Well-covered and uniformly well-covered graphs

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Abstract

A graph G is called well-covered if all maximal independent sets of vertices have the same cardinality. A well-covered graph G is called uniformly well-covered if there is a partition of the set of vertices of G such that each maximal independent set of vertices has exactly one vertex in common with each part in the partition. The problem of determining which graphs is well-covered, was proposed in 1970 by M.D. Plummer. Let $\mathcal{G}$ be the class of graphs with some disjoint maximal cliques covering all vertices. In this paper, some necessary and sufficient conditions are presented to recognize which graphs in the class $\mathcal{G}$ are well-covered or uniformly well-covered. This characterization has a nice algebraic interpretation according to zero-divisor elements of edge ring of graphs which is illustrated in this paper.

Key words: Well-covered graph, uniformly well-covered graph, maximal independent set, maximal clique, minimal vertex cover, zero-divisor element.
2010 MR Subject Classification: 05C25, 05E45.

Introduction

A graph G is said to be well-covered if every maximal independent sets of vertices have the same cardinality. In some texts, well-covered graphs are called *unmixed*. These graphs were introduced by M.D. Plummer [16] in 1970. Although the recognition problem of well-covered graphs in general is Co-NP-complete ([22]), it is characterized for certain classes of graphs. For instance, claw-free well-covered graphs [20], well-covered graphs which have girth at least 5 [4], (4-cycle, 5-cycle)-free [5] or chordal graphs [18] are all recognizable in polynomial time. An excellent survey of the work on well-covered graphs is given in Plummer [17] and a survey of recent activities is presented in Hartnell [8]. There are also many other papers studying special properties of well-covered or very well-covered graphs. For example see references [1], [2], [3], [9], [12], [13], [14], [18], [19], [20], [25] and [27].

Let G be a graph with no loop and multiple edge. Denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$. A subset A of $V(G)$ is called an *independent set* if there is no any edge between vertices of A. A subset C of $V(G)$ is called a *clique* if any two vertices in C are connected by an edge in $E(G)$. A subset $B \subseteq V(G)$ is called a *vertex cover* if any edge in $E(G)$ has at least one vertex in B.

Let A and B be subsets of $V(G)$. We say A *dominates* B if for any vertex v in B there is at least one vertex in A connected to v by an edge in $E(G)$. A subset A of $V(G)$ is called a *dominating set* if any vertex of G is in A or adjacent to some vertices in A. It is called *minimal dominating set* if there is no proper subset of A dominating G. A graph G is called *well-dominated* if all minimal dominating sets of G are of the same cardinality.

A subset of $E(G)$ is called a *matching* if there is not any common vertex in any two edges in this set. A matching is called *perfect matching* if it covers all vertices of G.

Let $[n] = \{1, 2, \ldots, n\}$. A (finite) simplicial complex Δ on n vertices, is a system of subsets of $[n]$ such that the following conditions hold:
a) $\{i\} \in \Delta$ for any $i \in [n]$,
b) if $E \in \Delta$ and $F \subseteq E$, then $F \in \Delta$.
An element of Δ is called a *face* and a maximal face with respect to inclusion is called a *facet*. The set of all facets is denoted by $\mathcal{F}(\Delta)$. The dimension of a face $F \in \Delta$ is defined to be $|F|-1$ and dimension of Δ is maximum of dimensions of its faces. A simplicial complex is called *pure* if all its facets have the same cardinality. For more details on simplicial complexes see [24].

Let G be a graph. The set of all independent sets of vertices of G is a simplecial complex, because, any single vertex is independent and any subset of an independent set is again independent. This simplicial complex is denoted by Δ_G. With the above definitions, a graph G is well-covered means that the simplicial complex Δ_G is pure.

A simplicial complex Δ is called *balanced* if it is pure and there is a partition of the set of vertices as $V_1, \ldots, V_s$ such that any facet of Δ has exactly one vertex in common with each V_i, $i = 1, \ldots, s$. This definition is introduced by R. Stanley in [23]. In literature of graph theory there is a classical definition for balanced graphs which is different with the above definition of balanced simplicial complex of independent sets of a graph. To avoid any confusion, we present the following definition.

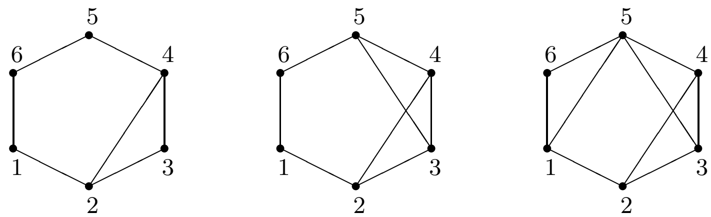
Definition 1. A graph G is called *uniformly well-covered* if the simplicial complex Δ_G is balanced. Equivalently, a graph G is uniformly well-covered if it is well-covered and there is a partition of $V(G)$ as $V_1, \ldots, V_s$ such that any maximal independent set of G has exactly one vertex in common with each V_i, $i = 1, \ldots, s$.

Example. As an example, in the following figure, from left to right,

i) The first graph is not well-covered: $\{1, 3, 5\}$ and $\{1, 4\}$ are among maximal independent sets with different cardinalities.

ii) The second is well-covered but not uniformly well-covered: $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 5\}$, $\{2, 6\}$, $\{3, 6\}$, $\{4, 6\}$ are all maximal independent sets which are all of size 2 but, there is no any partition of vertices to satisfy the definition of uniformly well-covered.

iii) The third graph is uniformly well-covered: $\{1, 3\}$, $\{1, 4\}$, $\{2, 5\}$, $\{2, 6\}$, $\{3, 6\}$, $\{4, 6\}$ are all maximal independent sets and $\{\{1, 5, 6\}, \{2, 3, 4\}\}$ is a partition of vertices such that each maximal independent set has exactly one vertex in intersection with each of these parts.



Well-covered and uniformly well-covered graphs

Lemma 2. Let G be a graph. Then G is uniformly well-covered if and only if there are maximal cliques $C_1, \ldots, C_s$ in G satisfying the following conditions.

i) $C_i \cap C_j = \varnothing$ for each i and j, $1 \leq i < j \leq s$, and $C_1 \cup \cdots \cup C_s = V(G)$.

ii) For each $1 \leq i \leq s$, if $A \subseteq V(G) \setminus C_i$ is a dominating set of C_i, then A is not an independent set.

Proof. Let G be uniformly well-covered . Then, there is a partition of $V(G)$ as $V_1, \ldots, V_s$ such that any maximal independent set of G has exactly one vertex in common with each V_i, $i = 1, \ldots, s$. Let $C_i = V_i$ for $i = 1, \ldots, s$. Let v be a vertex in V_i for some $1 \leq i \leq s$. By the fact that any independent set in G can be extended to a maximal independent set, it is clear that there is no other vertex in V_i independent to v. Therefore, each two vertices of V_i are adjacent and V_i is a clique. Also, V_i is a maximal clique because, in other case, there is a maximal clique C in G strictly containing V_i. Let $w \in C \setminus V_i$. Then, there is some $1 \leq j \leq s$, $j \neq i$ such that $w \in V_j$, and for any maximal independent set containing w, there is not any vertex in V_i belonging to the maximal independent set. Therefore, cardinality of this set is at most $s - 1$ which is a contradiction. This proves existence of the maximal cliques satisfying the condition i). To prove condition ii), let $1 \leq i \leq s$ be given and $A \subseteq V(G) \setminus C_i$ be a dominating set of C_i. If A is independent, then there is a maximal independent set B containing A. But, $B \cap C_i = \varnothing$ because any vertex of C_i has a common edge with some vertices in $A \subseteq B$. This is a contradiction with uniformly well-coveredness of G. Therefore, taking $C_i = V_i$, $i = 1, \ldots, s$, both conditions are satisfied.

Now, let there are maximal cliques $C_1, \ldots, C_s$ in G satisfying conditions i) and ii). Take $V_i = C_i$, $i = 1, \ldots, s$. It is clear that $V_1, \ldots, V_s$ is a partition of $V(G)$ and any maximal independent set in G has at most one vertex in common with each V_i. Let A be an independent set in G. If $A \cap V_i = \varnothing$ for some i, then $A \subseteq V(G) \setminus V_i$ and by hypothesis, A is not a dominating set of V_i and so, there is a vertex $v \in V_i$ not dominated by any vertex in A. Therefore, $A \cup \{v\}$ is an independent set. This means that any maximal independent set in G has exactly one vertex in common with each V_i. Therefore, G is uniformly well-covered. □

We interest to find conditions equivalent to well-covered in some classes of graphs. Using proof of the above lemma, we do such a classification in the following.

Corollary 3. Let G be a graph with a maximal independent set of size s and s maximal cliques $C_1, \dots, C_s$ satisfying condition i) of Lemma 2. The following conditions are equivalent.

i) G is well-covered.

ii) G is uniformly well-covered.

iii) For each $1 \leq i \leq s$, if $A \subseteq V(G) \setminus C_i$ is a dominating set of C_i, then A is not an independent set.

Proof. It is clear that conditions ii) and iii) are equivalent and each of them implies condition i). Therefore, it is enough to prove that condition i) implies one of the others. By the Lemma 2, the condition i) implies iii). $\square$

Lemma 4. Let G be a graph such that $\overline{G}$, the complement graph of G, is s-partite with partitions $V_1, \dots, V_s$. The following conditions are equivalent.

i) For each $1 \leq i \leq s$, V_i is a minimal dominating set of $\overline{G}$, and any minimal dominating set of G is of size s.

ii) G is uniformly well-covered.

iii) G is well-covered with a maximal independent set of size s.

Proof. The graph $\overline{G}$ is s-partite, means that G has s cliques $V_1, \dots, V_s$ which cover all vertices. Let i) holds. The first statement of condition i) implies that for a given vertex $v \notin V_i$, there is a vertex in V_i connected to v by an edge in $\overline{G}$. This means that V_i is a maximal independent set in $\overline{G}$ and hence a maximal clique in G. Let $A \subseteq V(G) \setminus V_i$ dominates V_i. If A is an independent set, then intersection of A with each V_j has at most one element. Adding an element from each V_j, $j \neq i$, with empty intersection with A, to A, finally yields a dominating set of G of cardinality less than s, which is a contradiction to the second statement of i). Therefore, A is not an independent set. By Corollary 3, This implies conditions ii) and iii). In other hand, we know that uniformly well-coveredness always implies well-coveredness. Therefore, to complete the proof, it is enough to prove that iii) implies i). Let iii) holds. Then, G is well-covered and any maximal independent set in G has cardinality s. This means that V_i is a maximal clique in G because otherwise, for some $i \neq j$, $V'_i \cap V'_j \neq \varnothing$, where V'_i and V'_j are maximal cliques containing V_i and V_j respectively. In this case, any maximal independent set containing an element of $V'_i \cap V'_j$, has cardinality of size less than s. Therefore, V_i is a maximal independent set in $\overline{G}$ and any vertex $v \in V(G) \setminus V_i$ is not independent to all vertices of V_i in $\overline{G}$ and so it is connected to at least one vertex of V_i by an edge in $\overline{G}$. This means that V_i is a dominating set of $\overline{G}$. It is minimal because it is independent. This proves the first statement of i). Let $A \subseteq V(G)$ be a minimal dominating set of G. A has at most one element from each V_i, so, it has at most s elements. If $|A| < s$, then $A \cap V_i = \varnothing$ for some $1 \leq i \leq s$, and A is an independent set in $V(G) \setminus V_i$ dominating V_i which is a contradiction by Corollary 3. □

Now, we summarize the above results in the following theorem.

Theorem 5. Let G be a graph with a maximal independent set of size s. The following conditions are equivalent.

i) G is well-covered and has a cliques cover $C_1, \ldots, C_s$.

ii) $\overline{G}$ is s-partite such that any part is a dominating set of $\overline{G}$, and, G is well dominated with domination number s.

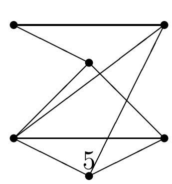
iii) G has a cliques cover $C_1, \ldots, C_s$, such that for each $1 \leq i < j \leq s$, $C_i \cap C_j = \varnothing$ and any dominating set of C_i in $V(G) \setminus C_i$ is not independent.

iv) G is uniformly well-covered.

Proposition 6. Let G be a s-partite well-covered graph such that all maximal cliques are of size s. Then all parts have the same cardinality and there is a perfect matching between each two parts.

Proof. Let the s parts of G be $V_1, \ldots, V_s$. Let $1 \leq i \leq s$ and $v \in V_i$. Each vertex belongs to some maximal clique and each maximal clique intersects each part in exactly one vertex. Therefore, the vertex v is adjacent to some vertices in each part V_j, $1 \leq j \leq s$, $j \neq i$. Then the part V_i is a maximal independent set because for each vertex out of V_i, there is an edge connecting it to some vertex in V_i. The graph G is well-covered therefore, cardinality of parts are the same. Let $1 \leq i < j \leq s$ be two given integers. Let $A \subseteq V_i$ be a nonempty set and $N_j(A)$ be the set of all vertices in V_j adjacent to some vertices in A. Suppose $|N_j(A)| < |A|$. There is no any edge between A and $V_j \setminus N_j(A)$. Therefore, $A \cup (V_j \setminus N_j(A))$ is an independent set and its size is strictly greater than size of V_j, which is a contradiction with well-coveredness of G. Therefore, $|N_j(A)| \geq |A|$ for each nonempty subset A of V_i. Therefore, by Theorem of Hall [7], there is a set of distinct representatives (SDR) for the set $\{N_j(\{v\}) : v \in V_i\}$, which is a perfect matching between V_i and V_j. □

Example 6. It is not true that in a well-covered graph, there are maximal cliques satisfying condition i) of Lemma 2. For instance, consider the following graph which is 3-partite, well-covered with maximal independent sets of size 2. But, there are no disjoint maximal cliques covering $V(G)$.



Stating many examples motivates to have the following conjecture.

Conjecture. Let G be a s-partite well-covered graph with all maximal cliques of size s. Then, G is uniformly well-covered.

At the end of this section, we restate the result of Ravindra about well-coveredness of bipartite graphs.

Proposition 7. [21] Let G be a bipartite graph with no vertex of degree zero. Then, G is well covered if and only if there is a perfect matching and for each $\{x, y\}$ in this matching, the induced subgraph on $N[\{x, y\}]$ is a complete bipartite graph.

Proof. By Proposition 6, cardinality of both parts are the same and there is a perfect matching in G. Then, condition i) of Lemma 2 is satisfied. Let $\{x, y\}$ be an edge in the matching. By Corollary 3, G is well-covered if and only if any dominating set of $\{x, y\}$ is dependent. The last statement is equal to say that any vertex in $N(x)$ is adjacent to any vertex in $N(y)$, i. e., the induced subgraph on $N[\{x, y\}]$ is a complete bipartite graph. $\square$

An algebraic interpretation

There is a very interesting algebraic interpretation of well-coveredness of a special class of graphs, which we state in this section. First we recall some definitions in commutative algebra.

Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. Let K be a field. In the polynomial ring $K[x_1, \ldots, x_n]$, consider $I(G)$ be the ideal generated by all monomials of the form $x_i x_j$ where v_i and v_j are adjacent in G. This ideal is called edge ideal of the graph G and the quotient ring $R(G) = K[x_1, \ldots, x_n]/I(G)$ is called edge ring of G. This ring is introduced by R. Villarreal [26] and has been extensively studied by several mathematicians.

Let R be a commutative ring. An element $a \neq 0$ in R is called zero-divisor if there is a nonzero element $b \in R$ such that $ab = 0$. An ideal in R is called monomial ideal if it can be generated by a set of monomials. For example, edge ideal of a graph is a monomial ideal. In a ring of polynomials, it is well known and easy to check that a polynomial belongs to a monomial ideal if and only if each monomial of the polynomial belongs to the ideal. If the monomial ideal is also square-free, then a monomial of $K[x_1, \ldots, x_n]$ belongs to I if and only if its square-free part (its radical) belongs to I. As an example of zero-divisor element, let $R(G)$ be the edge ring of a graph G. Let v_i be adjacent to v_j in G. The elements x_i and x_j are not zero in $R(G)$ but $x_i x_j = 0$ because $x_i x_j$ belongs to the ideal $I(G)$. Here, with abuse of notation, we have written x_i as same as its image in $R(G)$.

A term order on $K[x_1, \dots, x_n]$ is a linear order $\preceq$ on the set of terms $\{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} : a_i \in \mathbb{Z}_{\geq 0}, i = 1, 2, \dots n\}$, such that for each terms $\alpha, \alpha_1, \alpha_2$, the following conditions hold.

- $\alpha_1 \preceq \alpha_2$ then $\alpha_1 \alpha \preceq \alpha_2 \alpha$.
- $1 \preceq \alpha$.

Lexicographic, degree lexicographic and degree reverse lexicographic orders are examples of term order. There is a rich literature about term orderings, for instance see [11].

Lemma 8. Let K be a field, $I \subseteq K[x_1, \dots, x_n]$ an ideal generated by square-free monomials. Let f be a nonzero linear polynomial in $R = K[x_1, \dots, x_n]/I$. Then, f is zero-divisor on R if and only if there is a nonzero square-free monomial $m \in R$ such that $mf = 0$.

Proof. Let f be zero-divisor in R, then, there is a nonzero polynomial g in R such that $fg = 0$. We may rearrange variables such that $f = x_1 + a_2x_2 + \cdots + a_sx_s$, $a_j \in K$. Let $\prec$ be the lexicographic order on terms of $K[x_1, \dots, x_n]$ with respect to $x_1 \succ x_2 \succ \cdots \succ x_n$. Let $g = m_1 + m_2 + \cdots + m_t$ be decomposition of G to nonzero monomials such that $m_1 \succ m_2 \succ \cdots \succ m_t$. Then, in fg, the monomial x_1m_1 is strictly greater than all other monomials. Therefore, x_1m_1 must be zero in R. The ideal I is square-free and $x_1m_1 \in I$, therefore, we may assume that $x_1 \nmid m_1$. By the lexicographic order, we have $x_1 \nmid m_i$ for all $1 \leq i \leq t$. In other hand, $fg - x_1m_1 \in I$. The greatest term of $fg - x_1m_1$ is x_1m_2 and then $x_1m_2 \in I$ and $fg - (x_1m_1 + x_1m_2) \in I$. Continuing this process, we have $x_1m_i \in I$ for all $1 \leq i \leq t$. In the polynomial $fg - x_1g \in I$ the greatest term is x_2m_1 which must be in I. Similarly, $x_2m_i \in I$ for all $1 \leq i \leq t$. Finally, we get $x_im_j \in I$ for each $1 \leq i \leq s$ and $1 \leq j \leq t$. It means that $m_if = 0$ in R for each $1 \leq i \leq t$. Specially $m_1f = 0$, and as I is square-free and f is linear, we may assume m_1 to be square-free. □

Note that in the above lemma, assuming that I is square-free is essential. Because for example in $K[x_1, x_2]$ assume that $I = \langle x_1^2, x_1x_2, x_2^2 \rangle$. Then, $(x_1 + x_2)(x_1 - x_2) \in I$, that is $(x_1 + x_2)(x_1 - x_2) = 0$ in $K[x_1, x_2]/I$, and $(x_1 \pm x_2) \notin I$ but, there is no any nonzero monomial eliminating $(x_1 + x_2)$ or $(x_1 - x_2)$.

Proposition 9. Let G be a graph with a maximal independent set of size s and maximal cliques $C_1, \dots, C_s$ such that $C_i \cap C_j = \varnothing$ for each i and j, $1 \leq i < j \leq s$, and $C_1 \cup \cdots \cup C_s = V(G)$. Consider

$$\theta_i = \sum_{x_j \in C_i} x_j$$

for $i = 1, \dots, s$. Then, the following conditions are equivalent.

i) G is uniformly well-covered.

ii) G is well-covered.

iii) For each $i = 1, \dots, s$, the polynomial θ_i is not zero-divisor in the ring $R(G)$.

Proof. It is enough to show that the condition iii) of this proposition is equal to the condition iii) in the Corollary 3 above. Let θ_i is zero-divisor in $R(G)$. By the Lemma 8, there is a nonzero square-free monomial m in $R(G)$ such that $m\theta_i = 0$. According to the fact that I is a monomial ideal, then, for each x_j in C_i, we have $mx_j = 0$ in $R(G)$. Let $m = x_{i_1} \cdots x_{i_r}$. Therefore, $mx_j = 0$ means that there is a vertex x_{i_l} such that $x_{i_l} | m$ and x_{i_l} is adjacent to x_j in G. This means that the set of vertices deviding m is a dominating set of C_i. Note that this set is independent if and only of m is not zero in $R(G)$. This completes the proof. □

Let $\mathcal{G}$ be the class of uniformly well-covered graphs with some disjoint maximal cliques covering all vertices such that the number of these cliques is equal to cardinality of an independent set of vertices of G. The next natural question is when a graph in this class is Cohen-Macaulay. With the notations above, Cohen-Macaulayness of G is equal to regularity of the sequence $\theta_1, \theta_2, \ldots, \theta_s$ in $R(G)$. It means that θ_1 is not zero-divisor in $R(G)$ and for $i = 2, \ldots, s$, the element θ_i is not zero-divisor in $R(G)/\langle\theta_1, \ldots, \theta_{i-1}\rangle$. This is not an easy task and so far, only for class of bipartite Cohen-Macaulay graphs a nice combinatorial characterization has been presented [10].

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