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L^p-ESTIMATES OF THE BOTLZMANN EQUATION AROUND A TRAVELING LOCAL MAXWELLIAN

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Abstract. In this paper, we are interested in the L^p-estimates of the Boltzmann equation in the case that the distribution function stays around a travelling local Maxwellian. For this, we divide both sides of the Boltzmann equation by the velocity distribution function with a fractional exponent and reformulate the Boltzmann equation into a regularized one. This amounts to endowing additional integrability on the collision kernel, which in turn enables us to apply simple Hölder type inequalities. Our results cover the whole range of Lebesgue exponents: $0 < p \leq \infty$.

1. Introduction

In the kinetic theory of gases, it is postulated that all the relevant information is encoded in a velocity distribution function $f(x, v, t)$ representing the number density of particles located at position x with velocity v at time t. For non-ionized monatomic rarefied gas, the time evolution of f is governed by the celebrated Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad (x, v, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+. \tag{1.1}$$

The left hand side of (1.1) describes the free transport of non-interacting particles, whereas the collision operator $Q(f, f)$ captures collisions or interaction between particles. It can be written down explicitly as follows:

$$Q(f, f)(v) \equiv \frac{1}{\kappa} \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} B(v - v_*, \omega)(f' f'_* - f f_*) d\omega dv_*. \tag{1.2}$$

Here κ is the Knudsen number which is the ratio between the mean free path of molecules and the characteristic length of the flow and $\mathbb{S}^2_+ = \{\omega \in \mathbb{S}^2 \mid (v - v_*) \cdot \omega \geq 0\}$. For the simplicity of presentation, we adopt the following handy notations:

$$f' \equiv f(x, v', t), \quad f'_* \equiv f(x, v'_*, t), \quad f \equiv f(x, v, t) \quad \text{and} \quad f_* \equiv f(x, v_*, t),$$

where the pair (v', v'_*) denotes the post-collisional velocities which can be calculated explicitly from the pre-collisional pair of velocities (v, v_*) by

$$v' = v - [(v - v_*) \cdot \omega]\omega \quad \text{and} \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega. \tag{1.3}$$

The collision kernel $B(v - v_*, \omega)$ is determined by types of interaction between gas particles. For the precise form and relevant structural assumptions imposed on the collision kernel, see ($\mathcal{A}1$) below. For more detailed survey of mathematical and physical results of the Boltzmann equation, we refer to [4, 6, 7, 21, 22, 26].

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In this paper, we study the stability problem of the Boltzmann equation in L^p spaces when the velocity distribution function is bounded from above and below by a travelling local Maxwellian:

$$a_m\mathcal{M}_{\alpha,\beta}(x,v)\leq f^{\sharp}(x,v,t)\leq a_M\mathcal{M}_{\alpha,\beta}(x,v), \tag{1.4}$$

where a_m, a_M denotes positive constants and $\mathcal{M}_{\alpha,\beta}(x,v)$ is a travelling local Maxwellian solution:

$$\mathcal{M}_{\alpha,\beta}(x,v)\equiv e^{-\alpha|x|^2-\beta|v|^2}\quad \text{for positive constants}\quad \alpha,\beta>0. \tag{1.5}$$

For the stability problem of kinetic equations, L^1 space is the most natural setting in that it corresponds to the total mass of the system. The study of stability in L^1 space for the Boltzmann equation near vacuum was initiated by Ha [11, 12] who introduced a nonlinear functional approach motivated by the stability theory of hyperbolic conservation laws, and was studied extensively by Ha and his coworkers [8, 10, 14, 16]. See also [3, 19]. It is then quite natural to ask whether the stability results in L^1 can be extended to general L^p space. Considering that the asymptotic behavior of the Boltzmann equation in this regime is largely governed by the free transport equation:

$$\frac{\partial f}{\partial t}+v\cdot\nabla f=0,$$

for which the uniform L^p stability estimate trivially holds, it is reasonable to expect similar estimates to hold true for general L^p spaces. In this vein, there have been several results on the L^p-stability estimates of the Boltzmann equation near vacuum. In [15], Ha's nonlinear functional approach was extended to summational L^p setting. Then the Gronwall type argument also became available in [13] to obtain weighted L^p-stability estimates. Recently, Alonso et al. [1] resolved the uniform L^p stability problem for the Boltzmann equation with soft potential in the affirmative.

The usual difficulty encountered in the study of L^p type estimates of the collision operator is that even the simple Hölder inequality cannot be directly applied due to the singularity of the collision kernel. In [13], this difficulty was overcome by introducing polynomial weights in the velocity fields. In this paper, we attack this problem by dividing both sides of (1.1) by $\frac{1}{\mu}f^{1-\mu}$ and reformulating the the Boltzmann equation into the following form (See (3.2)):

$$\frac{\partial f^{\mu}}{\partial t}+v\cdot\nabla f^{\mu}=Q_{\mu}(f^{\mu},f^{\mu}).$$

In this way, the reformulated collision operator Q_{μ} gains additional integrability, and we are now able to apply Hölder type inequalities to obtain the following L^p-estimate:

$$\|f^{\mu}\|_p\leq C_{\mu,p}\|f^{\mu}\|_p,$$

which, upon adjusting the value of μ and p properly, leads to the main results. (See Theorem 1.1 and 1.2 below.) We mention that the parameter μ provides greater degree of freedom in determining the Lebesgue exponent, which is a key element in obtaining L^p estimates for $0<p<1$. Before we state our assumptions and main results, we introduce the notion of mild solutions.

Definition 1.1. *We say that a nonnegative function $f\in L^{\infty}(0,T;L^1(\mathbb{R}^3\times\mathbb{R}^3))$ is a mild solution if it satisfies the mild form:*

$$f^{\sharp}(x,v,t)=f_0(x,v)+\int_0^t Q^{\sharp}(f,f)(x,v,s)ds,\qquad (x,v,t)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+, \tag{1.6}$$

where the operator $\sharp$ *is defined by*

$$f^{\sharp}(x,v,t) \equiv f(x+tv,v,t)$$

The global existence of mild solutions for the Boltzmann equation in infinite vacuum was first established by Illner shinbrot [17] in the case that the solution decays exponentially in phase space by combining fixed point arguments with the celebrated Kaniel Shinbrot scheme [18]. Their result was then extended to more general settings including algebraically decaying data by several authors [5, 24, 25]. In [20, 23] the smallness assumption imposed on the upper travelling Maxwellian bound was replaced by a closedness condition to resolve the Cauchy problem of the Boltzmann equation close to a local Maxwellian regime, which is relevant to our case. We remark, however, that our stability analysis in this paper does not require any closedness nor smallness restrictions on the solutions. The main structural assumptions of this paper are as follows.

- ($\mathcal{A}1$). The collision kernel satisfies an inverse power potential and an angular cut-off assumption:

  $$B(v-v_*,\omega) = |v-v_*|^{\gamma} b_{\gamma}(\theta), \quad -3 < \gamma \leq 1,$$

  and

  $$\int_{\mathbb{S}^2_+} b_{\gamma}(\theta) d\omega = B_{\gamma} < \infty,$$

  where θ is the angle between $v - v_*$ and ω.
- ($\mathcal{A}2$). Mild solution f satisfies

  $$a_m \mathcal{M}_{\alpha,\beta}(x,v) \leq f^{\sharp}(x,v,t) \leq a_M \mathcal{M}_{\alpha,\beta}(x,v), \qquad \text{a.e.} \quad (x,v),$$

  for some strictly positive constants a_M, a_m.

Remark 1.1. *1. The existence of mild solution satisfying* ($\mathcal{A}2$) *with additional condition that* $a_M - a_m$ *is sufficiently small was established in* [20, 23]. *Recently, this result was extended to the classical solutions for soft potentials in* [1].

We are now in a position to state our main results. Below G_p denotes constants which depend on the Lebesgue exponent p, but not on x, v and t.

Theorem 1.1. *Suppose that main assumption* ($\mathcal{A}1$) *holds with* $-3 < \gamma \leq 1$ *and let* f *be a mild solution of* (1.1) *satisfying* ($\mathcal{A}2$) *corresponding to an initial datum* f_0. *Then we have*

$$||f(t)||_{L^p} \leq G_p ||f_0||_{L^p}, \qquad 0 < p \leq \infty. \tag{1.7}$$

Remark 1.2. *1. Alonzo et al.*[1] *has resolved* L^p*-stability problem of the Boltzmann equation with soft potentials for spatially decaying solutions. Our result is weaker in the sense that we cannot consider the difference of the two distribution functions:* $f - \bar{f}$, *but stronger in that it covers the hard potential case and the whole range of exponent:* $0 < p \leq \infty$.
2. We do not impose any smallness condition neither on a_M *nor on* $a_M - a_m$. *Although the existence result was established only when the distribution functions lie close to a local Maxwellian regime in the sense that* $a_M - a_m$ *is sufficiently small.*

The rest of this paper is organized as follows. In section 2, we present several estimates which will be crucial for the later sections. Through section 3 to section 4, we prove our main results. In the last section, we consider the stability problem of the difference of two distribution functions.

2. Preliminaries

2.1. **Basic estimates.**

In this part, we present several estimates to be used in later sections. For the proof, we refer readers to [11, 13, 16, 19].

Lemma 2.1. *Let $x \in \mathbb{R}^3$, $V \neq 0$ and $a > 0$. Then we have*

$$\int_0^\infty e^{-a|x+\tau V|^2} d\tau \quad \leq \quad \sqrt{\frac{\pi}{a}}\frac{1}{|V|}.$$

Lemma 2.2. *For $-3 < \gamma \leq 0$, we have*

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B(v-v_*,\omega)\mathcal{M}_{\alpha,\beta}(x+t(v-v_*),v_*)d\omega dv_* \\ \leq C(\gamma,\alpha,\beta)\cdot\frac{1}{(t+1)^{\gamma+3}},\end{aligned}$$

where $C(\gamma,\alpha,\beta) = B_\gamma\Big[\dfrac{2\pi}{\gamma+3} + \sqrt{\Big(\dfrac{\pi}{\alpha}\Big)^3} + \sqrt{\Big(\dfrac{\pi}{\beta}\Big)^3}\ \Big]$.

3. The proof of theorem 1.1 ($-3 < \gamma \leq 0$)

Let f be a mild solution of the Boltzmann equation satisfying the structural assumption ($\mathcal{A}2$). We then have from (1.6)

$$\frac{\partial f^\sharp}{\partial t} = \frac{1}{\kappa}Q^\sharp(f,f) \leq \frac{1}{\kappa}Q^{+\sharp}(f,f). \tag{3.1}$$

We divide both sides of (3.1) by $\frac{1}{\mu}(f^\sharp_\varepsilon)^{1-\mu}$ $(0<\mu<1)$ to get

$$\begin{aligned}\frac{\partial (f^\sharp)^\mu}{\partial t} &\leq \frac{1}{\kappa}\frac{\mu}{(f^\sharp)^{1-\mu}}Q^+(f,f)\\ &= \frac{\mu}{\kappa}\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B(v-v_*,\omega)\Big(\frac{f'^\sharp f'^\sharp_*}{f^\sharp}\Big)^{1-\mu}\big(f'^\sharp f'^\sharp_*\big)^\mu d\omega dv_*.\end{aligned}\tag{3.2}$$

We observe from the lower and upper bound estimate of ($\mathcal{A}2$)

$$\begin{aligned}\frac{f'^\sharp f'^\sharp_*}{f^\sharp} &\leq \frac{a_M^2 e^{-\alpha|x-t(v-v')|^2-\beta|v'_*|^2}e^{-\alpha|x-t(v-v'_*)|^2-\beta|v'_*|^2}}{a_m e^{\alpha|x|^2-\beta|v|^2}}\\ &= \frac{a_M^2 e^{-\alpha|x|^2+\beta|v|^2}e^{-\alpha|x-t(v-v_*)|^2-\beta|v_*|^2}}{a_m e^{-\alpha|x|^2-\beta|v|^2}}\\ &= \frac{a_M^2}{a_m}e^{-\alpha|x+t(v-v_*)|^2-\beta|v|^2}.\end{aligned}$$

We substitute the above estimate into (3.2) to obtain

$$\frac{\partial (f^\sharp)^\mu}{\partial t} \leq \mu e^\alpha\big(\frac{a_M^2}{a_m}\big)^{1-\mu}\int_{\mathbb{R}^3\times\mathbb{S}^2_+} A_{\mu,\alpha,\beta}(v-v_*)b(\theta)\big(f'^\sharp f'^\sharp_*\big)^\mu d\omega dv_*, \tag{3.3}$$

where $A_{\mu,\alpha,\beta}(v-v_*)$ denotes the regularized collision kernel defined by

$$A_{\mu,\alpha,\beta} \equiv A_{\mu,\alpha,\beta}(v-v_*) \equiv |v-v_*|^\gamma e^{-(1-\mu)(\alpha|x-t(v-v_*)|^2+\beta|v_*|^2)}.$$

Note that $A_{\mu,\alpha,\beta}$ now is an integrable function, which is a crucial ingredient in estimating the reformulated collision operator in L^p. We then multiply $p(f^\sharp)^{\mu(p-1)}$ to (3.3) and integrate over $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+$ with respect to (x,v,t) to obtain

$$\begin{aligned}\|(f^\sharp)^\mu(t)\|_p^p &\leq \|(f_0^\sharp)^\mu\|_p^p\\ &\quad + p\mu e^\alpha\Big(\frac{a_M^2}{a_m}\Big)^{1-\mu}\int_0^\infty\int_{\mathbb{R}^3\times\mathbb{S}^2_+}A_{\mu,\alpha,\beta}b(\theta)\big(f'^\sharp f'^\sharp_*\big)^\mu(f^\sharp)^{\mu(p-1)}d\omega dv_*dvdxdt.\end{aligned}\tag{3.4}$$

For brevity, we put

$$\mathcal{N}_1(t)\equiv\int_{\mathbb{R}^3\times\mathbb{S}^2_+}A_{\mu,\alpha,\beta}(v-v_*)b(\theta)\big(f'^\sharp f'^\sharp_*\big)^\mu\big(f^\sharp\big)^{\mu(p-1)}d\omega dxdvdv_*.$$

Lemma 3.1. *Let $\gamma\in(-2,0]$. Then for $q\geq 1$, $\mathcal{N}_1$ satisfies the following pointwise estimate:*

$$\mathcal{N}_1(t)\leq\frac{C_{\mathcal{N}_1}(a_M)^\mu}{(t+1)^{3+\gamma}}\|(f^\sharp)^\mu(t)\|_p^p,\tag{3.5}$$

for some constant $C_{\mathcal{N}_1}=C_{\mathcal{N}_1}(\mu,\alpha,\beta)$.

Proof. We apply Hölder inequality to $\mathcal{N}_1$ to obtain

$$\begin{aligned}\mathcal{N}_1 &\leq \int_{\mathbb{S}^2_+}b(\theta)\Big(\underbrace{\int_{\mathbb{R}^9}|v-v_*|^\gamma e^{-\frac{(1-\mu)p}{p-1}(\alpha|x+t(v-v_*)|^2+\beta|v_*|^2)}(f^\sharp)^{\mu p}dvdv_*dx}_{\mathcal{N}_{1A}}\Big)^{\frac{p-1}{p}}\\ &\quad\times\Big(\underbrace{\int_{\mathbb{R}^9}|v-v_*|^\gamma\big(f'^\sharp\big)^{p\mu}\big(f'^\sharp_*\big)^{p\mu}dvdv_*dx}_{\mathcal{N}_{1B}}\Big)^{\frac{1}{p}}d\omega.\end{aligned}\tag{3.6}$$

(i) The estimate of $\mathcal{N}_{1A}$: We observe from Lemma 2.2

$$\begin{aligned}\mathcal{N}_{1A}&\equiv\int_{\mathbb{R}^9}|v-v_*|^\gamma e^{-\frac{(1-\mu)p}{p-1}(\alpha|x+t(v-v_*)|^2+\beta|v_*|^2)}(f^\sharp(x,v,t))^{\mu p}dvdv_*dx\\ &=\int_{\mathbb{R}^6}(f^\sharp(x,v,t))^{\mu p}\Big(\int_{\mathbb{R}^3}|v-v_*|^\gamma e^{-\frac{(1-\mu)p}{p-1}(\alpha|x+t(v-v_*)|^2+\beta|v_*|^2)}dv_*\Big)dxdv\\ &\leq\frac{1}{(t+1)^{3+\gamma}}\Big[\frac{2\pi}{\gamma+3}+\sqrt{\Big(\frac{\pi(p-1)}{\alpha(1-\mu)p}\Big)^3}+\sqrt{\Big(\frac{\pi(p-1)}{\beta(1-\mu)p}\Big)^3}\;\Big]\|\big(f^\sharp(t)^\mu\big)\|_p^p\\ &\equiv\frac{C_{\mathcal{N}_{1A}}}{(t+1)^{3+\gamma}}\|\big(f^\sharp(t)\big)^\mu\|_p^p.\end{aligned}\tag{3.7}$$

(ii) The estimate of $\mathcal{N}_{1B}$: Applying a series of standard changes of variables, we have

$$\begin{aligned}\mathcal{N}_{1B}&=\int_{\mathbb{R}^9}|v-v_*|^\gamma(f(x+tv,v'))^{p\mu}(f(x+tv,v'_*))^{p\mu}dxdvdv_*\\ &=\int_{\mathbb{R}^9}|v-v_*|^\gamma(f(x,v'))^{p\mu}(f(x,v'_*))^{p\mu}dxdvdv_*\\ &=\int_{\mathbb{R}^9}|v-v_*|^\gamma(f(x,v))^{p\mu}(f(x,v_*))^{p\mu}dvdv_*dx\\ &=\int_{\mathbb{R}^9}|v-v_*|^\gamma(f^\sharp(x,v))^{p\mu}(f^\sharp(x+t(v-v_*),v_*))^{p\mu}dxdvdv_*.\end{aligned}$$

We then use Lemma 2.2 to see

$$\begin{aligned}\mathcal{N}_{1B} &= \int_{\mathbb{R}^6} (f^\sharp(x,v))^{p\mu}\Big(\int_{\mathbb{R}^3} |v-v_*|^\gamma (f^\sharp(x+t(v-v_*),v_*))^{p\mu} dv_*\Big) dvdx\\ &\leq (a_M)^{p\mu}\int_{\mathbb{R}^6} \big((f^\sharp(x,v))^{p\mu}\big)\Big(\int_{\mathbb{R}^3} |v-v_*|^\gamma e^{-p\mu(\alpha|x-t(v-v_*)|^2+\beta|v_*|^2)} dv_*\Big) dvdx\\ &\leq \frac{(a_M)^{p\mu}}{(t+1)^{3+\gamma}}\Big[\,\frac{2\pi}{\gamma+3}+\sqrt{\Big(\frac{\pi}{\alpha p\mu}\Big)^3}+\sqrt{\Big(\frac{\pi}{\beta p\mu}\Big)^3}\,\Big]\|(f^\sharp)^\mu(t)\|_p^p\\ &\equiv (a_M)^{p\mu}\frac{C_{\mathcal{N}_{1B}}}{(t+1)^{3+\gamma}}\|(f^\sharp)^\mu(t)\|_p^p.\end{aligned}\tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we obtain

$$\begin{aligned}\mathcal{N}_1 &\leq (a_M)^\mu \big(C_{\mathcal{N}_{1A}}\big)^{\frac{p}{p-1}}\big(C_{\mathcal{N}_{1B}}\big)^{\frac{1}{p}}(t+1)^{-(3+\gamma)(\frac{p-1}{p}+\frac{1}{p})}\int_{\mathbb{S}^2_+} b(\theta)\|(f^\sharp)^\mu\|_p^p d\omega\\ &\leq \big(C_{\mathcal{N}_{1A}}\big)^{\frac{p}{p-1}}\big(C_{\mathcal{N}_{1B}}\big)^{\frac{1}{p}}\frac{(a_M)^\mu B_\gamma}{(t+1)^{3+\gamma}}\|(f^\sharp)^\mu\|_p^p.\end{aligned}$$

We set

$$C_{\mathcal{N}_1}(\alpha,\beta,\mu) = (a_M)^\mu\big(C_{\mathcal{N}_{1A}}\big)^{\frac{p}{p-1}}\big(C_{\mathcal{N}_{1B}}\big)^{\frac{1}{p}}B_\gamma$$

to complete the proof. □

We now substitute the estimate (3.5) of Lemma 3.1 into (3.4) to obtain

$$\|(f^\sharp)^\mu(t)\|_p^p \leq \|(f_0^\sharp)^\mu\|_p^p + \mu p D_{\mu,p}\int_0^t \frac{1}{(t+1)^{3+\gamma}}\|(f^\sharp)^\mu(t)\|_p^p\,dt,\tag{3.9}$$

where

$$D_{\mu,p} = \ a_M^\mu C_{\mathcal{N}_1}B_\gamma\Big(\frac{a_M^2}{a_m}\Big)^{1-\mu}.$$

By Grownwall's lemma, this yields

$$\|f(t)\|_{\mu p}^{\mu p} \leq e^{2\mu p D_{\mu,p}}\|f_0\|_{\mu p}^{\mu p}$$

or, equivalently,

$$\|f(t)\|_{\mu p} \leq e^{D_{\mu,p}}\|f_0\|_{\mu p}.$$

We now adjust μ and p to complete the proof. For this, assume we are given a Lebesque exponent $P\in(0,\infty)$. We divide the argument into the following two cases:
(i) $P\in[1,\infty)$: we fix μ between 0 and 1 and set $p=\frac{P}{\mu}$ to obtain

$$\|f(t)\|_P \leq e^{D_{\mu,p}}\|f_0\|_P.$$

Letting $P\to\infty$, we get

$$\|f(t)\|_\infty \leq e^{D_{\mu,\infty}}\|f_0\|_\infty.$$

Here $D_{\mu,\infty}$ denotes

$$D_{\mu,\infty} = \lim_{P\to\infty} D_{\mu,\frac{P}{\mu}} < \infty.$$

(ii) $P\in(0,1)$: we fix p in $[1,\infty)$ and set $\mu=\frac{P}{p}$ to obtain

$$\|f(t)\|_P \leq e^{D_{\mu,p}}\|f_0\|_P.$$

Note that in both cases $0 < \mu < 1$ and $1 \leq p < \infty$ hold, which guarantee the relevance of the preceding argument.

4. The proof of theorem 1.1 ($-2 < \gamma \leq 1$)

If the intermolecular force is governed by hard potentials ($0 < \gamma \leq 1$), most of the crucial estimates in the previous sections are not relevant anymore due to the unboundedness of the collision kernel at infinity. We overcome this difficulty by incorporating the idea of Cho and Yu [9] into the reformulated setting. More precisely, we introduce a maximal distribution function $\sup_t f^\sharp$ and interchange the order of integration between time and velocity, to resolve the singularity of the collision kernel at infinity. We mention that the proof of this section is not restricted to the hard potential case and can be applied to the soft potential case either for $-2 < \gamma \leq 1$. We again start from the following inequality:

$$\frac{\partial (f^\sharp)^\mu}{\partial t} \leq \mu \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} A_{\mu,\alpha,\beta}(v - v_*) b(\theta) \big(f'^\sharp f'^\sharp_*\big)^\mu d\omega dv_*. \tag{4.1}$$

We integrate from 0 to t to obtain

$$\begin{aligned}(f^\sharp(t))^\mu &\leq (f_0^\sharp)^\mu + \mu \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} \int_0^t \int_{\mathbb{R}^9 \times \mathbb{S}^2_+} A_{\mu,\alpha,\beta} b(\theta) \big(f'^\sharp f'^\sharp_*\big)^\mu d\omega dv_* dt \\ &\leq (f_0^\sharp)^\mu + \mu \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} \int_0^\infty \int_{\mathbb{R}^9 \times \mathbb{S}^2_+} A_{\mu,\alpha,\beta} b(\theta) \big(f'^\sharp f'^\sharp_*\big)^\mu d\omega dv_* dt.\end{aligned} \tag{4.2}$$

We then take the supremum in time to obtain

$$\sup_t (f^\sharp)^\mu \leq (f_0^\sharp)^\mu + \mu \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} A_{\mu,\alpha,\beta} b(\theta) (f^\sharp f^\sharp_*)^\mu d\omega dv_* dt. \tag{4.3}$$

The reason why we do this will be clear in Lemma 4.1. We now take L^p norm directly on both sides, instead of multiplying $pf^{\sharp p-1}$ to both sides of (4.3) and integrating with respect to (x, v) as in the previous sections, to see

$$\begin{aligned}\|\sup_t (f^\sharp)^\mu\|_p &\leq \|(f_0^\sharp)^\mu\|_p \\ &\quad + \mu \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} \Big\| \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} A_{\mu,\alpha,\beta} b(\theta) \big(f'^\sharp f'^\sharp_*\big)^\mu d\omega dv_* dt \Big\|_p \\ &\equiv \|(f_0^\sharp)^\mu\|_p + \mu \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} \mathcal{N}_2.\end{aligned} \tag{4.4}$$

In the following lemma, we estimate $\mathcal{N}_2$. Note that $\mathcal{N}_2$ is bounded by the L^p-norm of $\sup_t (f^\sharp)$.

Lemma 4.1. *Let $\gamma \in (-2, 1]$. Then for $p \geq 1$ and $\mu \in (0, 1)$, we have*

$$\mathcal{N}_2 \leq C_{\mu,p} \big\| \big(\sup_t f^\sharp\big)^\mu \big\|_p \tag{4.5}$$

for some positive constant $C_{\mu,p}$

Proof. By Hölder inequality, we have

$$
\begin{aligned}
\mathcal{N}_2 \le \Big\| \int_{\mathbb{S}^2_+} b(\theta) & \Big(\underbrace{\int_0^\infty \int_{\mathbb{R}^3} |v-v_*|^\gamma e^{-\frac{p(1-\mu)}{p-1}(\alpha|x+t(v-v_*)|^2+\beta|v_*|^2)} dv_* ds}_{\mathcal{N}_{2A}} \Big)^{\frac{p-1}{p}} \\
& \times \Big(\int_0^\infty \int_{\mathbb{R}^3} |v-v_*|^\gamma \big(f'^\sharp\big)^{p\mu} (f'^\sharp_*)^{p\mu} dv_* ds \Big)^{\frac{1}{p}} d\omega \Big\|_{L^p(dx,dv)}.
\end{aligned} \tag{4.6}
$$

We use Lemma 2.1 and 2.2 to see

$$
\begin{aligned}
\mathcal{N}_{2A} &\equiv \int_0^\infty \int_{\mathbb{R}^3} |v-v_*|^\gamma e^{-\frac{p(1-\mu)}{p-1}(\alpha|x+t(v-v_*)|^2+\beta|v_*|^2)} dt dv_* \\
&= \int_{\mathbb{R}^3} |v-v_*|^\gamma e^{-\frac{p(1-\mu)}{p-1}\beta|v_*|^2} \Big(\int_0^\infty e^{-\frac{p(1-\mu)}{p-1}\alpha|x+t(v-v_*)|^2} dt \Big) dv_* \\
&\le \sqrt{\frac{\pi(p-1)}{\alpha p(1-\mu)}} \int_{\mathbb{R}^3} |v-v_*|^{\gamma-1} e^{-\frac{p(1-\mu)}{p-1}\beta|v_*|^2} dv_* \\
&\le \sqrt{\frac{\pi(p-1)}{\alpha p(1-\mu)}} \Big(\int_{|v_*|\le 1} |v-v_*|^{\gamma-1} dv_* + \int_{|v_*|>1} e^{-\frac{p(1-\mu)}{p-1}\beta|v_*|^2} dv_* \Big) \\
&\le \sqrt{\frac{\pi(p-1)}{\alpha p(1-\mu)}} \Big(\frac{2\pi}{\gamma+2} + \sqrt{\Big(\frac{p-1}{\beta p(1-\mu)}\Big)^3} \Big) \\
&\equiv C_{\mathcal{N}_{2A}}.
\end{aligned}
$$

Note that we performed integration in time first before the velocity integration. We plug the above estimate of $\mathcal{N}_{2A}$ into (4.6) to obtain

$$
\begin{aligned}
\mathcal{N}_2 \le (C_{\mathcal{N}_{2A}})^{\frac{p-1}{p}} \Big\| \Big(\int_0^\infty \int_{\mathbb{R}^3} & |v-v_*|^\gamma (f^\sharp(x-t(v-v'),v',t))^{p\mu} \\
& \times (f^\sharp(x-t(v-v'_*),v'_*,t))^{p\mu} dv_* dt \Big)^{\frac{1}{p}} \Big\|_{L^p(dx,dv)}.
\end{aligned}
$$

Applying a series of changes of variables: $x+tv \to x$, $(v',v'_*) \to (v,v_*)$ and $x \to x+tv$ gives

$$
\begin{aligned}
\mathcal{N}_2 \le (C_{\mathcal{N}_{2A}})^{\frac{p-1}{p}} \Big[\int_0^\infty \int_{R^9} & |v-v_*|^\gamma (f^\sharp(x,v,t))^{p\mu} \\
& \times (f^\sharp(x-t(v-v_*),v_*,t))^{p\mu} dx dv dv_* dt \Big]^{\frac{1}{p}}.
\end{aligned}
$$

We now introduce the maximal distribution $\sup_t f^\sharp(x,v)$ as follows

$$
\begin{aligned}
\mathcal{N}_2^p &\le a_M^{p\mu} (C_{\mathcal{N}_{2A}})^{p-1} \int_{\mathbb{R}^6} \big(\sup_t (f^\sharp(x,v))^{p\mu} \big) \\
&\qquad \times \Big(\int_{\mathbb{R}^3} \int_0^\infty |v-v|^\gamma e^{-p\mu(\alpha|x-t(v-v_*)|^2+\beta|v_*|^2)} dt dv_* \Big) dv dx \\
&\le a_M^{p\mu} (C_{\mathcal{N}_{2A}})^{p-1} \sqrt{\frac{\pi}{\alpha\mu p}} \int_{\mathbb{R}^6} \big(\sup_t (f^\sharp)^{p\mu} \big) \Big(\int_{\mathbb{R}^3} |v-v|^{\gamma-1} e^{-p\mu\beta|v_*|^2} dv_* \Big) dv dx \\
&\le a_M^{p\mu} (C_{\mathcal{N}_{2A}})^{p-1} \sqrt{\frac{\pi}{\alpha\mu p}} \Big(\frac{2\pi}{\gamma+2} + \sqrt{\Big(\frac{1}{\beta p\mu}\Big)^3} \Big) \| \sup_t (f^\sharp)^\mu \|_p^p,
\end{aligned}
$$

where we used

$$\begin{aligned} f^\sharp(x,v,t) &\le \sup_t (f^\sharp)(x,v) \quad \text{and} \\ f^\sharp(x,v_*,t) &\le a_M e^{-\alpha|x-t(v-v_*)|^2-\beta|v_*|^2}. \end{aligned}$$

Finally we put

$$C_{\mu,p} \equiv a_M^\mu (C_{\mathcal{N}_{2A}})^{\frac{p-1}{p}} \Big[\sqrt{\frac{\pi}{\alpha\mu p}}\Big(\frac{2\pi}{\gamma+2} + \sqrt{\Big(\frac{1}{\beta p\mu}\Big)^3}\ \Big)\Big]^{\frac{1}{p}}$$

to obtain the desired result. □

We now go back to the proof of the main theorem of this section. Substituting (4.5) into (4.4) and recalling

$$\|(f^\sharp)^\mu\|_p = \|(f^\sharp)\|_{\mu p}^\mu,$$

we have

$$\|\sup_t (f^\sharp)\|_{\mu p}^\mu \le \|(f_0^\sharp)\|_{\mu p}^\mu + \bar{C}_{\mu,p}\|\sup_t (f^\sharp)\|_{\mu p}^\mu, \tag{4.7}$$

where

$$\bar{C}_{\mu,p} \equiv \mu a_M^\mu (C_{\mathcal{N}_{2B}})^{\frac{p-1}{p}} \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} \Big[\sqrt{\frac{\pi}{\alpha\mu p}}\Big(\frac{2\pi}{\gamma+2} + \sqrt{\Big(\frac{1}{\beta p\mu}\Big)^3}\ \Big)\Big]^{\frac{1}{p}}.$$

As in the previous section, we first fix $\mu p = P$ for a given Lebesgue exponent $0 < P < \infty$. We then observe that

$$\bar{C}_{\mu,p} \le \mu \mathcal{O}(1)\Big[\sqrt{\frac{1}{P}}\Big(1 + \sqrt{\Big(\frac{1}{P}\Big)^3}\ \Big)\Big]^{\frac{\mu}{P}},$$

where we used the fact that $(C_{\mathcal{N}_{2A}})^{\frac{p-1}{p}}$ is uniformly bounded for $p \ge 1$, $0 < \mu < 1$, and

$$a_M^\mu \Big(\frac{a_M^2}{a_m}\Big)^{1-\mu} = a_M \Big(\frac{a_M}{a_m}\Big)^{1-\mu} < \frac{a_M^2}{a_m}.$$

Therefore, we can take μ sufficiently small (with P fixed) such that $\bar{C}_{\mu,p} < 1$, which gives from (4.7)

$$\|(f^\sharp)(t)\|_P^\mu \le \|\sup_t (f^\sharp)\|_P^\mu \le \frac{1}{1-\bar{C}_{\mu,p}}\|(f_0^\sharp)\|_P^\mu.$$

This implies the desired result.

5. On the stability of $f - \bar{f}$

Let f, $\bar{f}$ be two mild solutions of (1.1) which satisfy the upper bound estimate (but not necessarily lower bound estimate) of the main assumption $(\mathcal{A}2)$:

$$(\mathcal{A}2)': \quad 0 \le f^\sharp(x,v,t),\ \bar{f}^\sharp(x,v,t) \le a_M \mathcal{M}_{\alpha,\beta}(x,v), \qquad \text{a.e. } (x,v),$$

for some strictly positive constant a_M. Since the difference $f - \bar{f}$ does not satisfies the lower bound estimate of $(\mathcal{A}2)$ in general, the arguments given in section 3 and 4 are not directly applicable to the difference of two distribution functions. One way to circumvent this problem is to consider $g^\sharp(x,v,t) \equiv \mathcal{M}^{-1}_{\alpha,\beta} f^\sharp(x,v,t)$ instead of $f^\sharp(x,v,t)$. Substituting this into (1.1), we obtain

$$\frac{\partial g^\sharp}{\partial t} = \int_{\mathbb{R}^3\times\mathbb{S}^2_+} A_{\alpha,\beta}(v-v_*,\omega)(g'^\sharp g'^\sharp_* - g^\sharp g^\sharp_*)d\omega dv_*, \tag{5.1}$$

$$\frac{\partial \bar{g}^\sharp}{\partial t} = \int_{\mathbb{R}^3\times\mathbb{S}^2_+} A_{\alpha,\beta}(v-v_*,\omega)(\bar{g}'^\sharp \bar{g}'^\sharp_* - \bar{g}^\sharp \bar{g}^\sharp_*)d\omega dv_*, \tag{5.2}$$

where $A_{\alpha,\beta}$ denotes the regularized collision kernel as before:

$$A_{\alpha,\beta}(v-v_*,\omega) = |v-v_*|^\gamma e^{-\alpha|x-(v-v_*)t|^2-\beta|v_*|^2}.$$

We subtract (5.2) from (5.1) and multiply $sgn(f^\sharp - \bar{f}^\sharp)$ to both sides to see

$$\frac{\partial G^\sharp}{\partial t} \le \int_{\mathbb{R}^3\times\mathbb{S}^2_+} A_{\alpha,\beta}(v-v_*,\omega)(G'^\sharp D'^\sharp_* + D'^\sharp G'^\sharp_* + G^\sharp D^\sharp_* + D^\sharp G^\sharp_*)d\omega dv_*.$$

where $G = |g-\bar{g}|$ and $D = |g+\bar{g}|$. Then the exactly same arguments as in the previous sections yield

$$\|G\|_p \le C_p \|G_0\|_p, \qquad (-3 < \gamma \le 1), \tag{5.3}$$

where $\theta = 1$ for sufficiently small a_M. We now introduce the following notation for simplicity.

$$||f(t)||_{L^p_{\mathcal{M}}} \equiv \Big\{ \int_{\mathbb{R}^6} \big(f^\sharp(x,v,t)\mathcal{M}^{-1}_{\alpha,\beta}\big)^p dxdv \Big\}^{\frac{1}{p}},$$

then (5.3) leads to the following theorems.

Theorem 5.1. *Suppose that main assumption $(\mathcal{A}1)$ holds with $-3 < \gamma \le 1$. Let f and $\bar{f}$ be mild solutions satisfying $(\mathcal{A}2)'$ corresponding to initial data $f_0, \bar{f}_0$ respectively. Then we have*

$$||f(t) - \bar{f}(t)||_{L^p_{\mathcal{M}}} \le G_p ||f_0 - \bar{f}_0||_{L^p_{\mathcal{M}}}, \qquad t \ge 0.$$

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