

# DERIVED RESOLUTION PROPERTY FOR STACKS, EULER CLASSES AND APPLICATIONS

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ABSTRACT. By resolving any perfect derived object over a Deligne-Mumford stack, we define its Euler class. We then apply it to define the Euler numbers for a smooth Calabi-Yau threefold in  $\mathbb{P}^4$ . These numbers are conjectured to be the reduced Gromov-Witten invariants and to determine the usual Gromov-Witten numbers of the smooth quintic as speculated by J. Li and A. Zinger.

## 1. INTRODUCTION

Let  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$  be the DM stack of degree  $d$  genus- $g$  stable maps to a projective space  $\mathbf{P}$ . We let

$$(1.1) \quad \mathcal{X} \xrightarrow{\pi} \overline{\mathfrak{M}}_g(\mathbf{P}, d) \quad \text{and} \quad \mathcal{X} \xrightarrow{f} \mathbf{P}$$

be its universal family. For any integer  $k > 0$ , the derived object  $R\pi_*(f^*\mathcal{O}_{\mathbb{P}^m}(k))$  is quasi-isomorphic to a complex of locally free sheaves

$$\mathcal{E}^\bullet = [\mathcal{E}_0 \xrightarrow{\varphi} \mathcal{E}_1].$$

The main purpose of this article is to define the Euler class

$$\mathbf{e}(R\pi_*(f^*\mathcal{O}_{\mathbf{P}}(k)))$$

of this complex over the primary component of  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$ .

In fact, we consider any perfect derived object  $\mathcal{E}^\bullet$  in the bounded derived category  $D^b(M)$  of an integral DM stack  $M$  with cohomologies concentrated in the non-negative places. Our main theorem says that any such a perfect derived object  $\mathcal{E}^\bullet$ , such as  $R\pi_*(f^*\mathcal{O}_{\mathbb{P}^m}(k))$  in the above, can be resolved to have locally free sheaf cohomology  $\mathcal{H}^0$  after birational base change.

**Theorem 1.1.** (Existence of Resolution.) *Let  $\mathcal{E}^\bullet$  be any perfect derived object over an integral DM stack  $M$ . Assume that  $\mathcal{E}^\bullet$  can locally be represented by a complex of locally free sheaves of finite length supported only in non-negative degrees. Then there is another integral DM stack  $\widetilde{M}$  and a surjective birational morphism  $f : \widetilde{M} \rightarrow M$  such that  $\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)$  is locally free.*

Moreover, among the above stacks  $\widetilde{M}$ , there is also a unique one (up to isomorphism) that is minimal (see Theorem 4.3 and Proposition 4.5 for the precise statements). In the main text, we will prove the stronger Theorem 4.3 from which the above is a consequence. The main technical theorem for this purpose is Theorem 3.11.

**Definition 1.2.** *Suppose further that the cohomologies in positive places  $\mathcal{H}^{i>0}(\mathcal{E}^\bullet)$  are all torsion sheaves over  $M$ . Then, we define the Euler class  $\mathbf{e}(\mathcal{E}^\bullet)$  in the Chow group  $A_*M$  of cycles on  $M$  by,*

$$(1.2) \quad \mathbf{e}(\mathcal{E}^\bullet) := f_*(c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)) \cdot [\widetilde{M}]),$$

where  $r = \text{rank } \mathcal{H}^0(Lf^*\mathcal{E}^\bullet)$ .

**Proposition 1.3.** *Let  $M$  be an integral DM stack and  $\mathcal{E}^\bullet$  a perfect derived object as in Theorem 1.1. Suppose  $\mathcal{H}^{i<0}(\mathcal{E}^\bullet) = 0$  and  $\mathcal{H}^{i>0}(\mathcal{E}^\bullet)$  are torsion sheaves. Then the Euler class  $\mathbf{e}(\mathcal{E}^\bullet)$  is independent of the choice of the resolutions  $f : \widetilde{M} \rightarrow M$ .*

For more precise statements, see Proposition 5.3 and Corollary 5.4.

The above, when applied to the derived object  $R\pi_*\mathbf{f}^*\mathcal{O}_{\mathbb{P}^4}(5)$  restricted to the primary component  $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)'$  of the moduli stack  $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ , enables us to construct the modular Euler class when  $d > 2g - 2$ . (The general points of  $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)'$  are maps with smooth domains, and  $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)'$  is irreducible and of the expected dimension when  $d > 2g - 2$ . cf. 6.4.) In this case, letting  $\mathbf{f}'$  be the restriction of the universal family  $\mathbf{f}$  to  $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)'$ , then  $R^1\pi_*\mathbf{f}'^*\mathcal{O}_{\mathbb{P}^4}(5)$  is a torsion sheaf over  $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)'$ .

**Definition 1.4.** *For  $d > 2g - 2$ , we define the modular Euler class of  $R\pi_*\mathbf{f}^*\mathcal{O}_{\mathbb{P}^4}(5)$  over  $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)'$  to be*

$$(1.3) \quad \mathbf{e}(R\pi_*\mathbf{f}'^*\mathcal{O}_{\mathbb{P}^4}(5)) \in A_*(\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)');$$

for any smooth Calabi-Yau manifold  $Q$  in  $\mathbb{P}^4$ , we define

$$(1.4) \quad N'_{g,d}(Q) = \deg \mathbf{e}(R\pi_*\mathbf{f}'^*\mathcal{O}_{\mathbb{P}^4}(5)).$$

We believe that these numbers  $N'_{g,d}(Q)$  are the reduced GW-invariants speculated by Li and Zinger:

**Conjecture 1.5.** *Let  $N_{g,d}(Q)$  be the genus  $g$ , degree  $d$  GW-invariants of the smooth quintic  $Q \subset \mathbb{P}^4$ . There are universal constants  $c_h$  such that for  $d > 2g - 2$ ,*

$$N_{g,d}(Q) = \sum_{0 \leq h \leq g} c_h N'_{h,d}(Q).$$

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## 2. CONVENTIONS AND TERMINOLOGY

**2.1.** Throughout the paper, we fix an arbitrary algebraically closed base field  $\mathbf{k}$  of characteristic 0. All schemes and stacks in this paper are assumed to be noetherian over  $\mathbf{k}$ .

**2.2.** Paragraphs are enumerated, so are equations. For instance, 3.2 refers to Paragraph 3.2, while (3.2) refers to Equation (3.2). §3.2 refers to a subsection.

**2.3.** Unless otherwise stated, smoothness is in the sense of DM stack or Artin stack. On the coarse moduli level, this roughly means that the moduli space is locally in the étale topology a quotient of a smooth variety by a finite group or in the topology of smooth morphisms a quotient of a smooth variety by a group scheme.

**2.4.** All morphisms between stacks are assumed to be representable.

**2.5.** Let  $X$  be a scheme,  $D$  a Cartier divisor of  $X$  and  $Z$  a closed subscheme of  $X$ . We will write  $\mathcal{O}_Z(D)$  for the restriction  $\mathcal{O}_X(D)|_Z$ .

**2.6.** For any right exact functor  $F$  such as  $f^*$  from an Abelian category  $\mathcal{A}$  to another  $\mathcal{B}$ , we use  $LF$  to denote the left derived functor. Similarly, for a left exact functor  $F$  such as  $f_*$ ,  $RF$  is the right derived functor.

## 3. DIAGONALIZING SHEAF HOMOMORPHISM

**3.1.** Let  $M$  be a DM-stack. We denote by  $D^b(M)$  the derived category of bounded complexes of coherent sheaves over  $M$ . An object  $\mathcal{E}^\bullet \in D^b(M)$  is called *perfect* if locally it can be represented by a complex of locally free sheaves of finite length. Equivalently, the stack  $M$  admits an étale cover  $U$  by a scheme such that there is a finite length complex  $\mathcal{F}^\bullet$  of locally free sheaves over  $U$  such that  $\mathcal{E}^\bullet$  is represented by  $\mathcal{F}^\bullet$  in  $D^b(U)$ .

**3.2.** Let an integral DM stack  $M$  and a derived object  $\mathcal{E}^\bullet$  be as in Theorem 1.1. Since  $\mathcal{E}^\bullet$  is perfect, we can cover  $M$  by open charts  $\coprod U$  and for each open subset  $U$  there is a finite length complex  $\mathcal{F}^\bullet$  of locally free sheaves over  $U$  such that  $\mathcal{E}^\bullet|_U$  is represented by  $\mathcal{F}^\bullet$ . By our assumption, we may assume that  $\mathcal{F}^\bullet$  has the form

$$(3.1) \quad \mathcal{F}_0 \xrightarrow{\psi} \mathcal{F}_1 \longrightarrow \dots$$

Our aim is to resolve the sheaf  $\mathcal{H}^0(M, \mathcal{E}^\bullet)$ . Note that we have

$$\mathcal{H}^0(U, \mathcal{E}^\bullet|_U) \cong \ker \psi.$$

**3.3.** To resolve the sheaf  $\mathcal{H}^0(M, \mathcal{E}^\bullet)$ , we consider the following model. Let  $X$  be a scheme, not necessarily irreducible or reduced, and

$$\varphi : E \longrightarrow F$$

be a homomorphism between locally free sheaves over  $X$ . We let  $\bigwedge^i \varphi : \bigwedge^i E \longrightarrow \bigwedge^i F$  be the induced homomorphism between the wedge products,  $i \geq 0$ . We also view  $\bigwedge^i \varphi$  as a section of  $\text{Hom}(\bigwedge^i E, \bigwedge^i F)$ . Denote by  $m$  the rank of the image sheaf  $\text{Im} \varphi$ . This is the smallest integer such that  $\bigwedge^{m+1} \varphi \equiv 0$ .

**Definition 3.4.** For any  $0 \leq r \leq m - 1$ , we let  $Z_{\varphi, r} \subset X$  be the subscheme of vanishing of the section  $\bigwedge^{r+1} \varphi$  and call it the  $r$ -determinantal subscheme of  $X$  with respect to  $\varphi$ . Its scheme structure is given by the determinantal ideal  $\mathcal{I}_{\varphi, r}$  which is generated by all  $(r+1) \times (r+1)$  minor determinants of any local matrix representation of  $\varphi$ .

Note that the ideal sheaf  $\mathcal{I}_{\varphi, r}$  does not depend on the choice of local matrix representation; it is supported on the locus of points  $w$  such that  $\dim \text{Im}(\varphi(w)) \leq r$ .

**3.5.** Determinantal ideals have base change property. If  $f : Y \longrightarrow X$  is a morphism between schemes, then the  $r$ -determinantal ideal of  $f^* \varphi : f^* E \longrightarrow f^* F$  is the pullback of the  $r$ -determinantal ideal of  $\varphi : E \longrightarrow F$ . That is,  $\mathcal{I}_{f^* \varphi, r} = f^* \mathcal{I}_{\varphi, r}$ .

**3.6.** Now, we will define inductively the blowing ups of  $X$  along the determinantal ideal sheaves<sup>1</sup>. First, we let

$$b_0 : X_0 \longrightarrow X$$

be the blowing-up of  $X$  along the ideal sheaf  $\mathcal{I}_{\varphi, 0}$ . For any  $0 \leq r \leq m - 2$ , assume that

$$b_r : X_r \longrightarrow X_{r-1}$$

is already defined. We let

$$\varphi_r : E_r \longrightarrow F_r$$

be the pullback of  $\varphi : E \longrightarrow F$ . Then we define

$$b_r : X_{r+1} \longrightarrow X_r$$

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<sup>1</sup>We note here that similar idea has been used in [6].

to be the blowup of  $X_r$  along the ideal sheaf  $\mathcal{I}_{\varphi_r, r+1}$ . After the final blowing up

$$b_{m-1} : X_{m-1} \longrightarrow X_{m-2},$$

we define

$$b : \tilde{X} := X_{m-1} \longrightarrow X$$

to be the induced iterated blowing up. We also let  $D_r$  to denote the exceptional divisor of the birational morphism  $b_r : X_r \longrightarrow X_{r-1}$ .

**Theorem 3.7.** *The birational morphism  $b : \tilde{X} \longrightarrow X$  resolves the kernel sheaf of  $\varphi : E \longrightarrow F$ . That is, the kernel sheaf  $\ker(b^*\varphi|_{\tilde{X}'})$  is locally free where  $\tilde{X}'$  is any irreducible component of  $\tilde{X}$  with the reduced scheme structure.*

To prove this theorem, we will state and prove the technical but stronger Theorem 3.11, from which the above is an immediate consequence. To this end, we need the pivotal notion of locally diagonalizable homomorphism.

**Definition 3.8.** *A homomorphism  $\varphi : \mathcal{O}_X^{\oplus p} \longrightarrow \mathcal{O}_X^{\oplus q}$  is said to be diagonalizable if we have direct sum decompositions by trivial sheaves*

$$(3.2) \quad \mathcal{O}_X^{\oplus p} = G_0 \oplus \bigoplus_{i=1}^l G_i \quad \text{and} \quad \mathcal{O}_X^{\oplus q} = H_0 \oplus \bigoplus_{i=1}^l H_i$$

with  $\varphi(G_i) \subset H_i$  for all  $i$  such that

- (1)  $\varphi|_{G_0} = 0$ ;
- (2) for every  $1 \leq i \leq l$ ,  $\varphi|_{G_i}$  equals to  $p_i I_i$  for some  $0 \neq p_i \in \Gamma(\mathcal{O}_X)$  where  $I_i : G_i \rightarrow H_i$  is an isomorphism;
- (3)  $\langle p_i \rangle \supsetneq \langle p_{i+1} \rangle$

**Definition 3.9.** *A homomorphism  $\varphi : E \longrightarrow F$  between locally free sheaves of a scheme  $X$  is locally diagonalizable if there are trivializations of  $E$  and  $F$  over some open covering of  $X$  such that  $\varphi : E \longrightarrow F$  is diagonalizable over every open subset.*

It is routine to check the following useful observations.

**Proposition 3.10.** *Suppose that a homomorphism  $\varphi : E \longrightarrow F$  is locally diagonalizable, then*

- (1) for every  $0 \leq r \leq m-1$ , the determinantal ideal  $\mathcal{I}_{\varphi, r}$  is invertible;
- (2) for every irreducible component  $X'$  of  $X$  with the reduced scheme structure,  $\ker(\varphi|_{X'})$  is locally free<sup>2</sup>;

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<sup>2</sup>Note here that the rank of  $\ker \varphi$  depends on the properties of the functions  $p_i$ , hence may not be constant over  $X$ . Further, when  $X'$  is not reduced,  $\ker(\varphi|_{X'})$

- (3) if  $f : Y \rightarrow X$  is a morphism, then  $f^*\varphi$  is also locally diagonalizable (i.e., “locally diagonalizable” has base change property).

We now arrive at our main technical theorem.

**Theorem 3.11.** (Diagonalization.) *For any  $0 \leq r \leq m - 1$ , there is an open covering of  $X_r$  trivializing  $E_r$  and  $F_r$  such that over each open subset  $U$  we have direct sum decompositions by trivial sheaves*

$$E_r = \bigoplus_{i=0}^r G_i \oplus G_{r+1} \quad \text{and} \quad F_r = \bigoplus_{i=0}^r H_i \oplus H_{r+1}$$

with  $\varphi_r(G_i) \subset H_i$  for all  $0 \leq i \leq r$  and  $\varphi_r(G_{r+1}) \subset H_{r+1}$  making the following true

- (1)  $G_i \cong H_i \cong \mathcal{O}_U$  for all  $0 \leq i \leq r$ , hence  $\bigoplus_{i=0}^r G_i \cong \bigoplus_{i=0}^r H_i \cong \mathcal{O}_U^{\oplus(r+1)}$ ;
- (2) for every  $0 \leq i \leq r$ ,  $\varphi_r|_{G_i}$  equals to  $p_i I_i$  for some  $0 \neq p_i \in \Gamma(\mathcal{O}_X)$  where  $I_i : G_i \rightarrow H_i$  is an isomorphism;
- (3)  $p_i | p_{i+1}$  ( $0 \leq i \leq r - 1$ );
- (4)  $p_r$  divides  $\varphi_r|_{G_{r+1}}$ .

In particular, when  $r = m - 1$ , we obtain that the homomorphism  $b^*\varphi$  is locally diagonalizable, and hence the kernel sheaf  $\ker(b^*\varphi|_{\tilde{X}'})$  is locally free where  $\tilde{X}'$  is an irreducible component of  $\tilde{X}$  with the reduced scheme structure.

*Proof.* We will prove by induction on  $r$ .

First, consider the case of  $r = 0$ . Take any point  $\xi \in X$ . Locally around  $\xi$ , we can trivialize  $E$  and  $F$  over an open neighborhood  $U$  of  $\xi$  and choose bases of  $E$  and  $F$  such that  $\varphi$  is given by the matrix  $(\mu_{ij})$ .

If  $Z_{\varphi,0} = \emptyset$ , or equivalently  $\mathcal{S}_{\varphi,0} = \mathcal{O}_U$ , then there is a  $\mu_{ij}$  such that  $\mu_{ij}(0) \neq 0$ . By shrinking  $U$  we may assume that  $\mu_{ij} \in \Gamma(\mathcal{O}_U^*)$ . This way, by a basis change, we can arrange decompositions

$$E \cong G_1 \oplus G_2 \quad \text{and} \quad F \cong H_1 \oplus H_2$$

such that  $G_1, H_1 \cong \mathcal{O}_U$ ,  $\varphi(G_1) \subset H_1$ ,  $\varphi(G_2) \subset H_2$ , and  $\varphi|_{G_1} : G_1 \rightarrow H_1$  is an isomorphism. Since  $X_0 = X$  in this case, the statements of the theorem hold.

If  $Z_{\varphi,0}$  is a Cartier divisor, i.e., the ideal  $\mathcal{S}_{\varphi,0}$  is the principal ideal  $\langle p \rangle$  generated by some  $p \in \Gamma(\mathcal{O}_U)$ , then we can write  $(\mu_{ij})$  as  $(p \cdot \nu_{ij})$ . Since  $\langle \mu_{ij} \rangle = \langle p \rangle$ , we see that  $\langle \nu_{ij} \rangle = \mathcal{O}_U$ . This implies that the

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needs not to be locally free. These technical issues lead us to use the “integral” assumption whenever and only when we want to produce a locally free sheaf.

homomorphism  $\varphi$  factors as

$$\varphi : E \xrightarrow{\varphi'} F(-Z_{\varphi,0}) \xrightarrow{\text{inclusion}} F,$$

and  $\varphi' = (\nu_{ij})$  has that  $I_{\varphi'_0,0} = \mathcal{O}_U$ . Now apply the previous case to the homomorphism  $\varphi'$ , one checks that we obtain decompositions

$$E = G_1 \oplus G_2 \quad \text{and} \quad F = H_1 \oplus H_2$$

such that  $G_1, H_1 \cong \mathcal{O}_U$ ,  $\varphi(G_1) \subset H_1$ ,  $\varphi(G_2) \subset H_2$ , and  $\varphi|_{G_1} : G_1 \rightarrow H_1$  equals  $pI_1$  where  $I_1 : G_1 \rightarrow H_1$  is an isomorphism and  $p$  divides  $\varphi|_{G_2}$ . Again, since  $X_0 = X$  in this case, the statements of the theorem hold.

If  $Z_{\varphi,0}$  is not a Cartier divisor, that is,  $\mathcal{I}_{\varphi,0}$  is not principal, we blow up  $Z_{\varphi,0}$  to obtain

$$b_0 : X_0 \rightarrow X \quad \text{and} \quad b_0^* \varphi : b_0^* E \rightarrow b_0^* F.$$

By 3.5, we have that  $\mathcal{I}_{b_0^* \varphi,0} = b_0^* \mathcal{I}_{\varphi,0}$ . Since  $b_0^* \mathcal{I}_{\varphi,0}$  is principal, this reduces to the previous case. Note that  $b_0^* \varphi = \varphi_0$ . Thus, the case 0 is proved.

Assume now that the assertion holds for  $r$ . Since the question is local, we can restrict our focus on an affine open subset  $U$  of  $X_r$  such that  $E_r$  and  $F_r$  are all trivialized with the desired decomposition as in the theorem and  $\varphi_r : E_r \rightarrow F_r$  has the desired properties as in the theorem. In terms of the suitable bases of  $E_r$  and  $F_r$  as in the theorem granted by the inductive assumption, all these mean is that we can represent  $\varphi_r$  by the diagonal matrix

$$\text{diag}[p_0 I_0, \dots, p_r I_r, B]$$

where  $B$  is the matrix representation of  $\varphi_r|_{G_{r+1}}$  and  $p_r|B$ . Since  $p_r|B$ , we may write  $B = p_r B'$  and let  $\varphi'_r : G_{r+1} \rightarrow H_{r+1}$  be the homomorphism corresponding to  $B'$ . From the above representation, we see that

$$(3.3) \quad \mathcal{I}_{\varphi_r, r+1} = (p_0 \cdots p_r) p_r \mathcal{I}_{\varphi'_r, 0}.$$

Hence blowing up  $\mathcal{I}_{\varphi_r, r+1}$  is the same as blowing up  $\mathcal{I}_{\varphi'_r, 0}$ . Now we can apply the case 0 to the homomorphism

$$\varphi'_r : G_{r+1} \rightarrow H_{r+1}.$$

From here, it is routine to check against the three cases of case 0 (for  $\varphi'_r$ ) so that we will obtain the desired decompositions for  $E_{r+1}$  and  $F_{r+1}$  with the desired properties for  $\varphi_{r+1}$ .

By induction, this proves the statements (1)–(4) of the theorem.

To finish off, in the case of  $r = m - 1$ , because  $\bigoplus_{i=0}^r G_i \cong \mathcal{O}_U^{\oplus m}$  with  $m$  the maximal rank of  $\varphi_{m-1} = b^* \varphi$ , we must have that  $\varphi_{m-1}(G_{r+1}) = 0$ .

This implies that  $b^*\varphi$  is locally diagonalizable, and in particular, if we restrict  $b^*\varphi$  to an irreducible component with the reduced structure, its kernel sheaf is locally free.

This completes the proof.  $\square$

**Remark 3.12.** The requirement that each  $G_i$  is isomorphic to  $\mathcal{O}_U$  in the above theorem appears to be “stronger” than Definition 3.8, but in fact, they are equivalent since in this theorem we allow  $\langle p_i \rangle = \langle p_{i+1} \rangle$  for some pairs  $p_i$  and  $p_{i+1}$ .

**Proposition 3.13.** (Universality.) *Let  $\varphi : E \rightarrow F$  be a homomorphism between locally free sheaves over a scheme  $X$  and  $\tilde{X}$  the blowup of  $X$  along determinantal loci of  $\varphi$ . If  $f : Z \rightarrow X$  is any dominant morphism between schemes such that the pullback homomorphism  $f^*\varphi : f^*E \rightarrow f^*F$  is locally diagonalizable, then  $f$  factors uniquely through  $\tilde{X}$ .*

*Proof.* Let  $m$  be the rank of the image sheaf  $\text{Im}\varphi$ . For any  $0 \leq r \leq m - 1$ , we will show inductively that  $f$  factors uniquely through  $X_r$ .

When  $r = 0$ , consider any point  $\xi \in Z$  such that  $f(\xi)$  belongs to the 0-determinantal locus of  $\varphi$ . As in the proof of Theorem 3.11, locally around  $f(\xi) \in W$ , we can represent  $\varphi$  by a matrix  $(\mu_{ij})$  with  $\mu_{ij} \in \Gamma(\mathcal{O}_U)$ , where  $U$  is an open neighborhood of  $f(\xi)$ . The homomorphism  $f^*\varphi : f^*E \rightarrow f^*F$  is given by the pullbacks  $(f^*\mu_{ij})$ . Let  $\mathcal{I} = f^{-1}\mathcal{I}_{\varphi,0}$  be the ideal generated by  $(f^*\mu_{ij})$ . Since  $f$  is dominant,  $f(Z)$  is not entirely contained in the 0-determinantal locus of  $\varphi$ , hence  $\mathcal{I} \neq 0$ . But then, since  $f^*\varphi : f^*E \rightarrow f^*F$  is diagonalizable, we can represent  $f^*\varphi$  as a diagonal matrix  $\text{diag}[p_1I_1, \dots, p_lI_l, 0]$ . This shows that  $\mathcal{I}$  is principal. By the universal property of blowing up (Proposition 7.14 of [4]),  $f$  factors uniquely through  $X_0$ .

Now assume that the claim holds for  $r$ :  $f$  factors uniquely through  $X_r$

$$f : Z \xrightarrow{g} X_r \xrightarrow{b_r} X.$$

Since  $X_{r+1}$  is the blowup of  $X_r$  along  $\mathcal{I}_{\varphi'_r,0}$  (here we use the notation as in the proof Theorem 3.11), we need only to consider a small open neighborhood of any point  $\xi \in Z$  such that  $g(\xi)$  belongs to the 0-determinantal locus of  $\varphi'_r$ . Since  $f$  is dominant, by the base change property, we have

$$g^{-1}\mathcal{I}_{\varphi'_r,0} = \mathcal{I}_{g^*\varphi'_r,0} \quad \text{and} \quad f^{-1}\mathcal{I}_{\varphi,r+1} = \mathcal{I}_{f^*\varphi,r+1}.$$

By (3.3) in the proof of Theorem 3.11, we see that  $\mathcal{I}_{g^*\varphi'_r,0}$  and  $\mathcal{I}_{f^*\varphi,r+1}$  differ by an invertible sheaf. Since  $f^*\varphi : f^*E \rightarrow f^*F$  is diagonalizable, from its diagonal representation we conclude that  $\mathcal{I}_{f^*\varphi,r+1}$ , which is



$f^{-1}\mathcal{I}_{\varphi,r+1}$ , is invertible. Hence, so is  $\mathcal{I}_{g^*\varphi'_r,0} = g^{-1}\mathcal{I}_{\varphi'_r,0}$ . This implies that  $g$  factors uniquely through  $X_{r+1}$ , hence so does  $f$ .

By induction, this finishes the proof.  $\square$

**Definition 3.14.** For any homomorphism  $\psi : E \rightarrow F$  between two locally free sheaves over a scheme  $X$ , we say that a point  $x \in X$  is  $\psi$ -regular if  $\psi(x)$  is of maximal rank.

It is easy to see that if  $x \in X$  is  $\psi$ -regular, then locally around  $x$ ,  $\psi : E \rightarrow F$  is diagonalizable (and in fact, can be diagonalized to the form  $\text{diag}(I, 0)$ ).

**Remark 3.15.** The universality proposition 3.13 may also hold for any morphism  $f : Z \rightarrow X$  such that  $f(Z)$  contains  $\varphi$ -regular points and  $f^*\varphi$  is locally diagonalizable. But, we do not need this stronger version in this paper.

#### 4. RESOLUTION OF DERIVED OBJECT

**4.1.** Let  $\mathcal{G}$  be any coherent sheaf over a scheme  $X$ . A presentation of  $\mathcal{G}$  is an exact sequence

$$\mathcal{O}_X^{\oplus n} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus m} \rightarrow \mathcal{G} \rightarrow 0.$$

The  $h$ -th ( $h \leq m$ ) Fitting ideal of the above presentation, denoted  $\mathcal{I}_h(\mathcal{G})$ , is the  $(m-h)$ -determinantal ideal of the homomorphism  $\alpha$  (and is defined to be  $\mathcal{O}_X$  when  $h > m$ ). The basic property of Fitting ideals is that any two presentations of  $\mathcal{G}$  have the same Fitting ideals ([2, 8]). This enables us to define the Fitting ideals of the coherent sheaf  $\mathcal{G}$  without reference to any particular presentation. It can be shown that  $\mathcal{I}_h(\mathcal{G})$  are finitely generated and form an increasing sequence

$$\mathcal{I}_0(\mathcal{G}) \subset \mathcal{I}_1(\mathcal{G}) \subset \dots.$$

Taking Fitting ideal commutes with base change. That is, if  $g : Y \rightarrow X$  is a morphism, then the  $h$ -th Fitting ideal of the sheaf  $f^*\mathcal{G}$  is generated, as an  $\mathcal{O}_Y$ -module, by the  $h$ -th Fitting ideal of  $\mathcal{G}$ . This is because the pullback of a presentation of  $\mathcal{G}$  is a presentation of  $f^*\mathcal{G}$  since the tensor product is a right exact functor. In addition, taking Fitting ideal commutes with localization. Let  $S$  be any multiplicative closed subset of  $\mathcal{O}_X$  not containing the zero element. Then for every  $h \geq 0$  we have

$$\mathcal{I}_h(\mathcal{G})(\mathcal{O}_X)_S = \mathcal{I}_h(\mathcal{G}_S),$$

where by  $\mathcal{I}_h(\mathcal{G}_S)$  we mean the  $h$ -th Fitting ideal of the sheaf  $\mathcal{G}_S$  of  $(\mathcal{O}_X)_S$ -modules. For more details of Fitting ideals, the reader is referred to [2] and [8].

**4.2.** We now are ready to prove Theorem 1.1 which we restate below. Since  $\mathcal{E}^\bullet$  is perfect and can locally be represented by a complex of locally free sheaves of finite length supported only in non-negative degrees, we can assume that  $M$  admits an open cover  $\coprod U$ , and over each open subset  $U$ , its restriction  $\mathcal{E}^\bullet|_U$  is represented by the following complex of locally free sheaves

$$(4.1) \quad \mathcal{F}_0 \xrightarrow{\psi_0} \mathcal{F}_1 \xrightarrow{\psi_1} \cdots \longrightarrow \mathcal{F}_{n-1} \xrightarrow{\psi_{n-1}} \mathcal{F}_n.$$

**Theorem 4.3.** *Let  $M$  be an integral DM stack and  $\mathcal{E}^\bullet$  a perfect object in the derived category  $D^b(M)$  which can be locally represented by a complex of locally free sheaves of finite length supported only in non-negative degrees. Then there is another integral DM stack  $\widetilde{M}$  and a dominant birational morphism  $f : \widetilde{M} \rightarrow M$  such that over any open chart and for every  $0 \leq i \leq n-1$ , the homomorphism  $f^* \mathcal{F}_i \xrightarrow{f^* \psi_i} f^* \mathcal{F}_{i+1}$  is diagonalizable. In particular,  $\mathcal{H}^0(Lf^* \mathcal{E}^\bullet)$  is locally free.*

*Proof.* We will adopt the notations from 4.2 and assume that locally  $\mathcal{E}^\bullet$  is represented by the complex as in (4.1).

We first consider the coherent sheaf  $\mathcal{H}^n(\mathcal{F}^\bullet) = \text{coker } \psi_{n-1}$ . The sequence

$$\mathcal{F}_{n-1} \xrightarrow{\psi_{n-1}} \mathcal{F}_n \longrightarrow \mathcal{H}^n(\mathcal{F}^\bullet) \longrightarrow 0$$

is a locally free presentation of  $\mathcal{H}^n(\mathcal{E}^\bullet)$ . Since the Fitting ideals of  $\mathcal{H}^n(\mathcal{E}^\bullet)$  are the same as the determinantal ideals of  $\psi_{n-1} : \mathcal{F}_{n-1} \rightarrow \mathcal{F}_n$  and Fitting ideals are independent of presentations, we conclude that when applied to the homomorphism  $\psi_{n-1} : \mathcal{F}_{n-1} \rightarrow \mathcal{F}_n$  over the open subset  $U$ , the iterated blowup as described in §3 patch together to produce a well-defined iterated blowup of  $M$

$$f_{n-1} : M_{n-1} \longrightarrow M$$

such that  $f_{n-1}^* \psi_{n-1}$  is locally diagonalizable. Now observe that if  $M$  is integral and  $Z \subset M$  a closed substack, then  $L_Z M$ , the blowup of  $M$  along  $Z$ , is also integral. This follows from the fact that if  $I$  is any ideal in a domain  $A$ , then  $\bigoplus_n I^n$  is also a domain. Thus  $M_{n-1}$  is integral and hence  $\ker f_{n-1}^* \psi_{n-1}$  is locally free.

Now we apply the left derived functor  $Lf_{n-1}^*$  to  $\mathcal{E}^\bullet$  and obtain

$$Lf_{n-1}^* \mathcal{E}^\bullet \in D^b(M_{n-1}).$$

The stack  $M_{n-1}$  admits the open cover by  $\coprod f_{n-1}^{-1}(U)$ . Over the open subset  $f_{n-1}^{-1}(U)$ ,  $Lf_{n-1}^* \mathcal{E}^\bullet$  is represented by

$$f_{n-1}^* \mathcal{F}_0 \xrightarrow{f_{n-1}^* \psi_0} f_{n-1}^* \mathcal{F}_1 \longrightarrow \cdots \longrightarrow f_{n-1}^* \mathcal{F}_{n-1} \xrightarrow{f_{n-1}^* \psi_{n-1}} f_{n-1}^* \mathcal{F}_n.$$

Consider the short exact sequence

$$f_{n-1}^* \mathcal{F}_{n-2} \xrightarrow{f_{n-1}^* \psi_{n-2}} \ker f_{n-1}^* \psi_{n-1} \longrightarrow \mathcal{H}^{n-1}(Lf_{n-1}^* \mathcal{E}^\bullet) \longrightarrow 0.$$

This is a locally free presentation of  $\mathcal{H}^{n-1}(Lf_{n-1}^* \mathcal{E}^\bullet)$ . This allows us to apply the same blowing up process as in the previous step to the coherent sheaf  $\mathcal{H}^{n-1}(Lf_{n-1}^* \mathcal{E}^\bullet)$  to get

$$f_{n-2} : M_{n-2} \longrightarrow M_{n-1}$$

such that  $f_{n-2}^* f_{n-1}^* \psi_{n-2}$  is diagonalizable and for the same reason as explained earlier  $\ker f_{n-2}^* f_{n-1}^* \psi_{n-2}$  is locally free.

Applying the above repeatedly (or by induction), we will eventually arrive at a birational dominant morphism

$$f : \widetilde{M} \longrightarrow M,$$

factoring as

$$\widetilde{M} = M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} M_{n-1} \xrightarrow{f_{n-1}} M,$$

such that if we let  $f$  to be  $f_{n-1} \circ f_{n-2} \cdots \circ f_0$ , then for each  $0 \leq i \leq n-1$ ,  $f^* \psi_i$  is diagonalizable,  $M_i$  is integral and  $\ker f^* \psi_i$  is locally free. Now use that  $Lf^* \mathcal{E}^\bullet$  is locally represented by

$$f^* \mathcal{F}_0 \xrightarrow{f^* \psi_0} f^* \mathcal{F}_1 \longrightarrow \cdots \longrightarrow f^* \mathcal{F}_{n-1} \xrightarrow{f^* \psi_{n-1}} f^* \mathcal{F}_n,$$

we see that

$$\mathcal{H}^0(Lf^* \mathcal{E}^\bullet) = \ker f^* \psi_0,$$

the theorem is thus proved.  $\square$

**Corollary 4.4.** *Let  $M$  be a DM stack (not necessarily integral) and  $\mathcal{E}^\bullet$  a perfect object in the derived category  $D^b(M)$  with  $\mathcal{H}^i(\mathcal{E}^\bullet) = 0$  for  $i < 0$ . Assume further that locally  $\mathcal{E}^\bullet$  can be represented by a two-term complex  $\mathcal{F}_0 \xrightarrow{\psi} \mathcal{F}_1$  of locally free sheaves. Then there is another DM stack  $\widetilde{M}$  and a birational morphism  $f : \widetilde{M} \rightarrow M$  such that the homomorphism  $f^* \mathcal{F}_0 \xrightarrow{f^* \psi} f^* \mathcal{F}_1$  is diagonalizable. In particular, for any irreducible component  $\widetilde{M}'$  of  $\widetilde{M}$  endowed with the reduced stack structure,  $\mathcal{H}^0(Lf^* \mathcal{E}^\bullet|_{\widetilde{M}'})$  is locally free.*

*Proof.* As in the first step of the proof of the above theorem, we consider the locally free presentation

$$\mathcal{F}_0 \xrightarrow{\psi} \mathcal{F}_1 \longrightarrow \mathcal{H}^1(\mathcal{E}^\bullet) \longrightarrow 0$$

of the sheaf  $\mathcal{H}^1(\mathcal{E}^\bullet)$ , we then simply blow up  $M$  along its Fitting ideals and obtain  $f : \widetilde{M} \rightarrow M$ . It follows from Theorem 3.11 that the homomorphism  $f^* \mathcal{F}_0 \xrightarrow{f^* \psi} f^* \mathcal{F}_1$  is diagonalizable. Now Proposition 3.10 (2) implies the rest of the statements.  $\square$

As a direct consequence of Proposition 3.13, the stack  $\widetilde{M}$  is universal.

**Proposition 4.5.** (Universality.) *Let the situation be as in Theorem 4.3 or as in Corollary 4.4. If  $g : Z \rightarrow M$  is any dominant morphism such that for each  $0 \leq i \leq n-1$ ,  $g^*\psi : g^*\mathcal{F}_i \rightarrow g^*\mathcal{F}_{i+1}$  is diagonalizable, then  $g$  factors uniquely through  $f : \widetilde{M} \rightarrow M$ .*

But, for topological applications, the following base change property, although weaker, is more convenient to use and easy to prove.

**Proposition 4.6.** (Base Change Property.) *Suppose that  $M$  is integral. For any DM stack  $N$  and a dominant morphism  $g : N \rightarrow M$  such that  $\mathcal{H}^0(Lg^*\mathcal{E}^\bullet)$  is locally free, we can find another DM stack  $N'$  and a dominant morphism  $\tilde{g} : N' \rightarrow M$ , factoring through  $f$  and  $g$  (i.e.,  $\tilde{g} = f \circ f' = g \circ g'$ )*

$$\begin{array}{ccc} N' & \xrightarrow{f'} & \widetilde{M} \\ g' \downarrow & & \downarrow f \\ N & \xrightarrow{g} & M \end{array}$$

so that  $\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet)$  is locally free and is the pull back of  $\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)$  and  $\mathcal{H}^0(Lg^*\mathcal{E}^\bullet)$ .

*Proof.* Indeed, we let  $N'$  be the closure of the open subset of the graph of the rational map

$$N \longrightarrow \widetilde{M}$$

that is isomorphic to its image when projected to either  $N'$  and  $\widetilde{M}$ . Then we obtain the square as in the proposition such that  $\tilde{g} : N' \rightarrow M$  is a dominant morphism. The rest conclusions are local. So, locally we will represent  $Lf^*\mathcal{E}^\bullet$  by the complex of locally free sheaves

$$f^*\mathcal{F}_0 \xrightarrow{f^*\psi_0} f^*\mathcal{F}_1 \longrightarrow \dots$$

This implies that  $L\tilde{g}^*\mathcal{E}^\bullet = Lf'^* \circ Lf^*\mathcal{E}^\bullet$  is locally represented by

$$f'^*f^*\mathcal{F}_0 \xrightarrow{f'^*f^*\psi_0} f'^*f^*\mathcal{F}_1 \longrightarrow \dots$$

Now, note that

$$0 \longrightarrow \ker f^*\psi_0 \longrightarrow f^*\mathcal{F}_0 \xrightarrow{f^*\psi_0} f^*\mathcal{F}_1$$

is exact. Since  $f'$  is dominant,

$$0 \longrightarrow f'^*\ker f^*\psi_0 \longrightarrow f'^*f^*\mathcal{F}_0 \xrightarrow{f'^*f^*\psi_0} f'^*f^*\mathcal{F}_1$$

is also exact. Hence

$$\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet) = f'^*\mathcal{H}^0(Lf^*\mathcal{E}^\bullet).$$

Similarly,

$$\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet) = g'^*\mathcal{H}^0(Lg^*\mathcal{E}^\bullet).$$

□

**4.7.** Suppose that  $M$  is integral. One can also routinely verify the following.

- (1) Let  $\mathcal{E}^\bullet$  be a perfect derived object over an DM stack  $M$  and  $\mathcal{E}^{\bullet'}$  be a perfect derived object over an DM stack  $M'$ , both with vanishing  $\mathcal{H}^{i<0}$ . Let  $f : \widetilde{M} \rightarrow M$  and  $f' : \widetilde{M}' \rightarrow M'$  be given as in Theorem 4.3, then  $\mathcal{H}^0(L(f, f')^*(\mathcal{E}^\bullet \boxplus \mathcal{E}^{\bullet'}))$  is locally free over  $\widetilde{M} \times \widetilde{M}'$ ;
- (2) Let  $\mathcal{E}^\bullet$  and  $\mathcal{E}^{\bullet'}$  be perfect derived objects over a DM stack  $M$ , both with vanishing  $\mathcal{H}^{i<0}$ . Let  $f : \widetilde{M} \rightarrow M$  and  $f' : \widetilde{M}' \rightarrow M$  be given as in Theorem 4.3. We let  $\mathbf{M}$  be the graph of the rational map  $\widetilde{M} \rightarrow \widetilde{M}'$  and  $\tilde{f}$  the projection to  $M$ . Then  $\mathcal{H}^0(L\tilde{f}^*(\mathcal{E}^\bullet \oplus \mathcal{E}^{\bullet'}))$  is locally free over  $\mathbf{M}$ .

## 5. THE EULER CLASS OF PERFECT DERIVED OBJECT

**5.1.** In this subsection, we again make assumption that  $M$  (hence also  $\widetilde{M}$ ) is integral. To define the Euler class of the complex  $\mathcal{E}^\bullet$ , we will use the top Chern class of the locally free sheaf  $\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)$ . Let  $r = \text{rank } \mathcal{H}^0(\mathcal{E}^\bullet)$ . The Chern class  $c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet))$  is a homomorphism

$$c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)) : A_*(\widetilde{M}) \longrightarrow A_{*-r}(\widetilde{M}).$$

Here  $A_*(\widetilde{M})$  is the Chow group of cycles on  $\widetilde{M}$ . We will assume that  $\text{rank } \mathcal{H}^0(\mathcal{E}^\bullet) > 0$  and the higher cohomology sheaves  $\mathcal{H}^i(\mathcal{E}^\bullet)$  are all torsion for  $i > 0$ . This way, the Euler class of the complex  $\mathcal{E}^\bullet$ , as expected, should only depend on  $\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)$ .

**Definition 5.2.** For any integral DM stack  $M$  of dimension  $n$  and a derived object  $\mathcal{E}^\bullet$  as in Theorem 4.3, let  $r = \text{rank } \mathcal{H}^0(\mathcal{E}^\bullet)$ . Then we define its Euler class  $\mathbf{e}(\mathcal{E}^\bullet) \in A_{n-r}(M)$  as:

$$(5.1) \quad \mathbf{e}(\mathcal{E}^\bullet) := f_*(c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)) \cdot [\widetilde{M}]).$$

Here  $A_*(M)$  is the Chow group of cycles on  $M$ .

**Proposition 5.3.** For any integral DM stack  $N$  and a surjective birational morphism  $g : N \rightarrow M$  such that  $\mathcal{H}^0(Lg^*\mathcal{E}^\bullet)$  is locally free, we have

$$g_*(c_r(\mathcal{H}^0(Lg^*\mathcal{E}^\bullet)) \cdot [N]) = f_*(c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)) \cdot [\widetilde{M}]).$$

*Proof.* By Proposition 4.6, we have a square

$$\begin{array}{ccc} N' & \xrightarrow{f'} & \widetilde{M} \\ g' \downarrow & & \downarrow f \\ N & \xrightarrow{g} & M. \end{array}$$

The diagonal morphism  $N' \rightarrow M$ , denoted  $\tilde{g}$ , is a surjective birational morphism. From Proposition 4.6, we have that

$$\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet) = f'^*\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)$$

(note that all the sheaves involved are locally free). Observe that we also have  $f'_*[N'] = [\widetilde{M}]$  because  $f'$  is birational and surjective. Hence we obtain

$$\begin{aligned} f'_*(c_r(\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet)) \cdot [N']) &= f'_*(f'^*(c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)))) \cdot [N'] \\ &= c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)) \cdot [\widetilde{M}]. \end{aligned}$$

This implies that

$$\begin{aligned} f_*(c_r(\mathcal{H}^0(Lf^*\mathcal{E}^\bullet)) \cdot [\widetilde{M}]) &= f_*f'_*(c_r(\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet)) \cdot [N']) \\ &= \tilde{g}_*(c_r(\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet)) \cdot [N']). \end{aligned}$$

Similarly, we get

$$\begin{aligned} g_*(c_r(\mathcal{H}^0(Lg^*\mathcal{E}^\bullet)) \cdot [N]) &= g_*g'_*(c_r(\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet)) \cdot [N']) \\ &= \tilde{g}_*(c_r(\mathcal{H}^0(L\tilde{g}^*\mathcal{E}^\bullet)) \cdot [N']). \end{aligned}$$

This proves the proposition.  $\square$

**Corollary 5.4.** *Let  $\mathcal{E}^\bullet$  be a perfect object in  $D^b(M)$ . Suppose  $\mathcal{H}^{i < 0}(\mathcal{E}^\bullet) = 0$  and  $\mathcal{H}^{i > 0}(\mathcal{E}^\bullet)$  are torsion. Then the Euler class  $e(\mathcal{E}^\bullet)$  is well-defined and is independent of the choice of the bitational surjective morphisms  $f : \widetilde{M} \rightarrow M$ .*

## 6. APPLICATIONS TO GW-INVARIANTS

**6.1.** Let  $\mathbf{P}$  be a projective space, and let  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$  be the DM stack of degree  $d$  genus  $g$  stable maps to  $\mathbf{P}$  as before. Let

$$(6.1) \quad (\mathfrak{f}, \pi) : \mathcal{X} \rightarrow \mathbf{P} \times \overline{\mathfrak{M}}_g(\mathbf{P}, d)$$

be its universal family. For any positive integer  $k$ , the derived object  $R\pi_*(\mathfrak{f}^*\mathcal{O}_{\mathbf{P}}(k))$  in  $D^b(\overline{\mathfrak{M}}_g(\mathbf{P}, d))$  is perfect.

**6.2.** One way to see this is to pick two sufficiently large integers  $n$  and  $n'$  and form  $\mathcal{L} = \mathfrak{f}^* \mathcal{O}_{\mathbf{P}}(n) \otimes \omega_{\mathcal{X}/\overline{\mathfrak{M}}_g(\mathbf{P}, d)}^{\otimes n'}$ ; then form the tautological homomorphism

$$\mathcal{A}_0 = \pi^* \pi_* (\mathfrak{f}^* \mathcal{O}_{\mathbf{P}}(k) \otimes \mathcal{L}) \otimes \mathcal{L}^{-1} \longrightarrow \mathfrak{f}^* \mathcal{O}_{\mathbf{P}}(k).$$

Since  $n$  and  $n'$  are sufficiently large, it is surjective. We let  $\mathcal{A}_{-1}$  be the kernel of the above homomorphism. Then it is easy to see that we have an quasi-isomorphism

$$[R^1 \pi_* \mathcal{A}_{-1} \rightarrow R^1 \pi_* \mathcal{A}_0] = R\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbf{P}}(k).$$

Again since  $n$  and  $n'$  are sufficiently large,

$$[\mathcal{E}_0 \rightarrow \mathcal{E}_1] := [R^1 \pi_* \mathcal{A}_{-1} \rightarrow R^1 \pi_* \mathcal{A}_0]$$

is a complex of locally free sheaves. This proves that  $R\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbf{P}}(k)$  is perfect.

**Definition 6.3.** Assume  $d > 2g - 2$ . We let  $\mathfrak{M}_g(\mathbf{P}, d)_0 \subset \overline{\mathfrak{M}}_g(\mathbf{P}, d)$  be the open subset consisting of stable morphisms with irreducible domain curves. We define the primary part  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)'$  of  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$  to be the closure of  $\mathfrak{M}_g(\mathbf{P}, d)_0$  in  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$ .

The open subset  $\mathfrak{M}_g(\mathbf{P}, d)_0$  is non-empty, smooth and has the expected dimension. Thus  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)'$  is generically smooth and of the expected dimension.

**Remark 6.4.** Some remarks on the primary components are in order. Let  $C$  be an irreducible curve of genus  $g$ . A map  $u : C \rightarrow \mathbf{P}$  is given by  $(m+1)$ -sections  $u_0, \dots, u_m \in \Gamma(u^* \mathcal{O}_{\mathbf{P}}(1))$ , where  $m = \dim \mathbf{P}$ . We may assume that  $u^* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_C(D)$  for some effective divisor  $D$ . Assume that  $d > g$ . Then there are general divisors on the curve  $C$ . If  $D$  is general, by the geometric version of Riemann-Roch theorem,  $\dim \Gamma(\mathcal{O}_C(D)) = d + 1 - g \geq 2$ . From here, one checks that the dimension of  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$  at such a map is

$$3g - 3 + d + (d + 1 - g)m = d(m + 1) + (m - 3)(1 - g),$$

as expected. When  $d > 2g - 2$ , all divisors are general. If  $C$  is reducible and has more than one irreducible components (of positive genera, for instance) that are not contracted by the stable morphism, then conjecturally they do not contribute the GW number of quintic Calabi-Yaus. When  $g < d < 2g - 1$ , there are special divisors over the curve  $C$ . Such divisors give rise to  $\Gamma(\mathcal{O}_C(D))$  with  $\dim \Gamma(\mathcal{O}_C(D)) > d + 1 - g \geq 2$ . Hence they may produce a component of  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$  with dimension larger than expected. When  $0 < d \leq g$ , it may happen that none of the irreducible components of  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)$  have the expected dimension.

**6.5.** We now assume  $d > 2g - 2$ . We apply the constructions in the previous sections to the complex  $R\pi_*\mathfrak{f}^*\mathcal{O}_{\mathbf{P}}(k)$  restricted to  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)'$ . Let  $\mathfrak{f}'$  be the restriction of  $\mathfrak{f}$  to  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)'$ . By the vanishing of high cohomology,  $R^1\pi_*\mathfrak{f}'^*\mathcal{O}_{\mathbf{P}}(k)$  is trivial on a dense open subset of  $\overline{\mathfrak{M}}_g(\mathbf{P}, d)'$ . Then by §5.4, we have a well-defined *modular Euler class*

$$(6.2) \quad \mathbf{e}(R\pi_*(\mathfrak{f}'^*\mathcal{O}_{\mathbf{P}}(k))) \in A_*(\overline{\mathfrak{M}}_g(\mathbf{P}, d)').$$

We denote this number by  $N'_{g,d}$ , (cf. Definition 6.3).

**6.6.** When  $g = 0$ ,  $N'_{0,d} = N_{0,d}$ , the usual Gromov-Witten number of the quintic  $X$ ; when  $g = 1$ ,  $N'_{1,d}$  is the reduced genus-1 GW-invariants (see [7, 1]).

We believe these numbers are the reduced GW-invariants of quintics conjectured by Li-Zinger, (cf. Conj. 1.5).

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