LARGE DEVIATIONS FOR ZEROS OF $P(\varphi)_2$ RANDOM POLYNOMIALS

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ABSTRACT. We extend the results of [ZZ] on LDP's (large deviations principles) for the empirical measures

$$Z_s := \frac{1}{N} \sum_{\zeta : s(\zeta)=0} \delta_\zeta, \quad (N := \#\{\zeta : s(\zeta) = 0\})$$

of zeros of Gaussian random polynomials s in one variable to $P(\varphi)_2$ random polynomials. The speed and rate function are the same as in the associated Gaussian case. It follows that the expected distribution of zeros in the $P(\varphi)_2$ ensembles tends to the same equilibrium measure as in the Gaussian case.

The purpose of this note is to extend the LDP (large deviation principle) of [ZZ] for the empirical measure

$$Z_s := d\mu_\zeta := \frac{1}{N} \sum_{\zeta : s(\zeta)=0} \delta_\zeta, \quad N := \#\{\zeta : s(\zeta) = 0\} \tag{1}$$

of zeros of Gaussian random holomorphic polynomials s of degree N in one variable to certain non-Gaussian measures which we call $P(\varphi)_2$ random polynomials. These are finite dimensional analogues of (or approximations to) the ensembles of quantum field theory, where the probability measure on the space of functions (or distributions) has the form $e^{-S(f)} df$, with

$$S(f) = \int (|\nabla f|^2 + |f|^2 + Q(|f|^2)) d\nu, \tag{2}$$

where Q is a semi-bounded polynomial. A more precise definition is given below; we refer to [Si] for background on $P(\varphi)_2$ theories. Our main results are that the empirical measures of zeros for such $P(\varphi)_2$ random polynomials satisfies an LDP with precisely the same speed and rate functional as in the Gaussian case in [ZZ] where $Q = 0$. In fact, our proof is to reduce the LDP to that case. As a corollary, the expected distribution $\frac{1}{N} \mathbf{E}_N Z_s$ of zeros in the $P(\varphi)_2$ case tends to the same weighted equilibrium measure as in the Gaussian case. In the Gaussian case, the proof of the last statement is derived from the asymptotics of the two point function (see [SZ1, SZ2, B]); in the $P(\varphi)_2$ case, the large deviations proof is the first and only one we know.

To state the result precisely, we need some notation and terminology. By a random polynomial, one means a probability measure γ_N on the vector space $\mathcal{P}_N$ of polynomials $p(z) = \sum_{j=0}^N a_j z^j$ of degree N. As in [ZZ], we identify polynomials $p(z)$ on $\mathbb{C}$ with holomorphic sections $s \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$, where $\mathcal{O}(N)$ is the Nth power of the hyperplane section line bundle $\mathcal{O}(1)$; strictly speaking, in the local coordinate, $s = pe^N$ where p is the

polynomial of degree N and e^N is a frame for $\mathcal{O}(N)$. The geometric language is useful for compactifying the problem to $\mathbb{CP}^1$, and we refer to [SZ1, ZZ] for background.

In [ZZ], the authors chose γ_N to be a Gaussian measure,

$$\gamma_N = e^{-||s||^2_{(h^N,\nu)}} ds,$$

determined by an inner product on $\mathcal{P}_N$,

$$||s||^2_{(h^N,\nu)} := \int_{\mathbb{CP}^1} |s(z)|^2_{h^N} d\nu(z). \tag{3}$$

Here, ν is an auxiliary probability measure and h is a smooth Hermitian metric on $\mathcal{O}(1)$ and h^N is the induced metric on the powers $\mathcal{O}(N)$. In the local frame e, h takes the classical form of a weight $h = e^{-\varphi}$; the assumption is that it extends smoothly to $\mathcal{O}(1) \to \mathbb{CP}^1$. Thus in the local coordinate, we rewrite

$$||s||^2_{(h^N,\nu)} = \int_{\mathbb{C}} |p(z)|^2 e^{-N\varphi(z)} d\nu(z). \tag{4}$$

In this article, we study the probability measures

$$\gamma_N = e^{-S(s)} ds \quad \text{on } \mathcal{P}_N, \tag{5}$$

where ds denotes Lebesgue measure and the action S has the form,

$$S(s) = \int_{\mathbb{CP}^1} |\nabla s(z)|^2_{h^N \otimes g} d\nu + \int_{\mathbb{CP}^1} P(|s|^2_{h^N}) d\nu, \tag{6}$$

where

$$P(x) = \sum_{j=1}^{k} c_j x^j, \quad \text{with } c_k = 1 \tag{7}$$

is a semi-bounded polynomial. Here, $\nabla : C^\infty(\mathbb{CP}^1, \mathcal{O}(N)) \to C^\infty(\mathbb{CP}^1, \mathcal{O}(N) \otimes T^*)$ is a smooth connection on the line bundle $\mathcal{O}(N) \to \mathbb{CP}^1$, and g is a smooth Riemannian metric on $\mathbb{CP}^1$. We recall that connections are the first order derivatives which are well-defined on sections of line bundles. We will take ∇ to be the Chern connection of a smooth connection h on $\mathcal{O}(1)$ and its extension to the tensor powers $\mathcal{O}(N)$ (which strictly speaking should be denoted by ∇_N). Note that the more elementary holomorphic derivative $\partial p(z) = p'(z)$ defines a meromorphic connection on $\mathcal{O}(N)$ with a pole at infinity, rather than a smooth connection. We refer to §2 and [GH, ZZ] for further background.

The integral $\int_{\mathbb{CP}^1} |\nabla s(z)|^2_{h^N \otimes g} d\nu$ is expressed in (33) in local coordinates. We often denote the first integral in $S(s)$ as $||\nabla s||^2_{(h^N \otimes g,\nu)}$ and the second as $\int P(|s|^2_{h^N})$. In $P(\varphi)_2$ Euclidean quantum field theory, $S(s)$ is known as the action, $||\nabla s||^2$ is known as the kinetic energy term, P is the potential, and $\mathcal{L}(s) = |\nabla s|^2 + P(|s|^2)$ is the Lagrangian (see e.g. [GJ, Si]). The Gaussian case is the 'non-interacting' or free field theory with quadratic Lagrangian $\mathcal{L}_0 = |\nabla s|^2 + m|s|^2$; while in the general $P(\varphi)_2$ case, the non-quadratic part of P is known as the interaction term. The Gaussian case was studied in [ZZ] without the (also Gaussian) kinetic term.

The large deviations result for empirical measures of zeros concerns a sequence $\{\mathbf{Prob}_N\}$ of probability measures on the space $\mathcal{M}(\mathbb{CP}^1)$ of probability measures on $\mathbb{CP}^1$. Roughly, $\mathbf{Prob}_N(B)$ is the probability that the empirical measure of zeros of a random $p \in \mathcal{P}_N$ lies in

the set B. To be precise, we recall some of the definitions from [ZZ]. The zero set $\{\zeta_1, \ldots, \zeta_N\}$ of a polynomial of degree N is a point of the Nth configuration space,

$$(\mathbb{CP}^1)^{(N)} = Sym^N \mathbb{CP}^1 := \underbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}_{N}/S_N. \tag{8}$$

Here, S_N is the symmetric group on N letters. We push forward the measure γ_N on $\mathcal{P}_N$ under the 'zeros' map

$$\mathcal{D} : \mathcal{P}_N \to (\mathbb{CP}^1)^{(N)}, \quad \mathcal{D}(s) = \{\zeta_1, \ldots, \zeta_N\}, \tag{9}$$

where $\{\zeta_1, \ldots, \zeta_N\}$ is the zero set of s, to obtain a measure

$$\vec{K}^N(\zeta_1, \ldots, \zeta_N) := \mathcal{D}_* d\gamma_N \tag{10}$$

on $(\mathbb{CP}^1)^{(N)}$, known as the joint probability current (or distribution), which we abbreviate by JPC. We then embed the configuration spaces into $\mathcal{M}(\mathbb{CP}^1)$ (the space of probability measures on $\mathbb{CP}^1$) under the map,

$$\mu : (\mathbb{CP}^1)^{(N)} \to \mathcal{M}(\mathbb{CP}^1), \quad d\mu_\zeta := \frac{1}{N}\sum_{j=1}^{N} \delta_{\zeta_j}. \tag{11}$$

The measure $d\mu_\zeta$ is known as the empirical measure of zeros of p. We then push forward the joint probability current to obtain a probability measure

$$\mathbf{Prob}_N = \mu_* \mathcal{D}_* \gamma^N \tag{12}$$

on $\mathcal{M}(\mathbb{CP}^1)$. The sequence $\{\mathbf{Prob}_N\}$ is said to satisfy a large deviations principle with speed N^2 and rate functional (or rate function) I if (roughly speaking) for any Borel subset $E \subset \mathcal{M}(X)$,

$$\frac{1}{N^2} \log \mathbf{Prob}_N\{\sigma \in \mathcal{M} : \sigma \in E\} \to -\inf_{\sigma \in E} I(\sigma).$$

To be precise, the condition is that

$$-I(\sigma) := \limsup_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) = \liminf_{\delta \to 0} \liminf_{N \to \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)), \tag{13}$$

for balls in the natural (Wasserstein) metric (see Theorem 4.1.11 of [DZ]).

0.1. **Statement of results.** Our first results give an LDP for slightly simpler $P(\varphi)_2$ ensembles where the action does not contain the kinetic term, i.e., we choose the probability measure to be $\gamma_N = e^{-S(s)} ds$ where $S(s) = \int P(|s|^2_{h^N})$. In §2 we add the kinetic term.

To obtain a large deviations result, we need to impose some conditions on the probability measure ν that is used to defined the integration measure on $\mathbb{CP}^1$ in the inner product (3) and the $P(\varphi)_2$ measures (5). In the pure potential case in §1, it must satisfy the mild conditions of [ZZ]: (i) the Bernstein-Markov condition, and (ii) that the support K of ν must be 'regular' in the sense that it is non-thin at all of its points. We call such measures *admissible*. We refer to [B, ZZ] for background on Bernstein-Markov measures and regularity. When we include the kinetic term, we must assume more about ν (see below).

If γ_N is defined by an *admissible* measure ν, then we prove that the speed and the rate function are the same as in the associated Gaussian case [ZZ] where $P(x) = x$.

THEOREM **1.** *Let $h = e^{-\varphi}$ be a smooth Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^1$ and let $\nu \in \mathcal{M}(\mathbb{CP}^1)$ be an admissible measure. Let $P(|s|^2_{h^N})$ be a semi-bounded polynomial defined by (7), and let γ_N be the probability measure defined by the action $S(s) = \int_{\mathbb{CP}^1} P(|s|^2_{h^N})d\nu$ without the kinectic term. Then the sequence of probability measures $\{\mathbf{Prob}_N\}$ on $\mathcal{M}(\mathbb{CP}^1)$ defined by (12) satisfies a large deviations principle with speed N^2 and rate functional*

$$I^{h,K}(\mu) = -\frac{1}{2}\mathcal{E}_h(\mu) + \sup_K U_h^\mu + E(h). \tag{14}$$

This rate functional is lower semi-continuous, proper and convex, and its unique minimizer $\nu_{h,K} \in \mathcal{M}(\mathbb{CP}^1)$ is the Green's equilibrium measure of K with respect to h.

Here, $\mathcal{E}_h(\mu) = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_h(z,w)d\mu(z)d\mu(w)$ is the Green's energy, where $G_h(z,w)$ is the Green's function with respect to h (see [ZZ] (6)). Also, $U_h^\mu(\mu) = \int_{\mathbb{CP}^1} G_h(z,w)d\mu(w)$ is the Green's potential of μ.

Things become more complicated when the action includes the kinetic term. We could choose independently the integration measures in the kinetic and potential terms, but for the sake of simplicity we only use the same measure ν for both terms. We then impose an extra condition on ν (and ∇), namely that ∇ satisfies a weighted L^2 Bernstein inequality,

$$\|\nabla s\|^2_{(h^N \otimes g, \nu)} \le CN^k \|s\|^2_{(h^N, \nu)} \tag{15}$$

on all $H^0(\mathbb{CP}^1, \mathcal{O}(N))$, for some $k, C(h, g, \nu) > 0$. When ν is admissible and such bounds hold, we say that ν (or (h, ν, ∇)) is *kinetic admissible*. In Lemma 6, we show that if $h = e^{-\varphi}$ is a Hermitian metric on $\mathcal{O}(1)$ with positive curvature form ω_h and g is any fixed Riemannian metric, then $\nu = \omega_h$ is kinetic admissible, and in fact

$$\|\nabla s\|^2_{(h^N \otimes g, \nu)} \le CN^2 \|s\|^2_{(h^N, \nu)}. \tag{16}$$

We then extend Theorem 1 to the full $P(\varphi)_2$ case. Perhaps surprisingly, when (h, ν, ∇) is kinetic admissible, the kinetic term becomes a 'lower order term' if $P(x)$ contains non-quadratic terms.

THEOREM **2.** *Let (h, ν, ∇) be kinetic admissible in the sense that (15) holds. Let $P(|s|^2_{h^N})$ be a semi-bounded polynomial as above, and let γ_N be the associated $P(\varphi)_2$ measure defined by the action (6). Then the sequence of probability measures $\{\mathbf{Prob}_N\}$ on $\mathcal{M}(\mathbb{CP}^1)$ defined by (12) satisfies a large deviations principle with speed N^2 and the same rate functional $I^{h,K}(\mu)$ as in Theorem 1.*

The proofs of Theorems 1-2 are to relate the LDP for the $P(\varphi)_2$ ensemble to the LDP for the associated (quadratic) Gaussian ensemble without kinetic term studied in [ZZ]. To avoid duplication, we refer the reader to the earlier article for steps in the proof which carry over to $P(\varphi)_2$ measures with no essential change. There are two new steps that are not in [ZZ]. The first new step (Propositions 3 and 8) is the calculation of the JPC (joint probability current, or distribution) of zeros in the $P(\varphi)_2$ ensembles. The main observation underlying this note is that the calculation of the JPC in the Gaussian ensemble in [ZZ] extends easily to the $P(\varphi)_2$ case. The second new step (loc. cit.) is the reduction of the proof of the LDP to that of [ZZ] by bounding the approximate rate function in the $P(\varphi)_2$ case above and below by that in the Gaussian case.

As a direct consequence of Theorems 1-2 we obtain,

COROLLARY 1. *With all assumptions in Theorems 1-2 , let* $\mathbf{E}_N(Z_s)$ *be the expected value of the empirical measure with respect to* γ_N. *Then,* $\mathbf{E}_N(Z_s) \to \nu_{h,K}$ *which is the equilibrium measure determined by* h *and* K.

Indeed, the limit measure $\lim_{N\to\infty} \mathbf{E}_N(Z_s)$ must be the unique minimizer of the rate functional. Convergence of the expected distribution of zeros to the equilibrium measure was first proved for Gaussian random polynomials with 'subharmonic weights' in [SZ1] and for flat weights and real analytic K in [SZ2]. In [B], the flat result was generalized to admissible measures. Corollary 1 is the first result to our knowledge for probability measures of the form (5). In fact, we are not aware of prior results on these finite dimensional approximations to $P(\varphi)_2$ quantum field theories. The results may have an independent interest in illustrating a novel kind of high frequency cutoff for such theories (in a holomorphic sector).

In conclusion, we thank O. Zeitouni for discussions and correspondence on this note.

0.2. An example: Kac-Hammersley.

As an illustration of the methods and results, we consider a $P(\varphi)_2$ generalization of the Kac-Hammersley ensemble. The classical Kac-Hammersley ensemble is the Gaussian random polynomial

$$s(z) = \sum_{j=0}^{N} a_j z^j, \quad z \in \mathbb{C}$$

where the coefficients a_j are independent complex Gaussian random variable of mean 0 and variance 1. In this case, $\mathbf{E}(Z_s) \to \delta_{S^1}$ as the week limit.

In the Gaussian case, $d\nu = \delta_{S^1}$ (the invariant probability measure on the unit circle), the weight $e^{-\varphi} = 1$ and g is the flat metric. Hence the inner product (3) reads

$$\|s\|_{\delta_{S^1}}^2 = \frac{1}{2\pi} \int_0^{2\pi} |s(e^{i\theta})|^2 d\theta$$

where s is a polynomial of degree N.

We now use the same metrics and measures, together with any semi-bounded polynomial $P(|s|^2)$, to define the kinetic $P(\varphi)_2$ Kac-Hammersley ensemble. We note that δ_{S^1} is *admissible* [ZZ]. Second, inequality (16) holds for any polynomials: In the setting of Kac-Hammersley, the connection ∇ is equal to $d = \partial + \bar{\partial}$, thus

$$\nabla s = (\sum_{j=1}^{N} j a_j z^{j-1}) dz$$

thus

$$\|\nabla s\|_{\delta_{S^1}}^2 = \sum_{j=1}^{N} j^2 |a_j|^2 \leq N^2 \sum_{j=0}^{N} |a_j|^2 = N^2 \|s\|_{\delta_{S^1}}^2$$

Hence, Theorems 1 - 2 hold in this case and we have

COROLLARY 2. *In the setting of Kac-Hammersley, let Let* $\mathbf{E}_N(Z_s)$ *be the expected value of the empirical measure with respect to* γ_N *defined by* $P(\varphi)_2$ *action (6) with the kinetic term. Then,* $\mathbf{E}_N(Z_s) \to \delta_{S^1}$.

1. Proof of the Theorem 1

In this section, we drop the kinetic term $\|\nabla s\|^2_{(h^N \otimes g, \nu)}$ and only consider actions of the form $\int P(|s|^2_{h^N}) d\nu$. We assume that $c_k > 0$ and with no essential loss of generality we put $c_k = 1$ (the coefficient could be re-scaled in the calculation). The following calculation generalizes Proposition 3 of [ZZ].

PROPOSITION 3. *Let* $(\mathcal{P}_N, \gamma_N)$ *be the* $P(\varphi)_2$ *ensemble with* $S(s) = \int_{\mathbb{CP}^1} P(|s|_{h^N}) d\nu$, *where* $d\nu$ *is an admissible measure. Denote by* k *the maximal non-zero power occurring in* P *(7). Let* $\vec{K}^N$ *be the joint probability current (10). Then,*

$$\vec{K}^N(\zeta_1, \ldots, \zeta_N) = \frac{(\Gamma_N(\zeta_1, \ldots \zeta_N))}{Z_N(h)} \frac{|\Delta(\zeta_1, \ldots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N}{\left(\int_{\mathbb{CP}^1} \prod_{j=1}^N |(z - \zeta_j)|^{2k} e^{-kN\varphi(z)} d\nu(z)\right)^{\frac{N+1}{k}}} \quad (17)$$

$$= \frac{(\Gamma_N(\zeta_1, \ldots \zeta_N))}{\hat{Z}_N(h)} \frac{\exp\left(\sum_{i<j} G_h(\zeta_i, \zeta_j)\right) \prod_{j=1}^N e^{-2N\varphi(\zeta_j)} d^2\zeta_j}{\left(\int_{\mathbb{CP}^1} e^{kN \int_{\mathbb{CP}^1} G_h(z,w) d\mu_\zeta} d\nu(z)\right)^{\frac{N+1}{k}}}. \quad (18)$$

where

$$\sup_{\{\zeta_1, \ldots, \zeta_N\} \in (\mathbb{CP}^1)^{(N)}} \frac{1}{N^2} \log \Gamma_N(\zeta_1, \ldots, \zeta_N) \to 0$$

and where $Z_N(h)$, *resp.* $\hat{Z}_N(h)$, *is the normalizing constant in Proposition 3 of* [ZZ].

We note that (17) (resp. (18)) is almost the same as (23) (resp. (24)) in Proposition 3 of [ZZ] except that we raise $\|s\|_{h^N}$ to the power k instead of the power $k = 2$. It is shown in [ZZ] that $\frac{1}{\hat{Z}_N} = e^{-\frac{1}{2}N(N-1)+N(N+1))E(h)}$. The existence of such an explicit JPC in the general $P(\varphi)_2$ case is the reason why it is possible to prove Theorem 1.

Proof. We coordinatize $\mathcal{P}_N$ using the basis z^j and put

$$s = a_0 \prod_{j=1}^N (z - \zeta_j) = \sum_{j=0}^N a_{N-j} z^j.$$

Any smooth probability measure on $\mathcal{P}_N$ thus has a density $\mathcal{D}(a_0, \ldots, a_N) \prod_{j=0}^N d^2 a_j$, where $d^2 a = da \wedge d\bar{a}$ is Lebesgue measure.

As in [ZZ], the first step is to push this measure forward under the natural projection from $\mathcal{P}_N$ to the projective space $\mathbb{P}\mathcal{P}_N$ of polynomials, whose points consists of lines $\mathbb{C}s$ of polynomials. This is natural since Z_s is the same for all multiples of s. Monic polynomials with $a_0 = 1$ form an affine space of $\mathbb{P}\mathcal{P}_N$. As affine coordinates on $\mathbb{P}\mathcal{P}_N$ we use $[1 : b_1 : \cdots : b_N]$ with $b_j = a_j/a_0$.

We then change variables from the affine coordinates b_j to the zeros coordinates ζ_k. Since $a_{N-j} = e_{N-j}(\zeta_1, \ldots, \zeta_N)$ (the $(N-j)$th elementary symmetric polynomial), the pushed forward probability measure on $\mathbb{P}\mathcal{P}_N$ then has the form

$$\vec{K}^N(\zeta_1, \ldots, \zeta_N) = \left(\int \mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N) |a_0|^{2N} d^2 a_0\right) \times |\Delta(\zeta_1, \ldots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N, \quad (19)$$

where $\mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N)$ is the density of the JPC in the coordinates $(a_0, \ldots, a_N)$ followed by the change of coordinates, and $\Delta(\zeta_1, \ldots, \zeta_N) = \prod_{i<j}(\zeta_i - \zeta_j)$ is the Vandermonde determinant. We refer to [ZZ] (proof of Proposition 3) for further details.

For the $P(\varphi)_2$ measures (5) without a kinetic term,

$$\mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N) = e^{-\int_{\mathbb{CP}^1} P(|a_0|^2 |\prod_{j=1}^N (z-\zeta_j)|^2_{h^N}) d\nu(z)}. \tag{20}$$

Put

$$\alpha_i(\zeta_1 \ldots, \zeta_N) := \alpha_i = \int_{\mathbb{CP}^1} |\prod_{j=1}^N (z-\zeta_j)|^{2i}_{h^N} d\nu(z). \tag{21}$$

Then

$$\mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N) = e^{-(\alpha_k|a_0|^{2k}+\alpha_{k-1}c_{k-1}|a_0|^{2k-2}+\cdots+\alpha_1 c_1 |a_0|^2)}, \tag{22}$$

and the pushed-forward density is

$$\int \mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N)|a_0|^{2N} d^2 a_0$$

$$= \int_{\mathbb{C}} e^{-(\alpha_k|a_0|^{2k}+\alpha_{k-1}c_{k-1}|a_0|^{2k-2}+\cdots+\alpha_1 c_1 |a_0|^2)}|a_0|^{2N} da_0 \wedge d\bar{a}_0. \tag{23}$$

We change variables to $\rho = |a_0|^2 \to \alpha_k^{-\frac{1}{k}}\rho$ to get

$$\int_0^\infty e^{-(\alpha_k \rho^k + \alpha_{k-1}c_{k-1}\rho^{k-1}+\cdots+\alpha_1 c_1 \rho)}\rho^N d\rho = (\alpha_k)^{\frac{N+1}{k}} \Gamma_N, \tag{24}$$

where

$$\Gamma_N(\zeta_1, \ldots, \zeta_N) := \int_0^\infty e^{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1}+\cdots+\beta_1 c_1 \rho)}\rho^N d\rho. \tag{25}$$

with $\beta_i = \frac{\alpha_i}{\alpha_k^{\frac{i}{k}}}$. We observe that

$$(\alpha_k)^{\frac{N+1}{k}} = \left(\int_{\mathbb{CP}^1} |\prod_{j=1}^N (z-\zeta_j)|^{2k}_{h^N} d\nu(z) \right)^{\frac{N+1}{k}}, \tag{26}$$

so that (24) implies the identity (17). The identity (18) is derived from (17) exactly as in Proposition 3 of [ZZ], so we refer there for the details.

To complete the proof of the Proposition, we prove the key

LEMMA 4. *We have,*

$$\sup_{\{\zeta_1,\ldots,\zeta_N\}\in(\mathbb{CP}^1)^{(N)}} \frac{1}{N^2} \log \Gamma_N(\zeta_1, \ldots, \zeta_N) \to 0$$

Proof. By the Hölder inequality with exponent $\frac{k}{i}$, $\beta_i \le (\int_{\mathbb{CP}^1} d\nu)^{1-\frac{i}{k}} = 1$, hence β_i is bounded independent of N for any polynomial s or roots $\{\zeta_1, \ldots, \zeta_N\}$.

We first note that

$$\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + \beta_1 c_1 \rho \ge \rho^k - |c_{k-1}|\rho^{k-1} - \cdots - |c_1|\rho$$

$$\ge \tfrac{1}{2}\rho^k, \quad \text{for } \rho \ge \rho_k := \rho_k(c_1, \ldots, c_{k-1}),$$

where $\frac{|c_1|}{\rho} + \cdots + \frac{|c_1|}{\rho^{k-1}} \le \frac{1}{2}$ for $\rho \ge \rho_k$. It follows that

$$\Gamma_N(\zeta_1, \ldots, \zeta_N) \le \int_0^{\rho_k} e^{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + \beta_1 c_1 \rho)} \rho^N d\rho + \int_{\rho_k}^\infty e^{-\frac{1}{2}\rho^k} \rho^N d\rho$$

$$\le \int_0^{\rho_k} e^{-(\rho^k - |c_{k-1}|\rho^{k-1} - \cdots - |c_1|\rho)} \rho^N d\rho + \int_0^\infty e^{-\frac{1}{2}\rho^k} \rho^N d\rho.$$

But

$$\int_0^\infty e^{-\frac{1}{2}\rho^k} \rho^N d\rho = N^{\frac{N+1}{k}} \int_0^\infty e^{N(\log\rho - \frac{1}{2}\rho^k)} d\rho \sim N^{\frac{N+1}{k}} e^{N(\frac{1}{k}\log\frac{1}{k} - \frac{1}{k})} \frac{1}{\sqrt{N}}.$$

Also,

$$\int_0^{\rho_k} e^{-(\rho^k - |c_{k-1}|\rho^{k-1} - \cdots - |c_1|\rho)} \rho^N d\rho \le (\rho_k)^N C_k,$$

where C_k is a constant independent of N and $\{\zeta_1, \ldots, \zeta_N\}$. Hence,

$$\Gamma_N \le (\rho_k)^N C_k + N^{\frac{N+1}{k}} e^{N(\frac{1}{k}\log\frac{1}{k} - \frac{1}{k})} \frac{1}{\sqrt{N}}.$$

To obtain a lower bound, we write

$$\int_0^\infty e^{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + \beta_1 c_1 \rho)} \rho^N d\rho = \int_0^1 + \int_1^\infty$$

For $\rho \in [0, 1]$ we have,

$$\rho^k + \beta_{k-1}|c_{k-1}|\rho^{k-1} + \cdots + \beta_1|c_1|\rho \le kC, \quad C = \max\{|c_j|\}_{j=1}^k$$

since each β_i is bounded by 1, thus

$$\int_0^1 e^{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + \beta_1 c_1 \rho)} \rho^N d\rho \ge \int_0^1 e^{-Ck} \rho^N d\rho \ge e^{-Ck} \frac{1}{N+1}$$

For $\rho \ge 1$ we have,

$$\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + \beta_1 c_1 \rho \le kC\rho^k,$$

hence

$$\int_1^\infty e^{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + \beta_1 c_1 \rho)} \rho^N d\rho \ge \int_1^\infty e^{-Ck\rho^k} \rho^N d\rho$$

$$= (Ck)^{-(N+1)/k} \int_1^\infty e^{-\rho^k} \rho^N d\rho \ge (Ck)^{-(N+1)/k}.$$

Putting together the two bounds, we get

$$(Ck)^{-(N+1)/k} + e^{-k} \frac{1}{N+1} \le \Gamma_N \le (\rho_k)^N C_k + N^{\frac{N+1}{k}} e^{N(\frac{1}{k}\log\frac{1}{k} - \frac{1}{k})} \frac{1}{\sqrt{N}}.$$

This completes the proof of Lemma 4, and hence of the Proposition.   □

Remark: In retrospect, what we proved is that

$$\frac{1}{N^2} \log \int \mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N)|a_0|^{2N} d^2 a_0 \sim \frac{1}{N^2} \log \int_0^\infty e^{-\alpha_k \rho^k} \rho^N d\rho \qquad (27)$$

We could obtain the limit by a slight generalization of the saddle point method,

$$\frac{1}{N^2} \log \int_0^\infty e^{-\alpha_k \rho^k} \rho^N d\rho \sim -\frac{1}{N^2} \inf_{\rho \in \mathbb{R}_+} (\alpha_k \rho^k - N \log \rho)$$

$$\sim -\frac{1}{kN} \log \alpha_N = -\frac{1}{kN} \log \int_{\mathbb{CP}^1} |\prod_{j=1}^N (z - \zeta_j)|_{h^N}^{2k} d\nu, \qquad (28)$$

since the minimum occurs at $\rho_N = (\frac{N}{k})^{\frac{1}{k}}\alpha^{-\frac{1}{k}}$. This is the same answer we are about to get by the more rigorous argument in [ZZ].

1.1. Completion of the proof of Theorem 1 without kinetic term.

We now modify the calculations of [ZZ], Section 4.7, of the approximate rate function I_N. As in that section, we define

$$\mathcal{E}_N^h(\mu_\zeta) = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta} G_h(z,w)d\mu_\zeta(z)d\mu_\zeta(w),$$

where $\Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is the diagonal. We also define

$$\mathcal{J}_N^{h,\nu}(\mu_\zeta) = \log \|e^{U_h^{\mu_\zeta}}\|_{L^{kN}(\nu)}. \tag{29}$$

It is almost the same functional of the same notation in [ZZ], Section 4.7, except that the L^N norm there now becomes the L^{kN} norm.

We define the approximate rate functional by

$$-N^2 I_N(\mu_\zeta) := -\frac{1}{2}\mathcal{E}_N^h(\mu_\zeta) + \frac{N+1}{N}\mathcal{J}_N^{h,\nu}(\mu_\zeta). \tag{30}$$

The following is the analogue of Lemma 18 of [ZZ].

PROPOSITION 5. *With the same notation as in Proposition 3, we have*

$$\vec{K}^N(\zeta_1,\ldots,\zeta_N) = \frac{\Gamma_N(\zeta_1,\ldots,\zeta_N)}{\hat{Z}_N(h)}e^{-N^2\left(-\frac{1}{2}\mathcal{E}_N^h(\mu_\zeta)+\frac{N+1}{N}\mathcal{J}_N^{h,\nu}(\mu_\zeta)\right)}.$$

The proof is the same calculation as in [ZZ] and we therefore omit most of the details. Indeed, the remainder of the proof of Theorem 1 for $P(\varphi)_2$ measure without kinetic term is identical to that of Theorem 1 of [ZZ], since the only change in the approximate rate functional is the change $1 \to k$ in $\mathcal{J}_N^{h,\nu}$ and the factor Γ_N. The change in $\mathcal{J}_N^{h,\nu}$ cancels out in the limit, since (as in [ZZ]), we have

$$\lim_{N\to\infty}\mathcal{J}_N^{h,\nu}(\mu_\zeta) = \log \|e^{U_h^{\mu_\zeta}}\|_{L^{kN}}(\nu) \uparrow \log \|e^{U_h^{\mu_\zeta}}\|_{L^\infty}(\nu) = \sup_K U_h^{\mu_\zeta}.$$

We briefly re-do the calculation for the sake of completeness, referring to [ZZ] for further details:

$$\int_{\mathbb{CP}^1}\prod_{j=1}^N |(z-\zeta_j)|^{2k}e^{-kN\varphi}d\nu(z) = \left(\int_{\mathbb{CP}^1}e^{k\int_{\mathbb{CP}^1}G_h(z,w)dd^c\log\|s_\zeta(w)\|_{h^N}^2\,d\nu}\right)e^{k\int_{\mathbb{CP}^1}\log\|s_\zeta\|_{h^N}^2(z)\omega_h}$$

$$= \left(\int_{\mathbb{CP}^1}e^{kN\int_{\mathbb{CP}^1}G_h(z,w)d\mu_\zeta(w)}d\nu\right)e^{k\int_{\mathbb{CP}^1}\log\|s_\zeta\|_{h^N}^2(z)\omega_h}. \tag{31}$$

The right side is then raised to the power $-\frac{N+1}{k}$. If we take $\frac{1}{N^2}\log$ of the result we get the supremum of $\int_{\mathbb{CP}^1}G_h(z,w)d\mu_\zeta(w)$ on the support of $d\nu$.

Further, by Proposition 3 the Γ_N factor does not contribute to the rate function $I^{h,K}$. Therefore the special case of Theorem 1 for $P(\varphi)_2$ measures where the $\|\nabla s\|_{(h^N \otimes g,\nu)}^2$ term is omitted follows from Proposition 5 and from the proof of Theorem 1 in [ZZ]. $\qquad\square$

2. Large deviations for Lagrangians with kinetic term.

We now include the kinetic energy term. In order to define ∇s we need to introduce a connection $\nabla : C^\infty(\mathbb{CP}^1, \mathcal{O}(1)) \to C^\infty(\mathbb{CP}^1, \mathcal{O}(1) \otimes T^*)$. To define the norm-square $||\nabla s||^2_{(H^N \otimes g, \nu)}$ we introduce a metric g on $\mathbb{CP}^1$ and a Hermitian metric H on $\mathcal{O}(1)$ to define $|\nabla s|^2_{H^N \otimes g}$ pointwise and a measure $d\mu$ on $\mathbb{CP}^1$ to integrate the result. The kinetic term is independent of the potential term, and we could choose H, μ differently from h, ν in the potential term. But to avoid excessive technical complications, we choose the metrics and connections to be closely related to those in the potential term.

We first assume that $h = e^{-\varphi}$ is a hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^1$ with positive $(1,1)$ curvature, $\omega_h = \frac{i}{\pi} \partial \bar{\partial} \varphi > 0$. We then choose ∇ to be the Chern connection of h. Thus, $\nabla s \in C^\infty(\mathbb{CP}^1, \mathcal{O}(N) \otimes T^{*(1,0)})$ if $s \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$. We fix a local frame e over $\mathbb{C}$ and express holomorphic sections of $\mathcal{O}(N)$ as $s = pe^N$. The connection 1-form is defined by $\nabla e = e \otimes \alpha$ and in the case of the Chern connection for h it is given by $\alpha = h^{-1} \partial h = \partial \varphi$. We further fix a smooth Riemannian metric g on $\mathbb{CP}^1$ (which could be ω_h but need not be).

We assume that the auxiliary probability measure $d\nu$ on $\mathbb{CP}^1$ satisfies the following L^2-condition: There exists $r \geq 0$ so that

$$\int_{\mathbb{CP}^1} |p|^2 e^{-N\varphi} \omega_h \leq CN^r \int_{\mathbb{CP}^1} |p|^2 e^{-N\varphi} d\nu, \tag{32}$$

for all $p \in \mathcal{P}_N$. That is, the inner product defined by (h^N, ω_h) is polynomially bounded by the inner product defined by (h^N, ν). We say that (h, ν) is *kinetic admissible* if the data satisfies these conditions. The metrics h and g and the measure ν induce inner products on $\Gamma(L^N \otimes T^{*(1,0)})$ by

$$\langle s \otimes dz, s \otimes dz \rangle_{h^N \otimes g} = \int_{\mathbb{CP}^1} (s,s)_{h^N} (dz, dz)_g \, d\nu.$$

Since $\nabla pe^N = e^N \otimes \partial p + Npe^N \otimes \alpha$, the kinetic energy is given in the local coordinate as

$$
\begin{aligned}
\int_{\mathbb{CP}^1} |\nabla s|^2_{h^N \otimes g} d\nu : &= \int_{\mathbb{C}} (e^N \otimes \partial p + Npe^N \otimes \alpha, e^N \otimes \partial p + Npe^N \otimes \alpha)_{h^N \otimes g} d\nu \\
&= \int_{\mathbb{C}} \left(|\partial p|^2_g + Np(\alpha, \partial p)_g + N\bar{p}(\partial p, \alpha)_g + N^2 |p|^2 |\alpha|^2_g \right) e^{-N\varphi} d\nu
\end{aligned}
\tag{33}
$$

2.1. **Kinetic admissible** (h, ν, ∇)**.** We now show that some natural choices of (h, ν, ∇) are kinetic admissible.

We first observe that $\frac{1}{N}\nabla$ is a bounded operator on $H^0(M, L^N)$ for any positive line bundle L over the projective Kähler manifold M, when the inner product is defined by a smooth volume form. This is an obvious result of Toeplitz calculus but we provide a proof using the Boutet de Monvel-Sjöstrand parametrix for the Szegö kernel. It is at this point that we need the assumption that $\omega_h > 0$.

LEMMA **6.** *Assume* $h = e^{-\varphi}$ *is a Hermitian metric on a holomorphic line bundle* $L \to M$ *over any compact projective Kähler manifold with* $\omega_h = \partial \bar{\partial} \varphi > 0$. *Assume* $d\nu$ *is a smooth volume form and* g *is a Riemannian metric over* M. *Then we have*

$$\|\nabla s\|^2_{(h^N \otimes g, \nu)} \leq C(h, g, \nu) N^2 \|s\|^2_{(h^N, \nu)}$$

where s *is the holomorphic section of line bundle* L^N.

Proof. Let $\Pi_{N,\nu} : L^2(M, L^N) \to H^0(M, L^N)$ be the orthogonal projection with respect to the inner product,

$$\langle f_1 \otimes e^N, f_2 \otimes e^N \rangle = \int_M f_1 \bar{f}_2 e^{-N\varphi} d\nu,$$

Here in the local coordinate, we write $s = fe^N$ as the section of the line bundle L^N.

Let $\Pi_{N,\nu}(z, w)$ be its Schwartz kernel with respect to $d\nu$,

$$(\Pi_{N,\nu} s)(z) = \int_M \Pi_{N,\nu}(z, w) f(w) e^{-N\varphi(w)} d\nu(w)$$

Then Bergman kernel has the paramatrix [BBS, BS]

$$\Pi_{N,\nu}(z, w) = e^{N\varphi(z \cdot w)} A_N e^N(z) \otimes \bar{e}^N(w)$$

where A_N is a symbol of order $m = \dim M$ depending on h and ν and where $\varphi(z \cdot w)$ is the almost-analytic extension of $\varphi(z)$. It follows that the Schwartz kernel of $\frac{1}{N} \nabla \Pi_{N,\nu}$ has the local form,

$$\frac{1}{N} \nabla \Pi_{N,\nu}(z, w) = \left((\frac{1}{N} \partial + \partial\varphi dz) e^{N\varphi(z \cdot w)} A_N(z, w) \right) e^N(z) \otimes \bar{e}^N(w)$$

$$= \left((\partial\varphi + \partial_z \varphi(z \cdot w) + \frac{1}{N} \partial \log A_N) e^{N\varphi(z \cdot w)} A_N \right) e^N(z) \otimes \bar{e}^N(w).$$

Put $\Phi(z, w) := \partial\varphi + \partial_z \varphi(z \cdot w) + \partial \log A_N$. Denote by $\Phi\Pi_{N,\nu}$ the product of Φ and the Schwartz kernel of $\Pi_{N,\nu}$. Then,

$$\|\frac{1}{N} \nabla s\|^2_{(h^N \otimes g, \nu)} = \|\frac{1}{N} \nabla \Pi_{N,\nu} s\|^2_{(h^N \otimes g, \nu)} = \|(\Phi\Pi_{N,\nu}) s\|^2_{(h^N \otimes g, \nu)}$$

We now claim that

$$\|(\Phi\Pi_N) s\|_{L^2(h^N \otimes g, \nu)} \leq C \|s\|_{L^2(h^N, \nu)}.$$

This follows from the Schur-Young bound on the $L^2 \to L^2$ mapping norm of the integral operator $\Phi\Pi_N$,

$$\|\Phi\Pi_N\| \leq C \sup_M \int_M |\Pi_{N,\nu}(z, w)| d\nu(z), \tag{34}$$

since for any metric g on M, $|\Phi|_g \leq C$ uniformly on M. To estimate the norm, we use the following known estimates on the Bergman kernel (see [SZ4] for a similar estimate and for background): when $d(z, w) \leq CN^{-\frac{1}{3}}$, we have

$$|\Pi_{N,\nu}(z, w)|_{h^N \otimes h^N} \leq CN^m e^{-\frac{1}{4}Nd^2(z,w)} + O(N^{-\infty}),$$

and in general,

$$|\Pi_{N,\nu}(z, w)| \leq CN^m e^{-\lambda\sqrt{N}d(z,w)}$$

for some constant C and λ.

Since we assume $d\nu$ is a volume form on M, there exists a positive function $J \in C^\infty(M)$ such that $d\nu = J\omega_h^m$. We break up the right side of (34) into

$$\int_{d(z,w) \leq N^{-1/3}} + \int_{d(z,w) \geq N^{-1/3}}.$$

The first term is bounded by

$$\leq CN^m \int_{d(z,w)\leq N^{-1/3}} e^{-\frac{1}{4}Nd^2(z,w)} J\omega^m(z)$$

$$\leq C(h,\nu)N^m \int_0^\infty e^{-\frac{1}{4}N\rho^2} d\rho^{2m} + O(N^{-\infty}) \leq C'(h,\nu)$$

The second term is bounded by

$$\leq CN^m \int_{d(z,w)\geq N^{-1/3}} e^{-\lambda\sqrt{N}d(z,w)} J\omega^m$$

$$\leq CN^m \int_M e^{-\lambda N^{\frac{1}{6}}} d\nu \leq O(N^{-\infty})$$

as N large enough. Thus the operator norm $\Phi\Pi_N$ is bounded by some constant $C'(h,g,\nu)$.
$\square$

Remark: The assumption that $d\nu$ is a smooth volume form allows us to take the adjoint of ∇.

We now give a more general estimate. We assume again that $h = e^{-\varphi}$ has positive curvature $\omega_h > 0$. But we now relax the assumption that $d\nu$ is a smooth volume form, and only assume that $d\nu$ satisfies the L^2 condition:

$$\int_M |s|^2 e^{-N\varphi}\omega_h^m \leq CN^r \int_M |s|^2 e^{-N\varphi} d\nu$$

for any $s \in H^0(M, L^N)$ and for some $r \geq 0$.

LEMMA **7.** *Let* $\dim M = m$. *Under the above assumptions, we have*

$$\|\nabla s\|^2_{(h^N\otimes g,\nu)} \leq CN^{r+2m+2}\|s\|^2_{(h^N,\nu)}$$

where $s \in H^0(M, L^N)$.

Proof. First we consider the following Bergman kernel $\Pi_{N,\omega_h}(z,w)$ with respect to the inner product,

$$\Pi_N(fe^N)(z) = \int_M \Pi_{N,\omega_h}(z,w)f(w)e^{-N\varphi}\omega_h^m(w).$$

As above, we write $\Phi(z,w) = \partial\varphi + \partial_z\varphi(z\cdot w) + \partial\log A_N$.

By Schwartz' inequality, we have (in an obvious notation)

$$\|\tfrac{1}{N}\nabla\Pi_{N,\omega_h}s\|^2_{L^2(h^N,\nu)}$$

$$\leq (\int_M |f|^2 e^{-N\varphi}\omega_h^m)(\int_{M\times M} |\Phi|^2|\Pi_{N,\omega_h}|^2 e^{-N\varphi(w)-N\varphi(z)}|dz|_g^2\omega_h^m(w)d\nu(z))$$

Since $|\Phi\Pi_{N,\omega_h}|^2|dz|_g^2 \leq CN^{2m}$ uniformly, this implies

$$\|\nabla s\|^2_{L^2(h^N\otimes g,\nu)} = \|\nabla\Pi_{N,\omega_h}s\|^2_{L^2(h^N\otimes g,\nu)}$$

$$\leq CN^{2m+2}\int_M |f|^2 e^{-N\varphi}\omega_h \leq CN^{r+2m+2}\|s\|^2_{L^2(h^N,\nu)},$$

under the L^2 condition. $\square$

2.2. **Proof of Theorem 2.** We now prove Theorem 2. At first one might expect the kinetic term to dominate the action, since its square root is the $H_2^1(d\nu)$ norm of s and since that norm cannot be bounded by the L^p norm for any $p < \infty$, at least when ν is a smooth area form. However, we are only integrating over holomorphic sections of $\mathcal{O}(N)$ and with the admissibility assumption, the ratios of all norms are bounded above and below by positive constants depending on N. Taking logarithm asymptotics erases any essential difference between these norms.

The main step in the proof is the following generalization of Proposition 3.

PROPOSITION 8. *Let $(\mathcal{P}_N, \gamma_N)$ be the $P(\varphi)_2$ ensemble with action (6), where (h, ∇, ν) is kinetic admissible. Let $\vec{K}^N$ be the joint probability current (10). Then,*

$$\vec{K}^N(\zeta_1, \ldots, \zeta_N) = \frac{(\tilde{\Gamma}_N(\zeta_1, \ldots \zeta_N))}{\hat{Z}_N(h)} \frac{\exp\left(\sum_{i<j} G_h(\zeta_i, \zeta_j)\right) \prod_{j=1}^N e^{-2N\varphi(\zeta_j)} d^2\zeta_j}{\left(\int_{\mathbb{CP}^1} e^{kN \int_{\mathbb{CP}^1} G_h(z,w) d\mu_\zeta} d\nu(z)\right)^{\frac{N+1}{k}}}. \quad (35)$$

where

$$(**) \quad \sup_{\{\zeta_1, \ldots, \zeta_N\} \in (\mathbb{CP}^1)^{(N)}} \frac{1}{N^2} \log \tilde{\Gamma}_N(\zeta_1, \ldots, \zeta_N) \to 0$$

and where $Z_N(h)$, resp. $\hat{Z}_N(h)$, is the normalizing constant in Proposition 3 of [ZZ].

Proof. We closely follow the proof of Proposition 3, and do not repeat the common steps. For the $P(\varphi)_2$ measures (5) with kinetic term,

$$\mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N) = e^{-\int_{\mathbb{CP}^1} (|\nabla \prod_{j=1}^N (z-\zeta_j)|^2 + P(|a_0|^2| \prod_{j=1}^N (z-\zeta_j)|_{hN}^2) d\nu(z)}$$

$$= e^{-(\alpha_k |a_0|^{2k} + \alpha_{k-1} c_{k-1} |a_0|^{2k-2} + \cdots + \alpha_1 c_1 |a_0|^2 + \eta |a_0^2|)}, \quad (36)$$

where

$$\eta = |a_0|^{-2} \|\nabla s\|_{L^2(h^N \otimes g, \nu)}^2. \quad (37)$$

Thus, the addition of the kinetic term changes the pushed forward probability density from (23) to

$$\int \mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N) |a_0|^{2N} d^2 a_0$$

$$= \int_{\mathbb{C}} e^{-(\alpha_k |a_0|^{2k} + \alpha_{k-1} c_{k-1} |a_0|^{2k-2} + \cdots + c_1 \alpha_1 |a_0|^2 + \eta |a_0|^2)} |a_0|^{2N} da_0 \wedge d\bar{a}_0$$

$$= \int_0^\infty e^{-(\alpha_k \rho^k + \alpha_{k-1} c_{k-1} \rho^{k-1} + \cdots + c_1 \alpha_1 \rho + \eta \rho)} \rho^N d\rho,$$

where $\rho = |a_0|^2$ and α_i is defined by (21). We only need to understand the effect of the new η term.

We change variable $\rho \to \rho \alpha_k^{\frac{1}{k}}$, to get

$$\int \mathcal{D}(a_0; \zeta_1, \ldots, \zeta_N) |a_0|^{2N} d^2 a_0 = \alpha_k^{-\frac{N+1}{k}} \tilde{\Gamma}_N(\zeta_1, \ldots, \zeta_N),$$

where

$$\tilde{\Gamma}_N(\zeta_1, \ldots, \zeta_N) := \int_0^\infty e^{-(\rho^k + \beta_{k-1} c_{k-1} \rho^{k-1} + \cdots + c_1 \beta_1 \rho + \frac{\eta}{\alpha_k^{\frac{1}{k}}} \rho)} \rho^N d\rho.$$

This is the same expression as in Proposition 3 except that the Γ_N factor has changed. Hence to prove (**), it suffices to prove

$$\frac{1}{N^2} \log \int_0^\infty e^{\frac{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + c_1\beta_1\rho + \frac{\eta}{\alpha_k^{\frac{1}{k}}}\rho)}{\alpha_k^{\frac{1}{k}}}} \rho^N d\rho \to 0.$$

We first prove that the limit is bounded above by 0. Since the addition of the positive quantity $\eta\alpha_k^{-\frac{1}{k}}$ increases the exponent, we have

$$\frac{1}{N^2} \log \int_0^\infty e^{\frac{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + c_1\beta_1\rho + \frac{\eta}{\alpha_k^{\frac{1}{k}}}\rho)}{\alpha_k^{\frac{1}{k}}}} \rho^N d\rho$$

$$\leq \frac{1}{N^2} \log \int_0^\infty e^{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + c_1\beta_1\rho)} \rho^N d\rho,$$

so the integral is bounded above by its analogue in the pure potential case, and it follows from the proof in section 1 that the last integral tends to 0.

We now consider the lower bound. By Lemmas 6 and 7 (with $m = 1$) and by Hölder inequality, we have

$$\eta \leq CN^n |a_0|^{-2} \|s\|^2_{L^2(h^N, \nu)} \leq CN^n \alpha_k^{\frac{1}{k}},$$

in the cases $n = 2$ with ν a smooth volume form or $n \geq 4$ when ν satisfies the weighted L^2 Bernstein inequality (15). We then have,

$$\frac{1}{N^2} \log \int_0^\infty e^{\frac{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + c_1\beta_1\rho + \frac{\eta}{\alpha_k^{\frac{1}{k}}}\rho)}{\alpha_k^{\frac{1}{k}}}} \rho^N d\rho$$

$$\geq \frac{1}{N^2} \log \int_0^\infty e^{-(\rho^k + \beta_{k-1}c_{k-1}\rho^{k-1} + \cdots + c_1\beta_1\rho + CN^n\rho)} \rho^N d\rho$$

$$\geq \frac{1}{N^2} \log \int_0^\infty e^{-(\rho^k + \beta_{k-1}|c_{k-1}|\rho^{k-1} + \cdots + |c_1|\beta_1\rho + CN^n\rho)} \rho^N d\rho.$$

Hence, it suffices to prove that

$$\frac{1}{N^2} \log \int_0^\infty e^{-(\rho^k + \beta_{k-1}|c_{k-1}|\rho^{k-1} + \cdots + |c_1|\beta_1\rho + CN^n\rho)} \rho^N d\rho \geq 0.$$

We use the steepest descent method to show that the latter tends to zero. The maximum of the phase function occurs when

$$k\rho_N^k + (k-1)\beta_{k-1}|c_{k-1}|\rho_N^{k-1} + \cdots + |c_1|\beta_1\rho_N + CN^n\rho_N = N.$$

It follows first that $\rho_N \leq \frac{1}{CN^{n-1}} < 1$. Thus

$$N = k\rho_N^k + (k-1)\beta_{k-1}|c_{k-1}|\rho_N^{k-1} + \cdots + |c_1|\beta_1\rho_N + CN^n\rho_N$$

$$\leq k\rho_N + (k-1)\beta_{k-1}|c_{k-1}|\rho_N + \cdots + |c_1|\beta_1\rho_N + CN^n\rho_N$$

which implies

$$\rho_N \geq \frac{N}{C(k, c_{k-1}, \cdots, c_1) + CN^n},$$

and therefore

$$\rho_N \sim \frac{1}{C'N^{n-1}}$$

for N large enough. Thus by the formula of steepest descent,

$$\frac{1}{N^2} \log \int_0^\infty e^{-(\rho^k + \beta_{k-1}|c_{k-1}|\rho^{k-1} + \cdots + |c_1|\beta_1\rho + CN^n\rho)} \rho^N d\rho$$

$$\sim \frac{1}{N} \log \rho_N - \frac{1}{N^2}(\rho_N^k + \beta_{k-1}|c_{k-1}|\rho_N^{k-1} + \cdots + |c_1|\beta_1\rho_N + CN^n\rho_N)$$

$$\sim -\frac{(n-1)\log(C'N)}{N} - \frac{1}{N^2}\left((\frac{1}{C'N^{n-1}})^k + \cdots + \beta_1|c_1|\frac{1}{C'N^{n-1}}\right) - C\frac{1}{C'N}$$

which goes to 0 as $N \to \infty$, and (**) holds. $\qquad\square$

This completes the proof of Proposition 8. The rest of the proof proceeds exactly as in §1.1, completing the proof of Theorem 2.

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