p_a(x), q_b(y) >= c_a$ do not involve bimoments of the form $I_{i,j}$ where $i + j > a + b$. Recall that $p_0, q_0 \in R$. Let $I_{0,0} = p_0^{-1} c_0 q_0^{-1}$. Then $< p_0, q_0 >= p_0 I_{0,0} q_0 = 1$ as desired.

Let $n \geq 1$ and suppose for all a,b such that $a + b < n$, we have defined $I_{a,b}$ to satisfy the previous conditions. For each $0 \leq i \leq n$ define $I_{i,n-i}$ as follows:

Case 1: If $i < n - i$ then the equation $< x^i, q_{n-i} >$=0 is a linear equation whose variables (the bimoments) have all been defined except for $I_{i,n-i}$ due to the order in which the $I_{a,b}$'s are defined. Therefore there is a unique solution which we must define $I_{i,n-i}$ to be.

Case 2: Similarly, if $i > n-i$, the equation $< p_i, y^{n-i} >$=0 has only one unknown and thus has a unique solution which we define $I_{i,n-i}$ to be.

Case 3: If $i = n - i$ then, again, the equation $< p_i, q_{n-i} >= c_i$ has one unknown and we define $I_{i,n-i}$ to be the unique solution to this linear equation.

At each step we satisfy all the necessary conditions and have no choice so the bimoments constructed are the unique set for which $\{p_n\}, \{q_n\}$ is a biorthogonal system with $< p_k, q_k >= c_k$.

7 Connections to Favard's Theorem for Orthogonal Polynomials:

Claim: Suppose $\{p_n\}, \{q_n\}$ is a system of monic commutative polynomials so that $p_{-1} = 0, p_0 = 1, p_n = q_n$ and

$$xp_n(x) = p_{n+1}(x) + c_{n+1}p_n(x) + d_{n+1}p_{n-1}(x)$$

for all $n \in \mathbb{N}$. If the inner product $< \cdot, \cdot >$ is constructed as in the previous proof and so that

$$< p_n, q_n >= \Pi_{i=1}^{n+1} d_i$$

then the inner product will obey all the properties of an orthogonal inner product, namely that $I_{a+1,b} = I_{a,b+1}$. Then let $S_{a+b+1} = I_{a+1,b} = I_{a,b+1}$.

Proof:

We will proceed by induction on $a+b$.

Suppose $a=b=0$. We would like to show $I_{0,1}=I_{1,0}$. In the construction of the bimoments, $I_{0,1}$ was chosen so that $<1,p_1>=<1,x-c_1>=I_{0,1}-c_1I_{0,0}=0$. Thus $I_{0,1}=c_1I_{0,0}$. $I_{1,0}$ was chosen so that $<p_1,1>=<x-c_1,1>=I_{1,0}-c_1I_{0,0}$. Thus $I_{1,0}=c_1I_{0,0}=I_{0,1}$ as desired.

Now suppose $a+b>0$, and for all n,m so that $n+m<a+b$ we have that $I_{n+1,m}=I_{n,m+1}$.

<u>Case 1:</u> $a=b$. We want to show $I_{a+1,a}=I_{a,a+1}$. $I_{a+1,a}$ was chosen so that

$<p_{a+1},x^a>=<xp_a-c_{a+1}p_a-d_{a+1}p_{a-1},x^a>=<xp_a,x^a>-c_{a+1}<p_a,x^a>=0.$

Suppose $p_a=\sum_{i=0}^{a} r_ix^i$. Then $I_{a+1,a}$ was chosen so that

$$\sum_{i=0}^{a} r_iI_{i+1,a}-c_{a+1}\cdot\sum_{i=0}^{a} r_iI_{i,a}$$

$$=r_aI_{a+1,a}+\sum_{i=0}^{a-1} r_iS_{a+i+1}-c_{a+1}\cdot\sum_{i=0}^{a} r_iS_{i+a}=0.$$

Similarly, $I_{a,a+1}$ was chosen so that

$<x^a,p_{a+1}>=<x^a,xp_a-c_{a+1}p_a-d_{a+1}p_{a-1}>=<x^a,xp_a>-c_{a+1}<x^a,p_a>$

$=\sum_{i=0}^{a} r_iI_{a,i+1}-c_{a+1}\cdot\sum_{i=0}^{a} r_iI_{a,i}=r_aI_{a,a+1}+\sum_{i=0}^{a-1} r_iS_{i+1+a}-c_{a+1}\cdot\sum_{i=0}^{a} r_iS_{i+a}=0.$

We can see that $I_{a,a+1}$ and $I_{a+1,a}$ were determined by the same equation and are thus equal.

<u>Case 2:</u> $a+1=b$. Suppose also that $p_a=\sum r_ix^i$ and $p_b=\sum t_ix^i$. We chose $I_{a+1,b}=I_{a+1,a+1}$ so $<p_{a+1},p_{a+1}>=<xp_a-c_{a+1}p_a-d_{a+1}p_{a-1},p_{a+1}>=<xp_a,p_{a+1}>=\sum_{0\leq i\leq a,0\leq j\leq a+1} r_iI_{i+1,j}t_j=\sum r_iS_{i+j+1}t_j+r_aI_{a+1,a+1}t_a=\Pi_{i=0}^{a+1}d_i$. $I_{a,b+1}$ was chosen so that $<x^a,p_{b+1}>=0$. Multiplying the equation by the ath coefficient of p_a we chose $I_{a,b+1}$ so that $<p_a,p_{b+1}>=<p_a,p_{a+2}>=<p_a,xp_{a+1}-c_{a+2}p_{a+1}-d_{a+1}p_a>=<p_a,xp_{a+1}>-d_{a+1}<p_a,p_a>=<p_a,xp_{a+1}>$

$- < p_{a+1}, p_{a+1} >= 0$. Under our initial assumptions, $I_{a,b+1}$ was chosen so that $< p_a, xp_{a+1} >= \sum_{0 \leq i \leq a, 0 \leq j \leq a+1} r_i I_{i,j+1} t_j = \sum r_i S_{i+j+1} t_j + r_a I_{a,a+2} t_a = \Pi_{i=0}^{a+1} d_i$. Again, examining the equations which determine $I_{a,b+1}$ and $I_{a+1,b}$, we see that they are equal.

<u>Case 3:</u> $|a - b| \geq 2$ and $a < b$. We have that $< x^{a+1}, p_b >= 0$. Suppose again that $p_b = \sum t_i x^i$. $< x^{a+1}, p_b >= \sum_{i=0}^{b} I_{a+1,i} t_i = \sum_{i=0}^{b-1} S_{a+i+1} t_i + I_{a+1,b} t_b = 0$. We have that $< x^a, p_{b+1} >=< x^a, xp_b - c_{b+1} p_b - d_{b+1} p_{b-1} >=< x^a, xp_b >= \sum_{i=0}^{b} I_{a,i+1} t_i = \sum_{i=0}^{b-1} S_{a+i+1} t_i + I_{a,b+1} t_b = 0$. We can see once again that $I_{a+1,b} = I_{a,b+1}$ as desired.

The cases for $a > b$ are very similar to cases 2 and 3.

By induction we have shown that $I_{a+1,b} = I_{a,b+1}$ for all $a, b \in \mathbb{N}$.

Note that the mirror of this inner product $< a, b >_{-1} =< b, a >$ has all of the above properties and so, by uniqueness, the inner product $< \cdot, \cdot >$ is symmetric.

8 Conclusion

This paper has established the beginnings of a theory of noncommutative biorthogonal polynomials. The generalization of Favard's theorem given in section 7 suggests both that the theory could be of use and that this is as general as should ever be necessary. There are still many other ideas from the theory of commutative orthogonal polynomials that may apply to noncommutative biorthogonal polynomials. In particular, it maybe possible to identify other constraints that allow for a finite-term recurrence relation or it may be possible to generalize some identities that apply to orthogonal polynomials.

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