The algebraic structure of finitely generated $L^0(\mathcal{F}, K)$-modules

Guang Shi[1]

LMIB and School of Mathematics and Systems Science, Beihang University, Beijing 100191, P.R. China

Abstract

Let K be the scalar field of all real or complex numbers, $(\Omega, \mathcal{F}, P)$ a probability space, and $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of K–valued $\mathcal{F}$–measurable random variables on Ω. This paper proves that every finitely generated unitary $L^0(\mathcal{F}, K)$-module can be decomposed into a direct sum of finite quasi-free stratifications of finite rank.

Keywords:
Finitely generated $L^0(\mathcal{F}, K)$-modules, Stratification structure

1. Introduction

Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space, K the scalar field R of real numbers or C of complex numbers, and $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of K–valued $\mathcal{F}$–measurable random variables on Ω under the ordinary scalar multiplication, addition and multiplication operations on equivalence classes. In addition, all $L^0(\mathcal{F}, K)$-modules in this paper are supposed to be unitary[12, Chapter 4, Definition 1.1].

Random metric theory including the theory of random normed spaces originated from the theory of probabilistic metric spaces[13]. Since the notions of random normed modules(briefly, RN modules) and random conjugate spaces for RN modules were presented[2, 4], the theory of random locally convex modules(specially, RN modules) has undergone considerably much of development[2, 3, 5, 6, 7, 10] and been applied to not only solving a series of problems in measure theory[8, 9] but also introducing some new

[1]Supported by NNSF No. 10871016

frames to financial mathematics[11]. All these motivate the study of the algebraic structure of $L^0(\mathcal{F}, K)$ modules.

A significant peculiarity of an $L^0(\mathcal{F}, K)$ module E is that E has stratification structures[5, 6, 7]. For a set $A \in \mathcal{F}$, the A-stratification of E is defined by $\tilde{I}_A E \triangleq \{\tilde{I}_A x \mid x \in E\}$, where $\tilde{I}_A$ is the equivalence class of the characteristic function of A. The limitation of the module multiplication $L^0(\mathcal{F}, K) \times E \to E$ to $\tilde{I}_A L^0(\mathcal{F}, K) \times \tilde{I}_A E \to \tilde{I}_A E$ makes $\tilde{I}_A E$ an unitary $\tilde{I}_A L^0(\mathcal{F}, K)$-module for any $A \in \mathcal{F}$ and $P(A) > 0$. Moreover, $\tilde{I}_A E$ is called a quasi-free stratification of finite rank of E for any $A \in \mathcal{F}$ and $P(A) > 0$ if $\tilde{I}_A E$ is a free module of finite rank over $\tilde{I}_A L^0(\mathcal{F}, K)$[3, Definition 1.1].

Notice that $L^0(\mathcal{F}, K)$ is not a division ring unless $(\Omega, \mathcal{F}, P)$ is a trivial probability space. Thus modules over $L^0(\mathcal{F}, K)$ may not be free. Nevertheless, we will show that if E is finitely generated, i.e. there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of E such that $E = \{\sum_{i=1}^{n} \xi_i x_i \mid \xi_i \in L^0(\mathcal{F}, K), 1 \leqslant i \leqslant n\}$, then E is a direct sum of finite quasi-free stratifications of finite rank.

The paper is organized as follows: in Section 2 we introduce some necessary notions and preliminaries. In Section 3, we state and prove the main result about the algebraic structure of finitely generated $L^0(\mathcal{F}, K)$-modules. And section 4 gives two corollaries.

2. Preliminaries

It follows from [11, Lemma 2.9] that $\mathcal{F}$ is a complete lattice with respect to the partial order of almost sure set inclusion. For a nonempty collection $\mathcal{E} \subseteq \mathcal{F}$, we use ess.sup$(\mathcal{E})$ to denote the essential supremum of $\mathcal{E}$ as in [11]. Furthermore, if $\mathcal{E}$ is directed upwards, i.e. $A \cup B \in \mathcal{E}$ for any $A, B \in \mathcal{E}$, then there exists a nondecreasing sequence $\{A_n \in \mathcal{E} \mid n \in N\}$ such that ess.sup$\mathcal{E} = \bigcup_{n \in N} A_n$. And throughout this paper we distinguish characteristic functions from their equivalence classes in $L^0(\mathcal{F}, K)$ by means of symbols: for example, I_A denotes the characteristic function of an $\mathcal{F}$–measurable set A, then we use $\tilde{I}_A$ for its equivalence class in $L^0(\mathcal{F}, K)$.

Besides, for any $A \in \mathcal{F}$, "$\xi > \eta$ on A" means $\xi^0(\omega) > \eta^0(\omega)$ a.s. on A for any chosen representative ξ^0 and η^0 of ξ and η, respectively. As usual, $\xi > \eta$ means $\xi \geqslant \eta$ and $\xi \neq \eta$.

Moreover, for any $\xi \in L^0(\mathcal{F}, K)$, $|\xi|$ and ξ^{-1} respectively stand for the equivalence classes determined by the $\mathcal{F}$-measurable function $|\xi^0| : \Omega \to R$

defined by $|\xi^0|(\omega) = |\xi^0(\omega)|$, $\omega \in \Omega$ and $(\xi^0)^{-1}$ defined by

$$(\xi^0)^{-1}(\omega) = \begin{cases} (\xi^0(\omega))^{-1}, & \xi^0(\omega) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

where ξ^0 is an arbitrarily chosen representative of ξ. It is clear that $|\xi| \in L^0_+ = \{\eta \in L^0(\mathcal{F}, R) \mid \eta \geqslant 0\}$ and $\xi \cdot \xi^{-1} = \tilde{I}_{\{\omega \in \Omega | \xi^0(\omega) \neq 0\}}$.

All the $L^0(\mathcal{F}, K)$-modules E in the sequel of this paper are assumed to satisfy the following property: If x and y are two elements in E and there exists a countable partition $\{A_n \mid n \in N\}$ of Ω to $\mathcal{F}$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} y$ for each $n \in N$, then $x = y$. Here $\tilde{I}_A x$ is called the A-stratification of x for any $A \in \mathcal{F}$.

Definition 2.1 [1]. Let E be a left module over the algebra $L^0(\mathcal{F}, K)$. A countable concatenation of some sequence $\{x_n \mid n \in N\}$ in E with respect to some countable partition $\{A_n \mid n \in N\}$ of Ω is a formal sum $\Sigma_{n \in N} \tilde{I}_{A_n} x_n$. Moreover, a countable concatenation $\Sigma_{n \in N} \tilde{I}_{A_n} x_n$ is well defined or $\Sigma_{n \in N} \tilde{I}_{A_n} x_n \in E$ if there is $x \in E$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} x_n$ for any $n \in N$. A subset A of E is called having the countable concatenation property if every countable concatenation $\Sigma_{n \in N} \tilde{I}_{A_n} x_n$ with $x_n \in A$ for each $n \in N$ still belongs to A, namely $\Sigma_{n \in N} \tilde{I}_{A_n} x_n$ is well defined and there exists $x \in A$ such that $x = \Sigma_{n \in N} \tilde{I}_{A_n} x_n$.

Definition 2.2 [1, 4]. An ordered pair $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, P)$ if E is a left module over the algebra $L^0(\mathcal{F}, K)$ and $\|\cdot\|$ is a mapping from E to L^0_+ such that the following three axioms are satisfied:

(1) $\|x\| = 0$ iff $x = \theta$(the null element of E);
(2) $\|\xi x\| = |\xi| \|x\|$, any $\xi \in L^0(\mathcal{F}, K)$ and $x \in E$;
(3) $\|x + y\| \leqslant \|x\| + \|y\|$, any $x, y \in E$.

Definition 2.3 [1, 4]. An ordered pair $(E, \langle \cdot, \cdot \rangle)$ is called a random inner product module (briefly, an RIP module) over K with base $(\Omega, \mathcal{F}, P)$ if E is a left module over the algebra $L^0(\mathcal{F}, K)$ and $\langle \cdot, \cdot \rangle : E \times E \to L^0(\mathcal{F}, K)$ satisfies the following statements:

(1) $\langle x, x\rangle \in L^0_+$ and $\langle x, x\rangle = 0$ iff $x = \theta$;
(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, any $x, y \in E$ where $\overline{\langle y, x\rangle}$ denotes the complex conjugate of $\langle y, x\rangle$;
(3) $\langle \xi x, y\rangle = \xi\langle x, y\rangle$, any $\xi \in L^0(\mathcal{F}, K)$ and $x, y \in E$;
(4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, any $x, y, z \in E$.

where $\langle x, y\rangle$ is called the random inner product between x and y.

An RIP module $(E, \langle\cdot,\cdot\rangle)$ is also an RN module when $\|\cdot\| : E \to L^0_+$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in E$. And x is orthogonal to y if $\langle x, y\rangle = 0$ for $x, y \in E$.

Example 2.4. Denote by $L^0(\mathcal{F}, K^n)$ the linear space of equivalence classes of K^n–valued $\mathcal{F}$–measurable functions on Ω, where n is a positive integer. Define $\cdot : L^0(\mathcal{F}, K) \times L^0(\mathcal{F}, K^n) \to L^0(\mathcal{F}, K^n)$ by $\lambda \cdot x = (\lambda\xi_1, \lambda\xi_2, \cdots, \lambda\xi_n)$ and $\langle\cdot,\cdot\rangle : L^0(\mathcal{F}, K^n) \times L^0(\mathcal{F}, K^n) \to L^0(\mathcal{F}, K)$ by $\langle x, y\rangle = \Sigma^n_{i=1}\xi_i\bar{\eta}_i$, for any $\lambda \in L^0(\mathcal{F}, K)$ and $x = (\xi_1, \xi_2, \cdots, \xi_n)$, $y = (\eta_1, \eta_2, \cdots, \eta_n) \in L^0(\mathcal{F}, K^n)$. It is easy to check that $(L^0(\mathcal{F}, K^n), \langle\cdot,\cdot\rangle)$ is an RIP module over K with base $(\Omega, \mathcal{F}, P)$, and also an RN module. Specially, $L^0(\mathcal{F}, K)$ is an RN module and $\|\lambda\| = |\lambda|$ for any $\lambda \in L^0(\mathcal{F}, K)$.

3. The algebraic structure of finitely generated $L^0(\mathcal{F}, K)$-modules

The main purpose of this section is to prove the following theorem:

Theorem 3.1. *Suppose E is a finitely generated left module over the algebra $L^0(\mathcal{F}, K)$, namely there exists a subset $X = \{x_1, x_2, \cdots, x_n\}$ of E such that X generates E, where n is some fixed positive integer. Then there exists a partition $\{A_0, A_1, \cdots, A_n\}$ of Ω to $\mathcal{F}$ such that $\tilde{I}_{A_i}E$ is a quasi-free stratification of rank i of E for each i which satisfies $0 \leqslant i \leqslant n$ and $P(A_i) > 0$. Consequently, $E = \bigoplus^n_{i=0} \tilde{I}_{A_i}E$, where $\bigoplus$ denotes the direct sum of modules.*

Proof. The proof will be divided into three steps.

Step 1. Let N_n denote the set $\{1, 2, \cdots, n\}$. For any nonempty subset L of N_n, let $\mathcal{F}_L$ be the collection of all sets $A \in \mathcal{F}$ such that if $\{\lambda_i \in L^0(\mathcal{F}, K) \mid i \in L\}$ satisfies $\tilde{I}_A \sum_{i\in L} \lambda_i x_i = \theta$ then $\tilde{I}_A\lambda_i = 0$ for each $i \in L$. We begin by proving that ess.sup$(\mathcal{F}_L) \in \mathcal{F}_L$.

Suppose $A, B \in \mathcal{F}_L$ and $\{\lambda_i \in L^0(\mathcal{F}, K) \mid i \in L\}$ satisfies $\tilde{I}_{A\cup B} \sum_{i\in L} \lambda_i x_i = \theta$. Then we have $\tilde{I}_A \sum_{i\in L} \lambda_i x_i = \theta$ since $\tilde{I}_A \tilde{I}_{A\cup B} = \tilde{I}_A$. It follows from $A \in \mathcal{F}_L$ that $\tilde{I}_A \lambda_i = 0$ for each $i \in L$. Similarly, we have $\tilde{I}_B \lambda_i = 0$ for each $i \in L$. Hence $\tilde{I}_{A\cup B}\lambda_i = (\tilde{I}_A + \tilde{I}_{A^c}\tilde{I}_B)\lambda_i = 0$ for each $i \in L$, which implies $A \cup B \in \mathcal{F}_L$. Thus $\mathcal{F}_L$ is directed upwards. Consequently, there exists a nondecreasing sequence $\{A_k \mid k \in N\}$ in $\mathcal{F}_L$ such that ess.sup$(\mathcal{F}_L) = \bigcup_{k\in N} A_k$. Let I_L denote the characteristic function of ess.sup$(\mathcal{F}_L)$, then $\{\tilde{I}_{A_k} \mid k \in N\}$ converges to $\tilde{I}_L$ in probability P.

If $\{\xi_i \in L^0(\mathcal{F}, K) \mid i \in L\}$ satisfies $\tilde{I}_L \sum_{i\in L} \xi_i x_i = \theta$, then $\tilde{I}_{A_k}\xi_i = 0$ for each $k \in N$ and each $i \in L$. Since $\{\tilde{I}_{A_k}\xi_i\}_{k\in N}$ converges to $\tilde{I}_L \xi_i$ in probability P for $i \in L$, hence $\tilde{I}_L \xi_i = 0$ for each $i \in L$ which implies ess.sup$(\mathcal{F}_L) \in \mathcal{F}_L$.

Step 2. If L_1, L_2 are two nonempty subsets of N_n and $L_1 \subset L_2$, then it is easy to check that ess.sup$(\mathcal{F}_{L_1})$ almost surely includes ess.sup$(\mathcal{F}_{L_2})$, where $\mathcal{F}_{L_1}$ and $\mathcal{F}_{L_2}$ are defined as in Step 1. Let A_{N_n} =ess.sup$(\mathcal{F}_{N_n})$ and A_H =ess.sup$(\mathcal{F}_H) \setminus \bigcup\{$ ess.sup$(\mathcal{F}_L) \mid H \subsetneqq L \subseteq N_n\}$ for each nonempty and proper subset H of N_n. In this step, we will prove that if $P(A_L) > 0$ for some nonempty subset L of N_n, then as a $\tilde{J}_L L^0(\mathcal{F}, K)$-module, $\tilde{J}_L E$ has a basis $Y \triangleq \{\tilde{J}_L x_i \mid i \in L\}$, i.e. $\tilde{J}_L E$ is a free module of rank $|L|$ over $\tilde{J}_L L^0(\mathcal{F}, K)$. Here J_L denotes the characteristic function of A_L and $|L|$ denotes the cardinal number of L.

Obviously, Y is $\tilde{J}_L L^0(\mathcal{F}, K)$-independent. Now we turn to prove that Y spans $\tilde{J}_L E$. Notice that $\{\tilde{J}_L x_i \mid i \in N_n\}$ spans $\tilde{J}_L E$, thus to complete this step it remains to show that if $L \subsetneqq N_n$, then for any $j \in N_n \setminus L$, $\tilde{J}_L x_j$ can be written as a $\tilde{J}_L L^0(\mathcal{F}, K)$-combination of Y, i.e. there exists $\{\xi_i \in \tilde{J}_L L^0(\mathcal{F}, K) \mid i \in L\}$ such that $\tilde{J}_L x_j = \sum_{i\in L} \xi_i (\tilde{J}_L x_i)$.

Define

$$\mathcal{F}_{x_j} = \{B \in \mathcal{F} \mid B \subseteq A_L \text{ and } \tilde{I}_B x_j \text{ is a } \tilde{J}_L L^0(\mathcal{F}, K) - \text{combination of } Y\}.$$

Also $\mathcal{F}_{x_j}$ is a upward directed subset of $\mathcal{F}$; and there exists a nondecreasing sequence $\{B_k \in \mathcal{F}_{x_j} \mid k \in N\}$ such that ess.sup$(\mathcal{F}_{x_j}) = \bigcup_{k\in N} B_k$.

Let $\{C_k \mid k \in N\}$ be a sequence of $\mathcal{F}$-measurable sets such that $C_0 = B_0$ and $C_k = B_k \setminus B_{k-1}$ for $k \geqslant 1$. Clearly $C_k \in \mathcal{F}_{x_j}$ for each $k \in N$. Thus there exists $\{\xi_i^k \in \tilde{J}_L L^0(\mathcal{F}, K) \mid i \in L\}$ for each $k \in N$ such that $\tilde{I}_{C_k} x_j = \sum_{i\in L} \xi_i^k (\tilde{J}_L x_i)$. Suppose I_{x_j} is the characteristic function of ess.sup$(\mathcal{F}_{x_j})$ and

$\xi_i = \sum_{k\in N} \tilde{I}_{C_k}\xi_i^k$ for each $i \in L$, then

$$\tilde{I}_{x_j}x_j = \sum_{k\in N}\tilde{I}_{C_k}x_j = \sum_{k\in N}\sum_{i\in L}\tilde{J}_L\xi_i^k x_i = \sum_{i\in L}\xi_i(\tilde{J}_L x_i),$$

i.e. ess.sup$(\mathcal{F}_{x_j}) \in \mathcal{F}_{x_j}$.

Obviously ess.sup$(\mathcal{F}_{x_j})$ is almost surely included in A_L. Let B denote the set $A_L\backslash$ess.sup$(\mathcal{F}_{x_j})$. If $P(B) > 0$, then it is easy to verify that $\{\tilde{I}_B x_i \mid i \in L\cup\{j\}\}$ is $\tilde{I}_B L^0(\mathcal{F}, K)$-independent. This contradicts with the chosen of A_L. Thus $A_L =$ess.sup$(\mathcal{F}_{x_j})$, which completes Step 2.

Step 3. Define $A_k = \bigcup\{\text{ess.sup}A_L \mid L \subset N_n \text{ and } |L| = k\}$ for each $k \in N_n$. If $P(A_k) > 0$ for some fixed k, then $\tilde{I}_{A_k}E$ is a free module of rank k over the algebra $\tilde{I}_{A_k}L^0(\mathcal{F}, K)$. In fact, suppose A is an $\mathcal{F}$-measurable set such that $P(A) > 0$ and $\tilde{I}_A E$ is a free module of rank k over $\tilde{I}_A L^0(\mathcal{F}, K)$; $\{y_l \in \tilde{I}_A E \mid 1 \leqslant l \leqslant k\}$ is a basis for the $\tilde{I}_A L^0(\mathcal{F}, K)$-module $\tilde{I}_A E$. Take a subset $L = \{i_1, i_2, \cdots, i_k\}$ of N_n such that $P(A_L) > 0$, where A_L is defined as in step 2. Then $\{\tilde{I}_{A_L}x_l \mid l \in L\}$ is a basis for the $\tilde{I}_{A_L}L^0(\mathcal{F}, K)$-module $\tilde{I}_{A_L}E$. Let J be the characteristic function of $A\cup A_L$; $z_l = \tilde{I}_A y_l + \tilde{I}_{A_L\backslash A}x_{i_l}$, $1 \leqslant l \leqslant k$. It is easy to check that $\{z_1, z_2, \cdots, z_k\}$ is a basis for the $\tilde{J}L^0(\mathcal{F}, K)$-module $\tilde{J}E$, i.e. $\tilde{J}E$ is a free module of rank k over the algebra $\tilde{J}L^0(\mathcal{F}, K)$. Thus the assertion can be proved easily by using the induction method.

Let $A_0 = \Omega \setminus \bigcup_{k\in N_n} A_k$. It is easy to check that $\tilde{I}_{A_0}E = \{\theta\}$. The desired partition can be obtained easily once we prove that $P(A_i \cap A_j) = 0$ when $0 \leqslant i, j \leqslant n$ and $i \neq j$. Suppose $P(A_i \cap A_j) > 0$, $0 \leqslant i, j \leqslant n$. It is easy to verify that $\tilde{I}_{A_i\cap A_j}E$ is a free module of rank i also j over the algebra $\tilde{I}_{A_i\cap A_j}L^0(\mathcal{F}, K)$. Since $\tilde{I}_{A_i\cap A_j}L^0(\mathcal{F}, K)$ is a commutative ring with identity, it follows from [12, Chapter 4, Corollary 2.12] that $\tilde{I}_{A_i\cap A_j}E$ has the invariant dimension property, which implies $i = j$.

And it is easy to check that $E = \bigoplus_{i=0}^n \tilde{I}_{A_i}E$. The proof is completed. □

Remark 3.2. Clearly the partition we obtained in Theorem 3.1 is unique in the sense of P-equivalent. An $\mathcal{F}$-measurable set A is called a support of rank k for E, if $P(A) > 0$ and A is equivalent to A_k.

4. Two corollaries

In this section, we just generalize a couple of results about finite dimensional linear spaces to finitely generated $L^0(\mathcal{F}, K)$-modules.

Corollary 4.1. *An $L^0(\mathcal{F}, K)$-module E is finitely generated iff it is module isomorphic to a submodule of $L^0(\mathcal{F}, K^n)$ with the countable concatenation property for some positive integer n.*

Proof. (1)Necessity: By Theorem 3.1, there exists a positive integer n and a partition $\{A_0, A_1, \cdots, A_n\}$ of Ω to $\mathcal{F}$ such that $\tilde{I}_{A_i}E$ is a free module of rank i over the algebra $\tilde{I}_{A_i}L^0(\mathcal{F}, K)$ for each i which satisfies $0 \leqslant i \leqslant n$ and $P(A_i) > 0$. Let L be the collection of i such that $1 \leqslant i \leqslant n$ and $P(A_i) > 0$. Then for each $i \in L$, there exists a basis $\{x_k^i \in \tilde{I}_{A_i}E \mid 1 \leqslant k \leqslant i\}$ for the free $\tilde{I}_{A_i}L^0(\mathcal{F}, K)$-module $\tilde{I}_{A_i}E$. It follows that for any $x \in E$, there exist unique $\{\xi_k^i \in \tilde{I}_{A_i}L^0(\mathcal{F}, K) \mid i \in L, 1 \leqslant k \leqslant i\}$ such that $x = \sum_{i\in L}\sum_{1\leqslant k\leqslant i}\xi_k^i x_k^i$. For each k such that $1 \leqslant k \leqslant n$ let $\xi_k = \sum_{i\in L, i\geqslant k}\xi_k^i$ if $\{i \in L \mid i \geqslant k\} \neq \emptyset$; otherwise let $\xi_k = 0$. Define $T : E \to L^0(\mathcal{F}, K^n)$ by $T(x) = (\xi_1, \xi_2, \cdots, \xi_n)$. It is easy to check that T is an embedding module homomorphism and $T(E)$ has the countable concatenation property since $T(E)$ is also finitely generated.

(2)Sufficiency: Suppose M is a submodule of $L^0(\mathcal{F}, K^n)$ with the countable concatenation property. For any nonnegative integer k, define

$$\mathcal{F}_k = \{A \in \mathcal{F} \mid \tilde{I}_A M \text{ is a quasi-free statification of rank } k \text{ of } M\}.$$

It is easy to check that $\mathcal{F}_k = \emptyset$ for $k > n$. Let B_k =ess.sup$(\mathcal{F}_k)$ for $0 \leqslant k \leqslant n$ and $\mathcal{F}_k \neq \emptyset$, then $B_k \in \mathcal{F}_k$ since M has the countable concatenation property. Moreover, $P(B_j \cap B_k) = 0$ if $j \neq k$. Now we turn to prove that $P(\bigcup\{B_k \mid 0 \leqslant k \leqslant n \text{ and } \mathcal{F}_k \neq \emptyset\}) = 1$. Suppose this is not the case, i.e. $P(A) > 0$ for $A = \Omega \setminus \bigcup\{B_k \mid 0 \leqslant k \leqslant n \text{ and } \mathcal{F}_k \neq \emptyset\}$. Since $A \notin \mathcal{F}_0$, there exists $x_1 \in \tilde{I}_A M$ and an $\mathcal{F}$-measurable set $A_1 \subset A$ such that $P(A_1) > 0$ and $\xi x_1 = 0$ implies $\xi = 0$ for any $\xi \in \tilde{I}_{A_1}L^0(\mathcal{F}, K)$. Again, since $A_1 \notin \mathcal{F}_1$, there exists $x_2\tilde{I}_A M$ and $A_2 \subset A_1$ such that $P(A_2) > 0$ and $\tilde{I}_{A_2}x_1, \tilde{I}_{A_2}x_2$ is $\tilde{I}_{A_2}L^0(\mathcal{F}, K)$-independent. Consequently, we will obtain an $\mathcal{F}$-measurable set A_{n+1} and $\{x_1, x_2, \cdots, x_{n+1}\}$ such that $P(A_{n+1}) > 0$ and $\{\tilde{I}_{A_{n+1}}x_i \mid 1 \leqslant i \leqslant n+1\}$ is $\tilde{I}_{A_{n+1}}L^0(\mathcal{F}, K)$-independent. This is a contradiction. Thus M is a direct sum of finite quasi stratifications of finite rank, which implies M is finitely generated.□

Remark 4.2. There exist submodules of $L^0(\mathcal{F}, K^n)$ which do not have the countable concatenation property and consequently could not be finitely generated. For example, let $\Omega = [0, 1]$, $\mathcal{F}$ be the collection of all Lebesgue measurable subsets of $[0, 1]$ and P the Lebesgue measure on $[0, 1]$. Suppose $M =$

$\{\tilde{I}_{[2^{-(n+1)},2^{-n}]} \mid n \in N\}$ and $E = \{\sum_{i=1}^{n} \xi_i x_i \mid \xi_i \in L^0(\mathcal{F}, C), x_i \in M, 1 \leqslant i \leqslant n$ and $n \in N\}$. Clearly E is a submodule of $L^0(\mathcal{F}, C)$ without the countable concatenation property. And it is easy to check that E could not be a finitely generated $L^0(\mathcal{F}, C)$-module.

Recall that if X is a proper linear subspace of C^n, then there exists $y \in C^n$ such that $y \neq 0$ and $(x, y) = 0$ for each $x \in X$. Here $(\cdot, \cdot)$ denotes the usual inner product defined on C^n. A proper submodule of $L^0(\mathcal{F}, K^n)$ with the countable concatenation property has a similar property.

Corollary 4.3. *Suppose M is a proper submodule of $L^0(\mathcal{F}, K^n)$ with the countable concatenation property, then there exists $x \in L^0(\mathcal{F}, K^n)$ such that $x \neq 0$ and $\langle x, y\rangle = 0$ for any $y \in M$, where $\langle \cdot, \cdot \rangle$ is defined as in Example 2.4.*

Proof. Notice that M is a proper submodule of $L^0(\mathcal{F}, K^n)$ and has the countable concatenation property, thus there exists i, $0 \leqslant i < n$ such that M has a support A of rank i. Since the case $i = 0$ is trivial, suppose $i > 0$ and $\{z_k \in \tilde{I}_A M \mid 1 \leqslant k \leqslant i\}$ is a basis for the $\tilde{I}_A L^0(\mathcal{F}, K^n)$-module $\tilde{I}_A M$. Now we use Schmidt orthonomal process to obtain an orthornomal basis for $\tilde{I}_A M$.

Let $w_1 = z_1$, $v_1 = \|w_1\|^{-1} w_1$; and $w_k = z_k - \sum_{l=1}^{k-1} \langle z_k, v_l\rangle v_l$, $v_k = \|w_k\|^{-1} w_k$ for $2 \leqslant k \leqslant i$. To prove $\{v_k \mid 1 \leqslant k \leqslant i\}$ is an orthornomal basis for $\tilde{I}_A M$, it only needs to show $\|w_k\| > 0$ on A for $1 \leqslant k \leqslant i$. In fact, let $\|w_k\|^0$ be an arbitrarily chosen representative of $\|w_k\|$ and $B_k = \{\omega \in A \mid \|w_k\|^0(\omega) = 0\}$. Then $\tilde{I}_{B_1} z_1 = \tilde{I}_{B_1} w_1 = 0$ and $\tilde{I}_{B_k} z_k - \tilde{I}_{B_k} \sum_{l=1}^{k-1} \langle z_k, v_l\rangle v_l = \tilde{I}_{B_k} w_k = 0$ for $2 \leqslant k \leqslant i$. Since $\sum_{l=1}^{k-1} \langle z_k, v_l\rangle v_l$ is an $\tilde{I}_A L^0(\mathcal{F}, K)$-combination of $\{z_l\}_{l=1}^{k-1}$ for $2 \leqslant k \leqslant i$, and $\{z_l\}_{l=1}^{k}$ is $\tilde{I}_A L^0(\mathcal{F}, K)$-independent for $1 \leqslant k \leqslant i$, hence $\tilde{I}_{B_k} = 0$, $1 \leqslant k \leqslant i$.

Let x_0 be an element in $L^0(\mathcal{F}, K^n)$ such that $\tilde{I}_A x_0 \notin \tilde{I}_A M$. Define $x = \tilde{I}_A x_0 - \sum_{k=1}^{i} \langle x_0, v_k\rangle v_k$. Clearly $x \neq 0$, $\tilde{I}_{A^c} x = 0$ and $\langle x, z\rangle = 0$ for any $z \in \tilde{I}_A M$. Thus $\langle x, y\rangle = \langle \tilde{I}_A x, \tilde{I}_A y\rangle = 0$ for any $y \in M$. Hence x is the required element.□

References

[1] T.X. Guo, Relations between some basic results derived from two kinds of topologies for a random locally convex module, J. Funct. Anal. 258(2010) 3024–3047.

[2] T.X. Guo, Extension theorems of continuous random linear operators on random domains, J. Math. Anal. Appl. 193(1)(1995) 15–27.

[3] T.X. Guo, S.L. Peng, A characterization for an $L(\mu, K)$-topological module to admit enough canonical module homomorphisms, J. Math. Anal. Appl. 263(2001) 580–599.

[4] T.X. Guo, Some basic theories of random normed linear spaces and random inner product spaces, Acta Anal. Funct. Appl. 1(2)(1999) 160–184.

[5] T.X. Guo, The relation of Banach-Alaoglu theorem and Banach-Bourbaki-Kakutani-Šmulian theorem in complete random normed modules to stratification structure, Sci China(Ser A)51(2008) 1651–1663.

[6] T.X. Guo, H.X. Xiao, X.X. Chen, A basic strict separation theorem in random locally convex modules, Nonlinear Anal. 71(2009) 3794–3804.

[7] T.X. Guo, L.H. Zhu, A characterization of continuous module homomorphisms on random seminormed modules and its applications, Acta Math. Sinica English Ser. 19(1)(2003) 201–208.

[8] T.X. Guo, Representation theorems of the dual of Lebesgue-Bochner function spaces, Sci. China Ser. A-Math. 43(2000) 234–243.

[9] T.X. Guo, Several applications of the theory of random conjugate spaces to measurability problems, Sci. China Ser. A-Math. 50(2007) 737–747.

[10] T.X. Guo, and S.B. Li, The James theorem in complete random normed modules, J. Math. Anal. Appl. 308(2005) 257–265.

[11] D.Filipović, M.Kupper, N.Vogelpoth, Separation and and duality in locally L^0-convex modules, J. Funct. Anal. 256(2009) 3996–4029.

[12] T.W. Hungerford, Algebra, Springer,New York, 1974.

[13] B.Schweizer, A.Sklar, Probabilistic Metric Spaces, Dover Publications, New York, 2005.