CLASSIFYING C^*-ALGEBRAS WITH BOTH FINITE AND INFINITE SUBQUOTIENTS

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ABSTRACT. We give a classification result for a certain class of C^*-algebras $\mathfrak{A}$ over a finite topological space X in which there exists an open set U of X such that U separates the finite and infinite subquotients of $\mathfrak{A}$. We will apply our results to C^*-algebras arising from graphs.

1. INTRODUCTION

Just like a finite group, any C^*-algebra $\mathfrak{A}$ with finitely many ideals has a decomposition series

$$0 = \mathfrak{I}_0 \lhd \mathfrak{I}_1 \lhd \cdots \lhd \mathfrak{I}_n = \mathfrak{A}$$

in such a way that all subquotients are simple. As in the group case, the simple subquotients are unique up to permutation of isomorphism classes, but far from determine the isomorphism class of $\mathfrak{A}$.

If we further assert that all simple subquotients are classifiable by algebraic invariants such as K-theory we are naturally lead to the pertinent question of which algebraic invariants, if any, classify all of $\mathfrak{A}$. This question has previously been studied, leading to a complete solution when all subquotients are AF (cf. [14]), and a partial solution when all are purely infinite (cf. [21], [3], [25]), but in the case when some are of one type and some of another, only sporadic results have been found. It is the purpose of this paper to provide a general framework in which classification of C^*-algebras can be proved for a large class of C^*-algebras of mixed type.

We are able to do so by combining several recent important developments in classification theory, notably

- Kirchberg's isomorphism result [15]
- The corona factorization property [16]
- The universal coefficient theorem of Meyer and Nest [21]

with refinements of our previous work [11] inspired by an idea of Rørdam [27].

As useful as these problems may be to test the borders of our understanding of classification we are driven to this project by one class of examples. A **graph C^*-algebra** has the property that all simple subquotients are either AF or purely infinite, and examples of mixed type occur even for very small graphs. For instance, consider the graphs $\{E_n\}_{n \in \mathbb{N}}$ given below

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$$\begin{bmatrix} 0 & 0 & 0 \\ n & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$
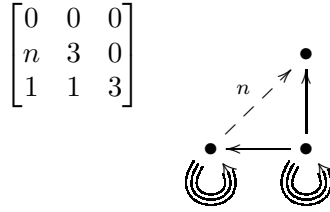
For any $n > 0$, $C^*(E_n)$ decomposes into a linear lattice with simple subquotients

$$\mathbb{K}, \mathcal{O}_3 \otimes \mathbb{K}, \mathcal{O}_3.$$

The results presented in this paper show that $C^*(E_n) \otimes \mathbb{K} \simeq C^*(E_{n+4}) \otimes \mathbb{K}$ and that there are three stable isomorphism classes

$$[C^*(E_1)] = [C^*(E_3)], [C^*(E_2)], [C^*(E_4)]$$

As in our previous paper [11] the technical focal point in this work is the general question of when one can deduce from the fact that $\mathfrak{A}$ and $\mathfrak{B}$ in the extension

$$0 \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{A} \longrightarrow 0$$

are classifiable by K-theory, that the same is true for $\mathfrak{E}$. We shall fix a finite (not necessarily Hausdorff) topological space X with an open subet $U \subseteq X$ and require that $\mathfrak{E}$ is a C^*-algebra over X – associating ideals in $\mathfrak{E}$ with open subsets of X – in such a way that $\mathfrak{B}$ is the C^*-algebra corresponding to U. Assuming then that $\mathfrak{A}$ and $\mathfrak{B}$ are KK-strongly classifiable by their filtered and ordered K-theories over $X \backslash U$ and U, respectively, we supply conditions on the extension securing that also $\mathfrak{E}$ is classifiable by filtered and ordered K-theory. Our key technical result to this effect, Theorem 3.7 below, provides stable isomorphism in this context under, among other things, the provision of fullness of the extension and KK-liftability of morphisms of filtered K-theory. KK-liftability follows in many cases by the UCT of Meyer and Nest [21] as generalized by Bentmann and Köhler [3], and we develop in Section 5 several useful tools to establish fullness.

Although we are confident that Theorem 3.7 will apply in other settings as well, we restrict ourselves to demonstrating how the results lead to classification of certain graph C^*-algebras up to stable isomorphism, generalizing results by the first named author and Tomforde in [13]. As a consequence of the results in [13], all graph C^*-algebras with exactly one non-trivial ideal are classifiable up to stable isomorphism by the six-term exact sequence in K-theory together with the positive cone of the ideal, algebra, and quotient, irrespective of the types of the simple subquotients. Indicating in a (Hasse) diagram of the ideal lattice a simple AF subquotient with a straight line, and a purely infinite subquotient with a curly line (the zero ideal indicated by "$\circ$"), the results in [13] solved all the cases
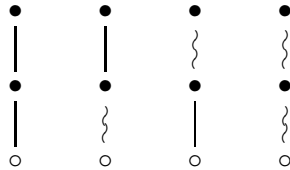
Further, for graph C^*-algebras with a maximal proper ideal which is AF, indicated diagrammatically as
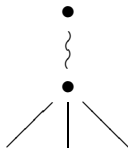
are classifiable up to stable isomorphism by the six-term exact sequence in K-theory together with the positive cone of the ideal, algebra, and quotient. In this paper we generalize to the settings
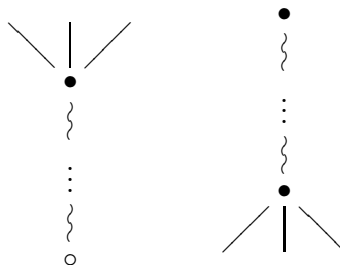
where the ideal lattice at one end (top or bottom) has a finite linear lattice of purely infinite C^*-algebras, and at the other has an AF algebra. In particular, we cover all graph C^*-algebras of finite linear lattices which has no more than one transition from finite to infinite subquotient, such as $C^*(E_n)$ defined above. We speculate that this condition is not necessary (although it certainly is for our approach) and in [12] give partial evidence for this, under extra conditions on the algebraic nature of the involved K-theory.

2. Notation and Conventions

We first start with some definitions and conventions that will be used throughout the paper.

2.1. C^*-**algebras over topological spaces.** Let X be a topological space and let $\mathbb{O}(X)$ be the set of open subsets of X, partially ordered by set inclusion $\subseteq$. A subset Y of X is called *locally closed* if $Y = U \setminus V$ where $U, V \in \mathbb{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of X will be denoted by $\mathbb{LC}(X)$. The set of all connected, non-empty locally closed subsets of X will be denoted by $\mathbb{LC}(X)^*$.

The partially ordered set $(\mathbb{O}(X), \subseteq)$ is a *complete lattice*, that is, any subset S of $\mathbb{O}(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset S of $\mathbb{O}(X)$,

$$\bigwedge_{U \in S} U = \left(\bigcap_{U \in S} U \right)^{\circ} \quad \text{and} \quad \bigvee_{U \in S} U = \bigcup_{U \in S} U$$

For a C^*-algebra $\mathfrak{A}$, let $\mathbb{I}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathbb{I}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset S of $\mathbb{I}(\mathfrak{A})$,

$$\bigwedge_{\mathfrak{I} \in S} \mathfrak{I} = \bigcap_{\mathfrak{I} \in S} \mathfrak{I} \quad \text{and} \quad \bigvee_{\mathfrak{I} \in S} \mathfrak{I} = \overline{\sum_{\mathfrak{I} \in S} \mathfrak{I}}$$

Definition 2.1. Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathrm{Prim}(\mathfrak{A})$ denote the *primitive ideal space* of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let X be a topological space. A C^*-*algebra over X* is a pair $(\mathfrak{A}, \psi)$ consisting of a C^*-algebra $\mathfrak{A}$ and a continuous map $\psi : \mathrm{Prim}(\mathfrak{A}) \to X$. A C^*-algebra over X, $(\mathfrak{A}, \psi)$, is *separable* if $\mathfrak{A}$ is a separable C^*-algebra. We say that $(\mathfrak{A}, \psi)$ is *tight* if ψ is a homeomorphism.

We always identify $\mathbb{O}(\mathrm{Prim}(\mathfrak{A}))$ and $\mathbb{I}(\mathfrak{A})$ using the lattice isomorphism

$$U \mapsto \bigcap_{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \setminus U} \mathfrak{p}$$

Let $(\mathfrak{A}, \psi)$ be a C^*-algebra over X. Then we get a map $\psi^* : \mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}(\mathfrak{A})) \cong \mathbb{I}(\mathfrak{A})$ defined by

$$U \mapsto \{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \: : \: \psi(\mathfrak{p}) \in U\} = \mathfrak{A}(U)$$

For $Y = U \setminus V \in \mathbb{LC}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By Lemma 2.15 of [20], $\mathfrak{A}(Y)$ does not depend on U and V.

In this paper, we will be mainly interested in the following examples:

Example 2.2. For any C^*-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \mathrm{id}_{\mathrm{Prim}(\mathfrak{A})})$ is a tight C^*-algebra over $\mathrm{Prim}(\mathfrak{A})$. For each $U \in \mathbb{O}(\mathrm{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap_{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \setminus U} \mathfrak{p}$.

Example 2.3. Let $X_n = \{1, 2, \ldots, n\}$ partially ordered with $\leq$. Equip X_n with the Alexandrov topology, so the non-empty open subsets are

$$[a, n] = \{x \in X \: : \: a \leq x \leq n\}$$

for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathfrak{A}, \phi)$ be a C^*-algebra over X_n. We will use the following notation throughout the paper:

$$\mathfrak{A}[k] = \mathfrak{A}(\{k\}), \ \mathfrak{A}[a, b] = \mathfrak{A}([a, b]), \ \text{and} \ \mathfrak{A}(i, j) = \mathfrak{A}[i + 1, j].$$

Using the above notation we have ideals $\mathfrak{A}[a, n]$ such that

$$\{0\} \trianglelefteq \mathfrak{A}[n] \trianglelefteq \mathfrak{A}[n-1, n] \trianglelefteq \cdots \trianglelefteq \mathfrak{A}[2, n] \trianglelefteq \mathfrak{A}[1, n] = \mathfrak{A}$$

Definition 2.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras over X. A homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is *X-equivariant* if $\phi(\mathfrak{A}(U)) \subseteq \mathfrak{B}(U)$ for all $U \in \mathbb{O}(X)$. Hence, for every $Y = U \setminus V$, ϕ induces a homomorphism $\phi_Y : \mathfrak{A}(Y) \to \mathfrak{B}(Y)$. Let $\mathfrak{C}^*\text{-}\mathfrak{alg}(X)$ be the category whose objects are C^*-algebras over X and whose morphisms are X-equivariant homomorphisms.

Remark 2.5. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are tight C^*-algebras over X_n. Then it is clear that a $*$-homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism if and only if ϕ is a X_n-equivariant isomorphism. We will use this fact in Theorem 6.9.*

Remark 2.6. *Let $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be an extension for $i = 1, 2$. Note that $\mathfrak{E}_i$ can be considered as a C^*-algebra over $X_2 = \{1, 2\}$ by sending $\emptyset$ to the zero ideal, $\{2\}$ to the image of $\mathfrak{B}_i$ in $\mathfrak{E}_i$, and $\{1, 2\}$ to $\mathfrak{E}_i$. Hence, there exists one-to-one correspondence between X_2-equivariant homomorphisms $\phi : \mathfrak{E}_1 \to \mathfrak{E}_2$ and homomorphisms from $\mathfrak{e}_1$ and $\mathfrak{e}_2$.*

2.2. **Filtered ordered K-theory.**

Definition 2.7. Let X be a finite topological space and let $\mathfrak{A}$ be a C^*-algebra over X. For open subsets U_1, U_2, U_3 of X with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \setminus U_1, Y_2 = U_3 \setminus U_1, Y_3 = U_3 \setminus U_1 \in \mathbb{LC}(X)$. Then we have a six term exact sequence

$$
\begin{array}{ccc}
K_0(\mathfrak{A}(Y_1)) \xrightarrow{\iota_*} K_0(\mathfrak{A}(Y_2)) \xrightarrow{\pi_*} K_0(\mathfrak{A}(Y_3)) \\
\partial_* \uparrow \qquad\qquad\qquad\qquad\qquad \downarrow \partial_* \\
K_1(\mathfrak{A}(Y_3)) \xleftarrow{\pi_*} K_1(\mathfrak{A}(Y_2)) \xleftarrow{\iota_*} K_1(\mathfrak{A}(Y_1))
\end{array}
$$

The *filtered K-theory* $\mathrm{FK}_X(\mathfrak{A})$ *of* $\mathfrak{A}$ is the collection of all K-groups thus occurring and the natural transformations $\{\iota_*, \pi_*, \partial_*\}$. The *filtered, ordered K-theory* $\mathrm{FK}_X^+(\mathfrak{A})$ *of* $\mathfrak{A}$ is $\mathrm{FK}_X(\mathfrak{A})$ of $\mathfrak{A}$ together with $K_0(\mathfrak{A}(Y))_+$ for all $Y \in \mathbb{LC}(X)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras over X, we will say that $\alpha : \mathrm{FK}_X(\mathfrak{A}) \to \mathrm{FK}_X(\mathfrak{B})$ is an *isomorphism* if for all $Y \in \mathbb{LC}(X)$, there exist group isomorphisms

$$
\alpha_{Y,*} : K_*(\mathfrak{A}(Y)) \to K_*(\mathfrak{B}(Y))
$$

preserving all natural transformations. We say that $\alpha : \mathrm{FK}_X^+(\mathfrak{A}) \to \mathrm{FK}_X^+(\mathfrak{B})$ is an *isomorphism* if there exists an isomorphism $\alpha : \mathrm{FK}_X(\mathfrak{A}) \to \mathrm{FK}_X(\mathfrak{B})$ in such a way that $\alpha_{Y,0}$ is an order isomorphism for all $Y \in \mathbb{LC}(X)$.

If $Y \in \mathbb{LC}(X)$ such that $Y = Y_1 \sqcup Y_2$ with two disjoint relatively open subsets $Y_1, Y_2 \in \mathbb{O}(Y) \subseteq \mathbb{LC}(X)$, then $\mathfrak{A}(Y) \cong \mathfrak{A}(Y_1) \times \mathfrak{A}(Y_2)$ for any C^*-algebra over X. Moreover, there is a natural isomorphism $K_*(\mathfrak{A}(Y))$ to $K_*(\mathfrak{A}(Y_1)) \oplus K_*(\mathfrak{A}(Y_2))$ which is a positive isomorphism from $K_0(\mathfrak{A}(Y))$ to $K_0(\mathfrak{A}(Y_1)) \oplus K_0(\mathfrak{A}(Y_2))$. If X is finite, any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace $\mathbb{LC}(X)$ by the subset $\mathbb{LC}(X)^*$. This observation reduces computation and will be used in Section 7.

3. The $KK(X_2; -, -)$ functor

Let a be an element of a C^*-algebra $\mathfrak{A}$. We say that a is *norm-full in* $\mathfrak{A}$ if a is not contained in any norm-closed proper ideal of $\mathfrak{A}$. The word "full" is also widely used, but since we will often work in multiplier algebras, we emphasize that it is the norm topology we are using, rather than the strict topology. We say that a sub-C^*-algebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is *norm-full* if the norm-closed ideal generated by $\mathfrak{B}$ is $\mathfrak{A}$.

Definition 3.1. An extension $\mathfrak{e}$ is said to be *full* if the associated Busby invariant $\tau_{\mathfrak{e}}$ has the property that $\tau_{\mathfrak{e}}(a)$ is a norm-full element of $\mathcal{Q}(\mathfrak{B}) = \mathcal{M}(\mathfrak{B})/\mathfrak{B}$ for every $a \in \mathfrak{A}\backslash\{0\}$.

Definition 3.2. A homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is *proper* (cf. [10]) if one of the following equivalent properties hold:

(1) $\mathfrak{B} = \overline{\phi(\mathfrak{A})\mathfrak{B}} = \overline{\mathfrak{B}\phi(\mathfrak{A})}$.

(2) $\phi(\mathfrak{A})$ is not contained in any proper, hereditary sub-C^*-algebra of $\mathfrak{B}$.

(3) The image under ϕ of an approximate identity of $\mathfrak{A}$ is an approximate identity of $\mathfrak{B}$.

Lemma 3.3. *Let $\mathfrak{A}$ be a stable, separable, nuclear, C^*-algebra satisfying the UCT with a norm-full projection. Then there exists a simple, stable, separable, nuclear, purely infinite C^*-algebra $\mathfrak{B}$ satisfying the UCT, and a proper injective homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $K_i(\phi)$ is an isomorphism for $i = 0, 1$.*

Proof. Let p be a norm-full projection in $\mathfrak{A}$. Since $p\mathfrak{A}p$ is norm-full in $\mathfrak{A}$ and $\mathfrak{A}$ is stable, by Theorem 2.6 of [6] there exists an isomorphism $\psi : \mathfrak{A} \to p\mathfrak{A}p \otimes \mathbb{K}$.

Since $p\mathfrak{A}p$ is a separable C^*-algebra, there exists a unital, separable, nuclear, simple, purely infinite C^*-algebra $\mathfrak{B}_0$ satisfying the UCT such that $(K_0(p\mathfrak{A}p), K_1(p\mathfrak{A}p), [p])$ is isomorphic to $(K_0(\mathfrak{B}_0), K_1(\mathfrak{B}_0), [1_{\mathfrak{B}_0}])$. Hence, by Theorem 6.7 of [18], there exists a unital injective homomorphism $\beta : p\mathfrak{A}p \to \mathfrak{B}_0$ such that $K_i(\beta)$ is an isomorphism for $i = 0, 1$.

Set $\phi = (\beta \otimes \mathrm{id}_{\mathbb{K}}) \circ \psi$. Then ϕ is proper injective homomorphism and $K_i(\phi)$ is an isomorphism for $i = 0, 1$. $\qquad\square$

Lemma 3.4. *Let $\mathfrak{B}$ be a stable, separable, nuclear, purely infinite, simple C^*-algebra and let $\mathfrak{A}$ be a stable, separable, nuclear, C^*-algebra satisfying the UCT. Let*

$$\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$$

be an essential extension. Then there exist a separable, stable, nuclear, purely infinite simple C^*-algebra $\mathfrak{A}'$ satisfying the UCT, an injective homomorphism $\beta : \mathfrak{A} \to \mathfrak{A}'$, and an essential extension

$$\mathfrak{e}' : 0 \to \mathfrak{B} \to \mathfrak{E}' \to \mathfrak{A}' \to 0$$

such that $K_i(\beta)$ is an isomorphism for $i = 0, 1$ and

$$\tau_{\mathfrak{e}} = \tau_{\mathfrak{e}'} \circ \beta$$

Consequently, we have a X_2-equivariant homomorphism $\phi : \mathfrak{E} \to \mathfrak{E}'$ such that $\phi_{\{2\}} = \mathrm{id}_{\mathfrak{B}}$ and $\phi_{\{1\}} = \beta$.

Proof. Let $\mathfrak{A}'$ be a separable, nuclear, stable, purely infinite simple C^*-algebra satisfying the UCT such that $K_i(\mathfrak{A}')$ is isomorphic to $K_i(\mathfrak{A})$. By Theorem 6.7 of [18], there exists an injective homomorphism $\beta : \mathfrak{A} \to \mathfrak{A}'$ such that $K_i(\beta)$ is an isomorphism.

Since $\mathfrak{A}$ and $\mathfrak{A}'$ satisfy the UCT and $K_i(\beta)$ is an isomorphism, $KK(\beta) \in KK(\mathfrak{A}, \mathfrak{A}')$ is invertible. Therefore, $KK(\beta)^{-1} \in KK(\mathfrak{A}', \mathfrak{A})$ exists. Note that $KK(\beta)^{-1} \times [\tau_{\mathfrak{e}}]$ is an element of $KK^1(\mathfrak{A}', \mathfrak{B})$. Therefore, there exists an essential extension $\tau : \mathfrak{A}' \to \mathcal{Q}(\mathfrak{B})$ such that $KK(\beta)^{-1} \times [\tau_{\mathfrak{e}}] = [\tau]$. Hence, $[\tau_{\mathfrak{e}}] = [\tau \circ \beta]$ in $KK^1(\mathfrak{A}, \mathfrak{B})$. Since $\mathfrak{B}$

is a stable, purely infinite simple C^*-algebra and $\tau_{\mathfrak{e}}$ and $\tau \circ \beta$ are non-unital essential extensions, by Theorem 3.2 (2) of [22], there exists a unitary $u \in \mathcal{Q}(\mathfrak{B})$ such that

$$\tau_{\mathfrak{e}} = \mathrm{Ad}(u) \circ \tau \circ \beta.$$

Let $\mathfrak{e}' : 0 \to \mathfrak{B} \to \mathfrak{E}' \to \mathfrak{A}' \to 0$ be the extension determined by $\mathrm{Ad}(u) \circ \tau$. The last statement follows from the construction of $\tau_{\mathfrak{e}'}$. $\qquad\square$

We now define some functors that will be used throughout the rest of the paper. Let X and Y be topological spaces. For every continuous function $f : X \to Y$ we have a functor

$$f : \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(Y), \quad (A, \psi) \mapsto (A, f \circ \psi)$$

(1) Define $g_X^1 : X \to X_1$ by $g_X^1(x) = 1$. Then g_X^1 is continuous. Note that the induced functor $g_X^1 : \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(X_1)$ is the forgetful functor.

(2) Let U be an open subset of X. Define $g_{U,X}^2 : X \to X_2$ by $g_{U,X}^2(x) = 1$ if $x \notin U$ and $g_{U,X}^2(x) = 2$ if $x \in U$. Then $g_{U,X}^2$ is continuous. Thus the induced functor

$$g_{U,X}^2 : \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(X_2)$$

is just specifying the extension $0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0$.

(3) We can generalize (2) to finitely many ideals. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X$ be open subsets of X. Define $g_{U_1,U_2,\ldots,U_n,X}^n : X \to X_n$ by $g_{U_1,U_2,\ldots,U_n,X}^n(x) = n - k + 1$ if $x \in U_k \setminus U_{k-1}$. Then $g_{U_1,U_2,\ldots,U_n,X}^n$ is continuous. Therefore, any C^*-algebra with ideals $0 \trianglelefteq \mathfrak{I}_1 \trianglelefteq \mathfrak{I}_2 \trianglelefteq \cdots \mathfrak{I}_n = \mathfrak{A}$ can be made into a C^*-algebra over X_n.

(4) For all $Y \in \mathbb{LC}(X)$, $r_X^Y : \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(Y)$ is the restriction functor defined in Definition 2.19 of [20]

Let $\mathfrak{KK}(X)$ be the category whose objects are separable C^*-algebras over X and the set of morphisms is $KK(X; \mathfrak{A}, \mathfrak{B})$. By Proposition 3.4 of [20], these functors induce functors from $\mathfrak{KK}(X)$ to $\mathfrak{KK}(Z)$, where $Z = Y, X_1, X_n$.

Lemma 3.5. *Let U be an open set of X and $Y = X \setminus U$. Then*

$$r_{X_2}^{\{2\}} \circ g_{U,X}^2 = g_U^1 \circ r_X^U \qquad \text{and} \qquad r_{X_2}^{\{1\}} \circ g_{U,X}^2 = g_Y^1 \circ r_X^Y$$

from $\mathfrak{C}^*\text{-}\mathfrak{alg}(X)$ to $\mathfrak{C}^*\text{-}\mathfrak{alg}(X_1)$. Consequently, the induced functors from $\mathfrak{KK}(X)$ to $\mathfrak{KK}(X_1)$ will be equal.

Proof. Let $\mathfrak{A}$ be a C^*-algebra over X. Then

$$r_{X_2}^{\{2\}} \circ g_{U,X}^2(\mathfrak{A}) = \mathfrak{A}(U)$$

and

$$g_U^1 \circ r_X^U(\mathfrak{A}) = g_U^1(\mathfrak{A}(U)) = \mathfrak{A}(U).$$

Suppose $\mathfrak{B}$ is a C^*-algebra over X and $\phi : \mathfrak{A} \to \mathfrak{B}$ is an X-equivariant homomorphism. Then

$$r_{X_2}^{\{2\}} \circ g_{U,X}^2(\phi) = \phi_U$$

and

$$g_U^1 \circ r_X^U(\phi) = g_U^1(\phi_U) = \phi_U.$$

Therefore, $r_{X_2}^{\{2\}} \circ g_{U,X}^2 = g_U^1 \circ r_X^U$.

Similar computation shows that $r_{X_2}^{\{1\}} \circ g_{U,X}^2 = g_Y^1 \circ r_X^Y$. $\qquad\square$

Lemma 3.6. *Let $\mathfrak{A}$, $\mathfrak{B}_1$, and $\mathfrak{B}_2$ be C^*-algebras over X_2. For $i = 1, 2$, let*

$$\mathfrak{e} \ : \ 0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0$$
$$\mathfrak{e}_i \ : \ 0 \to \mathfrak{B}_i[2] \to \mathfrak{B}_i \to \mathfrak{B}_i[1] \to 0$$

be extensions. Suppose there exists an X_2-equivariant homomorphism $\phi : \mathfrak{B}_1 \to \mathfrak{B}_2$ such that $\phi_{\{2\}}$ extends to $\mathcal{M}(\mathfrak{B}_1[2])$ to $\mathcal{M}(\mathfrak{B}_2[2])$ and suppose $KK(\phi_{\{2\}})$ and $KK(\phi_{\{1\}})$ are invertible elements.

(1) *Let $x \in KK(X_2; \mathfrak{A}, \mathfrak{B}_1)$. Then*

$$r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}_1}] = [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x)$$

if and only if

$$r_{X_2}^{\{1\}}(x \times KK(X_2; \phi)) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x \times KK(X_2; \phi))$$

(2) *Let $y \in KK(X_2; \mathfrak{B}_2, \mathfrak{A})$. Then*

$$r_{X_2}^{\{1\}}(y) \times [\tau_{\mathfrak{e}}] = [\tau_{\mathfrak{e}_2}] \times r_{X_2}^{\{2\}}(y)$$

if and only if

$$r_{X_2}^{\{1\}}(KK(X_2; \phi) \times y)) \times [\tau_{\mathfrak{e}}] = [\tau_{\mathfrak{e}_1}] \times r_{X_2}^{\{2\}}(KK(X_2; \phi) \times y)$$

Proof. Since $\phi_{\{2\}}$ extends to $\mathcal{M}(\mathfrak{B}_1[2])$ to $\mathcal{M}(\mathfrak{B}_2[2])$, as in the proof of Theorem 2.2 of [10], we have that $\tau_{\mathfrak{e}_2} \circ \phi_{\{1\}} = \widetilde{\phi}_{\{2\}} \circ \tau_{\mathfrak{e}_1}$, where $\widetilde{\phi}_{\{2\}} : \mathcal{Q}(\mathfrak{B}_1[2]) \to \mathcal{Q}(\mathfrak{B}_2[2])$ is the homomorphism induced by $\phi_{\{2\}}$. Hence, $KK(\phi_{\{1\}}) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times KK(\phi_{\{2\}})$.

Note that for $z \in KK(X_2, \mathfrak{A}, \mathfrak{B}_1)$ we have that

$$r_{X_2}^{\{i\}}(z \times KK(X_2, \phi)) = r_{X_2}^{\{i\}}(z) \times KK(\phi_{\{i\}})$$

Let $x \in KK(X_2; \mathfrak{A}, \mathfrak{B}_1)$. Suppose $r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}_1}] = [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x)$. Then

$$
\begin{aligned}
r_{X_2}^{\{1\}} &(x \times KK(X_2; \phi)) \times [\tau_{\mathfrak{e}_2}] \\
&= r_{X_2}^{\{1\}}(x) \times KK(\phi_{\{1\}}) \times [\tau_{\mathfrak{e}_2}] \\
&= r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}_1}] \times KK(\phi_{\{2\}}) \\
&= [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x) \times KK(\phi_{\{2\}}) \\
&= [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x \times KK(X_2; \phi))
\end{aligned}
$$

Suppose $r_{X_2}^{\{1\}}(x \times KK(X_2; \phi)) \times [\tau_{\mathfrak{e}2}] = [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x \times KK(X_2; \phi))$. Then

$$r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}1}] \times KK(\phi_{\{2\}})$$
$$= r_{X_2}^{\{1\}}(x) \times KK(\phi_{\{1\}}) \times [\tau_{\mathfrak{e}2}]$$
$$= r_{X_2}^{\{1\}}(x \times KK(X_2; \phi)) \times [\tau_{\mathfrak{e}2}]$$
$$= [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x \times KK(X_2; \phi))$$
$$= [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x) \times KK(\phi_{\{2\}})$$

Since $KK(\phi_{\{2\}})$ is invertible, we have that $r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}1}] = [\tau_{\mathfrak{e}}] \times r_{X_2}^{\{2\}}(x)$.
(2) is proved in a similar way and is left to the reader. $\qquad\square$

The following theorem is a generalization of Theorem 2.3 of [11]. This is the key technical theorem of the paper.

Theorem 3.7. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras over X_2. Suppose $\mathfrak{A}_i[k]$ is a separable, nuclear C^*-algebra satisfying the UCT and $\mathfrak{A}_i[2]$ contains at least one norm-full projection for $i, k = 1, 2$. Suppose $\mathfrak{e}_i : 0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is an essential extension for $i = 1, 2$. If $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ is invertible, then*

$$r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}2}] = [\tau_{\mathfrak{e}1}] \times r_{X_2}^{\{2\}}(x)$$

in $KK^1(\mathfrak{A}_1[1], \mathfrak{A}_2[2])$.

Proof. For any C^*-algebra $\mathfrak{C}$, let $\iota_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes \mathbb{K}$ be the embedding that sends x to $x \otimes e$, where e is a rank one projection. Since $\iota_{\mathfrak{C}}(\mathfrak{C}) = (1_{\mathcal{M}(\mathfrak{C})} \otimes e)(\mathfrak{C} \otimes \mathbb{K})(1_{\mathcal{M}(\mathfrak{C})} \otimes e)$ and $\iota_{\mathfrak{C}}$ sends an approximate identity of $\mathfrak{C}$ to an approximate identity of $(1_{\mathcal{M}(\mathfrak{C})} \otimes e)(\mathfrak{C} \otimes \mathbb{K})(1_{\mathcal{M}(\mathfrak{C})} \otimes e)$, $\iota_{\mathfrak{C}}$ extends to a homomorphism from $\mathcal{M}(\mathfrak{C})$ to $(1_{\mathcal{M}(\mathfrak{C})} \otimes e)\mathcal{M}(\mathfrak{C} \otimes \mathbb{K})(1_{\mathcal{M}(\mathfrak{C})} \otimes e) \subseteq \mathcal{M}(\mathfrak{C} \otimes \mathbb{K})$. Note also that $KK(\iota_{\mathfrak{C}})$ is invertible. Moreover, if $\mathfrak{J}$ is an ideal of $\mathfrak{C}$ then $\mathfrak{J} \otimes \mathbb{K}$ is an ideal of $\mathfrak{C} \otimes \mathbb{K}$ such that $\iota_{\mathfrak{C}}$ maps $\mathfrak{J}$ into $\mathfrak{J} \otimes \mathbb{K}$.

By the above observation, $\iota_{\mathfrak{A}_i}$ is an X_2-equivariant homomorphism such that $(\iota_{\mathfrak{A}_i})_{\{2\}}$ extends to a homomorphism from $\mathcal{M}(\mathfrak{A}_i[2])$ to $\mathcal{M}((\mathfrak{A}_i \otimes \mathbb{K})[2])$ and $KK(X_2; \iota_{\mathfrak{A}_i})$ is invertible. Therefore, by Lemma 3.6, we may assume that $\mathfrak{A}_i$ is a stable C^*-algebra.

By Lemma 3.3, there exists a stable, separable, nuclear, simple, purely infinite C^*-algebra $\widetilde{\mathfrak{B}}$ satisfying the UCT and there exists a proper injective homomorphism $\phi_0 : \mathfrak{A}_1[2] \to \widetilde{\mathfrak{B}}$ such that $K_i(\phi_0)$ is an isomorphism for $i = 0, 1$. Since ϕ_0 is a proper homomorphism, by Theorem 2.2 of [10], there exists a C^*-algebra $\mathfrak{A}_1'$ over X_2 such that $\mathfrak{A}_1'[2] = \widetilde{\mathfrak{B}}$ and $\mathfrak{A}_1'[1] = \mathfrak{A}_1[1]$ and there exists an X_2-equivariant homomorphism $\phi' : \mathfrak{A}_1 \to \mathfrak{A}_1'$ such that $\phi'_{\{1\}}$ is the identity map and $\phi'_{\{2\}} = \phi_0$.

By Lemma 3.4, there exist a separable, nuclear, stable, purely infinite simple C^*-algebra satisfying the UCT, $\widetilde{\mathfrak{D}}$, an injective homomorphism $\phi_2 : \mathfrak{A}_1'[1] \to \widetilde{\mathfrak{D}}$ with $K_i(\phi_2)$ an isomorphism, a tight C^*-algebra $\widetilde{\mathfrak{A}}_1$ over X_2 such that $\widetilde{\mathfrak{A}}_1[2] = \widetilde{\mathfrak{B}}$ and $\widetilde{\mathfrak{A}}_1[1] = \widetilde{\mathfrak{D}}$, and there exists an X_2-equivariant homomorphism $\phi'' : \mathfrak{A}_1' \to \widetilde{\mathfrak{A}}_1$ such that $\phi''_{\{2\}}$ is the identity and $\phi''_{\{1\}} = \phi_2$.

Set $\phi = \phi'' \circ \phi'$, then $\phi : \mathfrak{A}_1 \to \widetilde{\mathfrak{A}}_1$ is an X_2-equivariant homomorphism. Note that $\phi_{\{2\}} = \phi_0$ and $\phi_{\{1\}} = \phi_2$. Hence, the five lemma implies that $\mathrm{FK}_{X_2}(\phi) : \mathrm{FK}_{X_2}(\mathfrak{A}_1) \to \mathrm{FK}_{X_2}(\mathfrak{A}_2)$ is an isomorphism. Hence, by Bonkat's UCT [5], $KK(X_2; \phi)$ is invertible. Since $\phi_{\{2\}}$ is a proper map, $\phi_{\{2\}}$ extends to $\mathcal{M}(\mathfrak{A}_1[2])$ to $\mathcal{M}(\widetilde{\mathfrak{A}}_1[2])$.

By a similar argument, there exist a tight C^*-algebra $\widetilde{\mathfrak{A}}_2$ over X_2 and an X_2-equivariant homomorphism $\psi : \mathfrak{A}_2 \to \widetilde{\mathfrak{A}}_2$ such that $\widetilde{\mathfrak{A}}_2[i]$ is separable, stable, nuclear, purely infinite simple C^*-algebra satisfying the UCT, $KK(X_2, \psi)$ is invertible, and $\psi_{\{2\}}$ extends to a homomorphism from $\mathcal{M}(\mathfrak{A}_2[2])$ to $\mathcal{M}(\widetilde{\mathfrak{A}}_2[2])$

By the observations made in the previous two paragraphs and by Lemma 3.6, we may assume that $\mathfrak{A}_i$ is a stable, separable, nuclear, $\mathcal{O}_\infty$-absorbing tight C^*-algebra over X_2 such that $\mathfrak{A}_i[1]$ and $\mathfrak{A}_i[2]$ satisfy the UCT.

Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be invertible. Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are separable, stable, nuclear, $\mathcal{O}_\infty$-absorbing tight C^*-algebras over X_2, by a result of Kirchberg [15] there exists an X_2-equivariant isomorphism $\beta : \mathfrak{A}_1 \to \mathfrak{A}_2$ which induces x. Since $\beta_{\{2\}}$ is an isomorphism, $\beta_{\{2\}}$ is proper. Therefore, $\tau_{\mathfrak{e}_2} \circ \beta_{\{1\}} = \widetilde{\beta}_{\{2\}} \circ \tau_{\mathfrak{e}_1}$, where $\widetilde{\beta}_{\{2\}} : \mathcal{Q}(\mathfrak{A}_1[2]) \to \mathcal{Q}(\mathfrak{A}_2[2])$ is the homomorphism induced by $\beta_{\{2\}}$. Hence,

$$[\tau_{\mathfrak{e}_1}] \times r_{X_2}^{\{2\}}(x) = [\tau_{\mathfrak{e}_1}] \times KK(\beta_{\{2\}}) = KK(\beta_{\{1\}}) \times [\tau_{\mathfrak{e}_2}] = r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}_2}]$$

in $KK^1(\mathfrak{A}_1[1], \mathfrak{A}_2[2])$. $\qquad\square$

4. Classification

In this section we give a general classification result of certain class of extension algebras. The result of this section is a generalization of the classification results obtained by the authors in [11].

Definition 4.1. For a topological space X, we will be interested in classes $\mathcal{C}_X$ of separable, nuclear C^*-algebras in $\mathcal{N}$ such that

 (1) any element in $\mathcal{C}_X$ is a C^*-algebra over X;
 (2) if $\mathfrak{A}$ is in $\mathcal{C}_X$, then the stabilization of every full hereditary sub-C^*-algebra of $\mathfrak{A}$ is in $\mathcal{C}_X$.
 (3) if $\mathfrak{A}$ and $\mathfrak{B}$ are in $\mathcal{C}_X$ and there exists an invertible element α in $KK(X; \mathfrak{A}, \mathfrak{B})$ which induces an isomorphism from $\mathrm{FK}_X^+(\mathfrak{A})$ to $\mathrm{FK}_X^+(B)$, then there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $KK(\phi) = g_X^1(\alpha)$.

We concentrate on the following two classes satisfying (1)–(3) above:

Example 4.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ separable, nuclear, stable, $\mathcal{O}_\infty$-absorbing tight C^*-algebras over X. Let α be an invertible element in $KK(X; \mathfrak{A}, \mathfrak{B})$. By Kirchberg [15], there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $KK(X; \phi) = \alpha$. Hence, $KK(\phi) = g_X^1(KK(X; \phi)) = KK(\alpha)$. Thus, if $\mathcal{C}_X$ is the class of all stable, separable, nuclear C^*-algebras over X which are $\mathcal{O}_\infty$-absorbing, then $\mathcal{C}_X$ satisfies the properties of Definition 4.1.

Example 4.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be stable AF algebras. Let α be an invertible element in $KK(X_1; \mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B})$ which induces an isomorphism from $\mathrm{FK}_{X_1}^+(\mathfrak{A}) = K_0(\mathfrak{A})$ to $\mathrm{FK}_{X_1}^+(\mathfrak{B}) = K_0(\mathfrak{B})$. Then by the classification of AF algebras [14], there

exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $K_0(\phi) = K_0(\alpha)$. Since $KK(\mathfrak{A}, \mathfrak{B}) \cong$ $\mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B}))$, we have that $KK(\phi) = \alpha$. Thus, if $\mathcal{C}_{X_1}$ is the class of all stable, AF algebras, then $\mathcal{C}_{X_1}$ satisfies the properties of Definition 4.1.

Remark 4.4. *The condition*

(3') *if $\mathfrak{A}$ and $\mathfrak{B}$ are in $\mathcal{C}_X$ and there exists an isomorphism β from $\mathrm{FK}_X^+(\mathfrak{A})$ to $\mathrm{FK}_X^+(B)$, then there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi_* = \beta$.*

is more closely suited to our purposes, and (1),(2),(3') is true in general in Example 4.3, but not always in Example 4.2, as pointed out in [21]. In fact, there exists a space X with four points such that (3') fails in Example 4.2.

Lemma 4.5. *For $i = 1, 2$, let $\mathfrak{e}_i : 0 \to \mathfrak{I}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be non-unital full extensions. Suppose $\mathfrak{I}_i$ is a stable C^*-algebra satisfying the corona factorization property. If there exist an isomorphism $\phi_0 : \mathfrak{I}_1 \to \mathfrak{I}_2$ and an isomorphism $\phi_2 : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $KK(\phi_2) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times KK(\phi_0)$, then $\mathfrak{E}_1$ is isomorphic to $\mathfrak{E}_2$.*

Proof. Note that $\mathfrak{e}_1 \cong \mathfrak{e}_1 \cdot \phi_0$ and $\mathfrak{e}_2 \cong \phi_2 \cdot \mathfrak{e}_2$, where $\mathfrak{e}_1 \cdot \phi_0$ is the push-out of $\mathfrak{e}_1$ along ϕ_0 and $\phi_2 \cdot \mathfrak{e}_2$ is the pull-back of $\mathfrak{e}_2$ along ϕ_2 (cf. [27]). Since $[\tau_{\mathfrak{e}_1 \cdot \phi_0}] = [\tau_{\mathfrak{e}_1}] \times KK(\phi_0) = KK(\phi_2) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\phi_2 \cdot \mathfrak{e}_2}]$ in $KK^1(\mathfrak{A}_1, \mathfrak{I}_2)$, we have that $[\tau_{\mathfrak{e}_1 \cdot \phi_0}] = [\tau_{\phi_2 \cdot \mathfrak{e}_2}]$. Since $\tau_{\mathfrak{e}_1 \cdot \phi_0}$ and $\tau_{\phi_2 \cdot \mathfrak{e}_2}$ are non-unital full extensions and $\mathfrak{I}_2$ satisfies the corona factorization property, by Theorem 3.2(2) of [22], there exists a unitary u in $\mathcal{M}(\mathfrak{I}_2)$ such that $\mathrm{Ad}(\pi(u)) \circ \tau_{\mathfrak{e}_1 \cdot \phi_0} = \tau_{\phi_2 \cdot \mathfrak{e}_2}$. Hence, $(\mathrm{Ad}(u), \mathrm{Ad}(u), \mathrm{id}_{\mathfrak{A}_1})$ is an isomorphism between $\mathfrak{e}_1 \cdot \phi_0$ and $\phi_2 \cdot \mathfrak{e}_2$. Thus, $\mathfrak{E}_1$ is isomorphic to $\mathfrak{E}_2$. $\square$

We will apply the theorem below to a certain class of C^*-algebras arising from graphs. See Proposition 6.3, Corollary 6.4, Proposition 6.5, and Theorem 6.9.

Theorem 4.6. *Let X be a finite topological space and let $U \in \mathbb{O}(X)$. Set $Y = X \setminus U \in \mathbb{LC}(X)$. For $i = 1, 2$, let $\mathfrak{A}_i$ be a C^*-algebra over X such that $\mathfrak{A}_i$ is a stable, separable, nuclear C^*-algebra and every simple sub-quotient of $\mathfrak{A}_i$ is in the bootstrap category $\mathcal{N}$.*

Let $\mathcal{C}_{\mathcal{I},U}$ and $\mathcal{C}_{\mathcal{Q},Y}$ be classes of C^*-algebras that satisfy the properties of Definition 4.1. Suppose $\mathfrak{A}_i(U)$ is stable C^*-algebra in $\mathcal{C}_{\mathcal{I},U}$ containing a norm-full projection and satisfying the corona factorization property, and $\mathfrak{A}_i(Y)$ is a stable C^*-algebra in $\mathcal{C}_{\mathcal{Q},Y}$ containing a norm-full projection. Suppose for $i = 1, 2$

$$\mathfrak{e}_i : 0 \to \mathfrak{A}_i(U) \to \mathfrak{A}_i \to \mathfrak{A}_i(Y) \to 0$$

are full extensions. If there exists an isomorphism $\alpha : \mathrm{FK}_X(\mathfrak{A}_1) \to \mathrm{FK}_X(\mathfrak{A}_2)$ such that $\alpha_U : \mathrm{FK}_U^+(\mathfrak{A}_1(U)) \to \mathrm{FK}_U^+(\mathfrak{A}_2(U))$ and $\alpha_Y : \mathrm{FK}_Y^+(\mathfrak{A}_1(Y)) \to \mathrm{FK}_Y^+(\mathfrak{A}_2(Y))$ are isomorphisms, and α lifts to an invertible element in $KK(X; \mathfrak{A}_1, \mathfrak{A}_2)$, then $\mathfrak{A}_1 \cong \mathfrak{A}_2$.

Proof. Suppose there exists an isomorphism $\alpha : \mathrm{FK}_X(\mathfrak{A}_1) \to \mathrm{FK}_X(\mathfrak{A}_2)$ such that $\alpha_U : \mathrm{FK}_U^+(\mathfrak{A}_1(U)) \to \mathrm{FK}_U^+(\mathfrak{A}_2(U))$ and $\alpha_Y : \mathrm{FK}_Y^+(\mathfrak{A}_1(Y)) \to \mathrm{FK}_Y^+(\mathfrak{A}_2(Y))$ are isomorphisms, and α lifts to an invertible element in $KK(X; \mathfrak{A}_1, \mathfrak{A}_2)$. Let $x \in KK(X; \mathfrak{A}_1, \mathfrak{A}_2)$ be this lifting. Then $r_X^U(x)$ is an invertible element in $KK(U; \mathfrak{A}_1(U), \mathfrak{A}_2(U))$ and $r_X^Y(x)$ is an invertible element in $KK(Y; \mathfrak{A}_1(Y), \mathfrak{A}_2(Y))$. Since $\mathfrak{A}_1(U)$ and $\mathfrak{A}_2(U)$ are in $\mathcal{C}_{\mathcal{I},U}$ and $\mathfrak{A}_1(Y)$ and $\mathfrak{A}_2(Y)$ are in $\mathcal{C}_{\mathcal{Q},Y}$, there exists an isomorphism $\phi_0 : \mathfrak{A}_1(U) \to \mathfrak{A}_2(U)$ which induces $r_X^U(x)$ and there exists an isomorphism

$\phi_2 : \mathfrak{A}_1(Y) \to \mathfrak{A}_2(Y)$ which induces $r_X^Y(x)$. By Theorem 3.7 and by Lemma 3.5

$$KK(\phi_2) \times [\tau_{\mathfrak{e}_2}] = (g_Y^1 \circ r_X^Y(x)) \times [\tau_{\mathfrak{e}_2}] = (r_{X_2}^{\{1\}} \circ g_{U,X}^2(x)) \times [\tau_{\mathfrak{e}_2}]$$

$$= [\tau_{\mathfrak{e}_1}] \times (r_{X_2}^{\{2\}} \circ g_{U,X}^2)(x) = [\tau_{\mathfrak{e}_1}] \times (g_U^1 \circ r_X^U(x)) = [\tau_{\mathfrak{e}_1}] \times KK(\phi_0)$$

in $KK^1(\mathfrak{A}_1(Y), \mathfrak{A}_2(U))$. Since $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are full non-unital extensions and $\mathfrak{A}_i(U)$ has the corona factorization property, by Lemma 4.5 we have that $\mathfrak{A}_1 \cong \mathfrak{A}_2$.   □

5. Full Extensions

In this section, we prove that certain extensions arising from graph C^*-algebras are necessary full, allowing one to use the results in Section 4.

Let $\mathfrak{I}$ be an ideal of a C^*-algebra $\mathfrak{B}$. Set

$$\mathcal{M}(\mathfrak{B}; \mathfrak{I}) = \{x \in \mathcal{M}(\mathfrak{B}) \, : \, x\mathfrak{B} \subseteq \mathfrak{I}\}$$

It is easy to check that $\mathcal{M}(\mathfrak{B}; \mathfrak{I})$ is a (norm-closed, two-sided) ideal of $\mathcal{M}(\mathfrak{B})$.

Definition 5.1. Let $\{f_n\}$ be an approximate identity consisting of projections for $\mathbb{K}$, where $f_0 = 0$ and $f_n - f_{n-1}$ is a projection of dimension one. Let $\mathfrak{A}$ be a unital C^*-algebra and set $e_n = 1_{\mathfrak{A}} \otimes f_n$. Note that $\{e_n\}$ is an approximate identity of $\mathfrak{A} \otimes \mathbb{K}$ consisting of projections.

An element $X \in \mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$ is said to be *diagonal with respect to* $\{e_n\}$ if there exists a strictly increasing sequence $\{\alpha(n)\}$ of integers with $\alpha(0) = 0$ such that

$$X(e_{\alpha(n)} - e_{\alpha(n-1)}) - (e_{\alpha(n)} - e_{\alpha(n-1)})X = 0$$

for all $n \in \mathbb{N}$. We write $X = \mathrm{diag}(x_1, x_2, \dots)$, where

$$x_n = X(e_{\alpha(n)} - e_{\alpha(n-1)})$$

Conversely, if $\{x_n\}$ is a bounded sequence with $x_n \in \mathsf{M}_{k_n}(\mathfrak{A})$, then upon identifying $\mathsf{M}_{k_n}(\mathfrak{A})$ with

$$(e_{\alpha(n)} - e_{\alpha(n-1)})(\mathfrak{A} \otimes \mathbb{K})(e_{\alpha(n)} - e_{\alpha(n-1)})$$

for an appropriate $\alpha(n)$, we have that $X = \mathrm{diag}(x_1, x_2, \dots)$ for some $X \in \mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$.

Let $\epsilon > 0$. Define $f_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_\epsilon(t) = \begin{cases} 0, & \text{if } x \leq \epsilon \\ \epsilon^{-1}(t - \epsilon), & \text{if } \epsilon \leq x \leq 2\epsilon \\ 1, & \text{if } x \geq 2\epsilon \end{cases}$$

Theorem 5.2. *Let $\mathfrak{B}_0$ be a unital, $\mathcal{O}_\infty$-absorbing C^*-algebra. Let $\mathfrak{I}$ be the largest proper non-trivial ideal of $\mathfrak{B} = \mathfrak{B}_0 \otimes \mathbb{K}$. If $x \in \mathcal{M}(\mathfrak{B})$ such that x is not an element of $\mathcal{M}(\mathfrak{B}; \mathfrak{I}) + \mathfrak{B}$, then $\mathcal{I}(x) + \mathfrak{B} = \mathcal{M}(\mathfrak{B})$, where $\mathcal{I}(x)$ is the norm-closed ideal generated by x.*

Consequently, every nonzero element $x \in \mathcal{Q}(\mathfrak{B})$ that is not an element of $\mathcal{M}(\mathfrak{B}; \mathfrak{I})/\mathfrak{I}$ is norm-full in $\mathcal{Q}(\mathfrak{B})$.

Proof. First note that by the proof of Theorem 3.2 of [26], $\mathcal{M}(\mathfrak{B}; \mathfrak{I})$ is a proper ideal of $\mathcal{M}(\mathfrak{B})$. Let $x \in \mathcal{M}(\mathfrak{B}) \setminus \mathcal{M}(\mathfrak{B}; \mathfrak{I})$. By Proposition 2.8 (i) of [26], we may assume that $x = \mathrm{diag}(x_1, x_2, \dots)$ with respect to $\{e_n\}$, i.e. there exists a strictly increasing sequence of integers $\{\alpha(n)\}$ with $\alpha(0) = 0$ such that $x = \sum_{k=1}^{\infty} x_k$, where $x_k \in (e_{\alpha(k)} - e_{\alpha(k-1)}) \mathfrak{B} (e_{\alpha(k)} - e_{\alpha(k-1)})$ and the sum converges in the strict topology.

Let $m \in \mathbb{N}$. Since x is not an element of $\mathcal{M}(\mathfrak{B}; \mathfrak{I})$ and x is not an element of $\mathfrak{B}$, there exists $m' \geq m$ such that $\sum_{k=m}^{m'} x_k$ is not an element of $(e_{\alpha(m')} - e_{\alpha(m-1)}) \mathfrak{I} (e_{\alpha(m')} - e_{\alpha(m-1)})$. Hence, there exists $\delta > 0$ such that $\sum_{k=m}^{m'} f_\delta(x_k)$ is not an element of $(e_{\alpha(m')} - e_{\alpha(m-1)}) \mathfrak{I} (e_{\alpha(m')} - e_{\alpha(m-1)})$. Therefore, $\sum_{k=m}^{m'} f_\delta(x_k)$ is norm-full in $(e_{\alpha(m')} - e_{\alpha(m-1)}) \mathfrak{B} (e_{\alpha(m')} - e_{\alpha(m-1)})$. By Proposition 2.2 of [19], there exists $z \in (e_{\alpha(m')} - e_{\alpha(m-1)}) \mathfrak{B} (e_{\alpha(m')} - e_{\alpha(m-1)})$ such that

$$e_{\alpha(m')} - e_{\alpha(m-1)} = z \left(\sum_{k=m}^{m'} f_\delta(x_k) \right) z^*$$

Therefore,

$$1_{\mathfrak{B}_0} \leq e_{\alpha(m')} - e_{\alpha(m-1)} = z \left(\sum_{k=m}^{m'} f_\delta(x_k) \right) z^*$$

By Corollary 2.7 of [26], $\mathcal{I}(x) = \mathcal{M}(\mathfrak{B})$. $\qquad\square$

Corollary 5.3. *Let $\mathfrak{B}_0$ be a unital, $\mathcal{O}_\infty$-absorbing C^*-algebra. Let $\mathfrak{I}$ be the largest non-trivial proper ideal of $\mathfrak{B} = \mathfrak{B}_0 \otimes \mathbb{K}$. Suppose $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ such that $\mathfrak{e}' : 0 \to \mathfrak{B}/\mathfrak{I} \to \mathfrak{A}/\mathfrak{I} \to \mathfrak{A}/\mathfrak{B} \to 0$ is an essential extenion. Then the extension $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{B} \to 0$ is a full extension.*

Proof. Note that the canonical projection from $\mathfrak{A}$ to $\mathfrak{A}/\mathfrak{I}$ is an X_2-equivariant homomorphism. Therefore, by Theorem 2.2 of [10], the diagram

$$
\begin{array}{ccc}
\mathfrak{A}/\mathfrak{B} & \xrightarrow{\tau_\mathfrak{e}} & \mathcal{Q}(\mathfrak{B}) \\
& \searrow{\scriptstyle \tau_{\mathfrak{e}'}} & \downarrow \\
& & \mathcal{Q}(\mathfrak{B}/\mathfrak{I})
\end{array}
$$

is commutative.

We will first show that $\mathfrak{e}$ is an essential extension. Suppose $\mathfrak{D}$ is a nonzero ideal of $\mathfrak{A}$. Note that

$$((\mathfrak{I} + \mathfrak{D})/\mathfrak{I}) \cap \mathfrak{B}/\mathfrak{I} = 0 \quad \Longleftrightarrow \quad (\mathfrak{I} + \mathfrak{D})/\mathfrak{I} = 0$$

and the second equality occurs exactly when $\mathfrak{D} \subseteq \mathfrak{I}$. Suppose that $\mathfrak{D} \subseteq \mathfrak{I}$. Then it is clear that $\mathfrak{D} \cap \mathfrak{B} = \mathfrak{B} \neq 0$. Suppose $\mathfrak{D}$ is not a subset of $\mathfrak{I}$. By the above equivalence, $((\mathfrak{I} + \mathfrak{D})/\mathfrak{I}) \cap \mathfrak{B}/\mathfrak{I} \neq 0$. Hence, there exists $x \in \mathfrak{D}$ such that $x \in \mathfrak{B} \setminus \mathfrak{I}$. Therefore, $x \in \mathfrak{D} \cap \mathfrak{B}$ and $x \neq 0$. Hence, $\mathfrak{e}$ is an essential extension.

We now prove that $\mathfrak{e}$ is a full extension. Note that since $\mathfrak{B}/\mathfrak{I}$ is a stable purely infinite simple C^*-algebra and $\mathfrak{e}'$ is an essential extension, we have that $\mathfrak{e}'$ is a full extension. Let $a \in \mathfrak{A}/\mathfrak{B}$ be a nonzero element. Then the ideal generated by $\tau_{\mathfrak{e}'}(a)$ in $\mathcal{Q}(\mathfrak{B}/\mathfrak{I})$ is $\mathcal{Q}(\mathfrak{B}/\mathfrak{I})$.

Since

$$0 \to \mathcal{M}(\mathfrak{B};\mathfrak{I})/\mathfrak{I} \to \mathcal{Q}(\mathfrak{B}) \to \mathcal{Q}(\mathfrak{B}/\mathfrak{I}) \to 0$$

is an exact sequence, by the above commutative diagram, $\tau_{\mathfrak{e}}(a)$ is not an element of $\mathcal{M}(\mathfrak{B};\mathfrak{I})/\mathfrak{I}$. Hence, by Theorem 5.2, $\tau_{\mathfrak{e}}(a)$ is norm-full in $\mathcal{Q}(\mathfrak{B})$. $\square$

Proposition 5.4. *Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{I}$ and $\mathfrak{D}$ be ideals of $\mathfrak{A}$ with $\mathfrak{I} \subseteq \mathfrak{D}$. Suppose $\mathfrak{D}/\mathfrak{I}$ is an essential ideal of $\mathfrak{A}/\mathfrak{I}$. Then $\mathfrak{e}_1 : 0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ is a full extension if and only if $\mathfrak{e}_2 : 0 \to \mathfrak{I} \to \mathfrak{D} \to \mathfrak{D}/\mathfrak{I} \to 0$ is a full extension.*

Proof. Note that the natural embedding $\iota_{\mathfrak{D}} : \mathfrak{D} \to \mathfrak{A}$ is an X_2-equivariant homomorphism with $\iota_{\mathfrak{I}} = \mathrm{id}_{\mathfrak{I}}$. Hence, the following diagram is commutative

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathfrak{D} & \longrightarrow & \mathfrak{D}/\mathfrak{I} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow{\scriptstyle \iota_{\mathfrak{D}/\mathfrak{I}}} & & \\
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathfrak{A}/\mathfrak{I} & \longrightarrow & 0
\end{array}
$$

Therefore, the diagram

$$
\begin{array}{ccc}
\mathfrak{D}/\mathfrak{I} & \xrightarrow{\;\tau_{\mathfrak{e}_2}\;} & \mathcal{Q}(\mathfrak{I}) \\
{\scriptstyle \iota_{\mathfrak{D}/\mathfrak{I}}}\downarrow & & \| \\
\mathfrak{A}/\mathfrak{I} & \xrightarrow[\;\tau_{\mathfrak{e}_1}\;]{} & \mathcal{Q}(\mathfrak{I})
\end{array}
$$

is commutative.

Suppose $\mathfrak{e}_1$ is a full extension. Then it is clear from the above diagram that $\mathfrak{e}_2$ is a full extension. Suppose that $\mathfrak{e}_2$ is a full extension. Let $a \in \mathfrak{A}/\mathfrak{I}$ be a non-zero element. Let I be the ideal generated by a in $\mathfrak{A}/\mathfrak{I}$. Since $\mathfrak{D}/\mathfrak{I}$ is an essential ideal of $\mathfrak{A}/\mathfrak{I}$, there exists a non-zero $b \in \mathfrak{D}/\mathfrak{I}$ such that $\iota_{\mathfrak{D}/\mathfrak{I}}(b) \in I$. Hence, $\tau_{\mathfrak{e}_2}(b) = (\tau_{\mathfrak{e}_1} \circ \iota_{\mathfrak{D}/\mathfrak{I}})(b)$ is norm-full in $\mathcal{Q}(\mathfrak{I})$. Since $(\tau_{\mathfrak{e}_1} \circ \iota_{\mathfrak{D}/\mathfrak{I}})(b)$ is in the ideal generated by $\tau_{\mathfrak{e}_1}(a)$ in $\mathcal{Q}(\mathfrak{I})$, we have that $\tau_{\mathfrak{e}_1}(a)$ is norm-full in $\mathcal{Q}(\mathfrak{I})$. Thus, $\mathfrak{e}_1$ is a full extension. $\square$

Proposition 5.5. *Let $\mathfrak{A}$ be a graph C^*-algebra satisfying Condition (K). Suppose $\mathfrak{I}_1 \triangleleft \mathfrak{I}_2 \triangleleft \mathfrak{A}$ such that $\mathfrak{I}_1$ is an AF algebra, $\mathfrak{I}_1$ is the largest proper ideal of $\mathfrak{I}_2$, $\mathfrak{I}_2/\mathfrak{I}_1$ is purely infinite. Then $\mathfrak{e} : 0 \to \mathfrak{I}_1 \otimes \mathbb{K} \to \mathfrak{I}_2 \otimes \mathbb{K} \to \mathfrak{I}_2/\mathfrak{I}_1 \otimes \mathbb{K} \to 0$ is a full extension.*

Proof. By [9], $\mathfrak{A} \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$, where E is a graph satisfying Condition (K) and has no breaking vertices. Hence, by Theorem 3.6 of [1] and Proposition 3.4 of [1], $\mathfrak{I}_2 \otimes \mathbb{K}$ is isomorphic to a $C^*(E_1) \otimes \mathbb{K}$ where E_1 has no breaking vertices and satisfies Condition (K). Note that $C^*(E_1)$ has a largest ideal $\mathfrak{D}_1$ such that $C^*(E_1)/\mathfrak{D}_1$ is a purely infinite and $\mathfrak{D}_1$ is an AF algebra. Thus, by Proposition 3.10 of [13], there exists a projection $p \in C^*(E_1)$ such that $pC^*(E_1)p \otimes \mathbb{K} \cong C^*(E_1) \otimes \mathbb{K}$ and $p\mathfrak{I}_1 p$ is stable. Since $pC^*(E_1)p/p\mathfrak{I}_1 p$ is unital simple C^*-algebra and $0 \to p\mathfrak{I}_1 p \to pC^*(E_1)p \to pC^*(E_1)p/p\mathfrak{I}_1 p \to 0$ is a unital essential extension, the extension is full. Since $p\mathfrak{I}_1 p$ is stable, Proposition 1.6 of [11] implies that the extension

$$0 \to p\mathfrak{I}_1 p \otimes \mathbb{K} \to pC^*(E_1)p \otimes \mathbb{K} \to (pC^*(E_1)p/p\mathfrak{I}_1 p) \otimes \mathbb{K} \to 0$$

is full. The proposition now follows since isomorphisms take full extensions to full extensions. □

Corollary 5.6. *Let $\mathfrak{A}$ be a graph C^*-algebra satisfying Condition (K). Suppose that $\mathfrak{I}$ is an AF algebra such that $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, for all ideals $\mathfrak{J}$ of $\mathfrak{A}$ we have that $\mathfrak{J} \subseteq \mathfrak{I}$ or $\mathfrak{I} \subseteq \mathfrak{J}$, and $\mathfrak{A}/\mathfrak{I}$ is $\mathcal{O}_\infty$-absorbing. Then $\mathfrak{e} : 0 \to \mathfrak{I} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{I} \otimes \mathbb{K} \to 0$ is a full extension.*

Proof. Let $\{\mathfrak{C}_n : n \in \mathbb{N}\}$ be the set of all minimal ideals of $\mathfrak{A}/\mathfrak{I}$ and let $\mathfrak{A}_n$ be an ideal of $\mathfrak{A}$ such that $\mathfrak{I} \subseteq \mathfrak{A}_n$ and $\mathfrak{A}_n/\mathfrak{I} = \mathfrak{C}_n$.

Let $\mathfrak{J}$ be an ideal of $\mathfrak{A}_n$. Then $\mathfrak{J}$ is an ideal of $\mathfrak{A}$. Hence, $\mathfrak{J} \subseteq \mathfrak{I}$ or $\mathfrak{I} \subseteq \mathfrak{J}$. Suppose $\mathfrak{I} \subseteq \mathfrak{J}$ but $\mathfrak{I} \neq \mathfrak{J}$. Then, $\mathfrak{J}/\mathfrak{I} = \mathfrak{A}_n/\mathfrak{I} = \mathfrak{C}_n$ since $\mathfrak{C}_n$ is simple. Hence, $\mathfrak{I}$ is the largest ideal of $\mathfrak{A}_n$ such that $\mathfrak{I}$ is an AF algebra and $\mathfrak{A}_n/\mathfrak{I}$ is purely infinite. Therefore, by Proposition 5.5, $0 \to \mathfrak{I} \otimes \mathbb{K} \to \mathfrak{A}_n \otimes \mathbb{K} \to \mathfrak{C}_n \otimes \mathbb{K} \to 0$ is a full extension.

Let $\mathfrak{D} = \overline{\sum_{n=1}^\infty \mathfrak{A}_n}$. Then $\mathfrak{D}$ is an ideal of $\mathfrak{A}$ such that $\mathfrak{D}/\mathfrak{I}$ is an essential ideal of $\mathfrak{A}/\mathfrak{I}$. Since $\mathfrak{C}_i \cap \mathfrak{C}_j = \{0\}$ for $i \neq j$, we have that $0 \to \mathfrak{I} \otimes \mathbb{K} \to \mathfrak{D} \otimes \mathbb{K} \to \mathfrak{D}/\mathfrak{I} \otimes \mathbb{K} \to 0$. is a full extension. The corollary now follows from Proposition 5.4. □

6. Applications to graph C^*-algebras

Recall our definition of X_n from Example 2.3 above. We now apply the results of Section 4 and Section 5 to classify a certain class of graph C^*-algebras that are tight C^*-algebras over X_n.

Proposition 6.1. *Suppose $\mathfrak{A}$ is a C^*-algebra with finitely many ideals. If the stabilization of every simple sub-quotient of $\mathfrak{A} \otimes \mathbb{K}$ satisfies the corona factorization property, then $\mathfrak{A} \otimes \mathbb{K}$ satisfies the corona factorization property. Consequently, any graph C^*-algebra with finitely many ideals has the corona factorization property.*

Proof. We will prove the result of the proposition by induction. If $\mathfrak{A}$ is simple, then by our assumption, $\mathfrak{A} \otimes \mathbb{K}$ has the corona factorization property. Suppose that the proposition is true for any C^*-algebra $\mathfrak{B}$ with at most n ideals such that the stabilization of any simple sub-quotient of $\mathfrak{B} \otimes \mathbb{K}$ satisfies the corona factorization property.

Let $\mathfrak{A}$ be a C^*-algebra with $n + 1$ ideals such that the stabilization of every simple sub-quotient of $\mathfrak{A} \otimes \mathbb{K}$ satisfies the corona factorization property. Let $\mathfrak{I}$ be a proper non-trivial ideal of $\mathfrak{A} \otimes \mathbb{K}$. Then $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ are C^*-algebras with at most n ideals such that the stabilization of every simple sub-quotient of $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ satisfies the corona factorization property. Hence, $\mathfrak{I} \otimes \mathbb{K}$ and $\mathfrak{A}/\mathfrak{I} \otimes \mathbb{K}$ satisfies the corona factorization property. Therefore, by Theorem 3.1(1) of [17], $\mathfrak{A} \otimes \mathbb{K}$ satisfies the corona factorization property. □

The authors have been recently informed by Eduard Ortega that his joint work with Francesc Perera and Mikael Rørdam (see [23]) implies that the stabilization of any graph C^*-algebra has the corona factorization property.

Theorem 6.2. *(Meyer-Nest [21]) For the topological space X_n, if $\mathfrak{A}$ and $\mathfrak{B}$ are separable, nuclear, C^*-algebras over X_n such that $\mathfrak{A}[k]$ and $\mathfrak{B}[k]$ are in the bootstrap category $\mathcal{N}$, then any isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}) \to \mathrm{FK}_{X_n}(B)$ lifts to an invertible element in $KK(X_n; \mathfrak{A}, \mathfrak{B})$.*

Proposition 6.3. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ separable, nuclear, C^*-algebras over X_n. Suppose $\mathfrak{A}_i[1]$ is an AF algebra which contains a norm-full projection and $\mathfrak{A}_i[2,n]$ is a tight stable $\mathcal{O}_\infty$-absorbing C^*-algebra over $[2,n]$, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i[1,2]$. Then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$ if and only if there exists an isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}_1) \to \mathrm{FK}_{X_n}(\mathfrak{A}_2)$ such that $\alpha_{\{1\}}$ is positive.*

Proof. By Corollary 5.3, $0 \to \mathfrak{A}_i[2,n] \otimes \mathbb{K} \to \mathfrak{A}_i \otimes \mathbb{K} \to \mathfrak{A}_i[1] \otimes \mathbb{K} \to 0$ is a full extension. By Proposition 6.1, $\mathfrak{A}_i[2,n]$ has the corona factorization property. Since $\mathfrak{A}_i[2,n]$ is a tight, $\mathcal{O}_\infty$-absorbing C^*-algebra over $[2,n]$ we have that $\mathfrak{A}_i[2,n]$ contains a full projection. The theorem now follows from Theorem 4.6. $\square$

The following specialization of the result above is in a certain sense dual to Theorem 4.7 of [13].

Corollary 6.4. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be graph C^*-algebras satisfying Condition (K) and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C^*-algebras over X_2. Suppose $\mathfrak{A}_i[2]$ is $\mathcal{O}_\infty$-absorbing, $\mathfrak{A}_i[2]$ is the smallest ideal of $\mathfrak{A}_i$, and $\mathfrak{A}_i[1]$ is an AF algebra. Then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$ if and only if there exists an isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}_1) \to \mathrm{FK}_{X_n}(\mathfrak{A}_2)$ such that $\alpha_{\{1\}}$ is positive.*

Proposition 6.5. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be graph C^*-algebras satisfying Condition (K). Suppose $\mathfrak{A}_i$ is a C^*-algebra over X_n such that $\mathfrak{A}_i[n]$ is an AF algebra which contains a norm-full projection, for every ideal $\mathfrak{I}$ of $\mathfrak{A}_i$ we have that $\mathfrak{I} \subseteq \mathfrak{A}_i[n]$ or $\mathfrak{A}_i[n] \subseteq \mathfrak{I}$, and $\mathfrak{A}_i[1,n-1]$ is a tight, $\mathcal{O}_\infty$-absorbing C^*-algebra over $[1,n-1]$. Then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$ if and only if there exists an isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}_1) \to \mathrm{FK}_{X_n}(\mathfrak{A}_2)$ such that $\alpha_{\{n\}}$ is positive.*

Proof. By Corollary 5.6, $0 \to \mathfrak{A}_i[n] \to \mathfrak{A}_i \to \mathfrak{A}_i[1,n-1] \to 0$ is a full extension. By Lemma 3.10 of [11], $\mathfrak{A}_i[n]$ satisfies the corona factorization property. The result now follows from Theorem 4.6. $\square$

Remark 6.6. *Note that any AF algebra with finitely many ideals will have a norm-full projection. Hence, if $\mathfrak{A}_i$ in Proposition 6.3 and Proposition 6.5 has finitely many ideals, then $\mathfrak{A}_i[1]$ and $\mathfrak{A}[n]$ will have a norm-full projection.*

Remark 6.7. *Note that the above propositions do not assume that the ideal lattice of $\mathfrak{A}_i$ is linear. For example, Proposition 6.5 can be applied to C^*-algebras $\mathfrak{A}$ that are tight C^*-algebras over $X = \{1,2,3,4\}$ with $\mathbb{O}(X) = \{\emptyset\} \cup \{U \subseteq X : 4 \in U\}$ such that*

    (1) *$\mathfrak{A}(\{4\})$ is purely infinite*
    (2) *$\mathfrak{A}(\{1,2,3\})$ is an AF algebra.*

In a forthcoming paper, we study graph C^*-algebras with small ideal structures similar to the C^*-algebra described above. Proposition 6.3, Proposition 6.5, and related results will be used to classify these graph C^*-algebras.

Definition 6.8. We define a class $\mathcal{C}_n$ of graph C^*-algebras as follows: $\mathfrak{A}$ is in $\mathcal{C}_n$ if

    (1) $\mathfrak{A}$ is a graph C^*-algebra;
    (2) $\mathfrak{A}$ is a tight C^*-algebra over X_n; and
    (3) there exists $U \in \mathbb{O}(X_n)$ such that either $\mathfrak{A}(U)$ is an AF algebra and $\mathfrak{A}(X_n \backslash U)$ is $\mathcal{O}_\infty$-absorbing or $\mathfrak{A}(U)$ is $\mathcal{O}_\infty$-absorbing and $\mathfrak{A}(X \backslash U)$ is an AF algebra.

Note that if $C^*(E)$ is an element in $\mathcal{C}$, then by the proof of Lemma 3.1 of [13], E satisfies Condition (K).

Theorem 6.9. *Let E_1 and E_2 be graphs such that $C^*(E_1)$ and $C^*(E_2)$ are in $\mathcal{C}_n$. Then the following are equivalent:*

(1) $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$
(2) *There exists an isomorphism* $\alpha : \mathrm{FK}_{X_n}^+(C^*(E_1)) \to \mathrm{FK}_{X_n}^+(C^*(E_2))$

Proof. Suppose there exists an isomorphism $\alpha : \mathrm{FK}_{X_n}^+(C^*(E_1)) \to \mathrm{FK}_{X_n}^+(C^*(E_2))$. Note that by Cuntz [8], if $\mathfrak{A}$ is a $\mathcal{O}_\infty$-absorbing with a norm-full projection, then $K_0(\mathfrak{A}) = K_0(\mathfrak{A})_+$. Since $K_0(\mathfrak{B}) \neq K_0(\mathfrak{B})_+$ for any AF algebra, there are no positive isomorphism from the K_0-group of an AF algebra to the K_0-group of a $\mathcal{O}_\infty$-absorbing C^*-algebra with a norm-full projection.

With the above observation, one of the following four cases must happen:

(i) $C^*(E_1)$ and $C^*(E_2)$ are AF algebras;
(ii) $C^*(E_2)$ and $C^*(E_2)$ are $\mathcal{O}_\infty$-absorbing;
(iii) there exists $1 \leq k \leq n$ such that $C^*(E_i)[k, n]$ is an AF algebra and $C^*(E_i)[1, k-1]$ is $\mathcal{O}_\infty$-absorbing for $i = 1, 2$;
(iv) there exists $1 \leq k \leq n$ such that $C^*(E_i)[k, n]$ is $\mathcal{O}_\infty$-absorbing and $C^*(E_i)[1, k-1]$ is an AF algebra for $i = 1, 2$.

Case (i) follows from the classification of AF algebras. Case (ii) follows from Theorem 4.14 of [21]. Case (iii) follows from Proposition 6.5 and Case (iv) follows from Proposition 6.3. $\square$

7. EXAMPLES

7.1. **Case I.** Fix a prime p and consider the class of graph C^*-algebras given by adjacency matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ z & p+1 & 0 \\ y & x & p+1 \end{bmatrix}$$

for $y, z > 0$. Theorem 6.9 applies directly as the resulting graph C^*-algebra has a finite linear ideal lattice $0 \triangleleft \mathfrak{I}_1 \triangleleft \mathfrak{I}_2 \triangleleft \mathfrak{A}$ with subquotients $\mathfrak{I}_1 = \mathbb{K}, \mathfrak{I}_2/\mathfrak{I}_1 = \mathcal{O}_{p+1} \otimes \mathbb{K}$, and $\mathfrak{A}/\mathfrak{I}_2 = \mathcal{O}_{p+1}$. All K_1-groups in the filtered K-theory vanish, and the K_0-groups and the natural transformations

$$
\begin{array}{ccc}
K_0(\mathfrak{I}_1) \longrightarrow K_0(\mathfrak{I}_2) \longrightarrow K_0(\mathfrak{I}_2/\mathfrak{I}_1) \\
\| \qquad\qquad \downarrow \qquad\qquad \downarrow \\
K_0(\mathfrak{I}_1) \longrightarrow K_0(\mathfrak{A}) \longrightarrow K_0(\mathfrak{A}/\mathfrak{I}_1) \\
\downarrow \qquad\qquad \downarrow \\
K_0(\mathfrak{A}/\mathfrak{I}_2) =\!\!= K_0(\mathfrak{A}/\mathfrak{I}_2)
\end{array}
$$

may be computed as

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \mathrm{cok}\begin{bmatrix} z \\ p \end{bmatrix} & \longrightarrow & \mathrm{cok}\begin{bmatrix} p \end{bmatrix} \\
\| & & \downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & \mathrm{cok}\begin{bmatrix} z & y \\ x & p \\ p & 0 \end{bmatrix} & \longrightarrow & \mathrm{cok}\begin{bmatrix} x & p \\ p & 0 \end{bmatrix} \\
& & \downarrow & & \downarrow \\
& & \mathrm{cok}\begin{bmatrix} p \end{bmatrix} & = & \mathrm{cok}\begin{bmatrix} p \end{bmatrix}
\end{array}$$

with all maps induced by the canonical maps from $\mathbb{Z}^r$ into $\mathbb{Z}^s$ for suitably chosen r and s.

It is easy to see that two such filtered K-theories can only be isomorphic when

$$p \mid x \Leftrightarrow p \mid x' \qquad p \mid z \Leftrightarrow p \mid z'$$

but depending upon the invertibility of x and z in $\mathbb{Z}/p$ we get varying conditions on y. In fact, elementary (but long) computations yield

Example 7.1. With graphs E and E' given by matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ z & p+1 & 0 \\ y & x & p+1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ z' & p+1 & 0 \\ y' & x' & p+1 \end{bmatrix},$$

respectively, we have $C^*(E) \otimes \mathbb{K} \simeq C^*(E') \otimes \mathbb{K}$ precisely when

  (1) $p \mid x \Leftrightarrow p \mid x'$, and
  (2) $p \mid z \Leftrightarrow p \mid z'$, and
  (3) (a) $p \mid y \Leftrightarrow p \mid y'$ when $p \mid x$ and $p \mid z$
       (b) $p \mid [y - xz/p] \Leftrightarrow p \mid [y' - x'z'/p]$ when $p \nmid x$ and $p \mid z$

7.2. **Case II.** We now consider graphs given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & p+1 & 0 & 0 \\ y & 0 & p+1 & 0 \\ z & 0 & 0 & p+1 \end{bmatrix}$$

with $x, y, z > 0$. The resulting ideal lattice is not linear; in fact we have an extension

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathfrak{A} \longrightarrow \mathcal{O}_{p+1} \oplus \mathcal{O}_{p+1} \oplus \mathcal{O}_{p+1} \longrightarrow 0 \qquad (7.1)$$

showing that the ideal lattice is precisely of the type demonstrated by Meyer and Nest in [21] to not allow a UCT for filtered K-theory. However, the K-theory can

be seen directly to have projective dimension 1. Indeed, it has the form

$$
\begin{array}{ccccccc}
& & \operatorname{cok}\left[\begin{smallmatrix} x \\ p \end{smallmatrix}\right] & \longrightarrow & \operatorname{cok}\left[\begin{smallmatrix} x & y \\ p & 0 \\ 0 & p \end{smallmatrix}\right] & & \operatorname{cok}\left[p\right] \\
\end{array}
$$

which, using notation from [21], is the surjective image of $P_4 \oplus P_{14} \oplus P_{24} \oplus P_{34}$ with kernel $P_{14} \oplus P_{24} \oplus P_{34}$ and hence has projective dimension 1. Since the ideal $\mathfrak{I}_1 \cong \mathbb{K}$ is a least ideal with $\mathfrak{A}/\mathfrak{I}_1$ absorbing $\mathcal{O}_\infty$ we may apply Theorem 4.6.

It is straightforward to determine when the filtered K-theory for two such matrices are the same; indeed this amounts to

$$p \mid x \Leftrightarrow p \mid x' \qquad p \mid y \Leftrightarrow p \mid y' \qquad p \mid z \Leftrightarrow p \mid z'$$

and taking into account the homeomorphims of $\operatorname{Prim}(\mathfrak{A})$ we arrive at

Example 7.2. With graphs E and E' given by matrices

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
x & p+1 & 0 & 0 \\
y & 0 & p+1 & 0 \\
z & 0 & 0 & p+1
\end{bmatrix}
\qquad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
x' & p+1 & 0 & 0 \\
y' & 0 & p+1 & 0 \\
z' & 0 & 0 & p+1
\end{bmatrix},
$$

respectively, we have

$$C^*(E) \otimes \mathbb{K} \simeq C^*(E') \otimes \mathbb{K} \iff |\{r \in \{x,y,z\} \mid p \mid r\}| = |\{r \in \{x',y',z'\} \mid p \mid r\}|$$

Remark 7.3. *Note that $KK^1(\mathcal{O}_{p+1} \oplus \mathcal{O}_{p+1} \oplus \mathcal{O}_{p+1}, \mathbb{K})$ does not tell the full story about stable isomorphism among the possible extensions fitting in (7.1), as indeed, we have only 4 stable isomorphism classes among the p^3 different extensions.*

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