

REPRESENTATION STACKS, D-BRANES AND NONCOMMUTATIVE GEOMETRY

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ABSTRACT. In this note we prove that the $\mathbf{spec}(C)$ -points of the representation Artin-stack $[\mathbf{rep}_n R / PGL_n]$ of n -dimensional representations of an affine \mathbb{C} -algebra R correspond to \mathbb{C} -algebra morphisms $R \longrightarrow A_n$ where A_n is an Azumaya algebra of degree n over C . We connect this to the theory of D-branes and Azumaya noncommutative geometry, developed by Chien-Hao Liu and Shing-Tung Yau in a series of papers [9]-[14].

1. REPRESENTATION STACKS

Throughout, all algebras will be associative, unital \mathbb{C} -algebras which are finitely generated, that is, have a presentation

$$R = \frac{\mathbb{C}\langle x_1, \dots, x_k \rangle}{(p_i(x_1, \dots, x_k) \mid i \in I)}$$

With $\mathbf{rep}_n R$ we will denote the affine scheme of all n -dimensional representations of R . Its coordinate ring is the quotient of the polynomial algebra in all entries of k generic $n \times n$ matrices

$$X_i = \begin{bmatrix} x_{11}(i) & \dots & x_{1n}(i) \\ \vdots & & \vdots \\ x_{n1}(i) & \dots & x_{nn}(i) \end{bmatrix}$$

modulo the ideal generated by all the matrix-entries of the $n \times n$ matrices $p_i(X_1, \dots, X_k)$ for all $i \in I$. That is,

$$\mathcal{O}(\mathbf{rep}_n R) = \frac{\mathbb{C}[x_{ij}(l) : 1 \leq i, j \leq n, 1 \leq l \leq k]}{(p_i(X_1, \dots, X_k)_{uv} : 1 \leq u, v \leq n, i \in I)}$$

The affine group scheme GL_n (or rather PGL_n) acts via simultaneous conjugation on the generic matrices X_i and hence on the affine scheme $\mathbf{rep}_n R$, its orbits corresponding to isomorphism classes of n -dimensional representations of R . It is well-known that the GIT-quotient $\mathbf{rep}_n R / PGL_n$, that is the affine scheme corresponding to the ring of polynomial invariants $\mathcal{O}(\mathbf{rep}_n R)^{PGL_n}$, classifies isomorphism classes of *semi-simple* n -dimensional representations of R , see [18] or [7, Chp. 2].

In order to investigate all orbits, one considers the n -dimensional *representation stack* $[\mathbf{rep}_n R / PGL_n]$. By this we mean (following [2]) the category with objects all triples (Z, P, ϕ) where Z a scheme over \mathbb{C} , $\pi_P : P \longrightarrow Z$ a PGL_n -torsor in the

étale topology

$$\begin{array}{ccc} P & \xrightarrow{\phi} & \mathbf{rep}_n R \\ \downarrow \pi_P & & \\ Z & & \end{array}$$

and with ϕ a PGL_n -equivariant map. Morphisms in this category are pairs $(f, h) : (Z, P, \phi) \longrightarrow (Z', P', \phi')$ where $f : Z \longrightarrow Z'$ is a morphism

$$\begin{array}{ccccc} P & \xrightarrow{\phi} & \mathbf{rep}_n R & & \\ \downarrow \pi_P & \searrow h & & \nearrow \phi' & \\ Z & & P' & & \\ & \searrow f & \downarrow \pi_{P'} & & \\ & & Z' & & \end{array}$$

and h is a PGL_n -equivariant map such that $P \simeq Z \times_{Z'} P'$ and $\phi = \phi' \circ h$.

The full subcategory consisting of all triples (Z, P, ϕ) with fixed Z are called the Z -points of the stack and are denoted by $[\mathbf{rep}_n R / PGL_n](Z)$. Observe that this subcategory is a groupoid, that is, all its morphisms are isomorphisms.

Theorem 1. *For a commutative affine \mathbb{C} -algebra C , the $\mathbf{spec}(C)$ -points of the representation stack $[\mathbf{rep}_n R / PGL_n]$ are in natural one-to-one correspondence with \mathbb{C} -algebra morphisms $R \longrightarrow A_n$ where A_n is an Azumaya algebra of degree n with center C .*

Before we can prove this result, we need to recall some facts on Cayley-Hamilton algebras, see [18] and [7], and on Azumaya algebras, see [3] and [5].

Every \mathbb{C} -algebra R has a universal trace map $n_R : R \longrightarrow R/[R, R]_v$ where $[R, R]_v$ is the sub-vectorspace of R spanned by all commutators $[r, s] = rs - sr$ in R . It allows us to define the *necklace functor*

$$\oint : \mathbf{alg} \longrightarrow \mathbf{commalg}$$

which assign to any \mathbb{C} -algebra R its necklace algebra $\oint R = \text{Sym}(\frac{R}{[R, R]_v})$ where for any \mathbb{C} -vectorspace V we denote by $\text{Sym}(V)$ the symmetric algebra on V .

With $\mathbf{alg@}$ we denote the category of \mathbb{C} -algebras *with trace*. That is, a \mathbb{C} -algebra R belongs to $\mathbf{alg@}$ if it has a linear map $tr : R \longrightarrow R$ satisfying the following properties for all $r, s \in R$

$$\begin{cases} tr(r)s = str(r) \\ tr(rs) = tr(sr) \\ tr(tr(r)s) = tr(r)tr(s) \end{cases}$$

In particular, it follows that the image of the trace map is contained in the center $Z(R)$. Morphisms in $\mathbf{alg@}$ are trace preserving \mathbb{C} -algebra morphisms. The forgetful functor $\mathbf{alg@} \hookrightarrow \mathbf{alg}$ has a left adjoint, called the *trace algebra functor*

$$\int : \mathbf{alg} \longrightarrow \mathbf{alg@} \quad R \mapsto \int R = \oint R \otimes R$$

with the trace map on $\int R$ defined by $tr(c \otimes r) = cn_R(r) \otimes 1$, see [7]. That is, for any \mathbb{C} -algebra R and any \mathbb{C} -algebra with trace A , there is a natural one-to-one correspondence

$$Hom_{\mathbf{alg}\mathbb{Q}}(\int R, A) \longrightarrow Hom_{\mathbf{alg}}(R, A) \quad \text{given by} \quad f \mapsto f \circ n_R$$

Let $M \in M_n(\mathbb{C})$ be a diagonalizable matrix with distinct eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, then the characteristic polynomial has as coefficients the elementary symmetric functions $\sigma_i = \sigma_i(\lambda_1, \dots, \lambda_n)$. Another generating set of the ring of symmetric functions is given by the power sums $s_i = \lambda_1^i + \dots + \lambda_n^i$, hence there exist uniquely determined polynomials p_i such that

$$\sigma_i(\lambda_1, \dots, \lambda_n) = p_i(s_1, \dots, s_n) = p_i(Tr(M), \dots, Tr(M^i))$$

because $s_i = Tr(M^i)$.

This allows us to define for an algebra with trace $(R, tr_R) \in \mathbf{alg}\mathbb{Q}$ the formal Cayley-Hamilton polynomial of degree n for all elements $a \in R$ by

$$\chi_a^{(n)}(t) = t^n + p_1(tr_R(a))t^{n-1} + p_2(tr_R(a), tr_R(a^2))t^{n-2} + \dots + p_n(tr_R(a), \dots, tr_R(a^n))$$

With $\mathbf{alg}\mathbb{Q}$ we denote the category of all Cayley-Hamilton algebras of degree n , that is, having as its objects algebras with trace map (R, tr_R) satisfying $tr_R(1) = n$ and $\chi_a^{(n)}(a) = 0$ for all $a \in R$, and, trace preserving \mathbb{C} -algebra maps as morphisms. We have functors

$$\begin{cases} \int_n : \mathbf{alg} \longrightarrow \mathbf{alg}\mathbb{Q} & R \mapsto \frac{\int R}{(tr_R(1) - n, \chi_a^{(n)}(a) \ \forall a \in \int R)} \\ \oint_n : \mathbf{alg} \longrightarrow \mathbf{commalg} & R \mapsto tr_{\int_n R}(\int_n R) \end{cases}$$

The main structural results on the trace algebras $\int_n R$ and $\oint_n R$ and their invariant theoretic interpretation are summarized in the next theorem, due to Claudio Procesi. For a proof and more details the reader is referred to [7, Chp. 2] or to the original paper [18].

Theorem 2 (Procesi). *Let R be a finitely generated \mathbb{C} -algebra. Then, with notations as before*

- (1) $\oint_n R$ is an affine commutative \mathbb{C} -algebra
- (2) $\int_n R$ is a finitely generated module over $\oint_n R$
- (3) $\oint_n R$ is the coordinate ring of the GIT-quotient scheme $\mathbf{rep}_n R / PGL_n$
- (4) $\int_n R$ is the ring of all PGL_n -equivariant maps $\mathbf{rep}_n R \longrightarrow M_n(\mathbb{C})$

Azumaya algebras form an important class of Cayley-Hamilton algebras. For a C -algebra R denote its enveloping algebra $R^e = R \otimes_C R^{op}$, where R^{op} is the C -algebra with opposite multiplication map. There is a natural C -algebra morphism

$$j_R : R^e = R \otimes_C R^{op} \longrightarrow End_C(R) \quad j_R(a \otimes b)r = arb$$

Definition 1. A C -algebra A is called an Azumaya algebra iff A is a finitely generated projective C -module and the map j_A is an isomorphism of C -algebras.

It follows that the center $Z(A)$ of A is equal to C and that $A/\mathfrak{m}A \simeq M_n(\mathbb{C})$ for every maximal ideal \mathfrak{m} of C where n^2 is the local rank of A at \mathfrak{m} . If the local rank is constant and equal to n^2 we say that A is an Azumaya algebra of degree n . We recall the geometry associated to C -Azumaya algebras of degree n .

Lemma 1. *C -Azumaya algebras of degree n are classified up to C -algebra isomorphism by the étale cohomology group $H_{\text{ét}}^1(\text{spec}(C), PGL_n)$, that is, C -Azumaya algebras of degree n correspond to principal PGL_n -fibrations over $\text{spec}(C)$.*

Proof. It is well known (see for example [5]) that A is a C -Azumaya algebra of degree n if and only if there exist étale extensions $C \longrightarrow D$ splitting A , that is, such that $A \otimes_C D \simeq M_n(D)$. As $\text{Aut}(M_n(D)) = PGL_n(D)$, the isoclasses of such algebras are classified by the claimed cohomology group, see for example [16]. \square

The principal PGL_n -fibration corresponding to the degree n C -Azumaya algebra A is the representation scheme

$$\text{rep}_n A \longrightarrow \text{rep}_n A / PGL_n = \text{spec}(C)$$

as all finite dimensional simple A -representation have to be n -dimensional because $A/\mathfrak{m}A = M_n(\mathbb{C})$. By étale descent, A has a reduced trace map with $\text{tr}(A) = C$, and hence we deduce from theorem 2 that $A \in \text{alg}\mathfrak{n}$ and

$$\int_n A = A \quad \text{and} \quad \oint_n A = C$$

We have now all the tools needed to prove theorem 1.

Proof of theorem 1 : Every \mathbb{C} -algebra morphism $\beta : R \longrightarrow A_n$ induces a PGL_n -equivariant map

$$\beta^* : \text{rep}_n A_n \longrightarrow \text{rep}_n R$$

by composition. By the above remarks we know that the GIT-quotient

$$\text{rep}_n A_n \xrightarrow{\pi} \text{rep}_n A_n / PGL_n = \text{spec}(C)$$

is a PGL_n -torsor. Therefore $\beta : R \longrightarrow A_n$ determines the $\text{spec}(C)$ -point of the representation stack

$$\begin{array}{ccc} \text{rep}_n A_n & \xrightarrow{\beta^*} & \text{rep}_n R \\ \downarrow \pi & & \\ \text{spec}(C) & & \end{array}$$

Conversely, PGL_n -torsors over $\text{spec}(C)$ are classified by the pointed set

$$H_{\text{ét}}^1(\text{spec}(C), PGL_n)$$

which also classifies the isomorphism classes of Azumaya algebras of degree n with center C , see for example [16, p. 134]. Hence, any $\text{spec}(C)$ -point of the representation stack $[\text{rep}_n R / PGL_n]$ is of the form

$$\begin{array}{ccc} \text{rep}_n A_n & \xrightarrow{\phi} & \text{rep}_n R \\ \downarrow \pi & & \\ \text{spec}(C) & & \end{array}$$

Taking PGL_n -equivariant maps to $M_n(\mathbb{C})$ on both sides of the PGL_n -equivariant map ϕ gives us by theorem 2 and the remarks above a trace preserving algebra

morphism

$$\int_n R \xrightarrow{\phi_*} \int_n A_n = A_n$$

and composing this with the universal morphism $R \longrightarrow \int_n R$ we obtain the desired \mathbb{C} -algebra morphism $R \xrightarrow{\beta} A_n$ which induces $\phi = \beta^*$.

2. ALGEBRAIC D-BRANES

In this section we relate the above to the description of D-branes via Azumaya noncommutative geometry as developed by Chian-Hao Liu and Shing-Tung Yau in [9]-[14].

Let the affine variety X be an affine open piece of (a spatial slice of) space-time and let Z be a closed subscheme giving the boundary conditions for the endpoints of open strings (a D-brane). Wrapping one brane around Z gives a \mathbb{C}^* -bundle (or a $U(1)$ -symmetry) and the embedding of the D-brane in space-time X corresponds to the quotient map of the corresponding coordinate rings

$$\beta_1 : \mathcal{O}(X) \longrightarrow \mathcal{O}(Z)$$

However, if there are n branes wrapped around Z , then Z comes equipped with a GL_n -bundle P (or a $U(n)$ -symmetry) and the so called *Polchinski Ansatz*, see for example [9], asserts that the embedding of the n -stack of branes in space time is now governed by a \mathbb{C} -algebra morphism

$$\beta : \mathcal{O}(X) \longrightarrow \text{End}_{\mathcal{O}(Z)}(P)$$

where P is the so-called Chan-Paton bundle on Z . Observe that $\text{End}_{\mathcal{O}(Z)}(P)$ is a trivial Azumaya algebra of degree n with center $\mathcal{O}(Z)$ and hence the embedding morphism β gives a Z -point of the representation stack $[\mathbf{rep}_n \mathcal{O}(X)/PGL_n]$.

In more general situations, for example when the B -field is turned on, one may replace the trivial Azumaya algebra $\text{End}_{\mathcal{O}(Z)}(P)$ by a non-trivial Azumaya algebra A_n of degree n with center $\mathcal{O}(Z)$. Sometimes, one even allows for a noncommutative space time R . Motivated by this, we formalize D-branes in a purely algebraic context.

Definition 2. *For R an affine \mathbb{C} -algebra and Z an affine scheme we will call a Z -point of the representation stack*

$$\beta \in [\mathbf{rep}_n R/PGL_n](Z)$$

an algebraic D-brane of degree n on Z .

The upshot of this interpretation of a D-brane embedding as a Z -point in the representation stack $[\mathbf{rep}_n \mathcal{O}(X)/PGL_n]$ (or more generally $[\mathbf{rep}_n R/PGL_n]$) is that one expects good properties of D-branes whenever the representation stack is smooth, in particular when $\mathbf{rep}_n R$ is a smooth affine variety.

This connects the study of D-branes to that of Cayley-smooth orders and formally smooth algebras as described in detail in the book [7]. Observe that several of the examples worked out by Liu and Yau, such as the case of curves in [9, §4] and [10], or the case of the conifold algebra [12] (see also [8]) fall in this framework.

The dynamical aspect of D-branes, that is their Higgsing and un-Higgsing behavior as in [9] can also be formalized purely algebraically by enhancing the category \mathbf{alg} of all \mathbb{C} -algebras with a specific class of 2-morphisms.

Definition 3. A 2-morphism between two \mathbb{C} -algebra morphisms f and g

$$\begin{array}{ccc} & f & \\ R & \Downarrow \alpha & S \\ & g & \end{array}$$

is a $Z(S)$ -algebra mono-morphism between the centralizers in S of the images

$$\alpha : C_S(\text{Im}(f)) \hookrightarrow C_S(\text{Im}(g))$$

If such a 2-morphism exists we say that f degenerates to g or, equivalently, that g deforms to f .

In the case of algebraic D-branes, that is when $S = A_n$ is an Azumaya algebra of degree n with center $C = \mathcal{O}(Z)$, these 2-morphisms correspond to the notions of 'Higgsing' and 'un-Higgsing' as in [9]. Here, the idea is that the Lie-algebra structure of the centralizer $C_A(\text{Im}(\beta))$ can be interpreted as the gauge-symmetry group of the D-brane. Higgsing corresponds to symmetry-breaking, that is the gauge-group becomes smaller or, in algebraic terms, a 2-morphism deformation. Likewise, un-Higgsing corresponds to a 2-morphism degeneration of the algebraic D-brane.

Example 1. There are two extremal cases of algebraic D-branes. The 'maximal' ones coming from epimorphisms

$$R \xrightarrow{\beta} A$$

If C is the center of A such a D-brane determines (and is determined by) a $\text{spec}(C)$ -family of simple n -dimensional representations of R . At the other extreme, 'minimal' algebraic D-branes are given by a composition

$$\beta : R \xrightarrow{\pi} C \xhookrightarrow{i} A$$

where C is a commutative quotient of R and A is an Azumaya algebra over C . Such D-branes essentially determine a $\text{spec}(C)$ -family of one-dimensional representations of R .

For a maximal D-brane $\beta_{\max} : R \twoheadrightarrow A$ we have $C_A(\text{Im}(\beta_{\max})) = Z(A)$ whence for any other algebraic D-brane $g : R \twoheadrightarrow A$ we have a two-cell and hence a degeneration

$$\begin{array}{ccc} & \beta_{\max} & \\ R & \Downarrow \alpha & A \\ & g & \end{array}$$

Likewise, if $\beta_{\min} : R \twoheadrightarrow C \hookrightarrow A$ is a minimal D-brane we have $C_A(\text{Im}(\beta_{\min})) = A$ whence for any other D-brane we have a two-cell and corresponding deformation

$$\begin{array}{ccc} & g & \\ R & \Downarrow \alpha & A \\ & \beta_{\min} & \end{array}$$

Note however that these 2-morphisms do *not* equip \mathbf{alg} with a 2-category structure as defined for example in [15, XII.3]. Indeed, whereas we have an obvious vertical composition of 2-morphisms there is in general no horizontal composition of 2-morphisms.

Still, these 2-morphisms impose a natural compatibility structure on families of algebraic D-branes. In string-theory one often considers the limit $n \rightarrow \infty$ of n stacks of D-branes located at a subscheme $Z \hookrightarrow X$, that is, a family of algebra morphisms $\beta_n : \mathbb{C}[X] \longrightarrow A_n$ where A_n is a degree n Azumaya algebra with center $\mathbb{C}[Z]$. Our ringtheoretical considerations suggest one needs to impose a multiplicative compatibility requirement on such families.

Definition 4. *A family of algebraic D-branes*

$$\beta_n : R \longrightarrow A_n$$

such that $Z(A_n) = C$ for all n is said to be compatible if for every $n|m$ we have a C -monomorphism $i_{nm} : A_n \hookrightarrow A_m$ and a corresponding 2-morphism

$$\begin{array}{ccc} & \xrightarrow{\beta_m} & \\ R & \Downarrow \alpha & A_m \\ & \xleftarrow{i_{nm} \circ \beta_n} & \end{array}$$

That is, the algebraic D-brane β_m corresponding to the m stack of D-branes at $\text{spec}(C)$ is a Higgsing of all D-branes $R \longrightarrow A_n \hookrightarrow A_m$ for all divisors n of m .

We will give an example of a non-trivial example of a family of algebraic D-branes which are all neither minimal nor maximal. Consider the 2-dimensional torus $\mathbb{C}[s^\pm, t^\pm]$ and a primitive n -th root of unity $q_n = \sqrt[n]{1}$. The *quantum torus* is the non-commutative algebra generated by two elements U_n and V_n (and their inverses) satisfying the relations

$$V_n U_n = q_n U_n V_n \quad U_n^n = s \quad V_n^n = t$$

and we will denote this algebra as $\mathbb{C}_{q_n}[U_n^\pm, V_n^\pm]$. The center of this algebra is easily seen to be $\mathbb{C}[s^\pm, t^\pm]$ and in fact $\mathbb{C}_{q_n}[U_n^\pm, V_n^\pm]$ is an Azumaya algebra over it of degree n . When $m = n.k$ there are obvious embeddings of $\mathbb{C}[s^\pm, t^\pm]$ -algebras

$$i_{m,n} : \mathbb{C}_{q_n}[U_n^\pm, V_n^\pm] \hookrightarrow \mathbb{C}_{q_m}[U_m^\pm, V_m^\pm] \quad \begin{cases} U_n \longrightarrow U_m^k \\ V_n \longrightarrow V_m^k \end{cases}$$

Lemma 2. *Consider the coordinate ring of $GL_2 = \begin{bmatrix} s & u \\ v & t \end{bmatrix}$*

$$\mathcal{O}(GL_2) = \mathbb{C}[s, t, u, v, (st - uv)^{-1}]$$

then the \mathbb{C} -algebra morphisms $\beta_n : \mathcal{O}(GL_2) \longrightarrow \mathbb{C}_{q_n}[U_n^\pm, V_n^\pm]$ defined by

$$\beta = (\beta_n) : \begin{cases} u \mapsto 0 \\ v \mapsto 0 \\ s \mapsto s \\ t \mapsto V_n \end{cases}$$

is a compatible family of algebraic D-branes on the maximal torus $T_2 \hookrightarrow GL_2$.

Proof. Clearly the maps β_n are \mathbb{C} -algebra morphisms from $\mathcal{O}(GL_2)$ to the Azumaya algebra $\mathbb{C}_{q_n}[U_n^\pm, V_n^\pm]$ whence they are algebraic D-branes. Remains for every $m = n.k$ to compare the centralizers of the images at level m with those of the image with the inclusion $i_{m,n}$. One easily verifies that

$$\text{Im}(\beta_m) = \mathbb{C}[s^\pm, V_m^\pm] \quad \text{whereas} \quad \text{Im}(i_{m,n} \circ \beta_n) = \mathbb{C}[s^\pm, V_m^{\pm k}]$$

The centralizer of $Im(\beta_m)$ is equal to itself, whereas

$$C(Im(i_{m,n} \circ \beta_n)) = \mathbb{C}[s^\pm, V_m^{\pm k}] \otimes_{\mathbb{C}[s^\pm, t^\pm]} \mathbb{C}_{q_k}[U_m^{\pm n}, V_m^{\pm n}]$$

and so these form indeed a compatible family of algebraic D-branes on T_2 . \square

3. AZUMAYA NONCOMMUTATIVE GEOMETRY

The main problem studied in the papers [9]-[14] by Chien-Hao Liu and Shing-Tung Yau is to associate some noncommutative geometry to the morphism $\beta : \mathcal{O}(X) \longrightarrow A_n$ (or more generally $\beta : R \longrightarrow A_n$) describing the embedding of the n-stack of D-branes in space-time. That is, one wants to associate geometric objects (such as varieties, topological spaces and sheaves on these) that will allow us to reconstruct β from the geometric data.

3.1. Noncommutative structure sheaves. We will rephrase Liu and Yau's Azumaya noncommutative geometry, based on 'surrogates' and the 'noncommutative cloud', in classical noncommutative algebraic geometry of pi-algebras, as developed by Fred Van Oystaeyen and Alain Verschoren [23] in the early 80-ties.

Let S be a finitely generated Noetherian \mathbb{C} -algebra satisfying all polynomial identities of degree n . Observe that $\int_n R$, A_n and its subalgebras

$$A_c = Im(\beta)C \quad \text{and} \quad A_e = Im(\beta)C_A(Im(\beta))$$

all satisfy these requirements. With $\mathbf{spec}(S)$ we denote the set of all twosided prime ideals of S which becomes a topological space by equipping it with the Zariski topology having as typical closed sets $\mathbb{V}(I) = \{P \in \mathbf{spec}(S) \mid I \subset P\}$ for I a twosided ideal of S . However, this twosided prime spectrum is not necessarily functorial, that is, if $S \xrightarrow{f} T$ is a \mathbb{C} -algebra morphism between suitable algebras, then $f^{-1}(P)$ for a twosided prime ideal $P \triangleleft T$ does not have to be a prime ideal in S . Still, the twosided prime spectrum is functorial whenever $S \xrightarrow{f} T$ is an *extension*, that is, if $T = Im(f)C_T(Im(f))$ by [17, Thm. II.6.5].

In [23, V.3] Van Oystaeyen and Verschoren constructed a noncommutative structure sheaf \mathcal{O}_S^{bi} on $\mathbf{spec}(S)$ by means of localization in the category of S -bimodules. The main properties of this construction are [23, Thm. V.3.17 and Thm. V.3.36]

- One recovers the algebra back from the global sections $S = \Gamma(\mathbf{spec}(S), \mathcal{O}_S^{bi})$
- Every extension $S \xrightarrow{f} T$ defines a morphism of ringed spaces

$$(\mathbf{spec}(T), \mathcal{O}_T^{bi}) \longrightarrow (\mathbf{spec}(S), \mathcal{O}_S^{bi})$$

In fact, the results are stated there only for prime Noetherian rings satisfying all polynomial identities of degree n but extend to the case of interest here as our rings are finite modules over an affine center and hence we have enough central localizations. For more details, see [24].

An algebraic D-brane $R \xrightarrow{\beta} A$ of degree n is not necessarily an extension, but the morphisms below defined by it are :

$$\int_n R \xrightarrow{\beta} A_c = Im(\beta)C \quad \text{and} \quad \int_n R \xrightarrow{\beta} A_e = Im(\beta)C_A(Im(\beta))$$

Theorem 3. *Associate to the D-brane either of these noncommutative geometric data*

$$(\mathbf{spec}(A_*), \mathcal{O}_{A_*}^{bi}) \longrightarrow (\mathbf{spec}(\int_n R), \mathcal{O}_{\int_n R}^{bi})$$

It contains enough information to reconstruct $\beta : \int_n R \longrightarrow A$ by taking global sections, and hence by compositing with the universal morphism $R \longrightarrow \int_n R$ also the D-brane.

The fact that noncommutative prime spectra and bimodule structure sheaves behave only functorial with respect to \mathbb{C} -algebra morphisms which are extensions, explains the notion of a 'noncommutative cloud' in [9]. In our language, the noncommutative cloud of a C -Azumaya algebra A is the set

$$\text{cloud}(A) = \bigsqcup_{A_*} \text{spec}(A_*)$$

where the disjoint union of noncommutative prime spectra is taken over all subalgebras A_* satisfying

$$C \subset Z(A_*) \subset A_* \subset A$$

and such subalgebras are called 'surrogates' in [9]. The point being that we can now associate to an algebraic D-brane of degree n , $R \xrightarrow{\beta} A$ the partially defined map which is well defined because $\int_n R \longrightarrow A_e$ is an extension

$$\text{cloud}(A) \longleftarrow \text{spec}(A_e) \longrightarrow \text{spec}\left(\int_n R\right)$$

and by adorning $\text{cloud}(A)$ componentwise with noncommutative structure sheaves $\mathcal{O}_{A_*}^{bi}$, this partially defined map contains enough information to reconstruct the D-brane.

3.2. Noncommutative thin schemes. In [6, §I.2] Maxim Kontsevich and Yan Soibelman define a *noncommutative thin scheme* to be a covariant functor commuting with finite projective limits

$$\mathbf{X} : \mathbf{fd}\text{-}\mathbf{alg} \longrightarrow \mathbf{sets}$$

from the category $\mathbf{fd}\text{-}\mathbf{alg}$ of all finite dimensional \mathbb{C} -algebras to the category \mathbf{sets} of all sets. They prove [6, Thm. 2.1.1] that every noncommutative thin scheme is represented by a \mathbb{C} -coalgebra. That is, there is a \mathbb{C} -coalgebra $C_{\mathbf{X}}$ associated to the noncommutative thin scheme \mathbf{X} having the property that there is a natural one-to-one correspondence

$$\mathbf{X}(B) = \mathbf{alg}(B, C_{\mathbf{X}}^*)$$

for every finite dimensional \mathbb{C} -algebra B . Here, $C_{\mathbf{X}}^*$ is the dual algebra of all linear functionals on $C_{\mathbf{X}}$. $C_{\mathbf{X}}$ is called the *coalgebra of distributions* on \mathbf{X} and the *noncommutative algebra of functions* on the thin scheme \mathbf{X} is defined to be $\mathbb{C}[\mathbf{X}] = C_{\mathbf{X}}^*$. For a \mathbb{C} -algebra R , it is not true in general that the linear functionals R^* are a coalgebra, but the *dual coalgebra* R^o is, where

$$R^o = \{f \in R^* \mid \text{Ker}(f) \text{ contains a twosided ideal of finite codimension} \}$$

Kostant duality, see for example [21, Thm. 6.0.5], asserts that the functors

$$\begin{array}{ccc} & \circ & \\ \mathbf{alg} & \xrightleftharpoons{\quad} & \mathbf{coalg} \\ & * & \end{array}$$

are adjoint. That is, for every \mathbb{C} -algebra R and \mathbb{C} -coalgebra C there is a natural one-to-one correspondence between the homomorphisms

$$\mathbf{alg}(R, C^*) = \mathbf{coalg}(C, R^o)$$

For an affine \mathbb{C} -algebra R we define the contravariant functor

$$\mathbf{rep}_R : \mathbf{fd-coalg} \longrightarrow \mathbf{sets} \quad C \mapsto \mathbf{alg}(R, C^*)$$

describing the finite dimensional representations of R , see [6, Example 2.1.9]. As taking the linear dual restricts Koszul duality to an anti-equivalence between the categories $\mathbf{fd-alg}$ and $\mathbf{fd-coalg}$, we can describe \mathbf{rep}_R as the noncommutative thin scheme represented by the dual coalgebra R° as

$$\mathbf{rep}_R : \mathbf{fd-alg} \longrightarrow \mathbf{sets} \quad B = C^* \mapsto \mathbf{alg}(R, B) = \mathbf{coalg}(C, R^\circ)$$

Definition 5. *The noncommutative affine scheme \mathbf{rep}_R is the noncommutative thin scheme represented by the dual coalgebra R° of the affine \mathbb{C} -algebra R .*

By [21, Lemma 6.0.1] for every \mathbb{C} -algebra morphism $f \in \mathbf{alg}(R, B)$, the dual map determines a \mathbb{C} -coalgebra morphism $f^* \in \mathbf{coalg}(B^\circ, R^\circ)$. In particular, to an algebraic D-brane $\beta : R \longrightarrow A$ we can associate a morphism between their noncommutative affine schemes

$$\beta^* : \mathbf{rep}_A \longrightarrow \mathbf{rep}_R$$

determined by the coalgebra map $\beta^* : A^\circ \longrightarrow R^\circ$.

Theorem 4. *The morphism $\beta^* : \mathbf{rep}_A \longrightarrow \mathbf{rep}_R$ between the noncommutative thin schemes allows to reconstruct the algebraic D-brane $\beta : R \longrightarrow A$.*

In order to prove this we need to describe the dual coalgebras A° and R° . Recall that a coalgebra D is said to be simple if it has no proper nontrivial sub-coalgebras. Every simple \mathbb{C} -coalgebra is finite dimensional and as D^* is a simple \mathbb{C} -algebra, we have that $D \simeq M_n(\mathbb{C})^*$, the full matrix coalgebra that is, $\sum_{i,j} \mathbb{C} e_{ij}$ with

$$\Delta(e_{ij}) = \sum_{k=1}^n e_{ik} \otimes e_{kj} \quad \text{and} \quad \epsilon(e_{ij}) = \delta_{ij}$$

The coradical $\mathbf{corad}(C)$ of a coalgebra C is the direct sum of all simple subcoalgebras of C . It follows from Kostant duality that for any affine \mathbb{C} -algebra R we have

$$\mathbf{corad}(R^\circ) = \oplus_{S \in \mathbf{simp}R} M_n(\mathbb{C})_S^*$$

where $\mathbf{simp}R$ is the set of isomorphism classes of finite dimensional simple R -representations and the factor of $\mathbf{corad}(R^\circ)$ corresponding to a simple representation S is isomorphic to the matrix coalgebra $M_n(\mathbb{C})^*$ if $\dim(S) = n$.

In algebra, one can resize idempotents by Morita-theory and hence replace full matrices by the basefield. In coalgebra-theory there is an analogous duality known as *Takeuchi equivalence*, see [22]. The isotypical decomposition of $\mathbf{corad}(R^\circ)$ as an R° -comodule is of the form $\oplus_S C_S^{\oplus n_S}$, the sum again taken over all simple finite dimensional R -representations. Take the R° -comodule $E = \oplus_S C_S$ and its *coendomorphism coalgebra*

$$R^\dagger = \mathbf{coend}^{R^\circ}(E)$$

then Takeuchi-equivalence (see for example [1, §4, §5] and the references contained in this paper for more details) asserts that R° is Takeuchi-equivalent to the coalgebra R^\dagger which is *pointed*, that is, $\mathbf{corad}(R^\dagger) = \mathbb{C} \mathbf{simp}(R) = \oplus_S \mathbb{C} g_S$ with one *group-like* element g_S for every simple finite dimensional R -representation. Remains to describe the structure of the full basic coalgebra R^\dagger .

Example 2. For the affine Azumaya algebra A of degree n over its affine center C , we know that all finite dimensional simple A -representations are n -dimensional and are parametrized by the maximal ideals $\mathfrak{m} \in \mathbf{max}(C)$. That is, $\text{corad}(A^o) = \bigoplus_{\mathfrak{m}} M_n(\mathbb{C})^*$ and A^o is Takeuchi-equivalent to the pointed coalgebra

$$A^\dagger = C^o \quad \text{with} \quad \text{corad}(A^\dagger) = \bigoplus_{\mathfrak{m}} \mathbb{C} g_{\mathfrak{m}}$$

By [21, Prop. 8.0.7] we know that any cocommutative pointed coalgebra is the direct sum of its pointed irreducible components (at the algebra level this says that a commutative semi-local algebra is the direct sum of local algebras). Therefore,

$$A^\dagger = C^o = \bigoplus_{\mathfrak{m}} C_{\mathfrak{m}}^0$$

where $C_{\mathfrak{m}}^0$ is a pointed irreducible cocommutative coalgebra and as such is a sub-coalgebra of the enveloping coalgebra of the abelian Lie algebra on the Zariski tangent space $(\mathfrak{m}/\mathfrak{m}^2)^*$. That is, we recover the maximal spectrum $\mathbf{max}(C)$ of the center C from A^\dagger . But then, the dual algebra

$$A^{\dagger*} = C^{o*} = \prod_{\mathfrak{m}} \hat{C}_{\mathfrak{m}}$$

the direct sum of the completions of C at all maximal ideals \mathfrak{m} . Also, the double dual algebra

$$A^{o*} = \prod_{\mathfrak{m}} M_n(\hat{C}_{\mathfrak{m}})$$

For a general affine noncommutative \mathbb{C} -algebra R , the description of the pointed coalgebra R^\dagger is more complicated as there can be non-trivial extensions between non-isomorphic finite dimensional simple R -representations (note that this does not happen for Azumaya algebras).

For a (possibly infinite) quiver \vec{Q} we define the *path coalgebra* $\mathbb{C}\vec{Q}$ to be the vectorspace $\bigoplus_p \mathbb{C}p$ with basis all oriented paths p in the quiver \vec{Q} (including those of length zero, corresponding to the vertices) and with structural maps induced by

$$\Delta(p) = \sum_{p=p'p''} p' \otimes p'' \quad \text{and} \quad \epsilon(p) = \delta_{0,l(p)}$$

where $p'p''$ denotes the concatenation of the oriented paths p' and p'' and where $l(p)$ denotes the length of the path p . Hence, every vertex v is a group-like element and for an arrow $\odot \xrightarrow{a} \oslash$ we have $\Delta(a) = v \otimes a + a \otimes w$ and $\epsilon(a) = 0$, that is, arrows are skew-primitive elements.

For every natural number i , we define the *extⁱ-quiver* $\overrightarrow{\mathbf{ext}}_R^i$ to have one vertex v_S for every $S \in \mathbf{simp}(R)$ and such that the number of arrows from v_S to v_T is equal to the dimension of the space $\text{Ext}_R^i(S, T)$. With \mathbf{ext}_R^i we denote the \mathbb{C} -vectorspace on the arrows of $\overrightarrow{\mathbf{ext}}_R^i$.

The *Yoneda-space* $\mathbf{ext}_R^\bullet = \bigoplus \mathbf{ext}_R^i$ is endowed with a natural A_∞ -structure [4], defining a linear map (the *homotopy Maurer-Cartan map*, [20])

$$\mu = \bigoplus_i m_i : \mathbb{C}\overrightarrow{\mathbf{ext}}_R^1 \longrightarrow \mathbf{ext}_R^2$$

from the path coalgebra $\mathbb{C}\overrightarrow{\mathbf{ext}}_R^1$ of the ext^1 -quiver to the vectorspace \mathbf{ext}_R^2 , see [4, §2.2] and [20].

Theorem 5. *The dual coalgebra R^o is Takeuchi-equivalent to the pointed coalgebra R^\dagger which is the sum of all subcoalgebras contained in the kernel of the linear map*

$$\mu = \oplus_i m_i : \mathbb{C}\overrightarrow{\text{ext}}_R^1 \longrightarrow \text{ext}_R^2$$

determined by the A_∞ -structure on the Yoneda-space ext_R^\bullet .

Proof. We can reduce to finite subquivers as any subcoalgebra is the limit of finite dimensional subcoalgebras and because any finite dimensional R -representation involves only finitely many simples. Hence, the statement is a global version of the result on finite dimensional algebras due to B. Keller [4, §2.2].

Alternatively, we can use the results of E. Segal [20]. Let S_1, \dots, S_r be distinct simple finite dimensional R -representations and consider the semi-simple module $M = S_1 \oplus \dots \oplus S_r$ which determines an algebra epimorphism

$$\pi_M : R \twoheadrightarrow M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C}) = B$$

If $\mathfrak{m} = \text{Ker}(\pi_M)$, then the \mathfrak{m} -adic completion $\hat{R}_\mathfrak{m} = \varprojlim R/\mathfrak{m}^n$ is an augmented B -algebra and we are done if we can describe its finite dimensional (nilpotent) representations. Again, consider the A_∞ -structure on the Yoneda-algebra $\text{Ext}_R^\bullet(M, M)$ and the quiver on r -vertices $\overrightarrow{\text{ext}}_R^1(M, M)$ and the homotopy Maurer-Cartan map

$$\mu_M = \oplus_i m_i : \mathbb{C}\overrightarrow{\text{ext}}_R^1(M, M) \longrightarrow \text{Ext}_R^2(M, M)$$

Dualizing we get a subspace $\text{Im}(\mu_M^*)$ in the path-algebra $\mathbb{C}\overrightarrow{\text{ext}}_R^1(M, M)^*$ of the dual quiver. Ed Segal's main result [20, Thm 2.12] now asserts that $\hat{R}_\mathfrak{m}$ is Morita-equivalent to

$$\hat{R}_\mathfrak{m} \underset{M}{\sim} \frac{(\mathbb{C}\overrightarrow{\text{ext}}_R^1(M, M)^*)^\wedge}{(\text{Im}(\mu_M^*))}$$

where $(\mathbb{C}\overrightarrow{\text{ext}}_R^1(M, M)^*)^\wedge$ is the completion of the path-algebra at the ideals generated by the paths of positive length. The statement above is the dual coalgebra version of this. \square

Proof of theorem 4 : The morphism between the thin noncommutative schemes

$$\beta^* : \mathbf{rep}_A \longrightarrow \mathbf{rep}_R$$

corresponds to the coalgebra-map dual to β

$$\beta^o : A^o \longrightarrow R^o$$

Dualizing this map again we obtain a \mathbb{C} -algebra map, and composing with the natural map $R \longrightarrow R^{o*}$ and the observations of example 2 we obtain an algebra map

$$R \longrightarrow R^{o*} \xrightarrow{\beta^{o*}} A^{o*} = \prod_{\mathfrak{m}} M_n(\hat{C}_\mathfrak{m})$$

the components of which are the maps from the global sections of the structure sheaf of R in the étale topology to the stalks of the structure sheaf of A in the étale topology, induced by β . By étale descent we can therefore reconstruct the algebraic D-brane $\beta : R \longrightarrow A$ from them.

REFERENCES

- [1] William Chin, *A brief introduction to coalgebra representation theory*, in "Hopf Algebras" M. Dekker Lect. Notes in Pure and Appl. Math. (2004) 109-132. Online at <http://condor.depaul.edu/~wchin/crt.pdf>
- [2] Aise Johan de Jong, The Stacks Project, *Algebraic stacks - Examples*
- [3] Frank De Meyer and E. Ingraham, *Separable algebras over commutative rings*, Springer LNM **181** (1970)
- [4] Bernhard Keller, *A-infinity algebras in representation theory*, Contribution to the Proceedings of ICRA IX, Beijing (2000). Online at <http://www.math.jussieu.fr/~keller/publ/art.dvi>
- [5] Max-Albert Knus and Manuel Ojanguren, *Théorie de la Descente et Algèbres d'Azumaya*, Springer LNM **389** (1974)
- [6] Maxim Kontsevich and Yan Soibelman, *Notes on A_∞ -algebras, A_∞ -categories and non-commutative geometry I*, arXiv:math.RA/0606241 (2006)
- [7] Lieven Le Bruyn, *Noncommutative geometry and Cayley-smooth orders*, Pure and applied mathematics **290**, Chapman & Hall (2008)
- [8] Lieven Le Bruyn and Stijn Symens, *Partial desingularizations arising from non-commutative algebras*, arXiv:0507494 (2005)
- [9] Chien-Hao Liu and Shing-Tung Yau, *Azumaya-type noncommutative spaces and morphisms therefrom : Polchinski's D-branes in string theory from Grothendieck's viewpoint*, arXiv:0709.1515 (2007)
- [10] Si Li, Chien-Hao Liu, Ruifang Song and Shing-Tung Yau, *Morphisms from Azumaya prestackable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves*, arXiv:0809.2121 (2008)
- [11] Chien-Hao Liu and Shing-Tung Yau, *Azumaya structure on D-branes and resolution of ADE orbifold singularities revisited: Douglas-Moore vs. Polchinski-Grothendieck*, arXiv:0901.0342 (2009)
- [12] Chien-Hao Liu and Shing-Tung Yau, *Azumaya structure on D-branes and deformations and resolutions of a conifold revisited: Klebanov-Strassler-Witten vs. Polchinski-Grothendieck*, arXiv:0907.0268 (2009)
- [13] Chien-Hao Liu and Shing-Tung Yau, *Nontrivial Azumaya noncommutative schemes, morphisms therefrom, and their extension by the sheaf of algebras of differential operators: D-branes in a B-field background à la Polchinski-Grothendieck Ansatz*, arXiv:0909.2291 (2009)
- [14] Chien-Hao Liu and Shing-Tung Yau, *D-branes and Azumaya noncommutative geometry: From Polchinski to Grothendieck*, arXiv:1003.1178 (2010)
- [15] Saunders Mac Lane, *Categories for the working mathematician*, Springer Graduate Texts in Math. 5, 2nd edition (1997)
- [16] James S. Milne, *Étale cohomology*, Princeton Math. Series **33** (1980)
- [17] Claudio Procesi, *Rings with polynomial identities*, Pure and Appl. Math. **17** Marcel Dekker (1973)
- [18] Claudio Procesi, *A formal inverse to the Cayley-Hamilton theorem*, J. Alg. **107** (1987) 63-74
- [19] Claudio Procesi, *Deformations of representations*, Methods in ring theory (Levico Terme, 1997), 247-276, Lecture Notes in Pure and Appl. Math. **198**, Marcel Dekker (1998)
- [20] Ed Segal, *The A_∞ deformation theory of a point and the derived category of local Calabi-Yaus*, math.AG/0702539 (2007)
- [21] Moss E. Sweedler, *Hopf Algebras*, monograph, W.A. Benjamin (New York) (1969)
- [22] M. Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo **24** (1977) 629-644
- [23] Fred Van Oystaeyen and Alain Verschoren, *Noncommutative algebraic geometry*, Springer LNM **887** (1981)
- [24] Fred Van Oystaeyen, *Algebraic geometry for associative algebras*, Lecture Notes in Pure and Appl. Math. **232** Marcel Dekker (2000)

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