A generalized Kac-Moody algebra of rank 14

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Abstract

We construct a vertex algebra of central charge 26 from a lattice orbifold vertex operator algebra of central charge 12. The BRST-cohomology group of this vertex algebra is a new generalized Kac-Moody algebra of rank 14. We determine its root space multiplicities and a set of simple roots.

1 Introduction

So far, there are four generalized Kac-Moody algebras for which explicit vertex operator algebra constructions are known and the simple roots are determined. Besides the fake monster Lie algebra [Bor90] and monster Lie algebra [Bor92] constructed by Borcherds, these are the fake baby monster Lie algebra constructed by the authors [HS03] and the baby monster Lie algebra constructed by the first author [Höh03b]. The examples studied in [CKS07] depend on the existence of certain vertex operator algebras from [ANS93]. The general orbifold approach in [Car07] is more indirect.

The fake monster Lie algebra has rank 26 and the fake baby monster Lie algebra has rank 18. From several considerations [Sch06, Sch04, Bar03, ANS93], we expect that the largest possible rank besides 26 and 18 for which generalized Kac-Moody algebras with a natural vertex operator algebra construction exist is 14. In that case, we believe two such algebras exist: One is a $\mathbf{Z}_3$-twist of the fake monster Lie algebra which belongs to a series of generalized Kac Moody algebras investigated in [Bor92, Nie02, Sch04, CKS07]. The other can be a obtained from a $\mathbf{Z}_2$-twist of the fake monster Lie algebra corresponding to a class of involutions in the isomorphism group of the Leech lattice with a 12-dimensional fixed point lattice. In this note, we give a vertex operator algebra

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construction of this generalized Kac-Moody algebra and determine its simple roots. The approach of this paper is similar to the one in [HS03].

This new generalized Kac-Moody algebra together with the fake monster Lie algebra, the fake baby monster Lie algebra and the monster Lie algebra are the only generalized Kac-Moody algebras which can be obtained from a vertex operator algebra associated to a Niemeier lattice or the standard $\mathbf{Z}_2$-twist of such a vertex operator algebra. Furthermore, they seem to be all the generalized Kac-Moody algebras which can be described by framed vertex operator algebras.

The first three generalized Kac-Moody algebras mentioned in the first paragraph are obtained in the following way: Let V be the vertex operator algebra (VOA) V_Λ associated to the Leech lattice Λ, the Moonshine module VOA $V^\natural$ or the $\mathbf{Z}_2$-twist of V_K, where K is the Niemeier lattice with root lattice A_3^8. Let $V_{II_{1,1}}$ be the vertex algebra of the two-dimensional even unimodular Lorentzian lattice $II_{1,1}$. The tensor product $V\otimes V_{II_{1,1}}$ is a vertex algebra of central charge 26. By using the bosonic ghost vertex superalgebra $V_{\rm ghost}$ of central charge -26 one defines the Lie algebra $\mathbf{g}$ as the BRST-cohomology group $H^1_{\rm BRST}(V\otimes V_{II_{1,1}})$ (cf. [FGZ86]).

In the construction of our new Lie algebra $\mathbf{g}$, we take for V a VOA of central charge 24 which is obtained by glueing together the lattice VOA for the rescaled root lattice $\sqrt{2}D_{12}$ with the lattice $\mathbf{Z}_2$-orbifold V_K^+ of the extended rescaled root lattice $K=\sqrt{2}D_{12}^+$. The decomposition of V into $V_{\sqrt{2}D_{12}}$-modules can be described combinatorially using the theory of lattice $\mathbf{Z}_2$-orbifolds as developed in [AD04, ADL05, Shi04]. This combinatorial description together with the no-ghost theorem from string theory gives the root lattice and root multiplicities of $\mathbf{g}$. Then we construct an automorphic form on a Grassmannian of 2-planes in $\mathbf{R}^{14,2}$ using Borcherds' singular theta correspondence. The automorphic product can be interpreted as one side of the denominator identity of $\mathbf{g}$. This allows us to determine the simple roots.

One point which distinguishes $\mathbf{g}$ from the other three examples is that the Weyl group of the Lie algebra is not the full reflection group of the root lattice.

The paper is organized in the following way: In Section 2, the construction of a vertex operator algebra V is described and the $V_{\sqrt{2}D_{12}}$-module decomposition is used to express the U_1^{12}-equivariant character of V through theta series for the lattice $\sqrt{2}D_{12}$ and a vector valued modular function of weight -6. In the final section, the root lattice, the root multiplicities and the simple roots of $\mathbf{g}$ are determined.

2 The vertex operator algebra V of central charge 24

In this section, we define a vertex operator algebra V of central charge 24 by glueing together the lattice vertex operator algebra $V_{\sqrt{2}D_{12}}$ with the $\mathbf{Z}_2$-orbifold vertex operator algebra V_K^+ where $K = \sqrt{2}D_{12}^+$. Then we compute its character as a representation for the natural Heisenberg subalgebra of V.

2.1 The vertex operator algebra V_N and its intertwining algebra

Let $L \subset \mathbf{R}^n$ be an even integral lattice of rank n and let $L' = \{\lambda \in \mathbf{R}^n \mid (\lambda, \mu) \in \mathbf{Z}$ for all $\mu \in L\}$ be the dual lattice. The map $q_L : L'/L \to \mathbf{Q}/\mathbf{Z}$, $\lambda + L \mapsto (\lambda, \lambda)/2 \bmod \mathbf{Z}$, gives the discriminant group L'/L the structure of a finite quadratic space which is called the discriminant form of L. We sometimes write $\lambda^2 = (\lambda, \lambda)$ for the norm of λ.

The isomorphism classes of irreducible modules of the vertex operator algebra V_L associated to an even integral lattice L can be parameterized by the discriminant group L'/L of the lattice [DL93]. For each coset $\lambda + L \in L'/L$ there exists an an irreducible V_L-module which we denote by $V_{\lambda+L}$.

The fusion product between the irreducible modules is given by

$$V_{\lambda+L} \times V_{\mu+L} = V_{\lambda+\mu+L},$$

for $\lambda+L$, $\mu+L$ in L'/L, i.e., the fusion algebra of V_L is isomorphic to the group ring $\mathbf{C}[L'/L]$ and each simple module is a simple current.

On the direct sum of the irreducible modules of a lattice type vertex operator algebra V_L there is the structure of an abelian intertwining algebra [DL93], Th. 12.24, such that the cohomology class of the associated 3-cocycle is determined by the quadratic form q_L on L'/L. The conformal weights modulo $\mathbf{Z}$ of the irreducible V_L-modules $V_{\lambda+L}$, $\lambda + L \in L'/L$, are the values of the quadratic form q_L.

We collect these results in the following theorem.

Theorem 2.1 *The direct sum of the simple modules of V_L has the structure of an abelian intertwining algebra. The associated quadratic space can be identified with the discriminant form L'/L.* □

For the proof of some identities, it is useful to interpret an element f in $\mathbf{C}[L][[q^{1/k}]][q^{-1/k}]$, where L is a lattice and $k \in \mathbf{N}$, as a function on $\mathcal{H} \times (L \otimes \mathbf{C})$, where $\mathcal{H} = \{z \in \mathbf{C} \mid \operatorname{Im}(z) > 0\}$ is the complex upper half plane. This is done by the substitutions $q \mapsto e^{2\pi i\tau}$ and $e^{\mathbf{s}} \mapsto e^{2\pi i(\mathbf{s},\mathbf{z})}$ for $(\tau, \mathbf{z}) \in \mathcal{H} \times (L \otimes \mathbf{C})$ (in the case of convergence). We indicate this by writing $f(\tau, \mathbf{z})$.

Let $\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty}(1-q^k)$ be the Dedekind eta-function. We define the theta function of the coset $\lambda + L$ by

$$\theta_{\lambda+L} = \sum_{\mathbf{s}\in\lambda+L} q^{\mathbf{s}^2/2}\, e^{\mathbf{s}}.$$

The $\mathbf{Z}$-grading on a VOA $W = \bigoplus_{k=0}^{\infty} W_k$ is given by the eigenvalues of the Virasoro generator L_0. Suppose there is an action of a connected compact Lie group G on W respecting this grading. Let L be the weight lattice of a maximal torus of G. Then we denote by $W_k(s)$ the subspace of W_k of weight s. The character of W is defined by

$$\chi_W = q^{-c/24} \sum_{k\in\mathbf{Z}} \sum_{\mathbf{s}\in L} \dim W_k(\mathbf{s})\, q^k\, e^{\mathbf{s}},$$

where c is the central charge of W.

On V_L there is the action of $\mathbf{R}^n/L'$ by vertex operator algebra automorphisms. The L'/L-fold cover $T = \mathbf{R}^n/L$ acts also on the modules $V_{\lambda+L}$ and the weights form the coset $\lambda + L$.

From the construction of $V_{\lambda+L}$ one obtains the following description of the T-equivariant graded character.

Lemma 2.2 *The V_L-module $V_{\lambda+L}$ has the character $\theta_{\lambda+L}(\tau,\mathbf{z})/\eta(\tau)^n$.* □

We choose now for L the lattice $N = \sqrt{2}D_{12}$, i.e.

$$N = \left\{\sqrt{2}\,(x_1,\,\ldots,\,x_{12}) \in \mathbf{R}^{12} \mid \text{all } x_i \in \mathbf{Z} \text{ and } \sum_{i=1}^{12} x_i \equiv 0 \pmod 2\right\}.$$

Then the automorphism group of N is generated by the permutations of the coordinates and arbitrary sign changes. It has shape $2^{12}.\mathrm{Sym}_{12}$.

Lemma 2.3 *The discriminant group of N and the orbits under the induced action of $\mathrm{Aut}(N) \cong 2^{12}.\mathrm{Sym}_{12}$ on the discriminant group are described in Table 1. The lattice N has genus $II_{12,0}(2_{II}^{-10}4_{II}^{-2})$ in the notation of [CS93].*

Proof. The dual lattice of N is given by

$$N' = \left\{\tfrac{1}{\sqrt{2}}\,(x_1,\,\ldots,\,x_{12}) \in \mathbf{R}^{12} \mid \text{all } x_i \in \mathbf{Z} \text{ or all } x_i \in \mathbf{Z}+\tfrac{1}{2}\right\}.$$

It is easy to describe the decomposition of the discriminant group N'/N into orbits of $\mathrm{Aut}(N)$ and to determine representatives. The genus can be determined by diagonalizing a Gram matrix of N over the 2-adic integers. Note that the genus is uniquely determined by $(N'/N, q_N)$ and the rank of N. □

Table 1: Orbits of the discriminant group of N under $\mathrm{Aut}(N) \cong 2^{12}.\mathrm{Sym}_{12}$.

No.	representative	N'-orbit size	norm	orbit size	q_N	order
1	$\frac{1}{\sqrt{2}}(0^{12})$	1	0	1	0	1
2	$\frac{1}{\sqrt{2}}(2,0^{11})$	$2 \cdot 12$	2	1	0	2
3	$\frac{1}{\sqrt{2}}(1^{12})$	2^{12}	6	2	0	2
4	$\frac{1}{\sqrt{2}}(1^4,0^8)$	$2^4\binom{12}{4}$	2	990	0	2
5	$\frac{1}{\sqrt{2}}(1^8,0^4)$	$2^8\binom{12}{8}$	4	990	0	2
6	$\frac{1}{\sqrt{2}}(1^2,0^{10})$	$2^2\binom{12}{2}$	1	132	$1/2$	2
7	$\frac{1}{\sqrt{2}}(1^6,0^6)$	$2^6\binom{12}{6}$	3	1848	$1/2$	2
8	$\frac{1}{\sqrt{2}}(1^{10},0^2)$	$2^{10}\binom{12}{10}$	5	132	$1/2$	2
9	$\frac{1}{\sqrt{2}}(1,0^{11})$	$2 \cdot 12$	1/2	24	$1/4$	4
10	$\frac{1}{\sqrt{2}}(1^5,0^7)$	$2^5\binom{12}{5}$	5/2	1584	$1/4$	4
11	$\frac{1}{\sqrt{2}}(1^9,0^3)$	$2^9\binom{12}{9}$	9/2	440	$1/4$	4
12	$\frac{1}{\sqrt{2}}(\frac{3}{2},(\frac{1}{2})^{11})$	$2^{12} \cdot 12$	5/2	4096	$1/4$	4
13	$\frac{1}{\sqrt{2}}(1^3,0^9)$	$2^3\binom{12}{3}$	3/2	440	$3/4$	4
14	$\frac{1}{\sqrt{2}}(1^7,0^5)$	$2^7\binom{12}{7}$	7/2	1584	$3/4$	4
15	$\frac{1}{\sqrt{2}}(1^{11},0)$	$2^{11}\binom{12}{11}$	11/2	24	$3/4$	4
16	$\frac{1}{\sqrt{2}}((\frac{1}{2})^{12})$	2^{12}	3/2	4096	$3/4$	4

The first eight orbits of Table 1 form the 2-torsion subgroup of N'/N and the first three orbits consist of elements which are a multiple of 2 of another element. Thus the orbits in Table 1 separated by horizontal lines belong also to different orbits of N'/N under the action of the automorphism group of the discriminant form of N.

2.2 The vertex operator algebra V_K^+ and its intertwining algebra

As in the previous subsection, let $L \subset \mathbf{R}^n$ be an even integral lattice. We are interested in the VOA V_L^+, the fixed-point subspace of V_L under the involution induced from the -1 isomorphism of L. The irreducible modules of V_L^+ have been described in [AD04], their fusion rules in [ADL05] and the automorphism group in [Shi04, Shi06].

We specialize the discussion here to the case of the lattice $K = \sqrt{2}D_{12}^+$, i.e.

$$K = \left\{\sqrt{2}\,(x_1,\,\ldots,\,x_{12}) \mid \text{all } x_i \in \mathbf{Z} \text{ or all } x_i \in \mathbf{Z} + \tfrac{1}{2} \text{ and } \sum_{i=1}^{12} x_i \equiv 0 \pmod{2}\right\}.$$

The automorphism group of K is isomorphic to the Weyl group $W(D_{12})$ and has shape $2^{11}.\mathrm{Sym}_{12}$. The lattice D_{12}^+ is the unique indecomposable unimodular integral lattice in dimension 12.

The following lemma is easy to prove.

Lemma 2.4 *The discriminant group of K and the orbits under the induced action of $\mathrm{Aut}(K) \cong 2^{11}.\mathrm{Sym}_{12}$ on the discriminant group are described in Table 2. The lattice K has genus $II_{12,0}(2_4^{+12})$.* □

Let θ be the involution in $\mathrm{Aut}(V_K)$ which is up to conjugation the unique lift of the involution -1 in $\mathrm{Aut}(K)$ to $\mathrm{Aut}(V_K)$ (cf. [DGH98], Appendix D). Denote by V_K^+ the fixed point vertex operator subalgebra of V_K under the action of θ.

The isomorphism classes of irreducible modules of V_K^+ can be found in [AD04]. Since K is 2-elementary, that is $2K' \subset K$, the discussion can be simplified, cf. [Shi04], Section 3.2. The isomorphism classes of irreducible modules of V_K^+ consist of the so called untwisted modules $V_{\lambda+K}^{\pm}$ where $\lambda + K$ runs through the discriminant group K'/K and certain so called twisted modules $V_K^{T_\chi,\pm}$.

The fusion rules between the 2^{13} modules $V_{\lambda+K}^{\pm}$ are

$$V_{\lambda+K}^{\delta} \times V_{\mu+K}^{\epsilon} = V_{\lambda+\mu+K}^{\pm},$$

where δ, $\epsilon \in \{\pm\} \cong \mathbf{Z}_2$ and λ, $\mu \in K'$ and the exact sign in $V_{\lambda+\mu+K}^{\pm}$ can be determined from the discriminant form of K.

Since the fusion product $\times$ is commutative and associative we see that it induces on the set $\{V_{\lambda+K}^{\pm} \mid \lambda + K \in K'/K\}$ of isomorphism classes of untwisted

Table 2: Orbits of the discriminant group of K under $\mathrm{Aut}(K) \cong 2^{11}.\mathrm{Sym}_{12}$.

No.	representative	K'-orbit size	norm	orbit size	q_K	order
1	$\frac{1}{\sqrt{2}}(0^{12})$	1	0	1	0	1
2	$\frac{1}{\sqrt{2}}(2, 0^{11})$	$2 \cdot 12$	2	1	0	2
3	$\frac{1}{\sqrt{2}}(1^4, 0^8)$	$2^4 \cdot \binom{12}{4}$	2	990	0	2
4	$\frac{1}{\sqrt{2}}(1^2, 0^{10})$	$2^2 \cdot \binom{12}{2}$	1	132	$1/2$	2
5	$\frac{1}{\sqrt{2}}(1^6, 0^6)$	$2^6 \cdot \binom{12}{6}$	3	924	$1/2$	2
6	$\frac{1}{\sqrt{2}}(-\frac{3}{2}, (\frac{1}{2})^{11})$	$2^{11} \cdot 12$	5/2	1024	$1/4$	2
7	$\frac{1}{\sqrt{2}}((\frac{1}{2})^{12})$	2^{11}	3/2	1024	$3/4$	2

V_K^+-modules the structure of an abelian group of exponent 4. In fact, $V_{\lambda+K}^{\pm}$ is of order 4 if and only if λ has non-integral norm, cf. [Shi04], Remark 3.5. It follows from a careful examination of [ADL05], Theorem 5.1, (see also p. 216 loc. cit.) that the fusion product on the the set of all isomorphism classes of irreducible modules forms also an abelian group A of exponent 4 which is isomorphic to $\mathbf{Z}_2^{10} \times \mathbf{Z}_4^2$. In particular, all twisted modules are of order 4 in that group.

The VOA V_K^+ is rational and satisfies all the conditions which guarantee [Hua05] that its intertwining algebra defines a modular tensor category. Since all modules are simple currents one can conclude that the conformal weights modulo $\mathbf{Z}$ define a quadratic form q_A on A and (A, q_A) becomes a finite quadratic space. We avoid this argument by explicitly describing q_A in the following.

To determine the precise type of (A, q_A) and for later use, we consider first the characters of the irreducible V_K^+-modules. One has [FLM88]

$$\begin{aligned}
\chi_{V_K^{\pm}} &= \frac{1}{2}\left(\frac{\theta_K(q)}{\eta(q)^{12}} \pm \frac{\eta(q)^{12}}{\eta(q^2)^{12}}\right), \\
\chi_{V_{\lambda+K}^{\pm}} &= \frac{1}{2}\frac{\theta_{\lambda+K}(q)}{\eta(q)^{12}}, \quad \text{for } \lambda + K \neq K, \qquad (1) \\
\chi_{V_K^{T_\chi,\pm}} &= \frac{1}{2}\, q^{3/4}\left(\frac{\eta(q)^{12}}{\eta(q^{1/2})^{12}} \pm \frac{\eta(q^2)^{12}\eta(q^{1/2})^{12}}{\eta(q)^{24}}\right).
\end{aligned}$$

In particular, the characters of the V_K^+-modules $V_{\lambda+K}^{\pm}$ depend for $\lambda + K \neq K$ only on the orbit of $\lambda + K$ under $\mathrm{Aut}(K)$ in K'/K. We denote the character of $V_{\lambda+K}^{+}$ for $\lambda+K$ belonging to the orbit i in Table 2 by g_i. An explicit computation gives

$$\begin{aligned}
g_1 &= q^{-1/2}\,(1 + 210\,q^2 + 2752\,q^3 + 29727\,q^4 + 225408\,q^5 + \cdots),\\
g_2 &= q^{-1/2}\,(12\,q + 144\,q^2 + 2984\,q^3 + 29088\,q^4 + 227004\,q^5 \cdots),\\
g_3 &= q^{-1/2}\,(4\,q + 176\,q^2 + 2872\,q^3 + 29408\,q^4 + 226196\,q^5 + \cdots),\\
g_4 &= q^{-1/2}\,(\,q^{1/2} + 32\,q^{3/2} + 768\,q^{5/2} + 9600\,q^{7/2} + 83968\,q^{9/2} + \cdots),\\
g_5 &= q^{-1/2}\,(32\,q^{3/2} + 384\,q^{5/2} + 4992\,q^{7/2} + 49408\,q^{9/2} + \cdots),\\
g_6 &= q^{-1/2}\,(12\,q^{5/4} + 376\,q^{9/4} + 5316\,q^{13/4} + 50088\,q^{17/4} + \cdots),\\
g_7 &= q^{-1/2}\,(\,q^{3/4} + 78\,q^{7/4} + 1509\,q^{11/4} + 16966\,q^{15/4} + \cdots).
\end{aligned}$$

Now we discuss the automorphism group of V_K^+ (see [Shi04]) and its induced action on the quadratic space (A, q_A) although this information is not really necessary for the construction and understanding of the generalized Kac-Moody algebra $\mathbf{g}$. The centralizer H of θ in $\mathrm{Aut}(V_K)$ acts on V_K^+. H has shape $2^{12}.\mathrm{Aut}(K)$, where the 2^{12} can be identified with $\mathrm{Hom}(K, \mathbf{Z}_2)$. The element $\theta \in H$ acts trivially. The induced action of H on the set of isomorphism classes of irreducible V_K^+-modules stabilizes the set of untwisted modules. For $g \in H$ one has

$$g \circ \{V^{\pm}_{\lambda+K}\} = \{V^{\pm}_{\bar{g}(\lambda+K)}\}$$

where $\bar{g}$ is the image of g in $\mathrm{Aut}(K)$ regarded as an element of $\mathrm{Aut}(K')$. Moreover, if $g \in \mathrm{Hom}(K, \mathbf{Z}_2) \subset H$ then

$$g \circ V^{\pm}_{\lambda+K} = \begin{cases} V^{\pm}_{\lambda+K} & \text{if } g(2\lambda) = 0,\\ V^{\mp}_{\lambda+K} & \text{if } g(2\lambda) = 1\,. \end{cases}$$

Thus if we have an element $\lambda \in K'$ for which 2λ is not in $2K$, i.e. $\lambda \notin K$, we can find an $g \in \mathrm{Hom}(K, \mathbf{Z}_2) \subset H$ with $g(2\lambda) = 1$. It follows that the modules $V^{\pm}_{\bar{g}(\lambda+K)}$, where $\lambda + K$ runs through an $\mathrm{Aut}(K)$-orbit, belong for $\lambda \notin K$ all to the same H-orbit.

It was shown by Shimakura [Shi04] that the orbit of V_K^- under $\mathrm{Aut}(V_K^+)$ consists (since the dimension of K is different from 8 or 16) in addition of those $V^{\pm}_{\lambda+K}$, $\lambda + K \neq K$, for which the number of norm 2 vectors in $\lambda + K$ is exactly 24. This is exactly the case if $\lambda+K$ belongs to the orbit no. 2 of size 1 in Table 2. From this it can be deduced that $\mathrm{Aut}(V_K^+)$ has shape $2^{11}.2^{10}.\mathrm{Sym}_{12}.\mathrm{Sym}_3$. The point is that K can be constructed as L_C^+ for the binary code $C = \{0^{12}, 1^{12}\}$ (cf. Remark 2.7 below), but the code C cannot be constructed as C_Γ^+ for a Kleinian code Γ.

We collect the results in Table 3. Here we write $[n]^{\pm}$ for the set of modules $V^{\pm}_{\lambda+K}$ for which $\lambda + K$ belongs to the orbit no. n in Table 2. Note that g_n is the character of the V_K^+-modules in the n-th orbit in Table 3.

The only entry in Table 3 which remains to be discussed are the H-orbits of the twisted modules. If V_K^+ is extended by the unique module belonging to $[2]^+$

Table 3: Orbits of irreducible modules of V_K^+ under $\mathrm{Aut}(V_K^+) \cong 2^{11}.2^{10}.\mathrm{Sym}_{12}.\mathrm{Sym}_3$.

No.	H-orbits	orbit size	h	q	order	character
1	$[1]^+$	1	0	0	1	g_1
2	$[1]^-$, $[2]^+$, $[2]^-$	3×1	1	0	2	g_2
3	$[3]^+$, $[3]^-$	2×990	1	0	2	g_3
4	$[4]^+$, $[4]^-$	2×132	$1/2$	$1/2$	2	g_4
5	$[5]^+$, $[5]^-$	2×924	$3/2$	$1/2$	2	g_5
6	$[6]^+$, $[6]^-$, $\{[\chi]^-\}$	$3 \times 2 \times 1024$	$5/4$	$1/4$	4	g_6
7	$[7]^+$, $[7]^-$, $\{[\chi]^+\}$	$3 \times 2 \times 1024$	$3/4$	$3/4$	4	g_7

or $[2]^-$ then one obtains an extension V' of the VOA V_K^+ which is isomorphic to the lattice VOA V_K, but some twisted V_K^+-modules become now untwisted modules for V_K^+ considered as the fixed point VOA of $V' \cong V_K$. Under the extra automorphisms in $\mathrm{Aut}(V_K^+)$ which map $[1]^-$ to $[2]^+$ or $[2]^-$, a twisted V_K^+-module may be mapped to an untwisted one. In fact, this can be done for all the twisted V_K^+-modules, cf. [FLM88], Ch. 11. Now it follows from the given conformal characters that all twisted modules $V_K^{T_\chi,-}$ belong to the orbits $[6]^+$ and $[6]^-$ and all twisted modules $V_K^{T_\chi,+}$ belong to the orbits $[7]^+$ and $[7]^-$.

The above discussion and Lemma 2.3 imply:

Theorem 2.5 *The direct sum of representatives for the isomorphism classes of irreducible modules of V_K^+ can be given the structure of an abelian intertwining algebra. The associated quadratic space (A, q_A) is isomorphic to the discriminant form of the lattice N.* □

Remark 2.6 *There are exactly 6 orbits under the action of $\mathrm{Aut}(A, q_A)$ on A.*

Proof. It is enough to show that the orbits no. 4 and 5 in Table 3 fuse into a single orbit under $\mathrm{Aut}(A, q_A)$ since the remaining orbits of $\mathrm{Aut}(V_K^+)$ can be distinguished by the group structure of A and by q_A.

The genus $II_{12,0}(2_{II}^{-10}4_{II}^{-2})$ of N consists of the two classes $\sqrt{2}D_{12}$ and $\sqrt{2}(E_8 \oplus D_4)$ which have isomorphic discriminant forms. The automorphism group of $\sqrt{2}(E_8 \oplus D_4)$ is isomorphic to $W(E_8) \times W(D_4).\mathrm{Sym}_3$. Its induced action on the discriminant group decomposes the order 2 elements of norm $1/2$ into four orbits of size 12, 120, 360 and 1620. Since the corresponding $\mathrm{Aut}(V_K^+)$-orbits no. 4 and 5 have size 264 and 1848, the only possibility is that all these orbits fuse under the action of the larger group $\mathrm{Aut}(A, q_A)$ into one orbit of size 2112. $\square$

Remark 2.7 *V_K^+ is isomorphic to the framed code VOA $V_{\mathcal{C}}$ associated to the binary code $\mathcal{C}$ dual to the code $\mathcal{D}$ with generator matrix*

$$\begin{pmatrix} 1111\ 1111\ 0000\ 0000\ 0000\ 0000 \\ 0000\ 1111\ 1111\ 0000\ 0000\ 0000 \\ 0000\ 0000\ 1111\ 1111\ 0000\ 0000 \\ 0000\ 0000\ 0000\ 1111\ 1111\ 0000 \\ 0000\ 0000\ 0000\ 0000\ 1111\ 1111 \\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100 \\ 1010\ 1010\ 1010\ 1010\ 1010\ 1010 \end{pmatrix}.$$

Proof. The lattice K can be written in terms of the binary code $C = \{0^{12},\, 1^{12}\}$ of length 12 which is generated by the overall-one vector (1111 1111 1111) as

$$K = L_C^+ = \frac{1}{\sqrt{2}}\{c + x \mid c \in C,\ x \in (2\mathbf{Z})^{12} \text{ with } \sum_{i=1}^{12} x_i \equiv 0 \pmod 4\}.$$

Now the result follows from the Virasoro decomposition of $\widetilde{V}_{\widetilde{L}_C}$ given in [DGH98], Th. 4.10, by observing that the first term in the sum corresponds to $V_{L_C^+}^+$ so that V_K^+ and $V_{\mathcal{C}}$ have the same Virasoro decomposition and must therefore be isomorphic (see [DGH98], Prop. 2.16 and [Höh03a], Th. 4.3). Note that for C all markings are equivalent and that the proof of Th. 4.10 in [DGH98] shows that the self-duality assumption on C is unnecessary. $\square$

Theorem 2.5 can also be obtained from this remark and the results of [Miy98], where all irreducible modules of a framed code VOA $V_{\mathcal{C}}$ are described.

2.3 The gluing of V_N and V_K^+

The quadratic spaces (A, q_A) and $(A, -q_A)$ are isomorphic. We choose an isomorphism $i : N'/N \to A$ between the spaces $(N'/N, q_N)$ and $(A, -q_A)$. Let V be the $V_N \otimes V_K^+$-module

$$V \;=\; \bigoplus_{\lambda \in N'/N} V_\lambda \otimes V_K^+(i(\lambda)),$$

where $V_K^+(a)$ denotes the irreducible V_K^+-module labeled by $a \in A$.

Proposition 2.8 *The $V_N \otimes V_K^+$-module V has a unique simple VOA structure extending the VOA $V_N \otimes V_K^+$.*

Proof. The isomorphism i defines the isotropic subspace

$$C = \{(\lambda, i(\lambda)) \mid \lambda \in N'/N\} \subset (N'/N, q_N) \oplus (A, q_A).$$

It follows from Theorem 2.5 and [DL93] that the direct sum of the irreducible modules of the VOA $V_N \otimes V_K^+$ has the structure of an abelian intertwining algebra for the finite quadratic space $(N'/N, q_N) \oplus (A, q_A)$. The proposition follows now from [Höh03a], Theorem 4.3 (or [DM04]). □

Remark 2.9 The isomorphism type of V could (and in fact does) depend on the chosen isomorphism i. The reason is that neither the image of $\mathrm{Aut}(V_N)$ nor $\mathrm{Aut}(V_K^+)$ for the induced action on the set of isomorphism classes of irreducible modules is the full orthogonal group of the corresponding finite quadratic space. This follows from the observation that in both cases the six orbits of the orthogonal group split in 16 respectively 7 orbits. There are up to automorphisms six possibilities for i. They correspond to the VOAs with affine Kac-Moody subVOA $B_{12,2}$, $B_{6,2}^2$, $B_{4,2}^3$, $B_{3,2}^4$, $B_{2,2}^6$ or $A_{1,4}^{12}$ in Schellekens' list [ANS93] of self-dual VOA candidates of central charge 24.

The lattice N in the construction of V could be replaced by the lattice $\sqrt{2}(E_8 \oplus D_4)$ (cf. Remark 2.6). In this case the resulting VOAs would have the affine Kac-Moody subVOA $A_{8,2}F_{4,2}$, $C_{4,2}A_{4,2}^2$ or $D_{4,4}A_{2,2}^4$.

We extend the action of the torus T from section 2.1 on V_N and its modules to V by taking the trivial T-action on V_K^+ and its modules. Note that the T-action on V is compatible with the Virasoro module structure.

For the next theorem, we define $f_n = g_n/\eta^{12}$. Explicitly, one has

$$\begin{aligned}
f_1 &= q^{-1} + 12 + 300\,q + 5792\,q^2 + 84186\,q^3 + \cdots\,, \\
f_2 &= 12 + 288\,q + 5792\,q^2 + 84096\,q^3 + \cdots\,, \\
f_3 &= 4 + 224\,q + 5344\,q^2 + 81792\,q^3 + \cdots\,, \\
f_4 &= q^{-1/2} + 44\,q^{1/2} + 1242\,q^{3/2} + 22216\,q^{5/2} + \cdots \\
f_5 &= 32\,q^{1/2} + 1152\,q^{3/2} + 21696\,q^{5/2} + \cdots\,, \\
f_6 &= 12\,q^{1/4} + 520\,q^{5/4} + 10908\,q^{9/4} + \cdots\,, \\
f_7 &= q^{-1/4} + 90\,q^{3/4} + 2535\,q^{7/4} + 42614\,q^{11/4} + \cdots\,.
\end{aligned} \qquad (2)$$

For $\gamma \in N'/N$, we let $f_\gamma = f_n$ if $i(\gamma)$ belongs to the $\mathrm{Aut}(V_K^+)$-orbit no. n in Table 3.

The expression for the T-equivariant graded character of V at which we arrive is described in the following theorem.

Theorem 2.10

$$\chi_V(\tau, \mathbf{z}) = \sum_{\gamma \in N'/N} f_\gamma(\tau)\, \theta_\gamma(\tau, \mathbf{z}).$$

Proof. The theorem follows from Lemma 2.2 together with the definition of V and the f_γ. □

3 The generalized Kac-Moody algebra g

In this section we construct a generalized Kac-Moody algebra $\mathbf{g}$ from V. We determine its simple roots using the singular theta correspondence.

Let $II_{1,1}$ be the even unimodular Lorentzian lattice of rank 2 and $V_{II_{1,1}}$ the associated lattice vertex algebra. Let V be the VOA of the last section. Then the tensor product $W = V \otimes V_{II_{1,1}}$ is a vertex algebra of central charge 26.

Let $L = N \oplus II_{1,1}$. Since $II_{1,1}$ is unimodular this decomposition gives an isomorphism between the discriminant form of L and that of N.

Lemma 3.1 *The isomorphism type of the vertex algebra W does not depend on the isomorphism i used in the definition of V.*

Proof. From the isomorphism $i : (N'/N, q_N) \to (A, -q_A)$ we obtain an isomorphism $i' : (L'/L, q_L) \to (A, -q_A)$ and W has as $V_L \otimes V_K^+$-module the decomposition

$$V = \bigoplus_{\gamma \in L'/L} V_\gamma \otimes V_K^+(i'(\gamma)).$$

Since $\mathrm{Aut}(L)$ maps surjectively onto the automorphism group of $(L'/L, q_L)$ (Theorem 1.14.2, [Nik80]) the same holds for the induced action of $\mathrm{Aut}(V_L)$ on the set of isomorphism types of V_L-modules. Hence the result of the gluing depends up to an automorphism of V_L not on the chosen isomorphism i'. □

We remark that if in the construction of V the lattice N is replaced by $\sqrt{2}(E_8 \oplus D_4)$ then the resulting vertex algebra would also be isomorphic to W because $N \oplus II_{1,1} \cong \sqrt{2}(E_8 \oplus D_4) \oplus II_{1,1}$.

There is an action of the BRST-operator on the tensor product of the vertex algebra W of central charge 26 with the bosonic ghost vertex superalgebra V_{ghost} of central charge -26, which defines the BRST-cohomology groups $H^n_{\mathrm{BRST}}(W)$. The degree 1 cohomology group $H^1_{\mathrm{BRST}}(W)$ has additionally the structure of a Lie algebra [FGZ86, LZ93].

We can assume that V is defined over the field of real numbers. The same holds for the vertex algebra $V_{II_{1,1}}$, for V_{ghost} and hence for W.

Definition 3.2 We define the Lie algebra $\mathbf{g}$ as $H^1_{\mathrm{BRST}}(W)$.

Then the no-ghost theorem implies (cf. Prop. 3.2, [HS03]).

Proposition 3.3 *The Lie algebra* $\mathbf{g}$ *is a generalized Kac-Moody algebra graded by the lattice* $N' \oplus II_{1,1} = L'$*. Its components* $\mathbf{g}(\alpha)$*, for* $\alpha = (\mathbf{s}, r) \in N' \oplus II_{1,1}$ *are isomorphic to* $V_{1-r^2/2}(2\,\mathbf{s})$ *for* $\alpha \neq 0$ *and to* $V_1(0) \oplus \mathbf{R}^{1,1} \cong \mathbf{R}^{13,1}$ *for* $\alpha = 0$.

□

The subspace $\mathbf{g}(0)$ of degree $0 \in L'$ is a Cartan subalgebra for $\mathbf{g}$.

We denote the Fourier coefficient of f_γ at q^n by $[f_\gamma](n)$ and for $\alpha \in L'$ we let $\bar{\alpha} = \alpha \bmod L \in L'/L$.

Theorem 3.4 *For a nonzero vector* $\alpha \in L'$ *the dimension of* $\mathbf{g}(\alpha)$ *is given by*

$$\dim \mathbf{g}(\alpha) = [f_{\bar{\alpha}}](-\alpha^2/2).$$

The dimension of the Cartan subalgebra is 14.

Proof. Theorem 2.10 and Proposition 3.3. □

From the Fourier expansion of the f_γ we can read off the real roots of $\mathbf{g}$. Recall that we use the isomorphism i' to identify $(L'/L, q_L)$ with $(A, -q_A)$.

Corollary 3.5 *The real roots of* $\mathbf{g}$ *are the vectors*
$\alpha \in L$ *with* $\alpha^2 = 2$,
$\alpha \in L'$ *with* $\alpha^2 = 1$ *and* $i'(\bar{\alpha})$ *belongs to the orbit no. 4 in Table 3,*
$\alpha \in L'$ *with* $\alpha^2 = 1/2$.
They all have multiplicity 1.

The reflections in the real roots generate the Weyl group W of $\mathbf{g}$.

The Weyl group W has a Weyl vector, i.e. there is a vector ρ in $L' \otimes \mathbf{R}$ such that a set of simple roots of W are the roots α of W satisfying $(\rho, \alpha) = -\alpha^2/2$. The vector 2ρ is a primitive norm 0 vector in L' and 2ρ is in $2L' \bmod L$ (cf. Thm. 12.1 and 10.4 in [Bor98]).

Proposition 3.6 *The simple roots of the reflection group* W *form a set of real simple roots for* $\mathbf{g}$.

Proof. This follows from Cor. 2.4 in [Bor88]. □

Since L' is Lorentzian $L' \otimes \mathbf{R}$ has two cones of vectors of norm ≤ 0. The inner product of two nonzero vectors in one of the cones is at most 0 and equal to 0 if and only if both are positive multiples of the same norm 0 vector.

Proposition 3.7 *The vectors* $2n\rho$ *where* n *is a positive integer are imaginary simple roots of multiplicity* 12.

Proof. Since ρ has negative inner product with all real simple roots, ρ lies in the fundamental Weyl chamber C. We can choose imaginary simple roots lying in C (Prop. 2.1 in [Bor88]). It follows that ρ has inner product ≤ 0 with all simple roots. Now write $2n\rho$ as a sum of simple roots with positive integral coefficients, i.e. $2n\rho = \sum c_i \alpha_i$. Then $0 = \sum c_i(\alpha_i, 2n\rho) \leq 0$ so that $(\alpha_i, 2n\rho) = 0$ for all i. Since 2ρ is primitive in L' it follows that α_i is a positive multiple of 2ρ. Finally all positive multiples of 2ρ are simple roots because the support of a root is connected. The L-cosets of the $2n\rho$ are mapped by i' to the orbits no. 1 and 2 in Table 3. The constant coefficient of f_1 and f_2 is 12 so that the $2n\rho$ all have multiplicity 12. □

We will see that we have already found a complete set of simple roots for $\mathbf{g}$.

Proposition 3.8 *The function $F = \sum_{\gamma \in L'/L} f_\gamma e^\gamma$ is a vector valued modular function of weight -6 for the Weil representation of* $\mathrm{SL}_2(\mathbf{Z})$ *associated to L'/L.*

This follows in principle from the theory of VOAs since the f_γ are up to the factor $1/\eta^{12}$ the characters of the irreducible V_K^+-modules and the VOA V_K^+ has a modular tensor category associated to the finite quadratic space (A, q_A). However, we will give a direct proof.

Proof. Since we identify $(L'/L, q_L)$ with $(A, -q_A)$ by i' we have to show that $F = \sum_{a \in A} f_a e^a$, where $f_a = f_n$ if a belongs to the $\mathrm{Aut}(V_K^+)$-orbit no. n in Table 3, is a vector valued modular function of weight -6 with respect to the dual Weil representation of $\mathrm{SL}_2(\mathbf{Z})$ for the quadratic space (A, q_A).

The theta function $\Theta_K = \sum_{\mu \in K'/K} \theta_\mu e^\mu$ transforms under the dual Weil representation of K'/K. Hence $\Theta_K/(2\Delta)$ where $\Delta = \eta^{24}$ is a modular function of weight -6 for the dual Weil representation associated to K'/K.

Let $H = \{[1]^+, [1]^-\}$ be the order 2 subgroup of A corresponding to the two V_K^+-modules $[1]^+$ and $[1]^-$. Then the orthogonal complement $H^\perp$ of H in A consists of the set of untwisted V_K^+-modules denoted by $[n]^\pm$, $n = 1, \ldots, 7$, in Table 3 and the quotient $H^\perp/H$ is naturally isomorphic to K'/K.

Let $F_K = \sum_{a \in A} F_{K,a} e^a$ be the function with components $F_{K,a} = \theta_{\lambda+K}/(2\Delta)$ if $a \in H^\perp$ is mapped to $\lambda + K$ in $H^\perp/H \cong K'/K$ and $F_{K,a} = 0$ otherwise. It follows that F_K is a modular function of weight -6 for the dual Weil representation of (A, q_A).

Let $h(\tau) = 1/\eta(2\tau)^{12}$ and denote by $F_{h/2,0}$ and $F_{-h/4,H}$ the lifts of $h/2$ and $-h/4$ on the isotropic subgroups 0 and H, respectively (cf. [Sch09]). The liftings $F_{h/2,0}$ and $F_{-h/4,H}$ are also modular functions of weight -6 for the dual Weil representation of (A, q_A).

Explicit calculations using the equations (1) together with the identities arising from the induced action of $\mathrm{Aut}(V_K^+)$ on A show that

$$F = F_K + F_{h/2,0} + F_{-h/4,H}.$$

□

Now we use the singular theta correspondence to show that we have found above a set of simple roots for $\mathbf{g}$.

Theorem 3.9 *A set of simple roots for* $\mathbf{g}$ *is the following: The real simple roots are the vectors*

$\alpha \in L$ *with* $\alpha^2 = 2$,
$\alpha \in L'$ *with* $\alpha^2 = 1$ *and* $i'(\bar{\alpha})$ *belongs to the orbit no. 4 in Table 3,*
$\alpha \in L'$ *with* $\alpha^2 = 1/2$

and which satisfy $(\rho, \alpha) = -\alpha^2/2$. *The imaginary simple roots are the positive integral multiples of* 2ρ *each with multiplicity* 12.

Proof. Let $M = L \oplus II_{1,1} = N \oplus II_{1,1} \oplus II_{1,1}$. Then M'/M is isomorphic to N'/N and hence F defines a vector valued modular function for the Weil representation of M'/M. The singular theta correspondence associates to F an automorphic product Ψ on the Grassmannian of 2-dimensional negative definite subspaces of $M \otimes \mathbf{R}$. The level one expansion of Ψ is given by

$$e((\rho, Z)) \prod_{\alpha \in L'^{+}} \big(1 - e((\alpha, Z))\big)^{[f_{\bar{\alpha}}](-\alpha^2/2)} .$$

The automorphic form Ψ has singular weight so that the Fourier expansion is supported only on norm 0 vectors. Furthermore, Ψ is antisymmetric under the Weyl group W. It follows that Ψ has the sum expansion

$$\sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} \big(1 - e((2nw\rho, Z))\big)^{12} .$$

Now let $\mathbf{k}$ be the generalized Kac-Moody algebra with simple roots as stated in the theorem and Cartan subalgebra $L' \otimes \mathbf{R}$. Then the above argument shows that the denominator identity of $\mathbf{k}$ is given by

$$e^{\rho} \prod_{\alpha \in L'^{+}} (1 - e^{\alpha})^{[f_{\bar{\alpha}}](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, w\Big(e^{\rho} \prod_{n>0} (1 - e^{2n\rho})^{12}\Big) .$$

Hence $\mathbf{g}$ and $\mathbf{k}$ have the same root multiplicities. When we have fixed a Cartan subalgebra and a fundamental Weyl chamber the root multiplicities of a generalized Kac-Moody algebra determine the simple roots because of the denominator identity. It follows that $\mathbf{g}$ and $\mathbf{k}$ have the same simple roots and therefore are isomorphic. □

Corollary 3.10 *The denominator identity of* $\mathbf{g}$ *is*

$$e^{\rho} \prod_{\alpha \in L'^{+}} (1 - e^{\alpha})^{[f_{\bar{\alpha}}](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, w\Big(e^{\rho} \prod_{n>0} (1 - e^{2n\rho})^{12}\Big) .$$

□

We finish with two remarks.

Remark 3.11 The Lie algebra $\mathbf{g}$ can also be constructed by orbifolding the fake monster algebra [Bor92] with an automorphism in class $2C$ of Co_0. Such an automorphism has cycle shape 2^{12}.

Remark 3.12 The same method as for the construction of $\mathbf{g}$ can also be applied to the construction of the fake baby monster algebra [Bor92, HS03]. In this case one takes for K the rank 8 lattice $\sqrt{2}E_8$. The VOA V_K^+ has an abelian intertwining algebra based on a finite quadratic space (A, q_A) with A a 2-elementary group of order 2^{10}. The automorphism group $O^+(10,2)$ of V_K^+ equals in this case the isomorphism group of (A, q_A) [Gr98, Shi04]. For the lattice N, one can take any of the 17 lattices in the corresponding genus $II_{16,0}(2_{II}^{+10})$. The resulting VOAs V which here clearly do not depend on the chosen isomorphisms i are the 17 VOAs occurring in Schellekens' list of self-dual VOAs of central charge 24 having a Lie algebra V_1 of rank 16 [ANS93]. The resulting vertex algebras W and the corresponding Lie algebras are again isomorphic. In the paper [HS03], we started with V belonging to the affine Kac-Moody VOA $A_{1,2}^{16}$.

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