A remark on the Generalized Hodge Conjecture *

Dario Portelli

1 The results

Let X denote a smooth, projective, connected algebraic variety over $\mathbb{C}$, of dimension n.

The space $F'^{p}H^{i}(X,\mathbb{Q})$ of the arithmetic filtration of $H^{i}(X,\mathbb{Q})$ can be defined as follows. A class $\xi \in H^{i}(X,\mathbb{Q})$ is in $F'^{p}H^{i}(X,\mathbb{Q})$ if and only if there are a closed algebraic subset $Y \subset X$, of codimension $\geq p$, and a $(2n-i)$-cycle $\Gamma \subset Y$ (this forces $p \leq i/2$) such that the Poincaré dual $PD(\xi)$ of ξ is $[\Gamma]$. Hodge observed ([7],[8]) that for any cycle Γ satisfying these conditions we have the equality

$$\int_{\Gamma} \alpha = 0 \tag{1}$$

for every $[\alpha] \in F^{n-p+1}H^{2n-i}(X,\mathbb{C})$. This happens because $dim(Y) \leq n-p$ and α contains too many dz's to be supported by Y; in fact we already have $\alpha_{|_Y} \equiv 0$ (all the algebraic subsets $Y \subset X$ will be considered with their reduced structure, to have the set Y_{sm} of their smooth points everywhere dense in Y). Since α is closed, this is actually a property of $[\Gamma]$, or, equivalently, of ξ. Hodge also asked (loc. cit.) if this condition is sufficient for $\xi \in F'^{p}H^{i}(X,\mathbb{Q})$.

To express this differently, recall that, if ξ is the class of the closed form η, then for every closed $(2n-i)$-form α we have the equality

$$\int_{X} \alpha \wedge \eta = \int_{\Gamma} \alpha \tag{2}$$

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Putting toghether (1) and (2) we get

$$F'^p H^i(X,\mathbb{Q}) \subset (\, F^{n-p+1} H^{2n-i}(X,\mathbb{C})\,)^{\perp} = F^{\,p} H^i(X,\mathbb{C}) \tag{3}$$

where the orthogonal is with respect to the non degenerate pairing

$$H^{2n-i}(X,\mathbb{C}) \times H^i(X,\mathbb{C}) \to \mathbb{C} \qquad\qquad ([\alpha],[\beta]) \mapsto \int_X \alpha\wedge\beta$$

Hence

$$F'^p H^i(X,\mathbb{Q}) \subseteq F^p H^i(X,\mathbb{C}) \cap H^i(X,\mathbb{Q}) \tag{4}$$

Hodge's original problem was whether this inclusion is an equality or not ([7]). Grothendieck showed that the answer is negative ([5]).

This paper arises from the remark that the vanishing in (1) depends only on $dim(Y)$ and the type of α, not on the property for α of being closed. To formalize, we introduce the pairing (to simplify notation we set $r := 2n - i$)

$$A^r(X,\mathbb{C}) \times (\, \mathscr{Z}_r \otimes_{\mathbb{Q}} \mathbb{C}\,) \xrightarrow{\quad\int\quad} \mathbb{C} \tag{5}$$

still given by integration, where $A^r(X,\mathbb{C})$ denotes the $\mathbb{C}$-vector space of all complex valued r-forms, with $\mathscr{C}^\infty$ coefficients, *not necessarily closed*, and $\mathscr{Z}_r$ is the $\mathbb{Q}$-module of the topological r-cycles which are rational combinations of real-analytic, embedded r-simplexes (it is known that X can be triangulated by such simplexes; details will be given in the next section). Since there is the decomposition by types $A^r(X,\mathbb{C}) = \oplus_{a+b=r} A^{a,b}(X)$, inclusion (3) suggests to consider the orthogonal with respect to (5) of

$$G := \bigoplus_{a\,\geq\, n-p+1} A^{a,r-a}(X) \tag{6}$$

Then, if T denotes the image by the canonical map $\mathscr{Z}_r \otimes_{\mathbb{Q}} \mathbb{C} \twoheadrightarrow H_r(X,\mathbb{C})$ of $G^\perp$, set $S^{p,i} := PD(T)$. With this definition

$$F'^p H^i(X,\mathbb{Q}) \subseteq S^{p,i} \cap H^i(X,\mathbb{Q}) \tag{7}$$

Moreover, the identification of cohomology classes with the unique harmonic form they contain implies $F^{n-p+1}H^r(X,\mathbb{C}) \subseteq G$, hence $S^{p,i} \subseteq F^p H^i(X,\mathbb{C})$, which shows that (7) improves (4). The main result of the paper is

Theorem 1. *For any integers i and p we have the equality*

$$F'^p H^i(X,\mathbb{Q}) = S^{p,i} \cap H^i(X,\mathbb{Q}) \tag{8}$$

As a first step toward the proof of Theorem 1 we will characterize the classes into $S^{p,i}$ by a 'geometric' condition on the cycles representing their Poincaré duals. For any $\xi \in H^i(X,\mathbb{C})$, let

$$\Gamma := \sum_h m_h \, \sigma_h \in \mathscr{Z}_r \otimes_{\mathbb{Q}} \mathbb{C} \tag{9}$$

be such that $PD([\Gamma]) = \xi$. Recall that we assumed all the singular r-simplexes σ_h to be real-analytic embeddings. Fix such a h.

If Δ denotes the image of σ_h, then our assumptions ensure that the set of points of Δ which are in the image of some other σ_j is nowhere dense in Δ. Let $\mathscr{U} \subset \Delta$ denote the complementary set. For every $P \in \mathscr{U}$ the tangent space $T_P\Delta$ is a subspace of T_PX, where X is considered as a differentiable manifold, of real dimension $2n$. Recall that multiplication by $\sqrt{-1}$ on T_PX defines a complex structure $J : T_PX \to T_PX$ and $T_P\Delta + J(T_P\Delta)$ is the *smallest* complex subspace of T_PX which contains $T_P\Delta$.

Proposition 2. *We have $\xi \in S^{p,i}$ if and only if there is a singular cycle (9) representing $PD(\xi)$ such that, for every singular simplex σ_h in it and for every P in the corresponding open subset $\mathscr{U} \subset \Delta$, the following relation holds true*

$$dim_{\mathbb{C}}\big(T_P\Delta + J(T_P\Delta)\big) \le n - p \tag{10}$$

Notice that condition (10) is meaningful on the individual simplexes in (9). This will be the basic ingredient to prove

Proposition 3. *The spaces $S^{p,i}$ are rational, namely there is a $\mathbb{Q}$-subspace $W \subseteq H^i(X,\mathbb{Q})$ such that $S^{p,i} = W \otimes_{\mathbb{Q}} \mathbb{C}$.*

By general facts on the rationality of subspaces, equality (8) then implies $W = F'^pH^i(X,\mathbb{Q})$, hence

$$S^{p,i} = F'^pH^i(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{11}$$

In other words, the definition of $S^{p,i}$ is a geometric characterization of the complexification of $F'^pH^i(X,\mathbb{Q})$.

Finally, the proof of Theorem 1 never uses the fact that the homology and cohomology spaces considered are with rational coefficients. Hence the same argument yields

Corollary 4. *A r-cycle Γ with integral coefficients is contained into an algebraic subset $Y \subset X$, of codimension $\ge p$, if and only if (1) is true for every $\alpha \in \oplus_{a \ge n-p+1} A^{a,r-a}(X)$.*

Notice that, since the integral of a non closed form on a torsion cycle does not vanish a priori, the condition imposed on Γ is not vacuous also in the case of torsion cycles.

These results leave untouched Grothendiek's Generalized Hodge Conjecture. On the other hand, it seems to me that the method described here to build algebraic supports for suitable cohomology classes is of some interest.

After a section devoted to various preparations, the proofs of Proposition 2, Theorem 1 and Proposition 3 are given respectively in §§ 3 , 4 and 5 .

2 Preparations

Throughout this section we will fix an embedding $X \subset \mathbb{P}^N$ and a system of homogeneous coordinates $(\zeta_0 : \zeta_1 : \ldots : \zeta_N)$ for the ambient projective space.

First of all, we introduce some restriction on the systems of local holomorphic coordinates $(z_1, \ldots, z_n)$ on X we will use in the sequel.

Fix a point $P \in X$ and let $\Lambda \subset \mathbb{P}^N$ be any linear subvariety of dimension $N - n - 1$, which is disjoint from X and from the (projectivized) tangent space of X at P. Let $\pi : X \to \mathbb{P}^n$ denote the projection from Λ onto a general linear subvariety $\mathbb{P}^n \subset \mathbb{P}^N$. The differential map of π at P is an isomorphism by construction. Hence the restriction of π to a suitable euclidean neighborhood V of P is biholomorphic onto $\pi(V)$. We will use affine coordinates $z_1, \ldots, z_n$ on $\pi(V)$ as holomorphic coordinates on V. It is clear that these coordinates have an immediate algebro-geometricic meaning. In fact, assume that any point of a locus $L \subset V$ satisfies a set of algebraic equations $\varphi_j = 0$, with $j = 1, \ldots, k$, between $z_1, \ldots, z_n$. These equations define an algebraic subset Z inside $\mathbb{P}^n$, and L is clearly contained into the intersection of V with the cone C projecting Z from Λ. More precisely, if $codim_{\pi(P)}(Z, \mathbb{P}^n) = t$, then $codim_P(C \cap X, X) = t$. This follows easily from the fact that π is étale at P.

Assumption 1. *We will fix once and for all an open covering $\mathscr{V}$ of X, whose open sets are all domains of holomorphic coordinates as above, and every $V \in \mathscr{V}$ is contained for some k in the Zariski open set of $\mathbb{P}^N$*

$$U_k := \{ (\zeta_0 : \zeta_1 : \ldots : \zeta_N) \in \mathbb{P}^N \mid \zeta_k \neq 0 \}$$

Our next task will be the justification of the assumptions about the singular simplexes to be used on X we made in Introduction. They follow from the fact that every projective variety can be suitably triangulated. To clarify this and to further exploit the triangulation, we recall here for the reader convenience the complete statement of this theorem (see [6] and [11]).

The first step is the replacement of $\mathbb{P}^N$ and X with homeomorphic spaces.

If we assume that the homogeneous coordinates $(\zeta_0 : \zeta_1 : \ldots : \zeta_N)$ for a point P in $\mathbb{P}^N$ satisfy the normalization

$$\zeta_0\bar{\zeta}_0 + \zeta_1\bar{\zeta}_1 + \ldots + \zeta_N\bar{\zeta}_N = 1 \tag{12}$$

then the hermitian $(N+1) \times (N+1)$ matrix $(\zeta_h\bar{\zeta}_k)$ depends only on P. Therefore the real and imaginary parts of all the entries

$$\zeta_h\bar{\zeta}_k = \sigma_{hk} + i\tau_{hk} \qquad 0 \leq h \leq k \leq N$$

allow us to define a map $\rho : \mathbb{P}^N_{\mathbb{C}} \to \mathbb{R}^{(N+1)^2}$ by setting

$$\rho : (\zeta_0 : \zeta_1 : \ldots : \zeta_N) \mapsto (\sigma_{00}\,, \sigma_{01}, \ldots, \sigma_{NN}, \tau_{01}, \ldots, \tau_{N-1,N})$$

It is easily seen that ρ is a real-analytic embedding. The image $\mathfrak{R}$ of ρ is compact, and homeomorphic to $\mathbb{P}^N$. Moreover, $\mathfrak{R}$ is a real algebraic subset of $\mathbb{R}^{(N+1)^2}$. In fact, a complete set of equations for $\mathfrak{R}$ is given by

$$\sigma_{00} + \sigma_{11} + \ldots + \sigma_{NN} = 1$$

which translates (12), and the equations obtained by separating the real and imaginary part from all the obvious relations [1]

$$\zeta_j\,\bar{\zeta}_k\,\zeta_u\,\bar{\zeta}_v - \zeta_j\,\bar{\zeta}_v\,\zeta_u\,\bar{\zeta}_k = 0 \tag{13}$$

when they are written in terms of the σ's and τ's.

The image by ρ of any complex algebraic subset $Z \subset \mathbb{P}^N$ is a real algebraic subset of $\mathfrak{R}$. We will need this only when Z is the hypersurface $\zeta_k = 0$. In this simple case the k-th row and the k-th column of the matrix $(\zeta_h\bar{\zeta}_k)$ are both zero, and this supplies several algebraic equations in the σ's and τ's. Among them we find also $\sigma_{kk} = 0$, which in turn implies $\zeta_k = 0$.

What is actually triangulated is $\mathfrak{R}$ (both in [11] and [6]). For our set-up $X \subset \mathbb{P}^N$, the precise statement of the theorem (see [6]) says that we have a simplicial decomposition $\mathbb{R}^{(N+1)^2} = \cup_a \Delta_a$ and a *semi-algebraic* (see below) automorphism κ of $\mathbb{R}^{(N+1)^2}$ such that both $\mathfrak{R}$ and $\rho(X)$ are a finite union of some of the $\kappa(\Delta_a)$; hence the triangulation of $\mathbb{R}^{(N+1)^2}$ induces a

[1] Notice that these relations can be summarized by requiring that $rk\,(\zeta_h\bar{\zeta}_k) = 1$. In fact, a more conceptual (but not shorter) definition of $\mathfrak{R}$ and description of its properties can be given in terms of a suitable Segre embedding.

triangulations for $\mathfrak{R}$ and $\rho(X)$. Moreover, $\kappa(\Delta_a)$ is a locally closed smooth real-analytic submanifold of $\mathbb{R}^{(N+1)^2}$, and κ induces a real-analytic isomorphism $\Delta_a \simeq \kappa(\Delta_a)$, for any a . The whole set-up is as follows

$$
\begin{tikzcd}
\mathbb{R}^{(N+1)^2} \arrow[r, "\sim", "\kappa"'] & \mathbb{R}^{(N+1)^2} & \\
 & \mathfrak{R} \arrow[u, hook] \arrow[r, "\rho^{-1}"] & \mathbb{P}^N \\
\Delta_a \arrow[uu, hook] \arrow[r, "\kappa"'] & \rho(X) \arrow[u, hook] \arrow[r, "\rho^{-1}"'] & X \arrow[u, hook]
\end{tikzcd}
\tag{14}
$$

It is clear that all the singular simplexes $\sigma_a := \rho^{-1} \circ \kappa : \Delta_a \to X$ triangulate directly X. These simplexes are real-analytic embeddings. To see this, we have to write explicitly the map ρ^{-1}. It is enough to do it locally.

In fact, let us remark that, by iterated barycentric subdivision we can assume that the following requirement will be fulfilled from now on.

Assumption 2. *The image Δ_a of any singular simplex σ_a is contained in some open V of the covering $\mathscr{V}$ of X introduced in* Assumption 1.

By Assumption 1, any $V \in \mathscr{V}$ is contained in some $U_k = \{ \zeta_k \neq 0 \}$. Now, for any $(\sigma_{00}, \sigma_{01}, \dots, \sigma_{NN}, \tau_{01}, \dots, \tau_{N-1,N}) \in W_k := \rho(U_k)$ we can construct the hermitian matrix $(\zeta_a \bar{\zeta}_b)$. Therefore, on W_k the map ρ^{-1} is given by (we will not care anymore about the normalization (12))

$$
(\sigma_{00}, \sigma_{01}, \dots, \sigma_{NN}, \tau_{01}, \dots, \tau_{N-1,N}) \mapsto \left(\frac{\zeta_0}{\zeta_k} = \frac{\zeta_0 \bar{\zeta}_k}{\zeta_k \bar{\zeta}_k}, \dots, \widehat{\frac{\zeta_k}{\zeta_k}}, \dots, \frac{\zeta_N}{\zeta_k} \right)
$$

To use real coordinates on U_k we set as usual (just to simplify somewhat the notations we assume here $k = 0$)

$$
\frac{\zeta_j}{\zeta_0} =: z_j = x_j + i y_j \qquad j = 1, 2, \dots, N
$$

where x_j and y_j are real. Therefore we have

$$
x_j + i y_j = \frac{\zeta_j}{\zeta_0} = \frac{\zeta_j \bar{\zeta}_0}{\zeta_0 \bar{\zeta}_0} = \frac{\sigma_{0j} - i \tau_{0j}}{\sigma_{00}}
$$

hence

$$
x_j = \frac{\sigma_{0j}}{\sigma_{00}} \qquad y_j = -\frac{\tau_{0j}}{\sigma_{00}} \tag{15}
$$

A first consequence of these relations is that ρ^{-1} is real-analytic. Hence all the singular simplexes $\sigma_a = \rho^{-1} \circ \kappa : \Delta_a \to X$ are real-analytic as well; moreover, it is clear that they are embeddings.

To conclude this section we will exploit the fact that κ is a semi-algebraic automorphism of $\mathbb{R}^{(N+1)^2}$. The reader is referred to [1] for a detailed account about these sets; a brief summary of their main properties is in [6].

Semi-algebraic sets are the subsets of some $\mathbb{R}^M$ which can be obtained by finite union, finite intersection and complementation starting from the family of sets $\{\, x \in \mathbb{R}^M \mid f(x) \geq 0 \,\}$, where f is a polinomial with real coefficients. To see some elementary example, we checked above that both $\rho(X) \subset \mathbb{R}^{(N+1)^2}$ and $\rho(Z)$, where Z denotes the hyperplane $\zeta_k = 0$ of $\mathbb{P}^N$, are real algebraic. Hence they are, in particular, semi-algebraic. But then also $W_k = \rho(X) \setminus \rho(Z)$ is semi-algebraic.

A map between semi-algebraic sets is semi-algebraic if its graph is semi-algebraic. Not continuous semi-algebraic maps are not interesting, so usually one restricts to deal with the continuous ones.

Consider now a singular simplex $\sigma = \rho^{-1} \circ \kappa : \Delta_r \to X$ as above, except that here $\Delta_r \subset \mathbb{R}^{(N+1)^2}$ denotes a linear r-simplex. By Assumptions 2 and 1, the image Δ of σ is contained in some open $V \in \mathscr{V}$, which, in turn, is contained in some $U_k = \{\, \zeta_k \neq 0 \,\} \subset \mathbb{P}^N$. Here again we will assume $k = 0$ to simplify notations. Hence we can factorize σ more precisely as

$$\Delta_r \xrightarrow{\kappa} W_0 \xrightarrow{\rho^{-1}} U_0 = \mathbb{R}^{2N} \xrightarrow{p} V = \mathbb{R}^{2n}$$

where p is essentially a linear projection because of the peculiar local holomorphic coordinates we use on X.

Lemma 5. *The map $\sigma : \Delta_r \to \mathbb{R}^{2n}$ described above is semi-algebraic.*

Proof. First of all, Δ_r is clearly semi-algebraic and linear projections as p are known to be semi-algebraic. Since the composition of semi-algebraic maps is still semi-algebraic, it remains, therefore, to show that ρ^{-1} enjoys this property.

We remarked above that W_0 is semi-algebraic. Moreover, by (15) the graph of ρ^{-1} is contained inside the *algebraic* subset K of $W_0 \times U_0$, which is defined by the equations

$$\sigma_{00}\, x_j - \sigma_{0j} = 0 \qquad \sigma_{00}\, y_j + \tau_{0j} = 0 \qquad j = 1, 2, \ldots, N$$

Conversely, if $(A, B) \in K$, then trivially $B = \rho^{-1}(A)$ because of the (15). We conclude that K is the graph of ρ^{-1}, and the proof is complete. $\square$

Therefore, from now on we will strengthen our assumptions on the singular r-cycles we will deal with by requiring

Assumption 3. *$\mathscr{Z}_r$ is the $\mathbb{Q}$-module of the topological r-cycles which are rational combinations of real-analytic, embedded singular r-simplexes $\sigma : \Delta_r \to X$ such that $\sigma(\Delta_r)$ is contained in some $V \in \mathscr{V}$, and $\sigma : \Delta_r \to \mathbb{R}^{2n}$ is semi-algebraic.*

We have implicitly proved that the elements of $\mathscr{Z}_r$ as defined above still represent any homology class on X.

3 Proof of Proposition 2

If (10) is satisfied, then the pull-back to Δ of any form α in the space G defined in (6) is clearly $\equiv 0$, hence $\xi \in S^{p,i}$.

Conversely, assume that $\xi = PD([\Gamma])$, where

$$\Gamma \in \Big(\bigoplus_{a \geq n-p+1} A^{a,r-a}(X) \Big)^{\perp} \tag{16}$$

We already remarked that condition (10) is meaningful on the individual simplexes of Γ. Therefore we will deal with one of them $\sigma : \Delta_r \to X$, of image Δ. Here $\Delta_r \subset \mathbb{R}^r$ is the standard (open) r-simplex.

First of all, we write (10) in coordinates. Let V be a domain of holomorphic coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ on X. By decomposing any z_h into its real and imaginary parts we get the real coordinate chart $(V, x_1, \ldots, x_n, y_1 \ldots, y_n)$ for X. The complex structure J is given as follows. If we write the tangent vectors to X at P with respect to the base $\partial/\partial x_1, \ldots, \partial/\partial y_n$, then

$$J : (c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n}) \mapsto (-c_{n+1}, -c_{n+2}, \ldots, -c_{2n}, c_1, \ldots, c_n) \tag{17}$$

Moreover, let $(U, u_1, u_2, \ldots, u_r)$ be local coordinates for $\mathscr{U}$, and assume that $U \subset V$. Denoting by f the inclusion $\Delta \subset X$, we can assume more precisely that $U = \Delta \cap V$ because σ_h is an embedding. If $spt(\Gamma)$ denotes the closure of the union of all r-simplexes of Γ, it is then harmless to require that

$$U = spt(\Gamma) \cap V \tag{18}$$

Writing $f|_U$ in coordinates, for every $P \in \mathscr{U}$ the subspace $T_P\Delta$ is generated inside $T_P X$ by the columns of the jacobian matrix

$$\mathscr{J}_P = \frac{\partial\,(x_1, \ldots, x_n, y_1 \ldots, y_n)}{\partial\,(u_1, u_2, \ldots, u_r)}(P) = \begin{pmatrix} A \\ B \end{pmatrix} \tag{19}$$

where both A, B are $n \times r$ real matrices, whose entries are smooth functions on U. Now, by (17) the subspace $T_P\Delta + J(T_P\Delta)$ of T_PX is generated by the columns of

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

By elementary transformations on rows and columns, with complex coefficients, this matrix is transformed into

$$\begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}$$

Hence $dim_{\mathbb{R}}\big(T_P\Delta + J(T_P\Delta)\big) \,=\, 2\,rk(A+iB)$, and (10) becomes

$$rk(A+iB) \,\leq\, n-p \tag{20}$$

Before checking that this condition is satisfied at every point of U, we have to settle a minor point. If $p = 0$ there is nothing to prove, hence we assume $p \geq 1$. Moreover, if $i > n$, then by the Weak Lefschetz theorem we can always assume that $\Gamma \subseteq Y$, where Y is the complete intersection of $i-n$ general very ample divisors on X. Hence $p \geq i-n$ holds true in general in this case, and to have something to prove we assume $p > i-n$. In any case $k := n-i+p > 0$.

Fix an arbitrary $P \in U$, and let $\varphi : V \to \mathbb{R}$ be a function ≥ 0, of class $\mathscr{C}^\infty$, with compact support K and such that $\varphi(P) > 0$.

Fix also arbitrarily $n-p+1$ indices $1 \leq h_1 < \ldots < h_{n-p+1} \leq n$. Moreover, for any choice of natural numbers a, b such that $a+b = k-1$, set

$$\beta \,=\, dz_{k_1} \wedge dz_{k_2} \wedge \,\ldots\, \wedge dz_{k_a} \wedge d\bar{z}_{l_1} \wedge \,\ldots\, \wedge d\bar{z}_{l_b}$$

Then

$$\alpha := \begin{cases} \varphi\, dz_{h_1} \wedge dz_{h_2} \wedge \,\ldots\, \wedge dz_{h_{n-p+1}} \wedge \beta & \text{on } V \\[2ex] \equiv 0 & \text{outside } V \end{cases}$$

is an element of $\oplus_{a \geq n-p+1} A^{a, r-a}(X)$. By construction, the support of α is contained in V, and then by (18) and (16) we get

$$0 \,=\, \int_\Gamma \alpha \,=\, \int_U f^*\alpha$$

Now $f^* \alpha \,=\, \varphi|_U \det(M)\, du_1 \wedge \,\ldots\, \wedge du_r$, where M is a $r \times r$ matrix which can be written in block form as (recall the jacobian matrix (19))

$$M \;=\; \begin{array}{|c|} \hline \text{the rows} \quad h_1, \ldots, h_{n-p+1} \\ \text{from } A + iB \\ \hline \\ a \quad \text{rows from} \quad A + iB \\ \\ \hline \\ b \quad \text{rows from} \quad A - iB \\ \\ \hline \end{array}$$

Let us denote by H and I the functions $U \to \mathbb{R}$ which are respectively the real and imaginary part of $\det(M) : U \to \mathbb{C}$. Then

$$\int_U \varphi|_U \; H \; du_1 \wedge \,\ldots\, \wedge du_r \;+\; i \int_U \varphi|_U \; I \; du_1 \wedge \,\ldots\, \wedge du_r \;=\; 0$$

But the two integrals above are real numbers, hence

$$(21) \qquad \int_U \underbrace{\varphi|_U}_{\geq 0} \; H \; du_1 \wedge \,\ldots\, \wedge du_r \;=\; \int_U \varphi|_U \; I \; du_1 \wedge \,\ldots\, \wedge du_r \;=\; 0$$

Assume now that for the point $P \in U$ considered above we have

$$(\det M)(P) \;=\; H(P) + iI(P) \;\neq\; 0$$

To fix ideas, suppose that $H(P) \neq 0$. Then we can choose the support K of φ so small that the sign of H on $K \cap \Gamma$ is constant (notice that M does not depend upon α , but it depends only from the inclusion $f : \Delta \subset X$). This contradicts (21) because $\varphi(P) > 0$. Therefore necessarily $(\det M)(P) = 0$. This argument works for any $P \in U$, hence

$$(22) \qquad \det M \;\equiv\; 0 \qquad \text{on } \; U$$

To conclude the proof of the theorem we have to show that the determinants ν_I of all the maximal minors of the first $n - p + 1$ rows of M are zero. Because of (22), if we write the expansion of $det(M)$ based on the first $n - p + 1$ rows, we get (the signs are embodied into the ν_I 's)

$$\sum_{I = (\, 0 \leq i_1 < \ldots < i_{n-p+1} \leq r\,)} \nu_I \, \gamma_I \;=\; 0$$

This means that the last $a+b$ rows of M (which correspond to the $(k-1)$-form β) allow us to construct a solution of the homogeneous linear equation

$$(23) \qquad \sum_I \nu_{_I} X_{_I} \;=\; 0 \qquad \text{in the} \quad \binom{r}{n-p+1} \quad \text{indeterminates} \quad X_{_I}$$

To show that all the $\nu_{_I}$ vanish we have just to produce enough independent solutions of (23). This can be done by using different β's.

In fact, since σ_h is an embedding, the rank of $\mathscr{J}_{\!P}$ is r at any point, and therefore the rows of the matrix (19) generate $\mathbb{R}^r$. Hence they generate the complexified space $\mathbb{C}^r$ of $\mathbb{R}^r$ as well. This implies that $\mathbb{C}^r$ can be generated also by the rows of $A+iB$ toghether with those of $A-iB$.

Now, it is clear that $(\gamma_{_I})_{_I}$ can be thought as an element of $\wedge^{k-1}\mathbb{C}^r$, and the above remark says that this space is generated by all possible wedge products of $k-1$ among the rows of $A+iB$ and those of $A-iB$. In other words, by changing β we can construct $\binom{r}{n-p+1}$ independent solutions of (23), hence this equation is trivial and the proof of Proposition 2 is complete. $\square$

4 Proof of Theorem 1

Let $\xi \in S^{p,i} \cap H^i(X,\mathbb{Q})$, and let $\Gamma = \sum_h m_h \sigma_h \in \mathscr{Z}_r$ be a r-cycle whose class is the Poincaré dual of ξ, and which satisfies (16). We want to construct an algebraic subset of X, of codimension $\geq p$, containing Γ. For this it is sufficient to construct an algebraic subset of codimension $\geq p$ containing the image Δ_h of σ_h, for any singular simplex σ_h of Γ.

Let $\sigma : \Delta_r \to X$ be one of the singular simplexes of Γ, and denote its image by Δ. In the coordinates $u_1, u_2, \ldots, u_r$ of $\mathbb{R}^r$ the map σ is given by

$$x_j = f_j(u_1, u_2, \ldots, u_r) \qquad y_j = f_{j+n}(u_1, \ldots, u_r) \qquad j = 1, \ldots, n$$

where any $f_k : \Delta_r \to \mathbb{R}$ is real-analytic because σ is. Then we define a real-analytic map $g : \Delta_r \to \mathbb{C}^n$ by setting for any $j = 1, \ldots, n$

$$(24) \qquad z_j = g_j(u_1, u_2, \ldots, u_r) \;:=\; f_j(u_1, \ldots, u_r) + i\, f_{j+n}(u_1, \ldots, u_r)$$

It is well known that any g_j extends to a holomorphic function F_j, defined on a suitable neighborhood of Δ_r inside $\mathbb{C}^r$ (on $\mathbb{C}^r$ we will use complex coordinates $w_1, w_2, \ldots, w_r$ where $w_k = u_k + iv_k$ is the decomposition of w_k into its real and imaginary parts, and the u_k's are as above). Therefore, there is an open set $E \subset \mathbb{C}^r$ containing Δ_r, such that every F_j is defined on E, and the F_j are the components of a holomorphic map $F : E \to \mathbb{C}^n$. Moreover, we can also assume (with a slight abuse of notation) that $F(E) \subseteq V$. The proof of the following lemma is completely standard and left to the reader.

Lemma 6. *The jacobian matrix of F is $A + iB$ (see (19)), its rank is $\leq n - p$ at any point of E, and reaches its maximum at some point of Δ_r.*

Another basic property of the map $F : E \to \mathbb{C}^n$ is

Lemma 7. *If $E_0 \subseteq E$ is any open semi-algebraic set, then the restriction $F : E_0 \to \mathbb{C}^n = \mathbb{R}^{2n}$ is semi-algebraic.*

Proof. We know by Lemma 5 that $\sigma : \Delta_r \to \mathbb{R}^{2n}$ is semi-algebraic. It is easily seen that this is equivalent to have $f_j : \Delta_r \to \mathbb{R}$ semi-algebraic for every $j = 1, \ldots, 2n$. In turn, this amounts (see [1], Prop. 8.1.7) to the existence for any j of a polynomial $P_j(U_1, \ldots, U_r, T)$ with real coefficients and positive degree with respect to T, such that for every $u \in \Delta_r$

$$P_j(u, f_j(u)) = 0 \tag{25}$$

holds true; this is usually expressed by saying that f_j is a Nash function on Δ_r. The left hand side of the above relation can be thought as a real-analytic function of u. Hence, if we replace u by $u + iv$ in (25) we get, by a variant of the Identity Principle (see e.g. [10], pag. 21)

$$P_j(u + iv, f_j(u + iv)) = 0 \tag{26}$$

for every $u + iv \in E$.

This relation says that $f_j(u + iv)$ is algebraic over the field of fractions $\mathbb{C}(u, v)$ of the ring of polynomials $\mathbb{C}[u, v] = \mathbb{C}[u_1, \ldots, u_r, v_1, \ldots, v_r]$. But $\mathbb{C}(u, v)$ is algebraic over $\mathbb{R}(u, v)$, hence $f_j(u + iv)$ is algebraic over $\mathbb{R}(u, v)$ as well, and there is $Q_j \in \mathbb{R}[u, v, T]$, with positive degree with respect to T, such that for every $(u, v) = u + iv \in E$

$$Q_j(u, v, f_j(u + iv)) = 0$$

Since Q_j has real coefficients, this implies $Q_j(u, v, \overline{f_j(u + iv)}) = 0$, and also $\overline{f_j(u + iv)}$ is algebraic over $\mathbb{R}(u, v)$. If we set

$$f_j(u + iv) = \varphi_j(u + iv) + i\,\psi_j(u + iv)$$

where φ_j and ψ_j are real valued functions defined on E, then we can conclude that both $\varphi_j(u + iv)$ and $\psi_j(u + iv)$ are algebraic over $\mathbb{R}(u, v)$, i.e. they are Nash functions on E. Therefore these functions are semi-algebraic on every open semi-algebraic subset E_0 of $E \subset R^{2r}$ (loc. cit.). Finally, by (24) we have

$$F_j(u + iv) = f_j(u + iv) + i\, f_{j+n}(u + iv) =$$

$$= \varphi_j(u + iv) - \psi_{j+n}(u + iv) + i\left(\psi_j(u + iv) + \varphi_{j+n}(u + iv)\right)$$

Hence, for any $j = 1, \ldots, n$, both the real and imaginary components of the functions F_j are semi-algebraic on E_0, and the proof of Lemma 7 is complete. $\square$

Let $P_0 \in \Delta_r$ be a point where the rank of the Jacobian matrix of F reaches its maximum. By the Rank Theorem, locally at $P = F(P_0)$ the set $F(E)$ is a complex analytic manifold Y, locally closed in X. The dimension of Y is $n - t \leq n - p$, by Lemma 6.

The crucial point in the proof of Theorem 1 is to check that the Zariski closure of Y in X has codimension $\geq p$, or, roughly speaking, that 'Y is algebraic'. Several results are known which assert that a complex-analytic subset Z of $\mathbb{C}^m$ is algebraic, under suitable conditions. For example, this is true whenever $Z \subset \mathbb{C}^m = \mathbb{R}^{2m}$ is semi-algebraic ([3]).

In the beautyful paper [2] it was recognized that the property for a complex-analytic subset Z of some algebraic ambient complex manifold to be itself algebraic is of *local* nature. Below we will use locally the semi-algebraic condition to see that 'Y is algebraic' [2].

The semi-algebraic condition is concretely attained on Y as follows. The refined form of the Rank Theorem says that, with respect to suitable local holomorphic coordinates (not necessarily of the kind introduced in § 2), the map F can be written as a linear projection. Hence, in particular, $F : E \to F(E)$ is an open map locally at P_0, and the image $F(W)$ of a suitably small open polydisk W centered at P_0 will be open in Y, and semi-algebraic by Lemma 7. Since we are going to perform a local argument at P, we will just assume $Y = F(W)$. To simplify notations, we assume also that $P = (0, \ldots, 0)$.

Let $\mathcal{O}$ denote the ring of germs at P of holomorphic functions on X, and let $\mathscr{I} \subset \mathcal{O}$ be the ideal of germs representing functions whose restriction to Y vanishes identically around P. It is easily seen that $\mathscr{I}$ is a prime ideal. Moreover, $ht(\mathscr{I}) = t$ because the dimension of Y is $n - t$. Now, the ring of polynomials $\mathbb{C}[\,\underline{z}\,] = \mathbb{C}[z_1, \ldots, z_n]$ can be thought as a subring of $\mathcal{O}$. Consider the prime ideal $I := \mathscr{I} \cap \mathbb{C}[\,\underline{z}\,]$. To conclude the proof of Theorem 1 we have only to check that

$$ht(I) \geq t \tag{27}$$

(actually, also $ht(I) \leq t$ holds true, so that we have an equality). In fact, thanks to the restriction on the systems of local holomorphic coordinates on X we made in § 2, Y is contained in the irreducible *algebraic* subvariety Z of

[2] After the paper was completed, I learned that similar results were obtained in [4].

X defined locally at P by I, and (27) implies that this algebraic subvariety has codimension $t \geq p$ in X.

To summarize and conclude, we started from a singular simplex $\sigma : \Delta_r \to X$ of Γ. Then $\sigma(W \cap \Delta_r) \subseteq Z$ because W was defined above as an open polydisk centered at $P_0 \in \Delta_r$ such that $Y = F(W)$. But $W \cap \Delta_r$ is open into Δ_r and σ is real-analytic. Hence the whole image Δ of σ is contained into Z.

To use the additional information that Y is semi-algebraic, we have to complete the above algebraic set-up. Let A denote the ring of germs at P of real-analytic, complex valued functions on X. Notice that $\mathcal{O}$ is a subring of A. Both A and $\mathcal{O}$ are regular local rings, of dimension $2n$ and n respectively. Moreover, $\mathbb{C}[\underline{x}, \underline{y}] = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is a subring of A. Hence, to start we have the commutative diagram of rings

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & A \\ \uparrow & & \uparrow \\ \mathbb{C}[\underline{z}] & \longrightarrow & \mathbb{C}[\underline{x}, \underline{y}] \end{array} \tag{28}$$

where all the maps are inclusions. Now, denote by $\mathscr{J} \subset A$ the ideal of germs representing functions whose restriction to Y vanishes identically around P. It is easily seen that $\mathscr{J}$ is a prime ideal and that $\mathscr{I} = \mathscr{J} \cap \mathcal{O}$. Moreover, we set $J := \mathscr{J} \cap \mathbb{C}[\underline{x}, \underline{y}]$.

To prove (27), we start by applying the 'dimension formula' ([9], p. 119) to the ring extension $\mathbb{C}[\underline{z}] \subset \mathbb{C}[\underline{x}, \underline{y}]$, thus getting (notice that $\mathbb{C}[\underline{x}, \underline{y}] = \mathbb{C}[\underline{z}, \underline{x}]$)

$$ht(J) + tr.deg._{\kappa(I)} \kappa(J) = ht(I) + tr.deg._{\mathbb{C}[\underline{z}]} \mathbb{C}[\underline{x}, \underline{y}]$$

where the trascendence degree in the right hand side is that of the quotient field of $\mathbb{C}[\underline{x}, \underline{y}]$ over that of $\mathbb{C}[\underline{z}]$, and $\kappa(J)$ is the quotient field of $\mathbb{C}[\underline{x}, \underline{y}]/J$.

Lemma 8. $ht(J) = 2t$

Proof. The real-analytic manifold underlying Y has dimension $2(n-t)$, and this is the dimension of Y as a semi-algebraic set ([1], Prop. 2.8.13).

The ideal $K \subset \mathbb{R}[\underline{x}, \underline{y}]$ of all the polynomials vanishing on Y defines the Zariski closure $\overline{Y}$ of Y inside $\mathbb{R}^{2n}$. It is clear that $K = J \cap \mathbb{R}[\underline{x}, \underline{y}]$, which implies that K is a prime ideal. By [1], Prop. 2.8.2, the dimension of $\overline{Y}$ is $2(n-t)$, and then $ht(K) = 2t$.

Finally, since the extension $\mathbb{R}[\underline{x}, \underline{y}] \subset \mathbb{C}[\underline{x}, \underline{y}]$ is integral and flat, we can conclude $ht(J) = ht(K) = 2t$. $\square$

This lemma implies that (27) is equivalent to

$$tr.deg._{\kappa(I)}\,\kappa(J)\;\geq\;n-t \tag{29}$$

We will apply now the ‘dimension formula’ again, this time to the ring extension $\mathcal{O}\to\mathcal{O}[\,\underline{x}\,]$ $(\subset A)$. This is possible because $\mathcal{O}$ is universally catenarian, being a regular local ring, and $\mathcal{O}[\,\underline{x}\,]$ is an $\mathcal{O}$-algebra of finite type. Set $\mathscr{H}:=\mathscr{J}\cap\mathcal{O}[\,\underline{x}\,]$. Then

$$ht(\mathscr{H})\;+\;tr.deg._{\kappa(\mathscr{I})}\,\kappa(\mathscr{H})\;=\;ht(\mathscr{I})\;+\;tr.deg._{\mathcal{O}}\mathcal{O}[\,\underline{x}\,]$$

The rings $\mathcal{O}$ and A can be thought as the rings of convergent power series with complex coefficients, respectively in the variables $z_1,\ldots,z_n$ and $z_1,\ldots,z_n,x_1,\ldots,x_n$. This shows that $x_1,\ldots,x_n$ are algebraically independent over $\mathcal{O}$. Hence, if we assume for a moment

Lemma 9. $ht(\mathscr{H})\;=\;2\,t$

we conclude $tr.deg._{\kappa(\mathscr{I})}\,\kappa(\mathscr{H})=n-t$.

Finally, consider the commutative diagram of integral domains and injective ring maps, and the corresponding diagram of quotient fields

$$\begin{array}{ccc} \mathcal{O}/\mathscr{I} & \longrightarrow & \mathcal{O}[\,\underline{x}\,]/\mathscr{H} \\ \uparrow & & \uparrow \\ \mathbb{C}[\,\underline{z}\,]/I & \longrightarrow & \mathbb{C}[\,\underline{z}\,,\underline{x}\,]/J \end{array} \qquad\qquad \begin{array}{ccc} \kappa(\mathscr{I}) & \longrightarrow & \kappa(\mathscr{H}) \\ \uparrow & & \uparrow \\ \kappa(I) & \longrightarrow & \kappa(J) \end{array}$$

It is clear that a trascendence base for $\kappa(\mathscr{H})$ over $\kappa(\mathscr{I})$ can be extracted from the set of the residue classes mod $\mathscr{H}$ of $x_1,\ldots,x_n$. Say such a base is $\bar{x}_1,\ldots,\bar{x}_{n-t}$, the number of its elements was determined above. It is also clear that these elements are, a fortiori, algebraically independent over $\kappa(I)$. But $\bar{x}_1,\ldots,\bar{x}_{n-t}$ actually belong to $\mathbb{C}[\,\underline{z}\,,\underline{x}\,]/J$, and the proof of (29) is complete, except for the proof of Lemma 9.

Proof of Lemma 9. First of all, we construct a suitable system of generators for the ideal $J_{loc}\subset\mathbb{C}[\,\underline{z}\,,\underline{x}\,]_{loc}$, where ‘*loc*’ denotes the localization with respect to $(\,\underline{z}\,,\underline{x}\,)$. Recall that we assumed that $P=(0,\ldots,0)$; we can also assume that Y is defined locally at P by holomorphic equations like

$$z_1\;+\;\text{higher order terms}\;=\;0\qquad\ldots\ldots\qquad z_t\;+\;\text{h.o.t.}\;=\;0$$

Hence, if we consider $Y\subset V\subseteq\mathbb{C}^n$, then the embedded tangent space to the complex manifold Y at P is defined by $z_1=\ldots=z_t=0$. Therefore the embedded tangent space to the semi-algebraic set Y inside $\mathbb{R}^{2n} = \mathbb{C}^n$ is defined by

$$x_1 = 0 \quad \ldots\ldots \quad x_t = 0 \quad y_1 = 0 \quad \ldots\ldots \quad y_t = 0$$

It is then well known that we can define Y in a suitable Zariski neighborhood of P by a set of polynomial equations (with real coefficients) of the following kind, where j runs between 1 and t

$$(30) \qquad P_j = x_j + \text{h.o.t.} = 0 \qquad\qquad P_{t+j} = y_j + \text{h.o.t.} = 0$$

The polynomials $P_1, \ldots, P_t, P_{t+1}, \ldots, P_{2t}$ generate inside $\mathbb{C}[\underline{z}, \underline{x}]_{loc}$ a prime ideal of height $2t$ because their list can be completed by (30) to a minimal system of generators for the maximal ideal of the regular local ring $\mathbb{C}[\underline{z}, \underline{x}]_{loc}$. Now the ideal $(P_1, \ldots, P_{2t})$ is clearly contained into J_{loc}, hence the two ideals coincide having the same height, and we have the desired system of generators for J_{loc}.

Claim The polynomials $P_1, \ldots, P_{2t}$ generate the ideal $\mathscr{J}$ in A.

Clearly these polynomials are contained in $\mathscr{J}$. Conversely, let $\varphi : T \to \mathbb{C}$ a real-analytic function, with $T \subset \mathbb{R}^{2n}$ a neighborhood of P in X, such that its germ belongs to $\mathscr{J}$. By definition of $\mathscr{J}$ this means that the restriction of φ to Y vanishes identically.

Consider now the algebraic subset Z of $\mathbb{C}^{2n}$, defined by the equations (30). Locally at $\underline{0} := (0, \ldots, 0) \in \mathbb{C}^{2n}$ it is a complex manifold, of dimension $2n - 2t$. We already remarked that A can be thought as the ring of germs at $\underline{0}$ of holomorphic functions $\mathbb{C}^{2n} \to \mathbb{C}$. The holomorphic extension $\widetilde{\varphi}$ of φ is the holomorphic function defined in a suitable neighborhood of $\underline{0}$ inside $\mathbb{C}^{2n}$ by the same power series of φ. Therefore, to prove that the germ of φ belongs to the ideal of A generated by $P_1, \ldots, P_{2t}$ we have only to check that the restriction of $\widetilde{\varphi}$ to Z vanishes identically. In fact,

$$\sqrt{(P_1, \ldots, P_{2t})\, A} = (P_1, \ldots, P_{2t})\, A$$

because of the same argument used above shows that $P_1, \ldots, P_{2t}$ generate a prime ideal of the regular local ring A.

If we apply the Implicit Function Theorem for real-analytic functions to (30), we get a neighborhood H of $(0, \ldots, 0) \in \mathbb{R}^{2n-2t}$, with coordinates $x_{t+1}, \ldots, x_n, y_{t+1}, \ldots, y_n$, and a real-analytic map $\alpha : H \to \mathbb{R}^{2n}$ which parametrizes Y locally at P. Therefore the holomorphic extension $\widetilde{\alpha}$ of α parametrizes Z locally at $\underline{0}$.

The holomorphic function $\widetilde{\varphi} \circ \widetilde{\alpha} - \widetilde{\varphi \circ \alpha}$ is defined in a neighborhood H' of H inside $\mathbb{C}^{2n-2t}$; if H is connected, say an open ball, it is harmless to assume that H' is connected as well. But the restriction of $\widetilde{\varphi}\circ\widetilde{\alpha}-\widetilde{\varphi\circ\alpha}$ to H vanishes identically, hence $\widetilde{\varphi}\circ\widetilde{\alpha}=\widetilde{\varphi\circ\alpha}$ on H' ([10], pag. 21). Now, since $\varphi\equiv 0$ on Y, we have $\varphi\circ\alpha\equiv 0$, hence $\widetilde{\varphi}\circ\widetilde{\alpha}\equiv 0$. In other words, the restriction of $\widetilde{\varphi}$ to Z vanishes identically around $\underline{0}$. As we said above, this implies that the germ of φ belongs to $(P_1,\dots,P_{2t})\,A$, and therefore the Claim is completely proved.

We can complete the proof of Lemma 9.

Let $\mathscr{M}$ denote the maximal ideal of $\mathcal{O}$. Then $\mathscr{N}:=\mathscr{M}+(\,\underline{x}\,)$ is a maximal ideal of $\mathcal{O}[\,\underline{x}\,]$, and we have a local homomorphism $\mathcal{O}[\,\underline{x}\,]_{\mathscr{N}}\to A$. Both rings are regular local rings of dimension $2\,n$, whose respective maximal ideals are both generated by the germs of the coordinate functions $z_1,\dots,z_n,x_1,\dots,x_n$. We can then conclude that $\mathcal{O}[\,\underline{x}\,]_{\mathscr{N}}\to A$ is flat ([9], Thm. 23.1), hence it is a faithfully flat extension. But then, by the Claim and [9], Thm. 7.5

$$(P_1,\dots,P_{2t})\,\mathcal{O}[\,\underline{x}\,]_{\mathscr{N}}\;=\;(P_1,\dots,P_{2t})\,A\,\cap\,\mathcal{O}[\,\underline{x}\,]_{\mathscr{N}}\;=\;\mathscr{J}\,\cap\,\mathcal{O}[\,\underline{x}\,]_{\mathscr{N}}\;=\;\mathscr{H}_{\mathscr{N}}$$

The usual argument shows that $(P_1,\dots,P_{2t})\,\mathcal{O}[\,\underline{x}\,]_{\mathscr{N}}$ is a prime ideal of height $2\,t$, hence Lemma 9 is completely proved, as well as Theorem 1.

5 Proof of Proposition 3

The obvious actions of the Galois group $\mathscr{G}(\mathbb{C}/\mathbb{Q})$ on $H_r(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$ and $H^i(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$ can be transported respectively on $H_r(X,\mathbb{C})$ and $H^i(X,\mathbb{C})$ by the canonical isomorphisms of complex vector spaces

$$\mu_h : H_r(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}\to H_r(X,\mathbb{C}) \qquad\qquad \mu_c : H^i(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}\to H^i(X,\mathbb{C})$$

supplied by the corresponding universal coefficients theorems. More precisely, given $\varphi\in\mathscr{G}(\mathbb{C}/\mathbb{Q})$ we define $\varphi^i : H^i(X,\mathbb{C})\to H^i(X,\mathbb{C})$ by setting

$$\varphi^i\;:=\;\mu_c\circ(\,id\otimes\varphi\,)\circ\mu_c^{-1}$$

We will follow the usual convention to denote $\varphi^i(\xi)$ by ξ^{φ}. Similar definition and notational convention are also assumed for the homology spaces.

Now, the following diagram is clearly commutative

$$\begin{tikzcd}
H^i(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \arrow[r, "PD_{\mathbb{Q}}\otimes id_{\mathbb{C}}"] \arrow[d, "\mu_c"'] & H_r(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \arrow[d, "\mu_h"] \\
H^i(X,\mathbb{C}) \arrow[r, "PD_{\mathbb{C}}"'] & H_r(X,\mathbb{C})
\end{tikzcd}$$

and it is easily seen that this implies

$$PD_{\mathbb{C}} \circ \varphi^{i} \;=\; \varphi_r \circ PD_{\mathbb{C}} \tag{31}$$

i.e. the actions of $\mathscr{G}(\mathbb{C}/\mathbb{Q})$ on $H_r(X,\mathbb{C})$ and $H^{i}(X,\mathbb{C})$ commute with the Poincaré duality.

Take now $\xi \in S^{p,i}$, and assume that ξ is the Poincaré dual of $[\Gamma]$, where any singular simplex of Γ satisfies (10). Then, for every $\varphi \in \mathscr{G}(\mathbb{C}/\mathbb{Q})$

$$\xi^{\varphi} \;=\; PD\big([\Gamma]^{\varphi}\big)$$

because of (31). Moreover, the commutative diagram

$$\begin{tikzcd}
\mathscr{Z}_r \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, two heads, "can."] \arrow[d, "id\otimes\varphi"'] & H_r(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "\mu_h"] \arrow[d, "id\otimes\varphi"'] & H_r(X,\mathbb{C}) \arrow[d, "\varphi_r"] \\
\mathscr{Z}_r \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, two heads, "can."'] & H_r(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "\mu_h"'] & H_r(X,\mathbb{C})
\end{tikzcd}$$

clearly implies $[\Gamma]^{\varphi} \;=\; \big[\,\Gamma^{\varphi}\,\big]$. The cycle Γ can be written as $\Gamma = \sum_h z_h \Gamma_h$ where $\Gamma_h \in \mathscr{Z}_r$ and $z_h \in \mathbb{C}$ for any h. Note that we can use the same finite set of singular simplexes to represent any Γ_h, namely

$$\Gamma_h \;=\; a_{h1}\,\sigma_1 + \ldots + a_{hs}\,\sigma_s \qquad \text{where} \quad a_{hj} \in \mathbb{Q} \quad \text{for any} \quad h, j$$

Hence

$$\Gamma \;=\; \sum_{j=1}^{s} \big(a_{1j}\, z_1 + \ldots + a_{tj}\, z_t \big)\, \sigma_j$$

Therefore our assumption on Γ can be restated as: σ_j satisfies (10) whenever $a_{1j}\, z_1 + \ldots + a_{tj}\, z_t \neq 0$. Finally

$$\Gamma^{\varphi} \;=\; \sum_{h} \varphi(z_h)\, \Gamma_h \;=\; \sum_{j=1}^{s} \varphi\big(a_{1j}\, z_1 + \ldots + a_{tj}\, z_t \big)\, \sigma_j$$

which shows that also all the singular simplexes of Γ^{φ} satisfy (10). Hence $\xi^{\varphi} \in S^{p,i}$, and then, for every $\varphi \in \mathscr{G}(\mathbb{C}/\mathbb{Q})$ we have $(\,S^{p,i}\,)^{\varphi} \subseteq S^{p,i}$.

The rationality of $S^{p,i}$ follows by descente galoisienne.

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