BRAUER GROUP OF MODULI SPACES OF PAIRS

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Abstract. We show that the Brauer group of any moduli space of stable pairs with fixed determinant over a curve is zero.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ over the complex numbers. A *holomorphic pair* (also called a *Bradlow pair*) is an object of the form (E, ϕ), where E is a holomorphic vector bundle over X, and ϕ is a nonzero holomorphic section of E. The concept of stability for pairs depends on a parameter $\tau \in \mathbb{R}$. Moduli spaces of τ-stable pairs of fixed rank and degree were first constructed using gauge theoretic methods in [4], and subsequently using Geometric Invariant Theory in [3]. Since then these moduli spaces have been extensively studied.

Fix an integer $r \geq 2$ and a holomorphic line bundle Λ over X. Let $d = \deg(\Lambda)$. Let $\mathfrak{M}_\tau(r, \Lambda)$ be the moduli space of stable pairs (E, ϕ) such that $\mathrm{rk}\,(E) = r$ and $\det(E) = \bigwedge^r E = \Lambda$. This is a smooth quasiprojective variety; it is empty if $d \leq 0$. Therefore, $H^2_{ét}(\mathfrak{M}_\tau(r, \Lambda), \mathbb{G}_m)$ is torsion, and it coincides with the Brauer group of $\mathfrak{M}_\tau(r, \Lambda)$, defined by the equivalence classes of Azumaya algebras over $\mathfrak{M}_\tau(r, \Lambda)$. Let $\mathrm{Br}(\mathfrak{M}_\tau(r, \Lambda))$ denote the Brauer group of $\mathfrak{M}_\tau(r, \Lambda)$.

We prove the following (see Theorem 3.3 and Corollary 3.5):

Theorem 1.1. *Assume that* $(r, g, d) \neq (3, 2, 2)$. *Then* $\mathrm{Br}(\mathfrak{M}_\tau(r, \Lambda)) = 0$.

Let $M(r, \Lambda)$ be the moduli space of stable vector bundles over X of rank r and determinant Λ. There is a unique universal projective bundle over $X \times M(r, \Lambda)$. Restricting this projective bundle to $\{x\} \times M(r, \Lambda)$, where x is a fixed point of X, we get a projective bundle $\mathbb{P}_x$ over $M(r, \Lambda)$. We give a new proof of the following known result (see Corollary 3.4).

Corollary 1.2. *Assume* $(r, g, d) \neq (2, 2, \mathit{even})$. *The Brauer group of* $M(r, \Lambda)$ *is generated by the Brauer class of* $\mathbb{P}_x$.

This was first proved in [2]. We show that it follows as an application of Theorem 1.1.

For convenience, we work over the complex numbers. However, the results are still valid for any algebraically closed field of characteristic zero.

Date: 19 November 2010.

2000 *Mathematics Subject Classification.* 14D20, 14F22, 14E08.

Key words and phrases. Brauer group, moduli of pairs, stably rational.

Acknowledgements. The second and third authors are grateful to the Tata Institute of Fundamental Research (Mumbai), where this work was carried out, for its hospitality. Second author supported by (Spanish MICINN) research project MTM2007- 67623 and i-MATH. Third author partially supported by (Spanish MCINN) research project MTM2007-63582.

2. Moduli spaces of pairs

We collect here some known results about the moduli spaces of pairs, taken mainly from [4], [5], [9], [11] and [13].

Let X be a smooth projective curve defined over the field of complex numbers, of genus $g \geq 2$. A *holomorphic pair* (E, ϕ) over X consists of a holomorphic bundle on X and a nonzero holomorphic section $\phi \in H^0(E)$. Let $\mu(E) := \deg(E)/\operatorname{rk}(E)$ be the slope of E. There is a stability concept for a pair depending on a parameter $\tau \in \mathbb{R}$. A holomorphic pair (E, ϕ) is τ*-stable* whenever the following conditions are satisfied:

- for any subbundle $E' \subset E$, we have $\mu(E') < \tau$,
- for any subbundle $E' \subset E$ such that $\phi \in H^0(E')$, we have $\mu(E/E') > \tau$.

The concept of τ*-semistability* is defined by replacing the above strict inequalities by the weaker inequalities "$\leq$" and "$\geq$". A *critical value* of the parameter $\tau = \tau_c$ is one for which there are strictly τ-semistable pairs. There are only finitely many critical values.

Fix an integer $r \geq 2$ and a holomorphic line bundle Λ over X. Let d be the degree of Λ. We denote by $\mathfrak{M}_\tau(r, \Lambda)$ (respectively, $\overline{\mathfrak{M}}_\tau(r, \Lambda)$) the moduli space of τ-stable (respectively, τ-semistable) pairs (E, ϕ) of rank $\operatorname{rk}(E) = r$ and determinant $\det(E) = \Lambda$. The moduli space $\overline{\mathfrak{M}}_\tau(r, \Lambda)$ is a normal projective variety, and $\mathfrak{M}_\tau(r, \Lambda)$ is a smooth quasi-projective variety contained in the smooth locus of $\overline{\mathfrak{M}}_\tau(r, \Lambda)$ (cf. [11, Theorem 3.2]).

For non-critical values of the parameter, there are no strictly τ-semistable pairs, so $\mathfrak{M}_\tau(r, \Lambda) = \overline{\mathfrak{M}}_\tau(r, \Lambda)$ and it is a smooth projective variety. For a critical value τ_c, the variety $\overline{\mathfrak{M}}_{\tau_c}(r, \Lambda)$ is in general singular.

Denote $\tau_m := \frac{d}{r}$ and $\tau_M := \frac{d}{r-1}$. The moduli space $\mathfrak{M}_\tau(r, \Lambda)$ is empty for $\tau \notin (\tau_m, \tau_M)$. In particular, this forces $d > 0$ for τ-stable pairs. Denote by $\tau_1 < \tau_2 < \ldots < \tau_L$ the collection of all critical values in (τ_m, τ_M). Then the moduli spaces $\mathfrak{M}_\tau(r, \Lambda)$ are isomorphic for all values τ in any interval (τ_i, τ_{i+1}), $i = 0, \ldots, L$; here $\tau_0 = \tau_m$ and $\tau_{L+1} = \tau_M$.

However, the moduli space changes when we cross a critical value. Let τ_c be a critical value. Denote $\tau_c^+ := \tau_c + \epsilon$ and $\tau_c^- := \tau_c - \epsilon$ for $\epsilon > 0$ small enough such that (τ_c^-, τ_c^+) does not contain any critical value other than τ_c. We define the *flip loci* $\mathcal{S}_{\tau_c^\pm}$ as the subschemes:

- $\mathcal{S}_{\tau_c^+} = \{(E, \phi) \in \mathfrak{M}_{\tau_c^+}(r, \Lambda) \mid (E, \phi) \text{ is } \tau_c^-\text{-unstable}\}$,
- $\mathcal{S}_{\tau_c^-} = \{(E, \phi) \in \mathfrak{M}_{\tau_c^-}(r, \Lambda) \mid (E, \phi) \text{ is } \tau_c^+\text{-unstable}\}$.

When crossing τ_c, the variety $\mathfrak{M}_\tau(r,\Lambda)$ undergoes a birational transformation:

$$\mathfrak{M}_{\tau_c^-}(r,\Lambda)\setminus \mathcal{S}_{\tau_c^-} = \mathfrak{M}_{\tau_c}(r,\Lambda) = \mathfrak{M}_{\tau_c^+}(r,\Lambda)\setminus \mathcal{S}_{\tau_c^+}\,.$$

Proposition 2.1 ([10, Proposition 5.1])**.** *Suppose $r\geq 2$, and let τ_c be a critical value with $\tau_m<\tau_c<\tau_M$. Then*

- $\operatorname{codim}\mathcal{S}_{\tau_c^+}\geq 3$ *except in the case $r=2$, $g=2$, d odd and $\tau_c=\tau_m+\frac{1}{2}$ (in which case* $\operatorname{codim}\mathcal{S}_{\tau_c^+}=2$*),*
- $\operatorname{codim}\mathcal{S}_{\tau_c^-}\geq 2$ *except in the case $r=2$ and $\tau_c=\tau_M-1$ (in which case* $\operatorname{codim}\mathcal{S}_{\tau_c^-}=1$*). Moreover we have that* $\operatorname{codim}\mathcal{S}_{\tau_c^-}=2$ *only for $\tau_c=\tau_M-2$.*

The codimension of the flip loci is then always positive, hence we have the following corollary:

Corollary 2.2. *The moduli spaces $\mathfrak{M}_\tau(r,\Lambda)$, $\tau\in(\tau_m,\tau_M)$, are birational.*

The moduli spaces for the extreme values of the parameter τ_m^+ and τ_M^- are known explicitly. Let $M(r,\Lambda)$ be the moduli space of *stable* vector bundles or rank r and fixed determinant Λ. Define

$$\mathcal{U}_m(r,\Lambda)=\{(E,\phi)\in\mathfrak{M}_{\tau_m^+}(r,\Lambda)\,|\,E \text{ is a stable vector bundle}\}\,, \tag{2.1}$$

and denote

$$\mathcal{S}_{\tau_m^+}:=\mathfrak{M}_{\tau_m^+}(r,\Lambda)\setminus\mathcal{U}_m(r,\Lambda)\,.$$

Then there is a map

$$\pi_1:\mathcal{U}_m(r,\Lambda)\longrightarrow M(r,\Lambda),\qquad (E,\phi)\mapsto E\,, \tag{2.2}$$

whose fiber over E is the projective space $\mathbb{P}(H^0(E))$. When $d\geq r(2g-2)$, we have that $H^1(E)=0$ for any stable bundle, and hence (2.2) is a projective bundle (cf. [9, Proposition 4.10]).

Regarding the rightmost moduli space $\mathfrak{M}_{\tau_M^-}(r,\Lambda)$, we have that any τ_M^--stable pair (E,ϕ) sits in an exact sequence

$$0\longrightarrow\mathcal{O}\xrightarrow{\phi}E\longrightarrow F\longrightarrow 0\,,$$

where F is a semistable bundle of rank $r-1$ and $\det(F)=\Lambda$. Let

$$\mathcal{U}_M(r,\Lambda)=\{(E,\phi)\in\mathfrak{M}_{\tau_M^-}(r,\Lambda)\,|\,F \text{ is a stable vector bundle}\}\,,$$

and denote

$$\mathcal{S}_{\tau_M^-}:=\mathfrak{M}_{\tau_M^-}(r,\Lambda)\setminus\mathcal{U}_M(r,\Lambda)\,.$$

Then there is a map

$$\pi_2:\mathcal{U}_M(r,\Lambda)\longrightarrow M(r-1,\Lambda),\qquad (E,\phi)\mapsto E/\phi(\mathcal{O})\,, \tag{2.3}$$

whose fiber over $F\in M(r-1,\Lambda)$ is the projective spaces $\mathbb{P}(H^1(F^*))$ (cf. [6]). Note that $H^0(F^*)=0$ since $d>0$. So (2.3) is always a projective bundle.

In the particular case of rank $r=2$, the rightmost moduli space is

$$\mathfrak{M}_{\tau_M^-}(2,\Lambda)=\mathbb{P}(H^1(\Lambda^{-1}))\,, \tag{2.4}$$

since $M(1,\Lambda)=\{\Lambda\}$. In particular, Corollary 2.2 shows that all $\mathfrak{M}_\tau(2,\Lambda)$ are rational quasi-projective varieties.

We have the following:

Lemma 2.3 ([11, Lemma 5.3]). *Let S be a bounded family of isomorphism classes of strictly semistable bundles of rank r and determinant Λ. Then* $\dim M(r,\Lambda)-\dim S\geq (r-1)(g-1)$.

Proposition 2.4. *The following two statements hold:*

- $\operatorname{codim}\mathcal{S}_{\tau_m^+}\geq 2$ *except in the case* $r=2$, $g=2$, d *even (in which case* $\operatorname{codim}\mathcal{S}_{\tau_m^+}=1$*).*
- *Suppose $r\geq 3$. Then* $\operatorname{codim}\mathcal{S}_{\tau_M^-}\geq 2$ *except in the case* $r=3$, $g=2$, d *even (in which case the* $\operatorname{codim}\mathcal{S}_{\tau_M^-}=1$*).*

Proof. Let $(E,\phi)\in\mathcal{S}_{\tau_m^+}$, the vector bundle E is strictly semistable. Therefore, $\pi_1(\mathcal{S}_{\tau_m^+})$ is a bounded family of strictly semistable bundles of rank r and degree d, and hence Lemma 2.3 implies that $\operatorname{codim}\mathcal{S}_{\tau_m^+}\geq (r-1)(g-1)$. So the first statement follows.

As the dimension $\dim H^1(F^*)$ is constant, the codimension of $\mathcal{S}_{\tau_M^-}$ in $\mathfrak{M}_{\tau_M^-}(r,\Lambda)$ is at least the codimension of a locus of semistable bundles. Applying Lemma 2.3 to $M(r-1,\Lambda)$ we have $\operatorname{codim}\mathcal{S}_{\tau_M^-}\geq (r-2)(g-1)$. Now the second result follows. □

3. Brauer group

The Brauer group of a scheme Z is defined as the equivalence classes of Azumaya algebras on Z, that is, coherent locally free sheaves with algebra structure such that, locally on the étale topology of Z, are isomorphic to a matrix algebra $\mathrm{Mat}(\mathcal{O}_Z)$. If Z is a smooth quasiprojective variety, then the Brauer group $\mathrm{Br}(Z)$ coincides with $H^2_{\acute{e}t}(Z)$, and $H^2_{\acute{e}t}(Z)$ is a torsion group.

Theorem 3.1. [8, VI.5 (Purity)] *Let Z be a smooth complex variety and $U\subset Z$ be a Zariski open subset whose complement has codimension at least 2. Then* $\mathrm{Br}(Z)=\mathrm{Br}(U)$.

On the moduli space of stable vector bundles $M(r,\Lambda)$, there are three natural projective bundles. We will describe them.

We first note that there is a unique universal projective bundle over $X\times M(r,\Lambda)$. Fix a point $x\in X$. Restricting the universal projective bundle to $\{x\}\times M(r,\Lambda)$ we get a projective bundle

$$\mathbb{P}_x\longrightarrow M(r,\Lambda)\,. \tag{3.1}$$

Secondly, if $d\geq r(2g-2)$, then we have the projective bundle

$$\mathcal{P}_0\longrightarrow M(r,\Lambda)\,, \tag{3.2}$$

whose fiber over any $E\in M(r,\Lambda)$ is the projective space $\mathbb{P}(H^0(E))$; note that we have $H^1(E)=0$ because $d\geq r(2g-2)$.

Finally, assuming $d > 0$, let

$$\mathcal{P}_1 \longrightarrow M(r, \Lambda) \tag{3.3}$$

be the projective bundle whose fiber over any $E \in M(r, \Lambda)$ is the projective space $\mathbb{P}(H^1(E^*))$.

Proposition 3.2. *The Brauer class* $\mathrm{cl}(\mathbb{P}_x) \in \mathrm{Br}(M(r, \Lambda))$ *is independent of* $x \in X$. *Moreover,*

$$\mathrm{cl}(\mathbb{P}_x) = \mathrm{cl}(\mathcal{P}_0) = -\mathrm{cl}(\mathcal{P}_1) \,,$$

when they are defined.

Proof. The moduli space $M(r, \Lambda)$ is constructed as a Geometric Invariant Theoretic quotient of a Quot scheme $\mathcal{Q}$ by the action of a linear group $\mathrm{GL}_N(\mathbb{C})$ (see [12]). The isotropy subgroup for a stable point of $\mathcal{Q}$ is the center $\mathbb{C}^* \subset \mathrm{GL}_N(\mathbb{C})$. There is a universal vector bundle

$$\mathcal{E} \longrightarrow X \times \mathcal{Q} \,,$$

and any element λ of the center $\mathbb{C}^* \subset \mathrm{GL}_N(\mathbb{C})$ acts on $\mathcal{E}$ as multiplication by λ.

Let $\mathcal{Q}^s \subset \mathcal{Q}$ be the stable locus. The restriction of $\mathcal{E}$ to $X \times \mathcal{Q}^s$ will be denoted by $\mathcal{E}^s$. Let

$$\mathcal{E}_x := \mathcal{E}^s|_{\{x\} \times \mathcal{Q}^s} \longrightarrow \mathcal{Q}^s$$

be the restriction. Let $p_2 : X \times \mathcal{Q}^s \longrightarrow \mathcal{Q}^s$ be the natural projection. Define the vector bundles

$$\mathcal{E}_0 := p_{2*}\, \mathcal{E}^s \quad \text{and} \quad \mathcal{E}_1 := R^1 p_{2*}((\mathcal{E}^s)^*) \,.$$

The center $\mathbb{C}^* \subset \mathrm{GL}_N(\mathbb{C})$ acts trivially on $\mathcal{E}_x \otimes \mathcal{E}_1$. Hence $\mathcal{E}_x \otimes \mathcal{E}_1$ descends to a vector bundle over the quotient $M(r, \Lambda)$ of $\mathcal{Q}^s$. Therefore,

$$\mathrm{cl}(\mathbb{P}_x) = -\mathrm{cl}(\mathcal{P}_1) \,.$$

Similarly, the center $\mathbb{C}^*$ acts trivially on $\mathcal{E}_0 \otimes \mathcal{E}_1$. Hence $\mathcal{E}_0 \otimes \mathcal{E}_1$ descends to $M(r, \Lambda)$, implying

$$\mathrm{cl}(\mathcal{P}_0) = -\mathrm{cl}(\mathcal{P}_1) \,.$$

Finally, note that it follows that $\mathrm{cl}(\mathbb{P}_x)$ is independent of $x \in X$ for $d > 0$. For $d \leq 0$, $\mathcal{P}_0$ and $\mathcal{P}_1$ are not defined. In this case, we take a line bundle μ or large degree, and use the isomorphism $M(r, \Lambda \otimes \mu^r) \cong M(r, \Lambda)$. For any pair $x, x' \in X$, since $\mathrm{cl}(\mathbb{P}_x) = \mathrm{cl}(\mathbb{P}_{x'})$ in $\mathrm{Br}(M(r, \Lambda \otimes \mu^r))$, the same holds for $\mathrm{Br}(M(r, \Lambda))$. □

Theorem 3.3. *Assume that* $d \geq r(2g-2)$. *Then for the moduli space* $\mathfrak{M}_\tau(r, \Lambda)$ *of stable pairs, we have that*

$$\mathrm{Br}(\mathfrak{M}_\tau(r, \Lambda)) = 0 \,.$$

Proof. We will first prove it for $r = 2$. Recall from (2.4) that $\mathfrak{M}_{\tau_M^-}(2, \Lambda)$ is a projective space, hence

$$\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(2, \Lambda)) = 0 \,.$$

Moreover, all $\mathfrak{M}_\tau(2, \Lambda)$ are rational varieties. Thus

$$\mathrm{Br}(\mathfrak{M}_\tau(2, \Lambda)) = 0$$

for non-critical values $\tau \in (\tau_m, \tau_M)$, since the Brauer group of a smooth rational projective variety is zero [1, p. 77, Proposition 1]. For a critical value τ_c, we have

$$\mathfrak{M}_{\tau_c}(2,\Lambda) = \mathfrak{M}_{\tau_c^+}(2,\Lambda) \setminus \mathcal{S}_{\tau_c^+} .$$

By Proposition 2.1, $\operatorname{codim} \mathcal{S}_{\tau_c^+} \geq 2$, so the Purity Theorem implies that

$$\mathrm{Br}(\mathfrak{M}_{\tau_c}(2,\Lambda)) = 0 .$$

Now we assume that $r \geq 3$. From Proposition 2.1 and Theorem 3.1 it follows that the Brauer group $\mathrm{Br}(\mathfrak{M}_\tau(r,\Lambda))$ does not depend on the value of the parameter τ (for fixed r and Λ).

As we are assuming that $d \geq r(2g-2)$, we have a projective bundle

$$\pi_1 : \mathcal{U}_m(r,\Lambda) \longrightarrow M(r,\Lambda)$$

(see (2.2)). Note that this projective bundle coincides with the projective bundle $\mathcal{P}_0$ in (3.2). The projective bundle π_1 gives an exact sequence

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) \longrightarrow \mathrm{Br}(M(r,\Lambda)) \longrightarrow \mathrm{Br}(\mathcal{U}_m(r,\Lambda)) \longrightarrow 0 \tag{3.4}$$

(see [7, p. 193]). By Proposition 2.4 and the Purity Theorem,

$$\mathrm{Br}(\mathcal{U}_m(r,\Lambda)) \;=\; \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda)) ,$$

so we have

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) \longrightarrow \mathrm{Br}(M(r,\Lambda)) \longrightarrow \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda)) \longrightarrow 0 . \tag{3.5}$$

We will show that the theorem follows from (3.5) if we use [2]. From Proposition 3.2 we know that $\mathrm{cl}(\mathcal{P}_0) = \mathrm{cl}(\mathbb{P}_x)$, and from [2, Proposition 1.2(a)] we know that $\mathrm{cl}(\mathbb{P}_x)$ generates $\mathrm{Br}(M(r,\Lambda))$. Therefore, from (3.5) it follows that

$$\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda)) \;=\; 0 .$$

Since $\mathrm{Br}(\mathfrak{M}_\tau(r,\Lambda))$ is independent of τ, this completes the proof using [2]. But we shall give a different proof without using [2], because we want to show that the above mentioned result of [2] can be deduced from our Theorem 3.3 (see Corollary 3.4).

Consider the projective bundle $\pi_2 : \mathcal{U}_M(r-1,\Lambda) \longrightarrow M(r-1,\Lambda)$ from (2.3). Note that this projective bundle coincides with the projective bundle $\mathcal{P}_1$ in (3.3) for rank $r-1$. The projective bundle π_2 gives an exact sequence

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_1) \longrightarrow \mathrm{Br}(M(r-1,\Lambda)) \longrightarrow \mathrm{Br}(\mathcal{U}_M(r,\Lambda)) = \mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda)) \longrightarrow 0 , \tag{3.6}$$

using Proposition 2.4, with the exception of the case $(r,g,d) = (3,2,\text{even})$. Let us leave this "bad" case aside for the moment.

Let

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) \longrightarrow \mathrm{Br}(M(r-1,\Lambda)) \longrightarrow \mathrm{Br}(\mathcal{U}_m(r-1,\Lambda)) = \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r-1,\Lambda)) \longrightarrow 0 \tag{3.7}$$

be the exact sequence obtained by replacing r with $r-1$ in (3.5); the last equality holds as $(r-1,g,d) \neq (2,2,\text{even})$, by Proposition 2.4.

Since $\mathrm{cl}(\mathcal{P}_1) = -\mathrm{cl}(\mathcal{P}_0)$ (see Proposition 3.2), comparing (3.6) and (3.7) we conclude that the two quotients of $\mathrm{Br}(M(r-1,\Lambda))$, namely

$$\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda)) \quad \text{and} \quad \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r-1,\Lambda)),$$

coincide. In particular, $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda))$ is isomorphic to $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r-1,\Lambda))$. Therefore, using induction, the group $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,d))$ is isomorphic to $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(2,\Lambda))$. We have already shown that $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(2,\Lambda)) = 0$. Hence the proof of the theorem is complete for $d \geq r(2g-2)$ and $(r,g,d) \neq (3,2,\text{even})$.

Let us now investigate the missing case of $(r,g,d) = (3,2,2k)$. Take a line bundle ν of degree 1. Using (3.4) twice, we have

$$\begin{array}{ccccccc}
\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) & \longrightarrow & \mathrm{Br}(M(3,\Lambda)) & \longrightarrow & \mathrm{Br}(\mathcal{U}_m(3,\Lambda)) & \longrightarrow & 0 \\
\downarrow\cong & & || & & & & \\
\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) & \longrightarrow & \mathrm{Br}(M(3,\Lambda\otimes\nu^3)) & \longrightarrow & \mathrm{Br}(\mathcal{U}_m(3,\Lambda\otimes\nu^3)) & \longrightarrow & 0
\end{array}$$

The second vertical map is induced by the isomorphism $M(3,\Lambda) \longrightarrow M(3,\Lambda\otimes\nu^3)$ defined by $E \mapsto E\otimes\nu$, hence it is an isomorphism. This isomorphism preserves the class $\mathrm{cl}(\mathbb{P}_x)$, and hence the class $\mathrm{cl}(\mathcal{P}_0)$, by Proposition 3.2. Therefore, $\mathrm{Br}(\mathcal{U}_m(3,\Lambda)) = \mathrm{Br}(\mathcal{U}_m(3,\Lambda\otimes\nu^3))$. But $\deg(\Lambda\otimes\nu^3)$ is odd, hence

$$\mathrm{Br}(\mathcal{U}_m(3,\Lambda)) = \mathrm{Br}(\mathcal{U}_m(\Lambda\otimes\nu^3)) = 0\,.$$

By the Purity Theorem, $\mathrm{Br}(\mathfrak{M}_\tau(3,\Lambda)) = 0$ for any τ. □

Note that the proof of Theorem 3.3 works in the following cases:

- $r = 2$, any d ;
- $r = 3$, $g = 2$, $d \geq 6$; and
- $(r,g) \neq (3,2)$, $d \geq (r-1)(2g-2)$.

Before proceeding to remove the assumption $d \geq r(2g-2)$ in Theorem 3.3, we want to show that Theorem 3.3 implies Proposition 1.2(a) of [2].

Corollary 3.4. *Suppose that* $(r,g,d) \neq (2,2,\text{even})$*. The Brauer group* $\mathrm{Br}(M(r,\Lambda))$ *is generated by the Brauer class* $\mathrm{cl}(\mathbb{P}_x) \in \mathrm{Br}(M(r,\Lambda))$ *in* (3.1).

Proof. Without loss of generality we can assume that d is large (since we have an isomorphism $M(r,\Lambda) \xrightarrow{\sim} M(r,\Lambda\otimes\mu^r)$, $E \mapsto E\otimes\mu$, where μ is a line bundle.

First, we have $\mathrm{Br}(\mathcal{U}_m(r,\Lambda)) = \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda))$ by the Purity Theorem and Proposition 2.4. Second, $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda)) = 0$ by Theorem 3.3, so $\mathrm{Br}(\mathcal{U}_m(r,\Lambda)) = 0$. Finally, we use the exact sequence in (3.4) to see that $\mathrm{cl}(\mathcal{P}_0)$ generates $\mathrm{Br}(M(r,\Lambda))$. Now from Proposition 3.2 it follows that $\mathrm{cl}(\mathbb{P}_x)$ generates $\mathrm{Br}(M(r,\Lambda))$. □

Corollary 3.5. *Suppose* $(r,g,d) \neq (3,2,2)$*. Then we have that* $\mathrm{Br}(\mathfrak{M}_\tau(r,\Lambda)) = 0$.

Proof. For $r = 2$, this result is proved as in Theorem 3.3. As we know it for $d \geq r(2g-2)$, we assume that $d < r(2g-2)$

Let $r \geq 3$. Suppose first that $(r,g,d) \neq (3,2,\text{even})$ (that is, $(r,g,d) \neq (3,2,2),(3,2,4)$). As $d > 0$, we still have a projective bundle $\pi_2 : \mathcal{U}_M(r,\Lambda) \longrightarrow M(r-1,\Lambda)$. Therefore there is an exact sequence as in (3.6). Note that Proposition 2.4 and the Purity Theorem imply that $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda)) = \mathrm{Br}(\mathcal{U}_M(r,\Lambda))$. Now using Proposition 3.2 and Corollary 3.4 and (3.6) it follows that $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda)) = 0$. The result follows.

Finally, let us deal with the missing case $(r,g,d) = (3,2,4)$. Let

$$Z = \{E \in M(3,\Lambda) \mid H^1(E) \neq 0\}.$$

For $E \in M(3,\Lambda) \setminus Z$, we have that $\dim H^0(E) = 4 + 3(1-g) = 1$. So the projective bundle

$$\pi_1 : \mathcal{U}_m(3,\Lambda) \setminus \pi_1^{-1}(Z) \longrightarrow M(3,\Lambda) \setminus Z$$

is actually an isomorphism. In this situation, the exact sequence

$$\mathbb{Z} \cdot \mathrm{cl}(\mathcal{P}_0) \longrightarrow \mathrm{Br}(M(3,\Lambda) \setminus Z) \longrightarrow \mathrm{Br}(\mathcal{U}_m(3,\Lambda) \setminus \pi_1^{-1}(Z)) \longrightarrow 0 \tag{3.8}$$

satisfies that $\mathrm{cl}(\mathcal{P}_0) = 0$. The proof of Proposition 3.2 works also for $M(3,\Lambda) \setminus Z$, so $\mathrm{cl}(\mathbb{P}_x) = 0$. We shall see below that

$$\mathrm{codim}\, Z \geq 2 \qquad \text{and} \qquad \mathrm{codim}\, \pi_1^{-1}(Z) \geq 2\,.$$

From this, $\mathrm{Br}(M(3,\Lambda) \setminus Z) = \mathrm{Br}(M(3,\Lambda))$ and $\mathrm{Br}(\mathcal{U}_m(3,\Lambda) \setminus \pi_1^{-1}(Z)) = \mathrm{Br}(\mathcal{U}_m(3,\Lambda)) = \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(3,\Lambda))$. By Corollary 3.4, $\mathrm{cl}(\mathbb{P}_x) = 0$ generates $\mathrm{Br}(M(3,\Lambda))$, so $\mathrm{Br}(M(3,\Lambda)) = 0$ and $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(3,\Lambda)) = 0$, as required.

To see the codimension estimates, we work as follows. If $H^1(E) \neq 0$, then $H^0(E^* \otimes K_X) \neq 0$, so there is an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow E' = E^* \otimes K_X \longrightarrow F \longrightarrow 0\,, \tag{3.9}$$

for some sheaf F. Note that $\deg(F) = \deg(E') = 2$. Here F must be a rank 2 semistable sheaf, since any quotient $F \to Q$, with $\mu(Q) < \mu(F) = 1$, satisfies that $\mu(Q) < \mu(E') = \frac{2}{3}$, violating the stability of E'. In particular, F is a (semistable) bundle, and it is parametrized by an irreducible variety of dimension $3(g-1) = 3$. Now the bundle E' in (3.9) is given by an extension in $\mathbb{P}(H^1(F^*))$. As $H^0(F^*) = 0$ (by semistability), we have that $\dim \mathbb{P}(H^1(F^*)) = -(-2 + 2(1-g)) - 1 = 3$. So the bundles E' are parametrized by a 6-dimensional variety, and therefore $\dim Z = 6$ and $\mathrm{codim}\, Z = 3$.

Now let us see that $\dim \pi_1^{-1}(Z) \leq 7$. Let $E \in Z$ and F as in (3.9), and note that the determinant of F is fixed. Recalling that $\dim H^1(F^*) = 4$, we see that we have to check that

$$\dim \mathcal{F} + 3 + \dim H^0(E) - 1 \leq 7\,,$$

where $\mathcal{F}$ is the family of the bundles F. Now $\dim H^0(E) = \dim H^1(E) + 1 = \dim H^0(E') + 1 \leq \dim H^0(F) + 2$. Hence we only need to show that

$$\dim \mathcal{F}_i + \dim H^0(F) \leq 3\,, \tag{3.10}$$

for $F \in \mathcal{F}_i$, where $\mathcal{F} = \bigsqcup \mathcal{F}_i$ is the family (suitably stratified) of the possible bundles F.

We have the following possibilities:

(1) $F = L_1 \oplus L_2$, where L_1, L_2 are line bundles of degree one, $L_2 = \det(F) \otimes L_1^{-1}$. The generic such F moves in a 2-dimensional family, and $H^0(F) = 0$. If $\dim H^0(F) \neq 0$, then it should be either $L_1 = \mathcal{O}(p)$ or $L_2 = \mathcal{O}(q)$, $p, q \in X$. In this case F moves in a 1-dimensional family, and $\dim H^0(F) \leq 2$, so (3.10) holds.

(2) F is a non-trivial extension $L \to F \to L$, where L is a line bundle of degree one. As $\det(F) = L^2$ is fixed, then there are finitely many possible L. Now $\dim \operatorname{Ext}^1(L, L) = 2$, so the bundles F move in a 1-dimensional family. Again $\dim H^0(F) \leq 2$, so (3.10) is satisfied.

(3) F is a non-trivial extension $L_1 \to F \to L_2$, where L_1, L_2 are non-isomorphic line bundles of degree one. As $\dim \operatorname{Ext}^1(L_2, L_1) = 1$, we have that F moves in 2-dimensional family. If $\dim H^0(F) = 1$ then (3.10) holds. Otherwise, it must be $L_1 = \mathcal{O}(p)$ and $L_2 = \mathcal{O}(q)$, hence F moves in a 1-dimensional family and $\dim H^0(F) \leq 2$. So (3.10) holds again.

(4) F a rank 2 stable bundle and $H^0(F) = 0$. This is clear, since $\dim M(2, \Lambda) = 3$.

(5) F a rank 2 stable bundle and $H^0(F) = 1$. Then we have an exact sequence $\mathcal{O} \to F \to L$, where L is a (fixed) line bundle of degree two. As $\dim H^1(L^*) = 3$, we have that F moves in a 2-dimensional family and (3.10) holds.

(6) F a rank 2 stable bundle, $\mathcal{O} \to F \to L$, $\dim H^0(L) = 1$ and $\dim H^0(F) = 2$. The connecting map $H^0(L) = \mathbb{C} \to H^1(\mathcal{O})$ is given by multiplication by the extension class in $H^1(L^*)$ defining F. To have $\dim H^0(F) = 2$, this connecting map must be zero, hence the extension class is in $\ker(H^1(L^*) \to H^1(\mathcal{O}))$. This kernel is one-dimensional (since the map is surjective). So the family of such F is zero-dimensional, and (3.10) is satisfied.

(7) F a rank 2 stable bundle, $\mathcal{O} \to F \to L$, $\dim H^0(L) = 2$ and $\dim H^0(F) \geq 2$. Now it must be $L = K_X$. The connecting map
$$c_\xi : H^0(K_X) \to H^1(\mathcal{O}) = H^0(K_X)^*$$
is given by multiplication with the extension class ξ in $H^1(L^*) = H^0(K_X^2)^*$ defining F. Actually, $c_\xi \in H^0(K_X)^* \otimes H^0(K_X)^*$ corresponds to the element $\xi \in H^0(K_X^2)^* = \operatorname{Sym}^2 H^0(K_X)^*$.

If $\dim H^0(F) = 2$, c_ξ is not an isomorphism. The condition $\det(c_\xi) = 0$ gives a 2-dimensional family of $\xi \in H^1(L^*)$. So the family of such F is one-dimensional and (3.10) is satisfied. If $\dim H^0(F) = 3$, then $c_\xi = 0$, which is not possible.

This completes the proof of the corollary. □

Remark 3.6. Note that $\operatorname{Br}(\mathcal{U}_M(r, \Lambda)) = 0$ for $(r, g, d) \neq (3, 2, \text{even})$ (use Corollary 3.5 and Proposition 2.4).

Also, if $d \geq r(2g - 2)$, then $\operatorname{Br}(\mathcal{U}_m(r, \Lambda)) = 0$ for $(r, g, d) \neq (2, 2, \text{even})$ (use Corollary 3.5 and Proposition 2.4). Actually, in the range $d \geq r(2g - 2)$, the proof of Theorem 3.3 shows that $\operatorname{Br}(\mathcal{U}_m(r, \Lambda)) = \operatorname{Br}(\mathcal{U}_M(r + 1, \Lambda))$, for any (r, g, d).

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