BEHAVIOR OF QUILLEN (CO)HOMOLOGY WITH RESPECT TO ADJUNCTIONS

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Abstract. This paper aims to answer the following question: Given an adjunction between two categories, how is Quillen (co)homology in one category related to that in the other? We identify the induced comparison diagram, giving necessary and sufficient conditions for it to arise, and describe the various comparison maps. Examples are given. Along the way, we clarify some categorical assumptions underlying Quillen (co)homology: cocomplete categories with a set of small projective generators provide a convenient setup.

1. Introduction

1.1. **Motivation and goals.** D. Quillen [14, §II.5] introduced a notion of cohomology that makes use of homotopical algebra and simplicial methods to take derived functors in a non-abelian context, generalizing the derived functors of homological algebra. One of the goals was to solve problems in algebra using methods from homotopy theory, although Quillen cohomology later found many applications to homotopy theory and topology [12, Rem 4.35].

Quillen cohomology works in a broad context which includes many interesting categories. The case of commutative algebras, the celebrated André-Quillen cohomology [15, §4] [3] [12, §4.4], was one of the first examples studied. The analogue for associative algebras [15, §3] is related to another well studied theory, namely Hochschild cohomology. Quillen exhibited relations between the two [15, §8], which can be useful when cohomology is easier to compute in one category or the other.

This paper investigates the question of relating Quillen (co)homology in different categories, more specifically when two categories are related by an adjunction. The author's motivating example was to compute some Quillen cohomology groups of truncated Π-algebras controlling the obtructions to realization [7], which is done in section 4.3. However, the broader question seems natural, given that adjoint pairs abound in nature.

1.2. **Organization and results.** Section 2 clarifies the categorical assumptions underlying Quillen cohomology. It consists mostly of category theory, with a short excursion into universal algebra, all for the purposes of homotopical algebra. The main clarifications are propositions 2.32, 2.39, and 2.40. Propositions 2.41 and 2.43 clarify conditions related to Beck modules being abelian.

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Section 3 is the heart of the paper, describing the effect of an adjunction on Quillen (co)homology. We first describe the comparison diagram consisting of Quillen pairs, and work out various comparison maps from it. The main result is 3.8, from which 3.10 and 3.12 follow.

Section 4 studies examples of adjunctions where the right adjoint is the inclusion of a regular-epireflective full subcategory. In other words, the right adjoint forgets certain conditions satisfied by the objects, and the left adjoint is the quotient that freely imposes the conditions. The main results are 4.13 and 4.15.

1.3. **Notations and conventions.**

Definition 1.1. For an object X of C, the category $\mathbf{Mod}_X$ of **Beck modules** over X is the category $Ab(C/X)$ of abelian group objects in the slice category C/X.

Definition 1.2. If the forgetful functor $U_X \colon Ab(C/X) \to C/X$ has a left adjoint $Ab_X \colon C/X \to Ab(C/X)$, it is called **abelianization** over X.

Definition 1.3. For a map $f \colon X \to Y$ in C, the **direct image** functor $f_! \colon C/X \to C/Y$ is postcomposition by f, which is left adjoint to the **pullback** functor $f^* \colon C/Y \to C/X$. Since f^* preserves limits, it induces a functor $f^* \colon Ab(C/Y) \to Ab(C/X)$ also called **pullback**. The **pushforward** by f is the left adjoint $f_* \colon Ab(C/X) \to Ab(C/Y)$ of f^*, if it exists.

Definition 1.4. The **cotangent complex** $\mathbf{L}_X$ of X is derived abelianization of X, i.e. the simplicial module over X given by $\mathbf{L}_X := Ab_X(C_\bullet \to X)$, where $C_\bullet \to X$ is a cofibrant replacement of X in sC, the category of simplicial objects in C.

Definition 1.5. The **Quillen homology** of X is derived functors of abelianization, given by $\mathrm{HQ}_*(X) := \pi_*(\mathbf{L}_X)$. If the category $\mathbf{Mod}_X$ has a good notion of tensor product $\otimes$, then Quillen homology with coefficients in a module M over X is $\mathrm{HQ}_*(X; M) := \pi_*(\mathbf{L}_X \otimes M)$.

Definition 1.6. The **Quillen cohomology** of X with coefficients in a module M is (simplicially) derived functors of derivations, given by $\mathrm{HQ}^*(X; M) := \pi^* \operatorname{Hom}(\mathbf{L}_X, M)$.

Definition 1.7. The **abelian cohomology** of X with coefficients in a module M is derived functors of derivations in the sense of homological algebra, given by $\mathrm{HA}^*(X; M) := \mathrm{Ext}^*(Ab_X X, M)$. The **abelian homology** of X with coefficients in M is $\mathrm{HA}_*(X; M) := \mathrm{Tor}_*(Ab_X X, M)$. They can be viewed as abelian approximations of Quillen (co)homology, with comparison maps $\mathrm{HA}^*(X; M) \to \mathrm{HQ}^*(X; M)$ and $\mathrm{HQ}_*(X; M) \to \mathrm{HA}_*(X; M)$

Remark 1.8. For ease of reading, we often abbreviate the word epimorphism to “epi”, monomorphism to “mono”, isomorphism to “iso”. and weak equivalence to “weak eq”.

2. SETUP FOR QUILLEN (CO)HOMOLOGY

In this section, we study in more detail the categorical assumptions needed in order to work with Quillen cohomology. Most importantly, we want the prolonged adjunction $Ab_X \colon sC/X \rightleftarrows sAb(C/X) \colon U_X$ to be a Quillen pair.

2.1. **Prolonged adjunctions as Quillen pairs.**

Proposition 2.1. *Assume we have an adjunction $L \colon C \rightleftarrows \mathcal{D} \colon R$.*

1. *If R preserves regular epis, then L preserves projectives.*
2. *If, moreover, the category C has finite limits and enough projectives, then the converse holds as well.*

Proof. 1. Let P be a projective in $\mathcal{C}$. We want to show LP is projective in $\mathcal{D}$. Let $f\colon d \to d'$ be any regular epi in $\mathcal{D}$. Then we have:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(LP, d) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{D}}(LP, d') \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{C}}(P, Rd) \arrow[r, "(Rf)_*"'] & \mathrm{Hom}_{\mathcal{C}}(P, Rd').
\end{tikzcd}$$

By assumption, $Rf\colon Rd \to Rd'$ is a regular epi in $\mathcal{C}$, and P is projective, hence the bottom (and top) map is a surjection. Thus LP is projective.

2. Under the additional hypotheses, regular epis and projectives determine each other. Indeed, [14, §II.4, Prop 2] asserts that $f\colon c \to c'$ is a regular epi iff the map

$$f_*\colon\ \mathrm{Hom}(P, c) \to \mathrm{Hom}(P, c')$$

is a surjection for all projective P. Now we start with a regular epi $f\colon d \to d'$ in $\mathcal{D}$ and want to show $Rf\colon Rd \to Rd'$ is a regular epi in $\mathcal{C}$. Let P be any projective in $\mathcal{C}$ and consider:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(P, Rd) \arrow[r, "(Rf)_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{C}}(P, Rd') \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{D}}(LP, d) \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathcal{D}}(LP, d').
\end{tikzcd}$$

By assumption, LP is projective and f is a regular epi, hence the bottom (and top) map is a surjection. Thus, by the criterion given above, Rf is a regular epi. □

Proposition 2.2. *Assume $\mathcal{C}$ and $\mathcal{D}$ have finite limits and enough projectives, and satisfy extra assumptions so that Quillen's theorem 4 applies (e.g. they are cocomplete and have sets of small projective generators). Assume we have an adjunction as above, and hence an induced adjunction*

$$s\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} s\mathcal{D} \tag{1}$$

between model categories. If L preserves projectives, or equivalently, if R preserves regular epis, then this is a Quillen pair.

Proof. We show a slightly stronger statement: R preserves fibrations and weak equivalences. Recall that a map $f\colon X_\bullet \to Y_\bullet$ is a fibration (resp. weak eq) if the induced map $f_*\colon \mathrm{Hom}(P, X_\bullet) \to \mathrm{Hom}(P, Y_\bullet)$ is a fibration (resp. weak eq) of simplicial sets for all projective P. Take P a projective in $\mathcal{C}$ and consider:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(P, RX_\bullet) \arrow[r, "(Rf)_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{C}}(P, RY_\bullet) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{D}}(LP, X_\bullet) \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathcal{D}}(LP, Y_\bullet).
\end{tikzcd}$$

By assumption, LP is projective in $\mathcal{D}$ and f is a fibration (resp. weak eq) in $s\mathcal{D}$, hence the bottom and top maps are fibrations (resp. weak eq) of simplicial sets. Thus $Rf\colon RX_\bullet \to RY_\bullet$ is a fibration (resp. weak eq). □

Proposition 2.3. *The converse also holds: If the prolonged adjunction (1) is a Quillen pair, then R preserves regular epis.*

Proof. Take a regular epi $f\colon X \to Y$ in $\mathcal{D}$ and view it as a map between constant simplicial objects in $s\mathcal{D}$. Factoring it as a cofibration - acyclic fibration [14, §II.4, Prop 3], we produce an acyclic fibration $f_\bullet\colon X_\bullet \to Y_\bullet$ in $s\mathcal{D}$ satisfying $X_0 = X$, $Y_0 = Y$, and $f_0 = f$. Since R prolongs to a right Quillen functor, $Rf_\bullet$ is an acyclic fibration in $s\mathcal{C}$, and hence a regular epi in each level. In particular, $Rf = Rf_0$ is a regular epi in $\mathcal{C}$. □

Remark 2.4. We've seen that a prolonged right Quillen functor in 2.2 is particularly strong: it preserves fibrations and *all* weak equivalences, not just between fibrant objects. However, the prolonged left Quillen functor does not enjoy this additional property in general, i.e. it need not preserve all weak equivalences, only those between cofibrant objects.

Example 2.5. Let R be a commutative ring and consider the functor $R \otimes -$ from abelian groups to R-modules. It preserves projectives (i.e. sends a free abelian group to a free R-module), but the prolonged left Quillen functor does not preserve all weak equivalences if R is not flat over $\mathbb{Z}$.

2.2. **Slice categories.**

Proposition 2.2 gives a simple criterion for when a prolonged adjunction is a Quillen pair. We want to know if the induced adjunction on slice categories is also a Quillen pair. Let us first describe regular epis and projectives in the slice category. A map in $\mathcal{C}/X$ is a regular epi iff the map of total spaces is, and an object of $\mathcal{C}/X$ is projective iff the total space is.

Proposition 2.6. *If $f\colon Y \to Z$ is a regular epi in $\mathcal{C}$, then*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd}$$

is a regular epi in $\mathcal{C}/X$. The converse also holds if $\mathcal{C}$ has coequalizers.

Proof. See [4, Chap 1, Prop 8.12]. It follows from the fact that the "source" functor $\mathcal{C}/X \to \mathcal{C}$ creates colimits. □

Proposition 2.7. *1. If P is projective in $\mathcal{C}$, then $p\colon P \to X$ is projective in $\mathcal{C}/X$.*
2. The converse also holds if $\mathcal{C}$ has enough projectives.

Proof. 1. Start with a regular epi

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "y"'] & Z \arrow[d, "z"] \\ & X \end{tikzcd}$$

in $\mathcal{C}/X$, which means $f\colon Y \to Z$ is a regular epi in $\mathcal{C}$, by 2.6. We want to know if the map

$$f_*\colon \operatorname{Hom}_{\mathcal{C}/X}(P \overset{p}{\to} X, Y \overset{y}{\to} X) \to \operatorname{Hom}_{\mathcal{C}/X}(P \overset{p}{\to} X, Z \overset{z}{\to} X)$$

is surjective. Let α be a map in the right-hand side which we are trying to reach and consider the diagram:

$$\begin{tikzcd}
& Y \arrow[r, two heads, "f"] \arrow[dd, "y"] & Z \arrow[ddl, "z"] \\
P \arrow[ur, dashed, "\widetilde{\alpha}"] \arrow[urr, "\alpha"] \arrow[dr, "p"'] & & \\
& X &
\end{tikzcd}$$

Since P is projective in C, there is a lift $\widetilde{\alpha}$ in the top triangle, meaning $f\widetilde{\alpha} = \alpha$. If $\widetilde{\alpha}$ is in fact a map in $\mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y \xrightarrow{y} X)$, then it will be our desired lift. So it suffices to check that the triangle on the left commutes: $y\widetilde{\alpha} = zf\widetilde{\alpha} = z\alpha = p$.

2. Let $E \xrightarrow{e} X$ be projective in C/X. Since C has enough projectives, pick a regular epi $\pi \colon P \to E$ from a projective P. Consider the diagram

$$\begin{tikzcd}
& P \arrow[r, two heads, "\pi"] \arrow[dd, "e\pi"] & E \arrow[ddl, "e"] \\
E \arrow[ur, dashed, "s"] \arrow[urr, "\mathrm{id}"] \arrow[dr, "e"'] & & \\
& X &
\end{tikzcd}$$

where there exists a lift s since $E \xrightarrow{e} X$ is projective in C/X. The relation $\pi s = \mathrm{id}_E$ exhibits E as a retract of a projective in C, hence itself projective. □

Now we can describe the standard Quillen model structure on $s(C/X) \cong sC/X$. A map

$$\begin{tikzcd}
Y_\bullet \arrow[r, "f"] \arrow[dr, "y"'] & Z_\bullet \arrow[d, "z"] \\
& X
\end{tikzcd} \tag{2}$$

is a fibration (resp. weak eq) in $s(C/X)$ iff the map

$$\mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y_\bullet \xrightarrow{y} X) \xrightarrow{f_*} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Z_\bullet \xrightarrow{z} X)$$

is a fibration (resp. weak eq) of simplicial sets for all projective $P \xrightarrow{p} X$ in C/X. By proposition 2.7, we can rephrase the latter as: for all projective P in C and map $p \in \mathrm{Hom}_C(P, X)$.

However, in the framework of Quillen (co)homology, we decided to work with the "slice" model structure on sC/X, where the map (2) is a fibration (resp. weak eq) iff the map $f_* \colon \mathrm{Hom}_C(P, Y_\bullet) \to \mathrm{Hom}_C(P, Z_\bullet)$ is a fibration (resp. weak eq) of simplicial sets for all projective P in C. In fact, let us check that the two model structures agree.

Proposition 2.8. *There is a natural iso of simplicial sets:*

$$\coprod_{p \in \mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y_\bullet \xrightarrow{y} X) \xrightarrow{\cong} \mathrm{Hom}_C(P, Y_\bullet).$$

Proof. Idea : For a fixed $y \colon Y \to X$, the data of a map $g \colon P \to Y$ is the same as the data of the commutative diagram:

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[dr, "p"'] & Y \arrow[d, "y"] \\ & X \end{tikzcd}$$

and thus we can partition all maps $g \colon P \to Y$ according to their composite $p = yg \colon P \to X$. More precisely, we take the map

$$\coprod_{p \in \mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \overset{p}{\to} X, Y \overset{y}{\to} X) \to \mathrm{Hom}_C(P, Y)$$

which is readily seen to be surjective and injective, i.e. an iso of sets. Moreover, it is natural in $y \colon Y \to X$, i.e. the two sides define two naturally isomorphic functors from C/X to **Set**. By naturality, it prolongs to a natural iso of simplicial sets. Since colimits of simplicial objects are computed levelwise, the simplicial set whose n^{th} level is

$$\left\{ \coprod_{p \in \mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \overset{p}{\to} X, Y_n \xrightarrow{y_n} X) \right\}_n$$

equals the left-hand side in the statement. □

Proposition 2.9. *The standard model structures on $s(C/X)$ and sC/X are the same.*

Proof. The top row in the diagram

$$\begin{tikzcd} \mathrm{Hom}_C(P, Y_\bullet) \arrow[r, "f_*"] \arrow[d, equal] & \mathrm{Hom}_C(P, Z_\bullet) \arrow[d, equal] \\ \coprod_p \mathrm{Hom}_{C/X}(P \overset{p}{\to} X, Y_\bullet \overset{y}{\to} X) \arrow[r, "f_*"'] & \coprod_p \mathrm{Hom}_{C/X}(P \overset{p}{\to} X, Z_\bullet \overset{z}{\to} X) \end{tikzcd}$$

is a fibration (resp. weak eq) of simplicial sets iff each summand is so. This means f is a fibration (resp. weak eq) in sC/X iff it is so in $s(C/X)$. Moreover, the model structures are closed i.e. cofibrations are determined by fibrations and weak equivalences (as having the LLP with respect to acyclic fibrations). Therefore the two model structures agree. □

2.3. **Abelian group objects.** In this section, we study the properties of the category $Ab(C)$ of abelian group objects in a category C and the forgetful functor $U \colon Ab(C) \to C$.

It is convenient to work with regular categories, so we would like to know if $Ab(C)$ is regular whenever C is. The main feature of regular categories is that any map can be factored as a regular epi followed by mono; isos are precisely maps that are both a regular epi and a mono. We will check that all three classes of maps are preserved and reflected by U.

First, recall that U is faithful, it and creates limits, and it reflects isos: if Uf is an iso, then $(Uf)^{-1}$ lifts to $Ab(C)$.

Proposition 2.10. *Assume C has kernel pairs. Then U preserves monos.*

Proof. In a category with kernel pairs, a map $f \colon X \to Y$ is a mono iff the two projections $X \times_Y X \rightrightarrows X$ from its kernel pair are equal. Thus, any functor between categories with kernel pairs which preserves kernel pairs also preserves monos. □

In [4, Chap 6, Prop 1.7], M. Barr shows the following.

Proposition 2.11. *Assume C is regular. Then U lifts the regular epi - mono factorization in C. In other words, if $f\colon X \to Y$ is a map in $Ab(C)$ and $UX \twoheadrightarrow UZ \hookrightarrow UY$ is a regular epi - mono factorization of the underlying map Uf, then we can lift it (uniquely) to a factorization $X \to Z \to Y$ in $Ab(C)$.*

Corollary 2.12. *If C is regular, then U preserves regular epis.*

In addition, we'd like to know if U reflects regular epis.

Proposition 2.13. *If C is regular, then $Ab(C)$ has coequalizers of kernel pairs, created by U.*

Proof. Let $f\colon X \to Y$ be any map in $Ab(C)$ and take its kernel pair $X \times_Y X \rightrightarrows X$. Since U preserves limits, the underlying diagram is still a kernel pair, and we can take its coequalizer:

$$\begin{tikzcd}
UX \times_{UY} UX \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & UX \arrow[r, "Uf"] \arrow[dr, two heads] & UY \\
 & & C. \arrow[u, "h"']
\end{tikzcd}$$

Since C is regular, the map $h\colon C \to Y$ is a mono [4, Chap 1, Prop 8.10]. By 2.11, there is a unique lift $X \to \widetilde{C} \to Y$ of that regular epi - mono factorization. One can check that $X \to \widetilde{C}$ is the desired coequalizer in $Ab(C)$ of the kernel pair of f. ◻

Proposition 2.14. *If C is regular, then U reflects regular epis.*

Proof. Let $f\colon X \to Y$ be a map in $Ab(C)$ such that Uf is a regular epi in C. We want to show that f is a regular epi. Since U creates limits, the kernel pair of f is the unique lift of the kernel pair $UX \times_{UY} UX \rightrightarrows UX$ of Uf, and the latter has a coequalizer, namely $Uf\colon UX \to UY$. Since U creates coequalizers of kernel pairs, there is a unique cocone lifting $Uf\colon UX \to UY$ and it is a coequalizer of $X \times_Y X \rightrightarrows X$. But $f\colon X \to Y$ is such a lift, hence f is a regular epi. ◻

Corollary 2.15. *The lifted factorization of 2.11 is a regular epi - mono factorization in $Ab(C)$.*

Corollary 2.16. *If C is regular, then $Ab(C)$ is regular.*

Proof. $Ab(C)$ has kernel pairs (or any limits that C has) and coequalizers of kernel pairs. It remains to check that the pullback of a regular epi is a regular epi:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "f^*e"'] & X \arrow[d, two heads, "e"] \\
W \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Since U preserves regular epis, Ue is a regular epi. Since pullbacks are computed in C, we have $U(f^*e) = (Uf)^*(Ue)$, which is a regular epi since C is regular. Since U reflects regular epis, f^*e itself is a regular epi in $Ab(C)$. ◻

Now that we've discussed regularity, let us discuss more general colimits in $Ab(C)$; it will become useful later. Recall a few definitions [8, Def 4.1.1, 4.1.2].

Definition 2.17. A **subobject** of an object X in a category C is an equivalence class of monomorphisms $Z \hookrightarrow X$, up to isomorphism over X. The equivalence class of $Z \hookrightarrow X$ is denoted $[Z \hookrightarrow X]$. The collection of subobjects of X is denoted $\mathrm{Sub}(X)$.

Definition 2.18. A category C is **well-powered** if the subobjects of any object form a set.

Lemma 2.19. *If C has finite limits and is well-powered, then $Ab(C)$ is well-powered.*

Proof. For any object X of $Ab(C)$, the functor $U \colon Ab(C) \to C$ induces a map

$$U_* \colon \operatorname{Sub}(X) \to \operatorname{Sub}(UX)$$

where the right-hand side is a set and the left-hand side is (a priori) a collection. Let us show that U_* is injective.

Let $[A \hookrightarrow X]$ and $[B \hookrightarrow X]$ be subobjects of X satisfying $U_*[A \hookrightarrow X] = U_*[B \hookrightarrow X] = [Z \hookrightarrow UX]$, which means there is a diagram in C like such:

$$\begin{tikzcd}
UA \arrow[d, "\varphi"', "\simeq"] \arrow[dr, hook] & \\
Z \arrow[r, hook] & UX. \\
UB \arrow[u, "\psi", "\simeq"'] \arrow[ur, hook] &
\end{tikzcd}$$

One can check that $\psi^{-1}\varphi$ commutes with the structure maps; it follows from the fact that the structure maps of A and B are restricted from those of X. Hence $\psi^{-1}\varphi$ lifts to an iso in $Ab(C)$ and we have $[A \hookrightarrow X] = [B \hookrightarrow X]$. ◻

Lemma 2.20. *If C is complete and well-powered, then one can form the equivalence relation (or effective equivalence relation) generated by a set of relations on an object X. More generally, if C has a limit-preserving functor U to some category $\mathcal{S}$ (sometimes* **Set***), the same conclusion holds with relations (or pseudorelations) on the underlying object UX.*

Proof. Let $\{\rho_i\}$ be a set of relations on UX, i.e. each ρ_i is a subobject of $UX \times UX$. The notion of being an equivalence relation is a well defined property for subojects $[R \hookrightarrow X \times X]$. So is the property of "containing" the relation ρ_i, meaning there is a factorization $\rho_i \to UR \hookrightarrow UX \times UX$; we write $\rho_i \leq UR$ when this happens. Consider the collection

$$\mathcal{R} := \{R \in \operatorname{Sub}(X \times X) \mid R \text{ is an equivalence relation and } \rho_i \leq UR \text{ for all } i\}$$

which is a set since C is well-powered. Take the intersection $\bigcap R$ of all relations in $\mathcal{R}$, which is the limit of the diagram like such:

$$\begin{tikzcd}
R_{i_1} \arrow[dr, hook] & R_{i_2} \arrow[d, hook] & R_{i_3} \arrow[dl, hook] & \cdots \\
& X \times X & &
\end{tikzcd}$$

and hence exists, by completeness of C. The intersection is still an equivalence relation, and still contains all ρ_i (since U preserves limits). By construction, it is the desired equivalence relation generated by all ρ_i.

The analogous proof for effective equivalence relations works as well. Indeed, an intersection of effective equivalence relations, i.e. kernel pairs of maps $f_j \colon X \to Y_j$, is the kernel pair of the map $(f_j) \colon X \to \prod Y_j$.

Note that in both cases, the collection $\mathcal{R}$ is non-empty, as it contains the terminal equivalence relation $\mathrm{id} \colon X \times X \hookrightarrow X \times X$, which is the kernel pair of the map $X \to *$ to the terminal object. ◻

Proposition 2.21. *If C is complete, well-powered, and regular, then C has coequalizers (of parallel pairs).*

Proof. Let $f, g \colon X \rightrightarrows Y$ be two maps, which we view as a pseudorelation on Y, i.e. a map $(f, g) \colon X \to Y \times Y$. Since C is regular, factor it as a regular epi followed by a mono $X \twoheadrightarrow R \hookrightarrow Y \times Y$. Let $\overline{R}$ be the effective equivalence relation on Y generated by R (lemma 2.20). Then $\overline{R} \rightrightarrows Y$ has a coequalizer $Y \to C$, since C is regular. One readily checks that $Y \to C$ is also the coequalizer of $X \rightrightarrows Y$. □

Corollary 2.22. *Under the same assumptions, $Ab(C)$ also has coequalizers of parallel pairs.*

Proof. $Ab(C)$ is also complete (since $U \colon Ab(C) \to C$ creates limits), well-powered (by 2.19), and regular (by 2.16). □

2.4. **Algebraic categories.** In the classic [14, §II.4, Thm 4], Quillen introduces a standard simplicial model structure on the category sC of simplicial objects in a category C, assuming C is nice enough. For example, the theorem applies when C has finite limits, all (small) colimits, and a set of small projective generators (in particular, enough projectives). This leads us to the following definition.

Definition 2.23. A category is called *algebraic* if it is cocomplete and has a set of small projective generators.

Remark 2.24. The word "algebraic" is overused, and we are *not* using it as in [9, §3.4], namely Lawvere's models of algebraic theories. The difference is that our algebraic categories are not necessarily exact. Note that our algebraic categories are locally finitely presentable in the sense of [9, Def 5.2.1].

Algebraic categories have excellent properties: they are complete (by [1, Cor 2.12], using the fact that any set of small objects with coproducts is abstractly finite), well-powered [8, Prop 4.5.15], and regular [14, §II.4, Cor after Prop 2].

Our goal is to show that algebraic categories provide a good setup for Quillen cohomology in the following sense: abelianizations and pushforwards exist, and the abelianization adjunction is a Quillen pair.

2.4.1. *Slice categories are algebraic.*

Proposition 2.25. *Let C be an algebraic category with generator set $\mathcal{S}$ and let $f \colon X \to Y$ be a map in C.*

1. *f is a mono iff $f_* \colon$ $\mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is a injective (i.e. a mono in* **Set***) for all $P \in \mathcal{S}$.*
2. *f is a regular epi iff $f_* \colon$ $\mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is surjective (i.e. a regular epi in* **Set***) for all $P \in \mathcal{S}$.*

In particular, the family of functors $\mathrm{Hom}(P, -)$ (for all $P \in \mathcal{S}$) collectively reflects isos, as shown in [8, Prop 4.5.10].

Proof. Straightforward, using the fact that any object A of C receives a regular epi $\pi \colon \coprod P_i \twoheadrightarrow A$ from a coproduct of generators $P_i \in \mathcal{S}$. □

Proposition 2.26. *In an algebraic category C, filtered colimits commute with finite limits.*

Proof. Let L be a filtered category, N a finite category, and $F\colon L \times N \to C$ a functor. There is a natural comparison map $\varphi\colon\ \operatorname{colim}_L \lim_N F \to \lim_N \operatorname{colim}_L F$ which we want to show is an iso. By 2.25, it suffices to show $\operatorname{Hom}(P, \varphi)$ is an iso (of sets) for all generator P. From the definition of limit and the smallness of the generators, we obtain:

$$\begin{tikzcd}
\operatorname{Hom}(P, \operatorname{colim}_L \lim_N F) \arrow[r, "\varphi_*"] \arrow[d, equal] & \operatorname{Hom}(P, \lim_N \operatorname{colim}_L F) \arrow[d, equal] \\
\operatorname{colim}_L \operatorname{Hom}(P, \lim_N F) \arrow[d, equal] & \lim_N \operatorname{Hom}(P, \operatorname{colim}_L F) \arrow[d, equal] \\
\operatorname{colim}_L \lim_N \operatorname{Hom}(P, F) \arrow[r, "\cong"] & \lim_N \operatorname{colim}_L \operatorname{Hom}(P, F).
\end{tikzcd}$$

The bottom map (and hence φ_*) is an iso, since filtered colimits commute with finite limits in **Set**. □

Proposition 2.27. *Let C be an algebraic category. Then $U\colon Ab(C) \to C$ creates filtered colimits. In particular, $Ab(C)$ has filtered colimits and U preserves them.*

Proof. Essentially the same reason U creates limits. Let L be a filtered category and $F\colon L \to Ab(C)$ a diagram whose underlying diagram $UF\colon L \to C$ admits a colimit. Then there is a unique lift of the colimiting cocone in C to a cocone in $Ab(C)$. Indeed, there is at most one way to endow $\operatorname{colim}_L UF$ with structure maps, since they are prescribed on each summand:

$$\begin{tikzcd}
\operatorname{colim}_L(UF \times UF) \cong \operatorname{colim}_L UF \times \operatorname{colim}_L UF \arrow[r] & \operatorname{colim}_L UF \\
UF(l) \times UF(l) \arrow[u] \arrow[r, "\mu"'] & UF(l). \arrow[u]
\end{tikzcd}$$

Applying colim_L to the structure maps of UF produces those structure maps for $\operatorname{colim}_L UF$. The result is the colimit of F in $Ab(C)$. □

Proposition 2.28. *Let C be an algebraic category and X an object of C. Then the slice category C/X is algebraic.*

Proof. 1. C/X has small colimits, since they are created by the "source" functor $C/X \to C$.

2. Let S be a set of small projective generators for C. Then

$$\left\{ P \xrightarrow{p} X \mid P \in S, p \in \operatorname{Hom}_C(P, X) \right\}$$

is a set of small projective generators for C/X. Smallness is a straightforward verification; the rest follows from 2.7, 2.6, and the fact that $(\amalg P_i) \to X$ is the coproduct $\amalg(P_i \to X)$ in C/X. (By the same argument, if C has enough projectives, then so does C/X.) □

2.4.2. *Abelianizations exist.* To show that an algebraic category has abelianizations, we venture into universal algebra. By a characterization theorem [2, Thm 5.2], every algebraic category is equivalent to a many-sorted finitary quasivariety. That is, a category where objects have an underlying graded set indexed by some set S of "sorts", equipped with some operations, and satisfying some equations and implications [2, §1.1].

More precisely, let Σ be a (many-sorted) signature, a set of finitary operations with the data of their (many-sorted) arities. Let $\Sigma\mathbf{Alg}$ denote the category of Σ-algebras: objects are S-graded sets equipped with operations prescribed by Σ, and morphisms are Σ-homomorphisms, i.e. maps of graded sets that respect all the operations. It is known that $\Sigma\mathbf{Alg}$ is complete and cocomplete, the forgetful functor $\Sigma\mathbf{Alg} \to \mathbf{Set}^S$ creates limits and filtered colimits, and it has a left adjoint F_Σ, which freely adjoins the operations. A variety (resp. quasivariety) $\mathcal{K}$ is a full subcategory of $\Sigma\mathbf{Alg}$ whose objects are precisely those satisfying a given set of equations (resp. equations and implications).

Example 2.29. The (one-sorted) variety of abelian groups is the full subcategory of $\{e, \iota, \mu\}$-algebras satisfying the usual equations for the neutral element e, inverse ι, and addition μ, with arities 0, 1, and 2 respectively. The quasivariety of torsion-free abelian groups is defined by the additional implications $(nx = 0 \Rightarrow x = 0)$ for all $n \in \mathbb{N}$. Likewise, commutative rings form a (one-sorted) variety, while reduced commutative rings, i.e. those without nilpotents, form a quasivariety defined by the additional implications $(x^n = 0 \Rightarrow x = 0)$ for all $n \in \mathbb{N}$.

The inclusion $I_{\mathcal{K}} \colon \mathcal{K} \to \Sigma\mathbf{Alg}$ has a left adjoint $\pi_{\mathcal{K}} \colon \Sigma\mathbf{Alg} \to \mathcal{K}$, which is essentially quotienting by all the equations and implications that define $\mathcal{K}$. The unit maps are regular epis and the counit maps are isomorphisms. In particular, $\mathcal{K}$ is cocomplete.

Lemma 2.30. *If $\mathcal{C}$ is a variety (resp. quasivariety), then so is $Ab(\mathcal{C})$*

Proof. Let Σ be the signature of $\mathcal{C}$. Objects of $Ab(\mathcal{C})$ have the underlying graded set of their underlying object in $\mathcal{C}$, equipped with the additional structure maps e, ι, μ, satisfying the conditions of associativity and so on, and the conditions that the structure maps be maps in $\mathcal{C}$. Thus $Ab(\mathcal{C})$ is the full subcategory of $\Sigma'\mathbf{Alg}$ satisfying the equations and implications that define $\mathcal{C}$, plus an additional set of equations. Here Σ' is the signature $\Sigma \sqcup \{e, \iota, \mu\}$ where the additional operations have arities 0, 1, and 2 respectively. In the many-sorted case, Σ' is $\Sigma \sqcup \{e_s, \iota_s, \mu_s\}_{s \in S}$ where the additional operations have arities $(\emptyset; s)$, $(s; s)$, and $(s, s; s)$ respectively. ◻

Proposition 2.31. *If $\mathcal{C}$ is algebraic, then $U \colon Ab(\mathcal{C}) \to \mathcal{C}$ has a left adjoint.*

Proof. Let us forget the equations defining $\mathcal{C}$ and $Ab(\mathcal{C})$ while keeping all the structure. In other words, consider the diagram

$$\begin{tikzcd}
Ab(\mathcal{C}) \arrow[r, "U"] \arrow[d, shift left, "I_{Ab(\mathcal{C})}"] & \mathcal{C} \arrow[d, shift left, "I_{\mathcal{C}}"] \\
\Sigma'\mathbf{Alg} \arrow[u, shift left, "\pi_{Ab(\mathcal{C})}"] \arrow[r, shift left, "U_{\Sigma',\Sigma}"] & \Sigma\mathbf{Alg} \arrow[u, shift left, "\pi_{\mathcal{C}}"] \arrow[l, shift left, "F_{\Sigma,\Sigma'}"]
\end{tikzcd}$$

where we have adjoint pairs on the left, bottom, and right sides. The right adjoints commute. Let us check that the obvious candidate $\pi_{Ab(\mathcal{C})} F_{\Sigma,\Sigma'} I_{\mathcal{C}}$ is in fact left adjoint to U. For X in $\mathcal{C}$ and B in $Ab(\mathcal{C})$, we have

$$\begin{aligned}
\mathrm{Hom}_{Ab(\mathcal{C})}\left(\pi_{Ab(\mathcal{C})} F_{\Sigma,\Sigma'} I_{\mathcal{C}} X, B\right) &\cong \mathrm{Hom}_{\Sigma\mathbf{Alg}}\left(I_{\mathcal{C}} X, U_{\Sigma',\Sigma} I_{Ab(\mathcal{C})} B\right) \\
&= \mathrm{Hom}_{\Sigma\mathbf{Alg}}\left(I_{\mathcal{C}} X, I_{\mathcal{C}} U B\right) \\
&= \mathrm{Hom}_{\mathcal{C}}\left(X, UB\right)
\end{aligned}$$

since $\mathcal{C}$ is a full subcategory of $\Sigma\mathbf{Alg}$. ◻

From 2.28 and 2.31, we obtain the following.

Corollary 2.32. *An algebraic category C has all abelianizations $Ab_X \colon C/X \to Ab(C/X)$.*

2.4.3. *Beck modules are algebraic.* Since we want to put the standard model structure on the category $sAb(C/X)$ of simplicial Beck modules, we'd like to know that $Ab(C/X)$ is itself algebraic. By 2.28, it suffices to show that if C is algebraic, then so is $Ab(C)$.

Lemma 2.33. *Assume C is regular and $U \colon Ab(C) \to C$ has a left adjoint. If a map $f \colon X \twoheadrightarrow UB$ is a regular epi in C, then its adjunct map $f^\sharp \colon AbX \to B$ is a regular epi in $Ab(C)$. In particular, the counit $AbUA \twoheadrightarrow A$ is always a regular epi.*

Proof. Recall that $AbX \to B$ is a regular epi in $Ab(C)$ iff $UAbX \to UB$ is a regular epi in C. The regular epi f factors as $f = (Uf^\sharp) \circ \eta_X \colon X \to UAbX \to UB$, which implies $Uf^\sharp$ is a regular epi since C is regular [8, Cor 2.1.5 (2)]. ◻

Remark 2.34. The converse is false in general. For example, take $C = \mathbf{Set}$, $X = \{*\}$, $Y = \mathbb{Z}$, and $f(*) = 1$. The map f is far from being a regular epi (i.e. surjection), but its adjunct $f^\sharp \colon Ab(*) = \mathbb{Z} \to \mathbb{Z}$ is a regular epi, even an iso.

Lemma 2.35. *Assume C is regular and has enough projectives, and $U \colon Ab(C) \to C$ has a left adjoint. Then an object of $Ab(C)$ is projective iff it is a retract of AbP for some projective P of C.*

Proof. ($\Leftarrow$) Trying to lift a map $Ab(P) \to B$ along a regular epi $A \twoheadrightarrow B$ is the same as trying to lift the adjunct map:

$$\begin{tikzcd}
 & P \arrow[dl, dashed] \arrow[d] \\
UA \arrow[r, two heads] & UB.
\end{tikzcd}$$

The bottom map is a regular epi since U preserves them, and thus the lift exists. Therefore $Ab(P)$ is projective, and a retract of a projective is projective.

($\Rightarrow$) Let Q be a projective in $Ab(C)$. Since C has enough projectives, there is a projective P of C with a regular epi $P \twoheadrightarrow UQ$. Take its adjunct map $AbP \twoheadrightarrow Q$, which is still a regular epi by 2.33. Lifting the identity of Q along that regular epi exhibits Q as a retract of AbP. ◻

Proposition 2.36. *If C is algebraic, then $Ab(C)$ is also algebraic.*

Proof. 1. $Ab(C)$ is cocomplete since it is a quasivariety (2.30).

2. Let S be a set of small projective generators for C. Then $\{Ab(P) \mid P \in S\}$ is a set of small projective generators for $Ab(C)$. Smallness is a straightforward verification, using 2.27. Each $Ab(P)$ is projective, by 2.35. Let us show that they form a family of generators. For any object X of $Ab(C)$, take a regular epi $\amalg P_i \twoheadrightarrow UX$ from a coproduct of generators in S. Then the adjunct map $\amalg Ab(P_i) = Ab(\amalg P_i) \twoheadrightarrow X$ is a regular epi. (By the same argument, if C has enough projectives, then so does $Ab(C)$.) ◻

It would be worthwhile to know under which assumptions does cocompleteness of C guarantee cocompleteness of $Ab(C)$. One may want to avoid the universal-algebraic argument used in the proof of 2.36.

Proposition 2.37. *Assume C is cocomplete and $U \colon Ab(C) \to C$ has a left adjoint. If $Ab(C)$ has coequalizers, then $Ab(C)$ is cocomplete.*

Proof. Using Beck's monadicity theorem, one can show that U is monadic [13, §VI.8, Thm 1]. The result follows from [9, Prop 4.3.4]. ◻

From 2.22, we obtain the following.

Corollary 2.38. *Assume $\mathcal{C}$ is cocomplete and $U\colon Ab(\mathcal{C}) \to \mathcal{C}$ has a left adjoint. If moreover $\mathcal{C}$ is complete, well-powered, and regular, then $Ab(\mathcal{C})$ is cocomplete.*

Note that in the case of algebraic categories, we did use the universal-algebraic argument in 2.31 to show that U has a left adjoint.

2.4.4. *Pushforwards exist.*

Proposition 2.39. *Let $\mathcal{C}$ be an algebraic category. Then $\mathcal{C}$ has all pushforwards.*

Proof. Let $f\colon X \to Y$ be a map in $\mathcal{C}$. Consider the diagram

$$\begin{tikzcd}
Ab(\mathcal{C}/Y) \arrow[r, "f^*"] \arrow[d, shift left, "U_Y"] & Ab(\mathcal{C}/X) \arrow[d, shift left, "U_X"] \\
\mathcal{C}/Y \arrow[u, shift left, "Ab_Y"] \arrow[r, shift left, "f^*"] & \mathcal{C}/X \arrow[u, shift left, "Ab_X"] \arrow[l, shift left, "f_!"]
\end{tikzcd}$$

where the abelianizations exist by 2.32. The right adjoints commute. Starting from a Beck module M in $Ab(\mathcal{C}/X)$, one naive candidate would be $Ab_Y f_! U_X M$, which is much too big for our purposes. However, we can trim it down to the right size by modding out some relations. More precisely, we find a quotient of $Ab_Y f_! U_X M$ which satisfies the solution set condition of the adjoint functor theorem [13, §V.6, Thm 2].

We do *not* have a map $M \to f^* Ab_Y f_! U_X M$, although we DO have a map of underlying objects

$$\eta\colon U_X M \to U_X f^* Ab_Y f_! U_X M = (f^* U_Y)(Ab_Y f_!) U_X M \tag{3}$$

in $\mathcal{C}/X$, which is the unit of the adjunction $Ab_Y f_! \dashv f^* U_Y$. Let $q\colon Ab_Y f_! U_X M \to Q$ be the closest quotient in $Ab(\mathcal{C}/Y)$ which makes the map (3) lift to $Ab(\mathcal{C}/X)$, i.e. we have a diagram in $Ab(\mathcal{C}/X)$

$$\begin{tikzcd}
M \arrow[r, dashed] \arrow[rr, bend right, "\widetilde{\eta}"'] & f^* Ab_Y f_! U_X M \arrow[r, "f^* q"] & f^* Q
\end{tikzcd}$$

satisfying $U_X \widetilde{\eta} = (U_X f^* q) \circ \eta$, and Q is initial with that property. To show Q exists, note that the equations for η to lift to $Ab(\mathcal{C}/X)$ are a set of pseudorelations on $U_X f^* Ab_Y f_! U_X M$. Take the effective equivalence relation on $Ab_Y f_! U_X M$ generated by those pseudorelations as in 2.20. Its coequalizer is the desired quotient.

Let us check that $\widetilde{\eta}\colon M \to f^* Q$ satisfies the solution set condition. Take N an object of $Ab(\mathcal{C}/Y)$ and $h\colon M \to f^* N$ a map in $Ab(\mathcal{C}/X)$. Consider the underlying map

$$U_X h\colon U_X M \to U_X f^* N = f^* U_Y N$$

and its adjunct map $(U_X h)^\sharp\colon Ab_Y f_! U_X M \to N$. By adjunction, the composite

$$U_X M \xrightarrow{\eta} f^* U_Y Ab_Y f_! U_X M \xrightarrow{f^* U_Y (U_X h)^\sharp} f^* U_Y N = U_X f^* N$$

is $U_X h$, which lifts to $Ab(\mathcal{C}/X)$. By the universal property of Q, we obtain a factorization $(U_X h)^\sharp = \varphi q$ for some map $\varphi\colon Q \to N$ in $Ab(\mathcal{C}/Y)$ and, upon applying f^*, the desired factorization $h = (f^* \varphi)\widetilde{\eta}$. □

2.5. **The setup.** Putting the ingredients together, we obtain a good setup for Quillen cohomology. It is essentially an observation of Quillen [14, §II.5, (4) before Thm 5], which we state and prove in more detail.

Proposition 2.40. *Let $\mathcal{C}$ be an algebraic category and X an object of $\mathcal{C}$. Then $\mathcal{C}/X$ and $Ab(\mathcal{C}/X)$ are algebraic and the prolonged adjunction*

$$\begin{tikzcd} s\mathcal{C}/X \arrow[r, shift left, "Ab_X"] & sAb(\mathcal{C}/X) \arrow[l, shift left, "U_X"] \end{tikzcd}$$

is a Quillen pair.

Proof. Both $\mathcal{C}/X$ and $Ab(\mathcal{C}/X)$ are algebraic, by 2.28 and 2.36. Moreover, $\mathcal{C}$ is regular and therefore $\mathcal{C}/X$ is also regular [4, Chap 1, Prop 8.12]. By proposition 2.12, the right adjoint $U_X : Ab(\mathcal{C}/X) \to \mathcal{C}/X$ preserves regular epis, hence the prolonged adjunction is a Quillen pair, by 2.2 and 2.9. □

The setup above is not quite enough to work with Quillen cohomology. There are additional assumptions on the homotopy category $\mathbf{Ho}Ab(s\mathcal{C}/X_\bullet)$: conditions (A) and (B) at the beginning of [14, II.5]. The conditions are satisfied for example if $\mathcal{C}$ has abelian Beck modules, i.e. the category $Ab(\mathcal{C}/X)$ is abelian for any object X. One condition guaranteeing abelian Beck modules is exactness [4, Chap 2, Thm 2.4]. In [15], at the beginning of section 2, Quillen uses the word “algebraic” as in definition 2.23 and then refers to Lawvere’s work, in which the categories are assumed to be exact (and in particular have abelian Beck modules). This is not automatic.

Proposition 2.41. *An algebraic category does not necessarily have abelian Beck modules (and in particular is not necessarily exact).*

Proof. As a counterexample, take the category $\mathbf{Ab}^{tf}$ of torsion-free abelian groups, viewed as a full subcategory of abelian groups. The inclusion $\iota : \mathbf{Ab}^{tf} \to \mathbf{Ab}$ has a left adjoint, which quotients out the torsion subgroup. Thus $\mathbf{Ab}^{tf}$ is cocomplete, and has a small projective generator, namely $\mathbb{Z}$, the same generator as for $\mathbf{Ab}$.

However, $\mathbf{Ab}^{tf}$ is not exact: the map $n : \mathbb{Z} \to \mathbb{Z}$ is a mono which is not the kernel of its cokernel. Indeed, its cokernel is $\mathbb{Z} \to 0$, whose kernel is $1 : \mathbb{Z} \to \mathbb{Z}$. In other words, the equivalence relation $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y (n)\}$ on $\mathbb{Z}$ is not effective.

Moreover, $\mathbf{Ab}^{tf}$ doesn’t have abelian Beck modules. Since ι preserves limits, a Beck module $E \to G$ over a torsion-free abelian group G is in particular a Beck module viewed in $\mathbf{Ab}$, i.e. a direct sum $G \oplus M \twoheadrightarrow G$. The only additional condition is that $G \oplus M$ be torsion-free, which happens iff M itself is torsion-free. Hence for every object G, we have $Ab(\mathbf{Ab}^{tf}/G) \cong \mathbf{Ab}^{tf}$, which is not an abelian category. □

Remark 2.42. For an algebraic category, being exact is equivalent to the generators being exact projective, and not merely regular projective [1, Def 2.4]. Exact projective means preserving coequalizers of *all* equivalence relations, whereas regular projective means preserving coequalizers of effective equivalence relations.

Since exactness is a convenient way of guaranteeing abelian Beck modules, one may wonder if the two conditions are equivalent, perhaps with additional assumptions.

Proposition 2.43. *Having abelian Beck modules does not imply exactness, even for an algebraic category.*

Proof. As a counterexample, take the category $\mathbf{Com}^{red}$ of reduced commutative rings, i.e. those without nilpotents, viewed as a full subcategory of all commutative (associative, unital) rings. The inclusion $\iota\colon \mathbf{Com}^{red} \to \mathit{Com}$ has a left adjoint, which quotients out the nilradical. Thus $\mathbf{Com}^{red}$ is cocomplete, and the free commutative ring on one generator, the polynomial ring $\mathbb{Z}[x]$, is still a small projective generator.

However, $\mathbf{Com}^{red}$ is not exact. Consider the map $4\colon \mathbb{Z} \to \mathbb{Z}$ which induces the equivalence relation $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y(4)\}$ on the target $\mathbb{Z}$. The coequalizer of R in $\mathbf{Com}^{red}$ is $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$, whose kernel pair is $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y(2)\}$ so R is not effective.

A Beck module over a reduced commutative ring R is in particular a Beck module viewed in $\mathbf{Com}$, i.e. a square zero extension $R \oplus M \twoheadrightarrow R$ with multiplication $(r,m)(r',m') = (rr', rm' + mr')$, where the left and right actions of R on M coincide. The only additional condition is for $R \oplus M$ to be a reduced ring, which happens iff M is zero, since the nilradical is $\mathrm{Nil}(R \oplus M) = M$. Hence for every object R, we have $Ab(\mathbf{Com}^{red}/R) \cong 0$, which is an abelian category. ◻

In short, an algebraic category has most of the ingredients for Quillen cohomology. If moreover the category is exact, then it has all the ingredients.

3. Effect of an adjunction

In this section, we investigate the main question: What does an adjunction $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ do to Quillen (co)homology?

3.1. Effect on Beck modules.

The right adjoint R always passes to abelian group objects $R\colon Ab(\mathcal{D}) \to Ab(\mathcal{C})$ since it preserves limits. A priori, we don't know what its left adjoint $\widetilde{L}\colon Ab(\mathcal{C}) \to Ab(\mathcal{D})$ will look like, but if L preserves finite products, then it passes to abelian group objects $L\colon Ab(\mathcal{C}) \to Ab(\mathcal{D})$ and the induced functors still form an adjoint pair.

First, let us see how an adjunction passes to slice categories. There are two versions, depending if one starts with a ground object in $\mathcal{C}$ or in $\mathcal{D}$. A straightforward verification yields the following proposition.

Proposition 3.1. (1) *For any object c in $\mathcal{C}$, there is an induced adjunction*

(4)
$$\begin{tikzcd} \mathcal{C}/c \arrow[r, shift left, "L"] & \mathcal{D}/Lc \arrow[l, shift left, "\eta_c^* R"] \end{tikzcd}$$

where $\eta_c\colon c \to RLc$ is the unit map.

(2) *For any object d in $\mathcal{D}$, there is an induced adjunction*

(5)
$$\begin{tikzcd} \mathcal{C}/Rd \arrow[r, shift left, "\epsilon_{d!} L"] & \mathcal{D}/d \arrow[l, shift left, "R"] \end{tikzcd}$$

where $\epsilon_d\colon LRd \to d$ is the counit map.

Proposition 3.2. *Assume $L\colon \mathcal{C} \to \mathcal{D}$ preserves kernel pairs of split epis.*

(1) *For any object c in $\mathcal{C}$, there is an induced adjunction on Beck modules:*

$$\begin{tikzcd} Ab(\mathcal{C}/c) \arrow[r, shift left, "L"] & Ab(\mathcal{D}/Lc). \arrow[l, shift left, "\eta_c^* R"] \end{tikzcd}$$

(2) *For any object d in $\mathcal{D}$, there is an induced adjunction on Beck modules:*

$$\begin{tikzcd} Ab(\mathcal{C}/Rd) \arrow[r, shift left, "\epsilon_{d*}L"] & Ab(\mathcal{D}/d). \arrow[l, shift left, "R"] \end{tikzcd}$$

Proof. 1. The assumption guarantees that the left adjoint $L\colon \mathcal{C}/c \to \mathcal{D}/Lc$ preserves finite products, and hence the adjunction (4) passes to abelian group objects.

2. Start with the natural equivalence

$$\mathrm{Hom}_{Ab(\mathcal{D}/d)}\left(\epsilon_{d*}L(c' \to Rd), d' \to d\right) \cong \mathrm{Hom}_{Ab(\mathcal{D}/LRd)}\left(Lc' \to LRd, \epsilon_d^*(d' \to d)\right).$$

The right-hand side consists of maps $Lc' \to d'$ that make the diagram

$$\begin{tikzcd} Lc' \arrow[r] \arrow[d] & d' \arrow[d] \\ LRd \arrow[r, "\epsilon_d"'] & d \end{tikzcd}$$

commute and respect the structure maps of the columns. This is equivalent to maps $c' \to Rd'$ that make the adjoint diagram

$$\begin{tikzcd} c' \arrow[r] \arrow[d] & Rd' \arrow[d] \\ Rd \arrow[r, "\mathrm{id}"'] & Rd \end{tikzcd}$$

commute and respect the structure maps of the columns. These are precisely maps from $(c' \to Rd)$ to $R(d' \to d)$ in $Ab(\mathcal{C}/Rd)$. □

Remark 3.3. The assumption that L passes to Beck modules is not crucial. We only used it to identify the induced left adjoint.

3.2. **Effect on abelian cohomology.** Before introducing any homotopical algebra, let us study the problem at the level of homological algebra. Assume $\mathcal{C}$ and $\mathcal{D}$ have abelian Beck modules with enough projectives, which is the case for example if they are exact algebraic categories. We want to describe the effect of the adjunction on abelian cohomology. Again, assume the left adjoint L passes to Beck modules. As we have seen in 3.2, there are two induced adjunctions, depending if one starts with a ground object in $\mathcal{C}$ or in $\mathcal{D}$.

3.2.1. *Ground object in $\mathcal{C}$.* Pick a ground object c in $\mathcal{C}$. The induced adjunction on Beck modules fits into the diagram

$$\begin{tikzcd} \mathcal{C}/c \arrow[r, shift left, "Ab_c"] \arrow[d, shift right, "L"'] & Ab(\mathcal{C}/c) \arrow[l, shift left, "U_c"] \arrow[d, shift right, "L"'] \\ \mathcal{D}/Lc \arrow[r, shift left, "Ab_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] & Ab(\mathcal{D}/Lc) \arrow[l, shift left, "U_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] \end{tikzcd} \tag{6}$$

where the diagram of right adjoints commutes (on the nose), and thus the diagram of left adjoints commutes as well. In particular, applying the left adjoints to id_c, we obtain

$LAb_c c = Ab_{Lc} Lc$. Take a module N over Lc and consider:

$$\begin{aligned} \mathrm{HA}^*(c; \eta_c^* RN) &= \mathrm{Ext}^*(Ab_c c, \eta_c^* RN) \\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_c}(P_\bullet, \eta_c^* RN) \\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(LP_\bullet, N) \qquad (7) \end{aligned}$$

where $P_\bullet \to Ab_c c$ is a projective resolution. We want to compare this to:

$$\begin{aligned} \mathrm{HA}^*(Lc; N) &= \mathrm{Ext}^*(Ab_{Lc} Lc, N) \\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(Q_\bullet, N) \end{aligned}$$

where $Q_\bullet \to Ab_{Lc} Lc$ is a projective resolution. Assume the induced left adjoint $L \colon \mathbf{Mod}_c \to \mathbf{Mod}_{Lc}$ preserves projectives (which is the case for example when its right adjoint $\eta_c^* R$ preserves epis, i.e. is exact). Then $LP_\bullet$ is projective but is not a resolution of $LAb_c c$. However, the map factors as $LP_\bullet \hookrightarrow Q_\bullet \xrightarrow{\sim} LAb_c c = Ab_{Lc} Lc$ and the first map induces $\operatorname{Hom}_{\mathbf{Mod}_{Lc}}(Q_\bullet, N) \to \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(LP_\bullet, N)$ which, upon passing to cohomology, induces a well defined map. We sum up the argument in the following proposition.

Proposition 3.4. *If the left adjoint L induces a functor on Beck modules which preserves projectives, then we obtain a comparison map in abelian cohomology:*

$$\mathrm{HA}^*(Lc; N) \to \mathrm{HA}^*(c; \eta_c^* RN). \qquad (8)$$

Note that (7) exhibits $\mathrm{HA}^*(c; \eta_c^* RN)$ as derived functors of $\operatorname{Hom}_{\mathbf{Mod}_{Lc}}(-, N) \circ L$ applied to $Ab_c c$. Since L sends projectives to projectives, we obtain a Grothendieck composite spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}^s(L_t L(Ab_c c), N) \Rightarrow \mathrm{HA}^{s+t}(c; \eta_c^* RN)$$

which is first quadrant, cohomologically graded. The comparison map (8) is the edge morphism

$$\mathrm{HA}^s(Lc; N) = \mathrm{Ext}^s(LAb_c c, N) = E_2^{s,0} \twoheadrightarrow E_\infty^{s,0} \hookrightarrow \mathrm{HA}^s(c; \eta_c^* RN).$$

If $L \colon \mathbf{Mod}_c \to \mathbf{Mod}_{Lc}$ happens to be exact, then $LP_\bullet$ is a projective resolution of $LAb_c c = Ab_{Lc} Lc$ and the comparison map (8) is an iso.

Remark 3.5. Starting with a module M over c, there is a map

$$\operatorname{Hom}_{\mathbf{Mod}_c}(Ab_c c, M) \to \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(LAb_c c, LM) = \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(Ab_c c, \eta_c^* RLM)$$

given by applying L, or equivalently, induced by the unit $M \to \eta_c^* RLM$. One might want to compare $\mathrm{HA}^*(c; M)$ and $\mathrm{HA}^*(Lc; LM)$, but they both naturally map into $\mathrm{HA}^*(c; \eta_c^* RLM)$, respectively via the unit and the comparison map (8). There is no direct comparison.

3.2.2. *Ground object in $\mathcal{D}$.* Pick a ground object d in $\mathcal{D}$. The induced adjunction on Beck modules fits into the diagram

$$\begin{tikzcd} C/Rd \arrow[r, shift left, "Ab_{Rd}"] \arrow[d, shift right, "\epsilon_! L"'] & Ab(C/Rd) \arrow[l, shift left, "U_{Rd}"] \arrow[d, shift right, "\epsilon_{d*} L"'] \\ \mathcal{D}/d \arrow[r, shift left, "Ab_d"] \arrow[u, shift right, "R"'] & Ab(\mathcal{D}/d) \arrow[l, shift left, "U_d"] \arrow[u, shift right, "R"'] \end{tikzcd} \qquad (9)$$

where the diagram of right adjoints commutes, and thus the diagram of left adjoints commutes as well. Take a module N over d and consider:

$$\begin{aligned}\mathrm{HA}^*(d;N) &= \mathrm{Ext}^*(Ab_d d, N)\\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_d}(P_\bullet, N)\end{aligned}$$

where $P_\bullet \to Ab_d d$ is a projective resolution. We want to compare this to:

$$\begin{aligned}\mathrm{HA}^*(Rd;RN) &= \mathrm{Ext}^*(Ab_{Rd}Rd, RN)\\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_{Rd}}(Q_\bullet, RN)\\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_d}(\epsilon_{d*}LQ_\bullet, N) \qquad (10)\end{aligned}$$

where $Q_\bullet \to Ab_{Rd}Rd$ is a projective resolution. Here again, assume the induced left adjoint $\epsilon_{d*}L \colon \mathbf{Mod}_{Rd} \to \mathbf{Mod}_d$ preserves projectives. Then $\epsilon_{d*}LQ_\bullet$ is projective and we have a map:

$$\epsilon_{d*}LQ_\bullet \to \epsilon_{d*}LAb_{Rd}Rd = \epsilon_{d*}Ab_{LRd}LRd = Ab_d(Lrd \xrightarrow{\epsilon_d} d) \xrightarrow{Ad_d(\epsilon_d)} Ab_d d.$$

It admits a factorization $\epsilon_{d*}LQ_\bullet \hookrightarrow P_\bullet \xrightarrow{\sim} Ab_d d$ and the first map induces

$$\operatorname{Hom}_{\mathbf{Mod}_d}(P_\bullet, N) \to \operatorname{Hom}_{\mathbf{Mod}_d}(\epsilon_{d*}LQ_\bullet, N)$$

which, upon passing to cohomology, induces a well defined map. We sum up the argument in the following proposition.

Proposition 3.6. *If the left adjoint L passes to Beck modules and the induced left adjoint $\epsilon_{d*}L \colon \mathbf{Mod}_{Rd} \to \mathbf{Mod}_d$ preserves projectives, then we obtain a comparison map in abelian cohomology:*

$$\mathrm{HA}^*(d;N) \to \mathrm{HA}^*(Rd;RN). \qquad (11)$$

Note that (10) exhibits $\mathrm{HA}^*(Rd;RN)$ as derived functors of $\operatorname{Hom}_{\mathbf{Mod}_d}(-,N) \circ \epsilon_{d*}L$ applied to $Ab_{Rd}Rd$. Since $\epsilon_{d*}L$ sends projectives to projectives, we obtain a Grothendieck composite spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}^s\left(L_t(\epsilon_{d*}L)(Ab_{Rd}Rd), N\right) \Rightarrow \mathrm{HA}^{s+t}(Rd;RN)$$

which is first quadrant, cohomologically graded. The comparison map (11) is $Ab_d(\epsilon_d)^*$ followed by an edge morphism:

$$\begin{aligned}\mathrm{HA}^s(d;N) = \mathrm{Ext}^s(Ab_d d, N) &\xrightarrow{Ab_d(\epsilon_d)^*} \mathrm{Ext}^s(\epsilon_{d*}LAb_{Rd}Rd, N)\\ &= E_2^{s,0} \twoheadrightarrow E_\infty^{s,0} \hookrightarrow \mathrm{HA}^s(Rd;RN).\end{aligned}$$

If $\epsilon_{d*}L \colon \mathbf{Mod}_{Rd} \to \mathbf{Mod}_d$ happens to be exact, then $\epsilon_{d*}LQ_\bullet$ is a projective resolution of $\epsilon_{d*}LAb_{Rd}Rd$, and we obtain an iso $\mathrm{Ext}^*(\epsilon_{d*}LAb_{Rd}Rd, N) \cong \mathrm{HA}^*(Rd;RN)$. In that case, the comparison map (11) is simply $Ab_d(\epsilon_d)^*$, which is not necessarily an iso.

Remark 3.7. Starting with a module M over Rd, one might want to compare $\mathrm{HA}^*(Rd;M)$ and $\mathrm{HA}^*(d;\epsilon_{d*}LM)$. Again, there is no direct comparison. They both map naturally into $\mathrm{HA}^*(Rd;R\epsilon_{d*}LM)$, the former via the unit $M \to R\epsilon_{d*}LM$ and the latter via the comparison map (11).

3.3. **The comparison diagram.** Now let us check that the adjunction behaves well at the level of homotopical algebra, when we pass to simplicial objects.

Theorem 3.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be algebraic categories. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be an adjunction that prolongs to a Quillen pair (equivalently, R preserves regular epis; equivalently, L preserves projectives), and assume L passes to Beck modules. Then the commutative diagram (6) simplicially prolongs to four Quillen pairs:*

$$\begin{tikzcd} s\mathcal{C}/c \arrow[r, shift left, "Ab_c"] \arrow[d, shift right, "L"'] & sAb(\mathcal{C}/c) \arrow[l, shift left, "U_c"] \arrow[d, shift right, "L"'] \\ s\mathcal{D}/Lc \arrow[r, shift left, "Ab_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] & sAb(\mathcal{D}/Lc) \arrow[l, shift left, "U_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] \end{tikzcd} \tag{12}$$

and so does the commutative diagram (9):

$$\begin{tikzcd} s\mathcal{C}/Rd \arrow[r, shift left, "Ab_{Rd}"] \arrow[d, shift right, "\epsilon_! L"'] & sAb(\mathcal{C}/Rd) \arrow[l, shift left, "U_{Rd}"] \arrow[d, shift right, "\epsilon_{d*} L"'] \\ s\mathcal{D}/d \arrow[r, shift left, "Ab_d"] \arrow[u, shift right, "R"'] & sAb(\mathcal{D}/d). \arrow[l, shift left, "U_d"] \arrow[u, shift right, "R"'] \end{tikzcd} \tag{13}$$

Proof. **Case 1: Ground object c in $\mathcal{C}$.** The induced right adjoint on slice categories is $\eta_c^* R\colon \mathcal{D}/Lc \to \mathcal{C}/c$ and it preserves regular epis. Indeed, $R\colon \mathcal{D}/Lc \to \mathcal{C}/RLc$ preserves regular epis by assumption and 2.6. The pullback η_c^* also preserves regular epis since $\mathcal{C}$ is regular and again by 2.6.

The induced right adjoint on Beck modules $\eta_c^* R\colon Ab(\mathcal{D}/Lc) \to Ab(\mathcal{C}/c)$ preserves regular epis. It follows from the same argument, and the fact that regular epis in $Ab(-)$ are preserved and reflected by the forgetful functor U, by 2.12 and 2.14.

Case 2: Ground object d in $\mathcal{D}$. The induced right adjoint on slice categories is just $R\colon \mathcal{D}/d \to \mathcal{C}/Rd$, which preserves regular epis. The induced right adjoint on Beck modules $R\colon Ab(\mathcal{D}/d) \to Ab(\mathcal{C}/Rd)$ also preserves regular epis. ◻

Remark 3.9. The result holds whether or not the left adjoint L passes to Beck modules, since the proof only relies on properties of the induced right adjoints. If L does not pass to Beck modules, the induced left adjoint is something else.

3.4. **Effect on Quillen (co)homology.** In this section, we describe the comparison maps induced on Quillen (co)homology. The argument is similar to section 3.2, except we start with the comparison diagrams in 3.8.

3.4.1. *Ground object c in $\mathcal{C}$.*

Proposition 3.10. *Assume the setup of 3.8. Then the comparison diagram induces the following comparison maps.*

(1) *A natural (up to homotopy) comparison map of cotangent complexes:*

$$L(\mathbf{L}_c) \to \mathbf{L}_{Lc}. \tag{14}$$

(2) *A natural comparison map in Quillen homology:*

$$L\,(\mathrm{HQ}_*(c)) \to \mathrm{HQ}_*(Lc). \tag{15}$$

(3) *If L preserves pullbacks, then we have $L\left(\mathrm{HQ}_*(c)\right) \cong \pi_* L(\mathbf{L}_c)$ and the map (15) is just the effect of (14) on π_*.*

(4) *If L preserves all weak equivalences, then (14) is a weak equivalence and (15) is an iso.*

Proof. 1. Starting with a cofibrant replacement $q_c \colon Qc \xrightarrow{\sim} c$ of id_c, we can apply L to obtain $LQc \to Lc$, where the source is still cofibrant (since L is a left Quillen functor) but the map is not a weak equivalence anymore. However, it factors (uniquely and functorially up to homotopy) as $LQc \xrightarrow{\psi} QLc \xrightarrow{\sim} Lc$ and we obtain the comparison map

$$\begin{aligned} L(\mathbf{L}_c) &= LAb_c(Qc \to c) = Ab_{Lc}L(Qc \to c) \\ &= Ab_{Lc}(LQc \to Lc) \to Ab_{Lc}(QLc \to Lc) = \mathbf{L}_{Lc} \end{aligned}$$

which is in fact $Ab_{Lc}(\psi)$.

2. There is a homology comparison map [5, Thm 2.2 and 2.6] for the right exact functor L, which we apply to the chain complex $\mathbf{L}_c$ (using implicitly the Dold-Kan correspondence):

$$L\left(\mathrm{HQ}_*(c)\right) = LH_*(\mathbf{L}_c) \to H_*L(\mathbf{L}_c) = \pi_*L(\mathbf{L}_c).$$

Note that the map is an edge morphism in the composite spectral sequence of $L \circ Ab_c$ applied to id_c. Following this homology comparison by the effect of (14) on π_*, we obtain the Quillen homology comparison:

$$L\left(\mathrm{HQ}_*(c)\right) \to \pi_*L(L_c) \to \pi_*\mathbf{L}_{Lc} = \mathrm{HQ}_*(Lc).$$

3. If L preserves pullbacks, then the induced L on Beck modules also preserves finite limits, hence is left exact (and thus exact). In that case, the homology comparison is an iso.

4. If L preserves all weak equivalences, then the map ψ is a weak equivalence. Since Ab_{Lc} is a left Quillen functor, the map (14) is also a weak equivalence. The induced L also preserves weak equivalences, and in particular is exact so the homology comparison is an iso. ◻

Proposition 3.11. *Let N be a module over Lc.*

(1) *The comparison diagram induces a natural comparison map*

$$\mathrm{HQ}^*(Lc; N) \to \mathrm{HQ}^*(c; \eta_c^* RN). \tag{16}$$

(2) *If the comparison of cotangent complexes (14) is a weak equivalence, then (16) is an iso. This holds in particular when L preserves all weak equivalences.*

Proof. 1. Apply the functor $\mathrm{Hom}_{\mathbf{Mod}_{Lc}}(-, N)$ to the comparison map (14)

$$\mathrm{Hom}_{\mathbf{Mod}_{Lc}}(\mathbf{L}_{Lc}, N) \to \mathrm{Hom}_{\mathbf{Mod}_{Lc}}(L(\mathbf{L}_c), N) \cong \mathrm{Hom}_{\mathbf{Mod}_c}(L_c, \eta_c^* RN)$$

and upon passing to cohomology, we obtain the map (16).

2. Since $L(\mathbf{L}_c)$ and $\mathbf{L}_{Lc}$ are cofibrant, a weak equivalence (14) between them will induce a weak equivalence upon applying $\mathrm{Hom}(-, N)$. ◻

3.4.2. *Ground object d in $\mathcal{D}$.* A very similar reasoning yields the following propositions.

Proposition 3.12. *Assume the setup of 3.8. Then the comparison diagram induces the following comparison maps.*

(1) *A natural (up to homotopy) comparison map of cotangent complexes:*

$$\epsilon_{d*}L(\mathbf{L}_{Rd}) \to \mathbf{L}_d. \tag{17}$$

(2) *A natural comparison map in Quillen homology:*

$$\epsilon_{d*}L\,(\mathrm{HQ}_*(Rd)) \to \mathrm{HQ}_*(d). \tag{18}$$

(3) *If L preserves pullbacks and ϵ_{d*} is exact, then $\epsilon_{d*}L\,(\mathrm{HQ}_*(Rd)) \cong \pi_*\epsilon_{d*}L(\mathbf{L}_{Rd})$ holds and the map (18) is just the effect of (17) on π_*.*

(4) *If L preserves all weak equivalences and ϵ_d is an iso, then (17) is a weak equivalence and (18) is an iso.*

Proposition 3.13. *Let N be a module over d.*

(1) *The comparison diagram induces a natural comparison map*

$$\mathrm{HQ}^*(d;N) \to \mathrm{HQ}^*(Rd;RN). \tag{19}$$

(2) *If the comparison of cotangent complexes (17) is a weak equivalence, then (19) is an iso.*

4. Examples

In this section we study three examples. The first serves as a warmup. The second tries to relate André-Quillen cohomology to Hochschild cohomology (4.7). The third shows how Quillen cohomology of a Π-algebra with coefficients in a truncated module can be computed within the world of truncated Π-algebras (4.15), which have a much simpler structure than (non-truncated) Π-algebras.

4.1. **Abelian groups.** Consider the functor $Com\colon \mathbf{Gp} \to \mathbf{Ab}$ that kills commutators, i.e. $Com(G) = G/[G,G]$, whose right adjoint is the inclusion functor $\iota\colon \mathbf{Ab} \to \mathbf{Gp}$. Although Com does not preserve kernel pairs in general, it does pass to Beck modules. Recall that for a (left) G-module M, the semidirect product $G \ltimes M$ is the group with underlying set $G \times M$ and multiplication $(g,m)(g',m') = (gg', m + gm')$.

Proposition 4.1. *Com passes to Beck modules, on which it induces the coinvariants functor $(-)_G\colon \mathbf{Mod}_G \to \mathbf{Ab}$.*

Proof. Let us first compute $Com(G \ltimes M)$. Commutators in $G \ltimes M$ are given by

$$[(g_1,m_1),(g_2,m_2)] = \left([g_1,g_2], m_1 - g_1g_2g_1^{-1}m_1 + g_1m_2 - g_1g_2g_1^{-1}g_2^{-1}m_2\right).$$

Applying Com to the split extension $G \ltimes M \to G$ yields a split extension $Com(G \ltimes M) \to Com(G)$ in $\mathbf{Ab}$ whose kernel is M modulo the subgroup

$$\begin{aligned}&\left\langle m_1 - g_1g_2g_1^{-1}m_1 + g_1m_2 - g_1g_2g_1^{-1}g_2^{-1}m_2 \mid g_i \in G, m_i \in M\right\rangle\\ &= \langle m - gm \mid g \in G, m \in M\rangle .\end{aligned}$$

In other words, we have $Com(G \ltimes M) \cong Com(G) \oplus M_G$, where M_G is the abelian group of coinvariants of M.

Moreover, Com preserves the pullback that defines the multiplication structure map:

$$\begin{aligned}Com\,((G \ltimes M) \times_G (G \ltimes M)) &= Com\,(G \ltimes (M \times M))\\ &= Com(G) \oplus (M \times M)_G\\ &= Com(G) \oplus (M_G \oplus M_G)\\ &= (Com(G) \oplus M_G) \times_{Com(G)} (Com(G) \oplus M_G)\\ &= Com(G \ltimes M) \times_{Com(G)} Com(G \ltimes M).\end{aligned}$$

In $\mathbf{Gp}$ as well as in $\mathbf{Ab}$, we think of the module as the kernel of the split extension, and in this case, a G-module M is sent to the abelian group M_G. □

Remark 4.2. In **Ab**, a Beck module consists only of a split extension with the data of the splitting. Therefore, any functor $F\colon \mathcal{C} \to \mathbf{Ab}$ passes to Beck modules. We've shown it explicitly for Com and identified the induced functor.

Let us describe the effect of the adjunction $Com\colon \mathbf{Gp} \rightleftarrows \mathbf{Ab}\colon \iota$ on Quillen homology. Note that the right adjoint ι preserves regular epis, which are just surjections. Hence, the prolonged adjunctions are Quillen pairs.

Note also that the unit of the adjunction is $\eta_G\colon G \twoheadrightarrow G/[G,G]$ and the counit is the identity. We work with a ground object G in **Gp**, since we get nothing new from a ground object in **Ab**. The comparison diagram (6) becomes:

$$\begin{tikzcd}
s\mathbf{Gp}/G \arrow[r, shift left, "Ab_G"] \arrow[d, shift right, "Com"'] & s\mathbf{Mod}_G \arrow[l, shift left, "G\ltimes -"] \arrow[d, shift right, "(-)_G"'] \\
s\mathbf{Ab}/Com(G) \arrow[r, shift left, "Src"] \arrow[u, shift right, "\eta_G^*\iota"'] & s\mathbf{Ab}. \arrow[l, shift left, "Com(G)\oplus -"] \arrow[u, shift right, "Triv"']
\end{tikzcd} \tag{20}$$

and by 3.8, it prolongs to four Quillen pairs. Here Src is the "source" functor, which is the abelianization over any abelian group, and $Triv$ is the functor assigning to an abelian group the trivial G-action. Indeed, the right adjoint on Beck modules is $\eta_G^*\iota$. Given a Beck module $Com(G) \oplus A$, view it as a split extension of groups, which means A has a trivial $Com(G)$ action, and then pull the action back along $\eta_G\colon G \to G/[G,G]$, which endows A with the trivial G-action.

Remark 4.3. In 4.1, we checked explicitly that Com induces the functor $(-)_G$ on Beck modules. Per remark 3.3, we could also look at the induced right adjoint $\eta_G^*\iota = Triv$ and use its left adjoint to complete diagram (20). The left adjoint of $Triv = \epsilon^*$ is indeed $(-)_G = \epsilon_* = \mathbb{Z} \otimes_{\mathbb{Z}G} (-)$, where $\epsilon\colon \mathbb{Z}G \to \mathbb{Z}$ is the augmentation.

We now formulate the result about Quillen homology.

Proposition 4.4. *Let $C_\bullet \to G$ be a cofibrant replacement of G in groups and let $\mathbf{L}_G$ denote the cotangent complex of G. Then the following holds:*

$$\pi_*\left(C_\bullet/[C_\bullet, C_\bullet]\right) = \pi_*\left((\mathbf{L}_G)_G\right).$$

Proof. Starting from a cofibrant replacement of G in **Gp** (or equivalently, of id_G in $\mathbf{Gp}/G$) in the upper left corner of (20), going down then right yields

$$\begin{aligned} Src \circ Com(C_\bullet \to G) &= Src\,(Com(C_\bullet) \to Com(G)) \\ &= Com(C_\bullet) = C_\bullet/[C_\bullet, C_\bullet] \end{aligned}$$

whereas going right then down yields $(Ab_G(C_\bullet \to G))_G = (\mathbf{L}_G)_G$. Taking π_* gives a well defined equality, since the simplicial G-module $\mathbf{L}_G$ is defined up to homotopy. □

In fact, one can compute both sides explicitly and check that they coincide. For groups, abelianization is $Ab_G G = I_G = \ker(\mathbb{Z}G \to \mathbb{Z})$ and the cotangent complex is discrete, meaning $\mathbf{L}_G \to I_G$ is a cofibrant replacement, in particular a flat resolution. Taking coinvariants results in the derived functors thereof, namely group homology:

$$\pi_*\left((\mathbf{L}_G)_G\right) = L_*(-)_G(I_G) = \mathrm{H}_*(G; I_G).$$

Using the short exact sequence $0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ of G-modules, the connecting morphism $\mathrm{H}_{i+1}(G;\mathbb{Z}) \to \mathrm{H}_i(G; I_G)$ is an iso for all $i \geq 0$, from which we conclude

$\pi_i\left((\mathbf{L}_G)_G\right) = \mathrm{H}_{i+1}(G;\mathbb{Z})$ for all $i \geq 0$. On the other hand, [12, Ex 4.26] uses a different argument to show $\pi_i\left(C_\bullet/[C_\bullet, C_\bullet]\right) = \mathrm{H}_{i+1}(G;\mathbb{Z})$ for all $i \geq 0$. Proposition 4.4 is consistent with these computations.

4.2. **Commutative algebras.** Let R be some fixed commutative ring; denote by $\mathbf{Alg}_R$ the category of associative R-algebras and by $\mathbf{Com}_R$ the category of commutative R-algebras. (All our rings and algebras are assumed associative and unital.) Consider the functor $Com\colon \mathbf{Alg}_R \to \mathbf{Com}_R$ which kills the 2-sided ideal generated by commutators, that is $Com(A) = A/[A,A]$. It is left adjoint to the inclusion functor $\iota\colon \mathbf{Com}_R \to \mathbf{Alg}_R$, which preserves regular epis (i.e. surjections).

Recall that Beck modules over an associative R-algebra A are A-bimodules over R, meaning that scalars in R act the same way on the left and the right; we denote this category $A - \mathbf{Bimod}_R$. Beck modules over a commutative R-algebra A are A-modules in the usual sense, which we denote $A - \mathbf{Mod}$.

Proposition 4.5. *1. The functor* $Com\colon \mathbf{Alg}_R \to \mathbf{Com}_R$ *passes to Beck modules.*

2. It induces the "central quotient" functor $CQ\colon A - \mathbf{Bimod}_R \to Com(A) - \mathbf{Mod}$ *which coequalizes the two actions.*

Proof. Start with a Beck module over A in $\mathbf{Alg}_R$, i.e. a split extension $p\colon A \oplus M \to A$ satisfying $M^2 = 0$. Applying Com to it yields a split extension

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & Com(A \oplus M) \arrow[r, shift left, "Com(p)"] & Com(A) \arrow[l, shift left, "Com(s)"] \arrow[r] & 0 \end{tikzcd}$$

in $\mathbf{Com}_R$. It remains to show that its kernel has square zero.

Commutators in $A \oplus M$**.** Using the decomposition $(a, m) = (a, 0) + (0, m)$, commutators will be generated by those of the forms $[(a, 0), (a', 0)] = ([a, a'], 0)$ and $[(a, 0), (0, m)] = (0, a \cdot m - m \cdot a)$. Thus the kernel is

$$K \simeq M/\left\langle a \cdot m - m \cdot a \right\rangle \tag{21}$$

where we kill the sub-A-bimodule generated by all elements of that form.

K **has square zero.** Take two elements $x, x' \in K = \ker Com(p) \subset Com(A \oplus M)$ and choose representatives (c, m) and (c', m') in $A \oplus M$, for $c, c' \in [A, A]$. Then xx' is represented by $(c, m)(c', m') = (cc', c \cdot m' + m \cdot c')$. One readily checks that elements of the form $c \cdot m$ and $m \cdot c$ are zero in $Com(A \oplus M)$, for any $m \in M$ and $c \in [A, A]$. This proves the first assertion, and (21) proves the second. □

The adjunction $Com\colon \mathbf{Alg}_R \rightleftarrows \mathbf{Com}_R\colon \iota$ allows us to compare the two categories. According to 4.5, the comparison diagram 6 becomes

$$\begin{tikzcd} \mathbf{Alg}_R/A \arrow[r, shift left, "A \otimes I_{(-)} \otimes A"] \arrow[d, shift right, "Com"'] & A - \mathbf{Bimod}_R \arrow[l, shift left, "A \oplus -"] \arrow[d, shift right, "CQ"'] \\ \mathbf{Com}_R/Com(A) \arrow[r, shift left, "Com(A) \otimes \Omega_{(-)/R}"] \arrow[u, shift right, "\eta_A^* \iota"'] & Com(A) - \mathbf{Mod} \arrow[l, shift left, "Com(A) \oplus -"] \arrow[u, shift right, "\text{same action}"'] \end{tikzcd} \tag{22}$$

where "same action", the right adjoint on the right, means that we view a $Com(A)$-module as an A-bimodule by acting via the unit $A \to Com(A) = A/[A, A]$ both on the left and the right. Abelianization in associative algebras is $Ab_A(B \to A) = A \otimes_B I_B \otimes_B A$ where I_B

denotes the kernel of the multiplication map $m\colon B \otimes_R B \to B$. Abelianization in commutative algebras is $Ab_S(T \to S) = S \otimes_T \Omega_{T/R}$ where $\Omega_{T/R}$ denotes the module of Kähler differentials I_T/I_T^2. By 3.8, diagram (22) prolongs to four Quillen pairs.

Remark 4.6. One can view $A - \mathbf{Bimod}_R$ as the category of left $A \otimes_R A^{\mathrm{op}}$ modules, and the "same action" functor $Com(A) - \mathbf{Mod} \to A - \mathbf{Bimod}_R$ as the restriction $(\eta_A m)^*$ along $A \otimes_R A^{\mathrm{op}} \xrightarrow{m} A \xrightarrow{\eta_A} Com(A)$. Its left adjoint is the pushforward $(\eta_A m)_* = (A \otimes_R A^{\mathrm{op}}) \otimes_{Com(A)} -$ which is indeed the functor coequalizing the two actions.

Some special cases are of particular interest. When the R-algebra A is just R itself – and is in particular commutative – the comparison diagram (22) becomes

$$\begin{tikzcd}
\mathbf{Alg}_R/R \arrow[r, shift left, "R\otimes I_{(-)}\otimes R"] \arrow[d, shift right, "Com"'] & R - \mathbf{Bimod}_R \arrow[l, shift left, "R\oplus -"] \arrow[d, shift right, "\mathrm{id}"'] \\
\mathbf{Com}_R/R \arrow[r, shift left, "R\otimes\Omega_{(-)/R}"] \arrow[u, shift right, "\iota"'] & R - \mathbf{Mod}. \arrow[l, shift left, "R\oplus -"] \arrow[u, shift right, "\mathrm{id}"']
\end{tikzcd} \tag{23}$$

The diagram says that killing all products can be done in two steps, by killing all commutators first. One could try to use the Grothendieck composite spectral sequence for the non-abelian setting [6, Thm 4.4] to relate Quillen homology in $\mathbf{Alg}_R$ to Quillen homology in $\mathbf{Com}_R$, i.e. André-Quillen homology. This approach would require the knowledge of homotopy operations in $\mathbf{Com}_R$, which are known notably for $R = \mathbb{F}_2$ [10] [11].

More generally, another interesting case is when the cotangent complex in associative algebras is discrete, i.e. $\mathbf{L}_A \to Ab_A A$ is a weak equivalence. Quillen [15, Prop 3.6] shows that this happens under the condition $\mathrm{Tor}_i^R(A, A) = 0$ for all $i \geq 1$ (for example if R is a field), in which case $\mathrm{HA}^*(A; M) \cong \mathrm{HQ}^*(A; M)$ is essentially the same as the usual Hochschild cohomology, and likewise for homology.

Proposition 4.7. *Let A be a commutative R-algebra satisfying $\mathrm{Tor}_i^R(A, A) = 0$ for all $i \geq 1$. Then for every $j \geq 1$, the Hochschild homology of A can be written as*

$$\mathrm{HH}_{j+1}(A) = \pi_j \left(A \otimes_{Com(C_\bullet)} \Omega_{Com(C_\bullet)/R}\right)$$

where $C_\bullet \to A$ is a cofibrant replacement of A in $\mathbf{Alg}_R$. In particular, there is a comparison map $\mathrm{HH}_{j+1}(A) \to \mathrm{HQ}_j(A)$ for $j \geq 1$.

Proof. Starting from a cofibrant replacement $C_\bullet \to A$ in $\mathbf{Alg}_R$ and going right in (22), one obtains $\mathbf{L}_A \to I_A$, which is a weak equivalence because of the flatness assumption on A. Then going down yields $A \otimes_{A \otimes_R A^{\mathrm{op}}} \mathbf{L}_A$, whose π_* is $\mathrm{Tor}_*^{A \otimes_R A^{\mathrm{op}}}(A, I_A)$. Again by the flatness assumption, Hochschild homology $\mathrm{HH}_*(A)$ is not just a relative Tor but the (absolute) $\mathrm{Tor}_*^{A \otimes_R A^{\mathrm{op}}}(A, A)$. The short exact sequence of bimodules $0 \to I_A \to A \otimes_R A^{\mathrm{op}} \to A \to 0$ gives a natural iso $\mathrm{Tor}_{i+1}^{A \otimes_R A^{\mathrm{op}}}(A, A) \cong \mathrm{Tor}_i^{A \otimes_R A^{\mathrm{op}}}(A, I_A)$ for all $i \geq 1$.

On the other hand, going down in the diagram yields $Com(C_\bullet) \to A$ and then going right yields $A \otimes_{Com(C_\bullet)} \Omega_{Com(C_\bullet)/R}$. The comparison map is π_* of (17), which measures the failure of $Com\colon \mathbf{Alg}_R \to \mathbf{Com}_R$ to preserve weak equivalences. □

4.3. **Truncated Π-algebras.** A Π-algebra is the algebraic structure best describing the homotopy groups of a pointed space X. More details can be found in [7, § 4] [16, § 4]; we recall the essentials. Let $\mathbf{\Pi}$ denote the homotopy category of pointed spaces with the homotopy type of a finite (possibly empty) wedge of spheres of positive dimensions.

Definition 4.8. A **$\boldsymbol{\Pi}$-algebra** is a contravariant functor $A\colon \boldsymbol{\Pi} \to \mathbf{Set}$ that sends wedges to products, i.e. a product-preserving functor $\boldsymbol{\Pi}^{op} \to \mathbf{Set}$ (or equivalently to pointed sets).

The prototypical example is the functor $[-, X]_*$, the homotopy Π-algebra of a pointed space X. A Π-algebra A can be viewed as a graded group $\{\pi_i = A(S^i)\}$ (abelian for $i \geq 2$) equipped with primary homotopy operations induced by maps between wedges of spheres, such as precomposition operations $\alpha^*\colon \pi_k \to \pi_n$ for every $\alpha \in \pi_n(S^k)$. The additional structure is determined by operations of that form, Whitehead products, and the π_1-action on higher π_i, and there are classical relations between them.

Let $\boldsymbol{\Pi}\mathbf{Alg}$ denote the category of Π-algebras, that is $\mathrm{Fun}^{\times}(\boldsymbol{\Pi}^{\mathrm{op}}, \mathbf{Set})$, where $\mathrm{Fun}^{\times}$ denotes product-preserving functors.

4.3.1. *Postnikov truncation.* We want to make precise the notion of Postnikov truncation for Π-algebras.

Definition 4.9. A Π-algebra A is called **n-truncated** if for all $i > n$, we have $A(S^i) = *$, the trivial pointed set.

Denote by $\boldsymbol{\Pi}\mathbf{Alg}_1^n$ the full subcategory of $\boldsymbol{\Pi}\mathbf{Alg}$ consisting of n-truncated Π-algebras. Denote by $\boldsymbol{\Pi}_n$ the full subcategory of $\boldsymbol{\Pi}$ consisting of spaces with the homotopy type of a wedge of spheres of dimension at most n, and let $I_n\colon \boldsymbol{\Pi}_n \to \boldsymbol{\Pi}$ be the inclusion functor. One can go the other way, by removing spheres above a certain dimension. Glossing over technicalities, define a "truncation" functor $T_n\colon \boldsymbol{\Pi} \to \boldsymbol{\Pi}_n$ by $T_n\left(\bigvee_{i=1}^k S^{n_i}\right) = \bigvee_{n_i \leq n} S^{n_i}$. It sends a map $f\colon \bigvee_i S^{n_i} \to \bigvee_j S^{m_j}$ to the homotopy lift

$$\begin{tikzcd}
\bigvee_{n_i \leq n} S^{n_i} \arrow[r, hook] \arrow[rrd, dashed] & \bigvee_i S^{n_i} \arrow[r, "f"] & \bigvee_j S^{m_j} \\
 & & \bigvee_{m_j \leq n} S^{m_j} \arrow[u, hook]
\end{tikzcd}$$

which exists and is unique since $\bigvee_{m_j \leq n} S^{m_j} \hookrightarrow \bigvee_j S^{m_j}$ is an iso on π_k for $k \leq n$. By the same argument, I_n is left adjoint to T_n. The unit $1 \to T_n I_n$ is the identity, and the counit $I_n T_n \to 1$ is the inclusion of wedge summands of small dimension. Note that both I_n and T_n preserve coproducts (wedges).

Proposition 4.10. *There is an equivalence of categories* $I_n^*\colon \boldsymbol{\Pi}\mathbf{Alg}_1^n \cong \mathrm{Fun}^{\times}(\boldsymbol{\Pi}_n^{op}, \mathbf{Set})\colon T_n^*$.

Proof. If F is a product-preserving functor $\boldsymbol{\Pi}_n^{op} \to \mathbf{Set}$, then we have $I_n^* T_n^* F = (T_n I_n)^* F = F$, since $T_n I_n$ is the identity. On the other hand, if A is an n-truncated Π-algebra, we have $T_n^* I_n^* A = (I_n T_n)^* A \cong A$. Indeed, A sends all counit maps

$$I_n T_n(\bigvee_i S^{n_i}) = \bigvee_{n_i \leq n} S^{n_i} \hookrightarrow \bigvee_i S^{n_i}$$

to isos since A is n-truncated. □

Since $I_n\colon \boldsymbol{\Pi}_n^{\mathrm{op}} \to \boldsymbol{\Pi}^{\mathrm{op}}$ and $T_n\colon \boldsymbol{\Pi}^{\mathrm{op}} \to \boldsymbol{\Pi}_n^{\mathrm{op}}$ preserve products, they induce restriction functors $I_n^*\colon \mathrm{Fun}^{\times}(\boldsymbol{\Pi}^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{Fun}^{\times}(\boldsymbol{\Pi}_n^{\mathrm{op}}, \mathbf{Set})$ and T_n^*. Write $P_n\colon \boldsymbol{\Pi}\mathbf{Alg} \to \boldsymbol{\Pi}\mathbf{Alg}_1^n$ for I_n^*, which is the Postnikov n-truncation of Π-algebras, and $\iota_n\colon \boldsymbol{\Pi}\mathbf{Alg}_1^n \to \boldsymbol{\Pi}\mathbf{Alg}$ for T_n^*, which is the inclusion of n-truncated Π-algebras.

Proposition 4.11. *P_n is left adjoint to ι_n.*

Proof. (Functor point of view) $I_n \colon \mathbf{\Pi}_n \to \mathbf{\Pi}$ is the left adjoint, and thus $I_n \colon \mathbf{\Pi}_n^{\mathrm{op}} \to \mathbf{\Pi}^{\mathrm{op}}$ is the right adjoint. Note that $\mathrm{Fun}(-, \mathbf{Set})$ is a (strict) 2-functor $\mathbf{Cat}^{\mathrm{op}} \to \mathbf{Cat}$, where the superscript in $\mathbf{Cat}^{\mathrm{op}}$ means that 1-cells have been reversed but 2-cells do not change. The same holds for $\mathrm{Fun}^{\times}(-, \mathbf{Set})$, as long as we take only categories and product-preserving functors between them. Therefore $P_n = I_n^*$ is left adjoint to $\iota_n = T_n^*$. □

Proof. (Graded group point of view) A map $f \colon A \to \iota_n B$ of Π-algebras into an n-truncated Π-algebra is determined by the map of graded group up to degree n. The additional conditions are that f respect the additional structure (π_1-action, Whitehead products, and precomposition operations). The latter preserves or increases degree, which means all the conditions coming from or landing in degree greater than n are vacuous. In other words, the data of a map f is the same data as the corresponding map $P_n A \to B$ in $\mathbf{\Pi Alg}_1^n$. □

Both $\mathbf{\Pi Alg}$ and $\mathbf{\Pi Alg}_1^n$ are categories of universal algebras – finitary many-sorted varieties, to be more precise. The free Π-algebra on a graded set $\{X_i\}$ is $F\{X_i\} = \pi_*(\bigvee_i \bigvee_{j \in X_i} S^i)$. By combining the two adjunctions

$$\begin{tikzcd}
\mathbf{GrSet} \arrow[r, shift left, "F"] & \mathbf{\Pi Alg} \arrow[l, shift left, "U"] \arrow[r, shift left, "P_n"] & \mathbf{\Pi Alg}_1^n \arrow[l, shift left, "\iota_n"]
\end{tikzcd}$$

we see that the free n-truncated Π-algebra on $\{X_i\}$ is

$$F_n\{X_i\} = P_n \pi_*\Big(\bigvee_i \bigvee_{j \in X_i} S^i\Big) = \pi_*\Big(P_n \bigvee_i \bigvee_{j \in X_i} S^i\Big).$$

In both categories, projective objects are retracts of free objects and regular epis are surjections of underlying graded sets [14, II.4, Rem 1 after Prop 1]. In particular, the left adjoint P_n preserves projectives and prolongs to a left Quillen functor. Note also that $\{\pi_*(P_n S^1), \pi_*(P_n S^2), \ldots, \pi_*(P_n S^n)\}$ is a set of small projective generators for $\mathbf{\Pi Alg}_1^n$, which exhibits $\mathbf{\Pi Alg}_1^n$ as an algebraic category.

4.3.2. *Standard model structure.* The standard model structure on the category $s\mathbf{\Pi Alg}$ of simplicial Π-algebras is described in [7, § 4.5] and the same description holds for $s\mathbf{\Pi Alg}_1^n$. A map $f \colon X_\bullet \to Y_\bullet$ is a fibration (resp. weak eq) if it is so at the level of underlying graded sets or graded groups. Cofibrations are maps with the left lifting property with respect to acyclic fibrations and can be characterized as retracts of free maps.

Proposition 4.12. *The left Quillen functor $P_n \colon s\mathbf{\Pi Alg} \to s\mathbf{\Pi Alg}_1^n$ preserves weak equivalences and fibrations. In particular, it preserves cofibrant replacements.*

Proof. (Functor point of view) Let $f \colon X_\bullet \to Y_\bullet$ be a fibration (resp. weak eq) in $s\mathbf{\Pi Alg}$. Let P be a projective of $\mathbf{\Pi Alg}_1^n$, exhibited as a retract of a free by $P \xrightarrow{s} F \xrightarrow{p} P$. Then $(P_n f)_* \colon \mathrm{Hom}(P, P_n X_\bullet) \to \mathrm{Hom}(P, P_n Y_\bullet)$ is a retract of $\mathrm{Hom}(F, P_n f)$ so it suffices that the latter be a fibration (resp. weak eq) of simplicial sets.

Note that $F = F_n(S)$ is free on a graded set S empty above dimension n, so we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n X_\bullet) &= \mathrm{Hom}_{\mathbf{GrSet}}(S, U P_n X_\bullet) \\
&= \mathrm{Hom}_{\mathbf{GrSet}}(S, U X_\bullet) \\
&= \mathrm{Hom}_{\mathbf{\Pi Alg}}(F(S), X_\bullet).
\end{aligned}$$

Using this, we obtain:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n X_\bullet) \arrow[r, "(P_n f)_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n Y_\bullet) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathbf{\Pi Alg}}(F(S), X_\bullet) \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathbf{\Pi Alg}}(F(S), Y_\bullet).
\end{tikzcd}$$

Since f is a fibration (resp. weak eq) in $s\mathbf{\Pi Alg}$, the bottom row is a fibration (resp. weak eq) of simplicial sets. ◻

Proof. (Graded group point of view) The map $f\colon X_\bullet \to Y_\bullet$ is a fibration (resp. weak eq) of simplicial sets in each degree, hence the map $P_n f$ is a fibration (resp. weak eq) of simplicial sets in each degree, that is in degrees 1 through n. ◻

Corollary 4.13. *1. For any Π-algebra A, the comparison map of cotangent complexes $P_n(\mathbf{L}_A) \xrightarrow{\sim} \mathbf{L}_{P_nA}$ induced by the adjunction $P_n \dashv \iota_n$ is a weak equivalence.*

2. If N is a module over P_nA, then the comparison map in Quillen cohomology

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^n}(P_nA; N) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; \eta_A^* \iota_n N) \tag{24}$$

is a natural iso.

Proof. By 3.10, 3.11, and 4.12. ◻

Here $\eta_A\colon A \to \iota_n P_n A$ is the Postnikov truncation map. We would like a better description of the module $\eta_A^*\iota_n N$ in (24). Think of a module over A as an abelian Π-algebra on which A acts (cf. [7, § 4.11]), namely the kernel of the split extension as opposed to its "total space".

Lemma 4.14. *The category $\mathbf{Mod}_{P_nA}$ of modules over P_nA is isomorphic to the full subcategory $\mathbf{Mod}_A^{n\text{-}tr}$ of $\mathbf{Mod}_A$ of modules that happen to be n-truncated.*

Proof. Consider the adjunction on modules:

$$\begin{tikzcd}
\mathbf{Mod}_A \arrow[r, shift left, "P_n"] & \mathbf{Mod}_{P_nA} \arrow[l, shift left, "\eta_A^*\iota_n"]
\end{tikzcd}$$

from 3.2. The composite $P_n\eta_A^*\iota_n$ is the identity. Moreover, $\eta_A^*\iota_n$ lands in $\mathbf{Mod}_A^{n\text{-tr}}$. By restricting P_n to the latter, we obtain an adjunction $\mathbf{Mod}_A^{n\text{-tr}} \rightleftarrows \mathbf{Mod}_{P_nA}$ where both composites $P_n\eta_A^*\iota_n$ and $\eta_A^*\iota_n P_n$ are the identity, i.e. an iso of categories. ◻

The lemma justifies the abuse of notation in the following repackaged statement.

Theorem 4.15. *(Truncation isomorphism) Let A be a Π-algebra and N a module over A that is n-truncated. Then there is a natural isomorphism*

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^n}(P_nA; N) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; N).$$

The following example is of interest in light of theorems 1.3 and 9.6 in [7].

Example 4.16. Let A be an n-truncated Π-algebra. For k a positive integer, the k-fold loops $\Omega^k A$ form a module over A (which is zero if $k \geq n$) and we are interested in the cohomology groups $\mathrm{HQ}^*(A; \Omega^k A)$. Since $\Omega^k A$ is $(n-k)$-truncated, theorem 4.15 says $\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^{n-k}}(P_{n-k}A; \Omega^k A) \cong \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; \Omega^k A)$.

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