# Vector Bundles over Normal Varieties Trivialized by Finite Morphisms

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**Abstract.** Let $Y$ be a normal and projective variety over an algebraically closed field $k$ and $V$ a vector bundle over $Y$. We prove that if there exist a $k$-scheme $X$ and a finite surjective morphism $g : X \to Y$ that trivializes $V$ then $V$ is essentially finite.

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## Contents



## 1 Introduction

Essentially finite vector bundles over a reduced, connected and proper scheme $Y$ over a perfect field $k$ have been defined by Nori in [5] and [6]. They turn out to be those vector bundles $V$ over $Y$ which are trivialized by a principal $G$ bundle $f : Z \to Y$ for a certain finite $k$-group scheme $G$ (i.e. $f^*(V)$ is trivial). In [2] Biswas and Dos Santos prove that if $Y$ is smooth and projective and $k$ is algebraically closed then $V$ is essentially finite if and only if there exist a scheme $X$ and a finite surjective morphism $X \to Y$ trivializing $V$. Here we prove the same property only assuming $Y$ to be normal and projective.

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## 2 The theorem

Throughout the whole paper $k$ will be an algebraically closed field and $Y$ a normal and projective variety over $k$. Let us denote by $EF(Y)$ the neutral tannakian category of essentially finite vector bundles over $Y$. The aim of this paper is to prove the following

**Theorem 2.1.** *Assume there exist a normal projective variety $X$ over $k$ and a finite surjective morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

**Remark 2.2.** *This theorem holds in both zero and positive characteristic.*

First we consider two important special cases: the case where $g : X \to Y$ is purely inseparable (i.e. the extension $K(Y) \subset K(X)$ of their function fields is purely inseparable, which only occurs when $char(k) > 0$) and the case where it is separable. Then it will only remain to reduce to these two cases.

**Lemma 2.3.** *Assume there exist a normal projective variety $X$ over $k$ and a finite, surjective, purely inseparable morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

*Proof.* We are in the case $char(k) = p > 0$. So let us denote by $F_X : X \to X$ and $F_Y : Y \to Y$ respectively the absolute Frobenius morphisms of $X$ and $Y$. Since $K(Y) \subset K(X)$ is purely inseparable then there exists a positive integer $n$ such that $K(X)^{(p^n)} \subset K(Y)$. This implies that there is a morphism $h : Y \to X$ such that $gh = F_Y^n$ (i.e. the Frobenius iterated $n$ times) and $hg = F_X^n$. By assumption $g^*(V)$ is trivial on $X$, thus $h^*g^*(V) = (gh)^*(V) = (F_Y^n)^*(V)$ is trivial hence $V$ is essentially finite (cf. [4]). □

**Lemma 2.4.** *Assume there exist a normal projective variety $X$ over $k$ and a finite, surjective, separable morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

*Proof.* We may assume that $K(X)$ is normal (then Galois) over $K(Y)$ with Galois group $G$ (if it is not simply consider the normal closure of the extension $K(Y) \subset K(X)$). Let $W := (g_*\mathcal{O}_X)_{max}$ be the maximal semistable subsheaf of $g_*\mathcal{O}_X$ (i.e. the first term of the Harder-Narasimhan filtration of $g_*\mathcal{O}_X$) then its slope $\mu(W) = \mu_{max}(g_*\mathcal{O}_X) = 0$: indeed since there is at least the canonical morphism $\mathcal{O}_Y \to g_*\mathcal{O}_X$ then in particular we have

$$0 = \mu(\mathcal{O}_Y) \leq \mu_{max}(g_*\mathcal{O}_X);$$

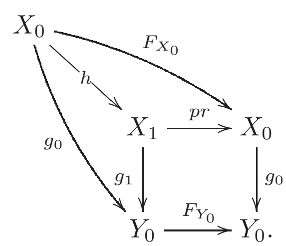
but $g$ is seperable then $g^*(W)$ is still semistable; now consider the isomorphism

$$Hom_X(g^*(W), \mathcal{O}_X) \simeq Hom_Y(W, g_*\mathcal{O}_X) \neq 0$$

from which we deduce $\mu(g^*(W)) \leq 0$ hence $\mu(W) \leq 0$ (recall that $\mu(W) = \mu(g^*(W))/deg(g))$.

The coherent sheaf $W$ is in general only torsion free over $Y$. But it is locally free if restricted to a big open subset $Y_0 \subset Y$, i.e. $codim_Y(Y \backslash Y_0) \geq 2$. Let $W_0 := W_{|Y_0}$ denote the vector bundle over $Y_0$, $Sym^*(W_0^*)$ the symmetric algebra of the dual of $W_0$ and consider $X_0 := \mathbf{Spec}(Sym^*(W_0^*))$ with its canonical map $g_0 : X_0 \to Y_0$.

The vector bundle $W_0$ is strongly semistable of degree 0 over $Y_0$: let us denote by $F_{X_0}$ and $F_{Y_0}$ respectively the absolute Frobenius morphisms of $X_0$ and $Y_0$ and assume $W$ is not strongly semistable then there exists a subsheaf $U$ of $F_{Y_0}^*(W)$ such that $deg(U) > 0$. Let $X_1$ be the fiber product of $g_0 : X_0 \to Y_0$ and $F_{Y_0}$. It is an integral scheme. We denote by $pr : X_1 \to X_0$ and $g_1 : X_1 \to Y_0$ the projections and also $h : X_0 \to X_1$ the normalization map:




Now $U \subseteq F_{Y_0}^*(g_{0*}(\mathcal{O}_{X_0})) = g_{1*}(pr^*(\mathcal{O}_{X_0})) = g_{1*}(\mathcal{O}_{X_1})$. But from $\mathcal{O}_{X_1} \hookrightarrow h_*(\mathcal{O}_{X_0})$ we obtain $g_{1*}(\mathcal{O}_{X_1}) \hookrightarrow g_{1*}(h_*(\mathcal{O}_{X_0})) = g_{0*}(\mathcal{O}_{X_0})$ the latter being semistable whence a contradiction. As a consequence we have a homomorphism of $\mathcal{O}_{Y_0}$-algebras $g_{0*}(\mathcal{O}_{X_0}) \simeq W_0$ (cf. also [1], §6).

Since $g_{0*}(\mathcal{O}_{X_0})$ is semistable of slope 0 over $Y_0$ then $X_0$ is a Galois-étale cover over $Y_0$, the Galois group of $g_0$ still being $G$. Now let us fix some notations: recall that by assumption $V$ is a vector bundle over $Y$ such that $T := g^*(V)$ is trivial on $X$; we set $V_0 := V_{|Y_0}$ and $T_0 := g_0^*(V_0)$ so the latter is also trivial on $X_0$. Since $g_0$ is a Galois-étale cover then $T_0$ is a $G$-bundle on $X_0$. But $X_0$ is a big open set in $X$ thus $G$ acts on $X$ and then $G$ acts also on $T$. Since $T$ is a $G$-bundle then we go on as follows: we have $X/G \simeq Y$ and the trivial bundle $T$ on $X$ descends to $Y$. So by Kempf's lemma (cf. for example [3], Théorème 2.3), for all $x$ in $X$, the stabilizer $G_x$ acts trivially on the fibre $T_x$. But $T$ is trivial and both $X$ and $X_0$ have no global sections except constants, this means that there is a map

$$\rho : G \to GL(T_x) = GL_r$$

over $X$, where $r := rank(T)$. Now let us assume for a moment that $G$ acts faithfully on $X$ so that the map $\rho : G \to GL_r$ is injective. We already know that $G$ acts freely on $X_0$. So let us take $x \in X \backslash X_0$: since $G_x$ is a subgroup of $G$ then $G_x$ has to be trivial. This proves that $G$ acts freely on $X$. So $g : X \to Y$ is a Galois-étale cover. So $V$ is in $EF(Y)$. Up to now we have assumed $\rho$ to be injective. If it is not, i.e. if $G$ does not act faithfully on $X$ then just consider $H := im(\rho)$, then consider the contracted product $X' := X \times^G H$, i.e. $X' = X/(ker(\rho))$, which is provided with a faithfull $H$-action and clearly

$Y \simeq X'/H$. Hence $H \to GL_r$ is injective, $V$ is trivial over $X_0' := X_0 \times^G H$ and we proceed as before.

□

We are now ready to prove the principal result:

*Proof. of Theorem* 2.1: if $char(k) = 0$ then lemma 2.4 is sufficient to conclude. So let us assume $char(k) = p > 0$: if $g$ is purely inseparable then lemma 2.3 is enough to conclude. Otherwise, if $g$ is arbitrary, we argue as follows: again we may assume that $K(X)$ is normal over $K(Y)$ with Galois group $G$. It is known that $L := K(X)^G$ is a proper purely inseparable field extension of $K(Y)$ while $K(X)$ is separable over $L$, then Galois. Let $Z$ be the integral closure of $Y$ in $L$, then $g : X \to Y$ factors through the maps $s : X \to Z$ and $t : Z \to Y$ (i.e. $ts = g$) where $t : Z \to Y$ is purely inseparable and $s : X \to Z$ is separable. By lemma 2.4 the vector bundle $W := t^*(V) \in EF(Z)$ because $s^*(W)$ is trivial on $X$. As we did for lemma 2.3, there exists a morphism $h : Y \to Z$ such that $h^*t^*(V) = (th)^*(V) = (F_Y^n)^*(V)$ for some integer $n$; but $h^*(W) \in EF(Y)$ thus $(F_Y^n)^*(V) \in EF(Y)$ then there exists $m \geq n$ such that $(F_Y^m)^*(V)$ is Galois-étale trivial (i.e. there exists a Galois-étale cover $j : Y' \to Y$ such that $j^*((F_Y^m)^*(V))$ is trivial on $Y'$) and that is enough to conclude that $V$ is essentially finite on $Y$. □

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