

ON EXPLICIT HOLMES-THOMPSON AREA FORMULA IN INTEGRAL GEOMETRY

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ABSTRACT. In this article, we give an exposition on the Holmes-Thompson theory developed by Alvarez. The space of geodesics in Minkowski space has a symplectic structure which is induced by the projection from the sphere-bundle. we show that it can be also obtained from the symplectic structure on the tangent bundle of the Riemannian manifold, the tangent bundle of the Minkowski unit sphere. We give detailed descriptions and expositions on Holmes-Thompson volumes in Minkowski space by the symplectic structure and the Crofton measures for them. For the Minkowski plane, a normed two dimensional space, we express the area explicitly in an integral geometry way, by putting a measure on the plane, which gives an extension of Alvarez's result for higher dimensional cases.

1. INTRODUCTIONS

1.1. Minkowski Space and Geodesics. A Minkowski space is a vector space with a Minkowski norm, and a Minkowski norm is defined in [8] as

Definition 1.1. A function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Minkowski norm if

- (1) $F(x) > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$ and $F(0) = 0$.
- (2) $F(\lambda x) = |\lambda|F(x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.
- (3) $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and the symmetric bilinear form

$$g_x(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x + su + tv)|_{s=t=0} \quad (1.1)$$

is positively definite on \mathbb{R}^n for any $x \in \mathbb{R}^n \setminus \{0\}$.

We denote a Minkowski space by (\mathbb{R}^n, F) . By the way, (2) and (3) in Definition 1.1 imply the the convexity of F , see Chapter 1 of [8].

First of all, we can infer the following theorem about geodesics in Minkowski space from Definition 1.1.

Theorem 1.2. *The straight line joining two points in Minkowski space is the only shortest curve joining them.*

Proof. For any $p, q \in (\mathbb{R}^n, F)$, let $r(t)$, $t \in [a, b]$ with $F(r'(t)) = 1$, be a curve joining p and q , which has the minimum length. Then $r(t)$ is the minimizer of the functional $\int_a^b F(r'(t))dt$.

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Note that F is smooth. By the fundamental lemma of calculus of variation (Let $V(h) := \int_a^b F(r'(t) + h\delta'(t))dt$, where $\delta(a) = \delta(b) = 0$). Then

$$\begin{aligned}
V'(0) &= \left. \frac{\partial}{\partial h} \right|_{h=0} \int_a^b F(r'(t) + h\delta'(t))dt \\
&= \int_a^b \left. \frac{\partial}{\partial h} \right|_{h=0} F(r'(t) + h\delta'(t))dt \\
&= \int_a^b \nabla F(r'(t)) \cdot \delta'(t)dt \\
&= \nabla F(r'(t)) \cdot \delta(t) \Big|_a^b - \int_a^b \delta(t) \cdot \frac{d}{dt} \nabla F(r'(t))dt \\
&= - \int_a^b \delta(t) \cdot \frac{d}{dt} \nabla F(r'(t))dt.
\end{aligned} \tag{1.2}$$

Thus we can obtain $\frac{d}{dt} \nabla F(r'(t)) = 0$ since $V'(0) = 0$ as $V(0) \leq V(h)$ for any $\delta(t)$, or by the Euler–Lagrange equation directly, we have

$$\frac{d}{dt} \nabla F(r'(t)) = 0. \tag{1.3}$$

Using chain rule, (1.3) becomes

$$Hess(F) \frac{d^2 r(t)}{dt^2} = 0. \tag{1.4}$$

On the other hand, we have

$$\nabla F(r'(t)) \frac{d^2 r(t)}{dt^2} = 0 \tag{1.5}$$

by differentiating $F(r'(t)) = 1$, and then by product rule, (1.4) and (1.5),

$$\begin{aligned}
\frac{1}{2} Hess(F^2) \frac{d^2 r(t)}{dt^2} &= F(r'(t)) Hess(F) \frac{d^2 r(t)}{dt^2} + (\nabla F(r'(t)))^T \nabla F(r'(t)) \frac{d^2 r(t)}{dt^2} \\
&= Hess(F) \frac{d^2 r(t)}{dt^2} \\
&= 0.
\end{aligned} \tag{1.6}$$

Hence we get $\frac{d^2 r(t)}{dt^2} = 0$ because $\frac{1}{2} Hess(F^2)$ is non-degenerated by (3) in Definition 1.1, and then it implies $r(t)$, $t \in [a, b]$, is a straight line segment connecting p and q . \square

Thus the space of geodesics in (\mathbb{R}^n, F) actually is the space of affine lines, denoted by $\overline{Gr_1(\mathbb{R}^n)}$. More generally, one can define

Definition 1.3. The affine Grassmannian $\overline{Gr_k(\mathbb{R}^n)}$ is the space of affine k -planes in (\mathbb{R}^n, F) .

1.2. Symplectic Structures on Cotangent Bundle. The Minkowski space (\mathbb{R}^n, F) , as a differentiable manifold, has a canonical symplectic structure on its cotangent bundle $T^*\mathbb{R}^n$, from which a symplectic structure on its tangent bundle $T\mathbb{R}^n$ can be derived as well.

The canonical contact form α on $T^*\mathbb{R}^n$ is defined as $\alpha_\xi(X) := \xi(\pi_{0*}X)$ for $X \in T_\xi T^*\mathbb{R}^n$, where $\pi_0 : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$ is the natural projection. And then the canonical symplectic form on $T^*\mathbb{R}^n$ is defined as $\omega := d\alpha$.

On the other hand, we know that the dual of Minkowski metric is defined as

$$F^*(\xi) := \sup \{ |\xi(v)| : v \in T\mathbb{R}^n, F(v) \leq 1 \} \tag{1.7}$$

for $\xi \in T^*\mathbb{R}^n$, and there is a natural correspondence between the sphere bundle $S\mathbb{R}^n$ and the cosphere bundle $S\mathbb{R}^n = \{ \xi \in T^*\mathbb{R}^n : F^*(\xi) = 1 \}$ of the Minkowski space (\mathbb{R}^n, F) .

By the convexity and the positive homogeneity of F , see [11], we can obtain $F^*(dF(\bar{\xi})) = 1$ and $dF^*(dF(\bar{\xi})) = \bar{\xi}$ for any $\bar{\xi} \in S_x\mathbb{R}^n$ and $x \in \mathbb{R}^n$, where dF is the gradient of F and similarly for dF^* . Thus dF is a diffeomorphism from $S_x\mathbb{R}^n$ to $S^*\mathbb{R}^n$, which induces another diffeomorphism

$$\begin{aligned} \varphi_F : S\mathbb{R}^n &\rightarrow S^*\mathbb{R}^n \\ \varphi_F((x, \bar{\xi}_x)) &= (x, dF(\bar{\xi}_x)) \end{aligned} \quad (1.8)$$

for any $\bar{\xi}_x \in S_x\mathbb{R}^n$. More generally, there is another diffeomorphism $\frac{1}{2}dF^2$ from $T_x\mathbb{R}^n \setminus \{0\}$ to $T_x^*\mathbb{R}^n \setminus \{0\}$ for any $x \in (\mathbb{R}^n, F)$, thus we obtain a diffeomorphism

$$\begin{aligned} \bar{\varphi}_F : T\mathbb{R}^n &\rightarrow T^*\mathbb{R}^n \\ \bar{\varphi}_F((x, \bar{\xi}_x)) &= (x, \frac{1}{2}dF^2(\bar{\xi}_x)) \end{aligned} \quad (1.9)$$

by ignoring the 0-sections.

The diffeomorphism (1.8) induces a 2-form $\bar{\omega} := \varphi_F^*\omega$ on $S\mathbb{R}^n$. Without loss of elegance, we can express it more concretely. Since $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^{n*}$, for $(x, \xi) \in T^*\mathbb{R}^n$ the canonical symplectic form ω on $T^*\mathbb{R}^n$ is actually $\omega = tr(dx \wedge d\xi)$, here we denote $dx \wedge d\xi := (dx_i \wedge d\xi_j)_{n \times n}$ and similarly $dx \wedge d\bar{\xi} := (dx_i \wedge d\bar{\xi}_j)_{n \times n}$, $n \times n$ matrices with 2-forms as entries, where $\xi_j(\xi) = \xi(\frac{\partial}{\partial x_j})$, $\bar{\xi}_j(\bar{\xi}) = dx_j(\bar{\xi})$ and $F^*(\bar{\xi}) = 1$. Then using chain rule, we can obtain

$$\bar{\omega} = \varphi_F^*(\omega|_{S^*\mathbb{R}^n}) = Hess(F) \star dx \wedge d\bar{\xi}|_{S\mathbb{R}^n}, \quad (1.10)$$

where \star is the Frobenius inner product which is the sum of the entries of the entrywise product of two matrices.

1.3. Gelfand Transform. Gelfand transform on a double fibration as a generalization of Radon transform plays an important role in making use of the symplectic form of Section 1.2 in integral geometry of Minkowski space.

Definition 1.4. Let $M \xleftarrow{\pi_1} \mathcal{F} \xrightarrow{\pi_2} \Gamma$ be double fibration where M and Γ are two manifolds, $\pi_1 : \mathcal{F} \rightarrow M$ and $\pi_2 : \mathcal{F} \rightarrow \Gamma$ are two fibre bundles, and $\pi_1 \times \pi_2 : \mathcal{F} \rightarrow M \times \Gamma$ is an submersion. Let Φ be a density on Γ , then the Gelfand transform of Φ is defined as $GT(\Phi) := \pi_{1*}\pi_2^*\Phi$. In the case Φ is a differential form and the fibres are oriented, then we also have a well-defined Gelfand transform $GT(\Phi) := \pi_{1*}\pi_2^*\Phi$, noting that the pushforward of a form is the integral of contracted form over the fibre.

To make it clear, let's see how the degree of a density or form changes by the transform. Suppose Φ is a density or form of degree m on Γ and the dimension of fibre π_1 is q , then $\pi_2^*\Phi$ has degree m , and then $GT(\Phi) = \pi_{1*}\pi_2^*\Phi = \int_{\pi_1^{-1}(x)} \pi_2^*\Phi$ for $x \in M$ has degree $m - q$.

An application of Gelfand transforms in integral geometry is the following fundamental theorem [5], whose proof is quite simple.

Theorem 1.5. *Suppose $M_\gamma := \pi_1(\pi_2^{-1}(\gamma))$ are smooth submanifolds of M for $\gamma \in \Gamma$, $\bar{M} \subset M$ is a immersed submanifold, and Φ is a top degree density on Γ . Then*

$$\int_\Gamma \#(\bar{M} \cap M_\gamma) \Phi(\gamma) = \int_{\bar{M}} GT(\Phi). \quad (1.11)$$

Proof. Working on the transitions of measures on manifolds and the transformations of intersection numbers, we have

$$\begin{aligned} \int_{\overline{M}} GT(\Phi) &= \int_{\overline{M}} \pi_{1*} \pi_2^* \Phi = \int_{\pi_1^{-1}(\overline{M})} \pi_2^* \Phi \\ &= \int_{\Gamma} \#(\pi_2^{-1}(\gamma) \cap \pi_1^{-1}(\overline{M})) \Phi(\gamma) \\ &= \int_{\Gamma} \#(\overline{M} \cap M_\gamma) \Phi(\gamma). \end{aligned} \quad (1.12)$$

□

2. THE SYMPLECTIC STRUCTURE ON THE SPACE OF GEODESICS

The symplectic structure on the space of geodesics in a Minkowski space is induced naturally from the canonical symplectic structure on its cotangent bundle.

The process of construction of symplectic form on $\overline{Gr_1(\mathbb{R}^n)}$ in Minkowski space (\mathbb{R}^n, F) is based on the following diagram

$$\begin{array}{ccccc} S\mathbb{R}^n & \xrightarrow{\varphi_F} & S^*\mathbb{R}^n & \xrightarrow{i} & T^*\mathbb{R}^n \\ \downarrow p & & & & \\ \overline{Gr_1(\mathbb{R}^n)} & & & & \end{array} \quad (2.1)$$

where p is the projection from $S\mathbb{R}^n$ onto $\overline{Gr_1(\mathbb{R}^n)}$ defined by

$$p((x, \bar{\xi})) := l(x, \bar{\xi}), \quad (2.2)$$

where $l(x, \bar{\xi})$ is the line passing through x with direction $\bar{\xi}$.

Consider the geodesic vector field $\overline{\mathcal{X}}(\bar{\xi}_x) := (\bar{\xi}_x, 0)$ on $T\mathbb{R}^n$ for any $\bar{\xi}_x \in S\mathbb{R}^n$, φ_F in (1.8) induces another vector field $\mathcal{X} := d\varphi_F(\overline{\mathcal{X}})$ on $T^*\mathbb{R}^n$ with

$$\mathcal{X}(dF(\bar{\xi}_x)) = (d\varphi_F(\overline{\mathcal{X}})(\varphi_F(\bar{\xi}_x)) = (\bar{\xi}_x, 0) \quad (2.3)$$

for $\bar{\xi}_x \in S\mathbb{R}^n$.

We have the following vanishing property about \mathcal{X} and ω on $S^*\mathbb{R}^n$.

Lemma 2.1. $i_{\mathcal{X}}\omega = 0$ on $S^*\mathbb{R}^n$.

Proof. Noting that $\omega(X, Y) = \langle X_1, Y_2 \rangle - \langle Y_1, X_2 \rangle$ for any $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ in $T_{\xi_x} S^*\mathbb{R}^n \subset T_{\xi_x} T^*\mathbb{R}^n$ because $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^{n*}$, where the inner product is the dual space action, by (2.3) we have

$$\omega_{\xi_x}(\mathcal{X}, Y) = \langle \bar{\xi}_x, Y_2 \rangle = \langle dF^*(\xi_x), Y_2 \rangle = 0 \quad (2.4)$$

because $Y_2 \in T_{\xi_x} S^*\mathbb{R}^n$ is “normal” to $dF^*(\xi_x)$, precisely, that can be obtained by differentiating $F^*(\xi_x) = 1$ and noting $Y_2 \in T_{\xi_x} S^*\mathbb{R}^n$. □

Furthermore, the Lie derivative of ω along geodesic vector field \mathcal{X} is

$$\mathcal{L}_{\mathcal{X}}\omega = di_{\mathcal{X}}\omega + i_{\mathcal{X}}d\omega = 0 \quad (2.5)$$

by Lemma 2.1. Then (2.5) implies $(\varphi_F)^*\omega|_{S^*\mathbb{R}^n}$ is invariant under $\overline{\mathcal{X}}$.

Based on the invariance of ω we can construct a symplectic structure on $\overline{Gr_1(\mathbb{R}^n)}$. However, in order to do that, we need to give a manifold structure for $\overline{Gr_1(\mathbb{R}^n)}$ first.

In fact, we can build a bijection ψ between $\overline{Gr_1(\mathbb{R}^n)}$ and TS_F^{n-1} , where S_F^{n-1} is the unit sphere in (\mathbb{R}^n, F) . For any $l(x, \bar{\xi}) \in \overline{Gr_1(\mathbb{R}^n)}$, let $\bar{\eta}$ be the tangent vector pointing at $l(x, \bar{\xi}) \cap T_{\bar{\xi}} S_F^{n-1}$, in fact, $\bar{\eta} = x - dF(\bar{\xi})(x)\bar{\xi} \in T_{\bar{\xi}} S_F^{n-1}$, see Figure 2.1 on page 5, and one can define

$$\psi(l(x, \bar{\xi})) := (\bar{\xi}, \bar{\eta}) = (\bar{\xi}, x - dF(\bar{\xi})(x)\bar{\xi}). \quad (2.6)$$

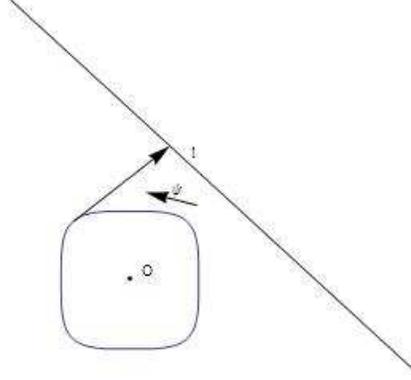


FIGURE 2.1. $\overline{Gr_1(\mathbb{R}^n)}$ Diffeomorphic to TS_F^{n-1}

Thus we have a homeomorphism ψ from $\overline{Gr_1(\mathbb{R}^n)}$ to TS_F^{n-1} , and then the manifold structure on TS_F^{n-1} provides one for $\overline{Gr_1(\mathbb{R}^n)}$.

Let us again consider the projection (2.2) with the manifold structure on $\overline{Gr_1(\mathbb{R}^n)}$, and then we can obtain the following lemma

Lemma 2.2. $\bar{\mathcal{X}}$ is in the kernel of dp , in other words, $p_*(\bar{\mathcal{X}}) = 0$.

Proof. Using the basic equality

$$dF(\bar{\xi})(\bar{\xi}) = F(\bar{\xi}) = 1 \quad (2.7)$$

obtained by the positive homogeneity of F for any $\bar{\xi} \in S_x \mathbb{R}^n$, we have

$$\begin{aligned} p_*(\bar{\mathcal{X}}) &= dp((\bar{\xi}, 0)) = d(\bar{\xi}, x - dF(\bar{\xi})(x)\bar{\xi})((\bar{\xi}, 0)) \\ &= \bar{\xi} - dF(\bar{\xi})(\bar{\xi})\bar{\xi} \\ &= (1 - dF(\bar{\xi})(\bar{\xi}))\bar{\xi} \\ &= 0. \end{aligned} \quad (2.8)$$

□

One can compute the rank of the Jacobian of p which is $2n - 2$, that implies $\dim(dp|_{\bar{\xi}_x}) = 1$ and then

$$\ker(dp|_{\bar{\xi}_x}) = \text{span}(\bar{\mathcal{X}}(\bar{\xi}_x)) \quad (2.9)$$

by Lemma 2.2.

Now we can obtain the following theorem

Theorem 2.3. *There exists a symplectic form ω_0 on $\overline{Gr_1(\mathbb{R}^n)}$, such that $p^*\omega_0 = \bar{\omega} = (\varphi_F)^*\omega|_{S^*\mathbb{R}^n}$.*

Proof. By (2.5), (2.9) and Lemma 2.1, we know that $\bar{\omega}_{\xi_x}(X, Y)$ is independent of the choices of preimages under the pushforward induced by projection p . Thus we have a well-defined two form ω_0 on $\overline{Gr_1(\mathbb{R}^n)}$,

$$\omega_{0_{p(\xi_x)}}(\tilde{X}, \tilde{Y}) := \bar{\omega}_{\xi_x}(X, Y), \quad (2.10)$$

where $(p_*)_{\xi_x}(X) = \tilde{X}$ and $(p_*)_{\xi_x}(Y) = \tilde{Y}$, such that

$$p^*\omega_0 = \bar{\omega} = (\varphi_F)^*i^*\omega. \quad (2.11)$$

□

That finishes the construction of symplectic structure on the space of geodesics in Minkowski space.

On the other hand, since $T^*S_F^{n-1}$ as a cotangent bundle on Riemannian manifold S_F^{n-1} has a canonical symplectic structure denoted as $\tilde{\omega}$, and we have a canonical diffeomorphism

$$\begin{aligned} \tilde{\varphi}_F : TS_F^{n-1} &\rightarrow T^*S_F^{n-1} \\ \tilde{\varphi}_F(\bar{\eta}_{\bar{\xi}}) &= \langle \bar{\eta}_{\bar{\xi}}, \cdot \rangle_{g_F}, \end{aligned} \quad (2.12)$$

in which g_F is the Riemannian metric on S_F^{n-1} , which is actually the bilinear form

$$\langle \bar{u}, \bar{v} \rangle_{g_F} := \frac{\partial^2}{\partial s \partial t} F(\bar{\xi} + s\bar{u} + t\bar{v})|_{s=t=0} \quad (2.13)$$

for any $\bar{u}, \bar{v} \in T_{\bar{\xi}}S_F^{n-1}$, see [8], and then $\tilde{\varphi}_F^*\tilde{\omega}$ is the the symplectic form induced on $TS_F^{n-1} \cong \overline{Gr_1(\mathbb{R}^n)}$. Also, we have another symplectic form ω_0 on $\overline{Gr_1(\mathbb{R}^n)}$ from Theorem 2.3. A natural question is whether the two symplectic structures on $\overline{Gr_1(\mathbb{R}^n)}$ are the same, the answer is yes, see the following theorem

Theorem 2.4. $\omega_0 = \tilde{\varphi}_F^*\tilde{\omega}$.

Let us first draw a diagram for this theorem by combining (2.1)

$$\begin{array}{ccccc} S\mathbb{R}^n & \xrightarrow{\varphi_F} & S^*\mathbb{R}^n & \xrightarrow{i} & T^*\mathbb{R}^n \\ \downarrow p & & & & \\ \overline{Gr_1(\mathbb{R}^n)} & \xrightarrow{\psi} & TS_F^{n-1} & \xrightarrow{\tilde{\varphi}_F} & T^*S_F^{n-1}. \end{array} \quad (2.14)$$

Proof. First, differentiating (2.7) and using chain rule, one can get

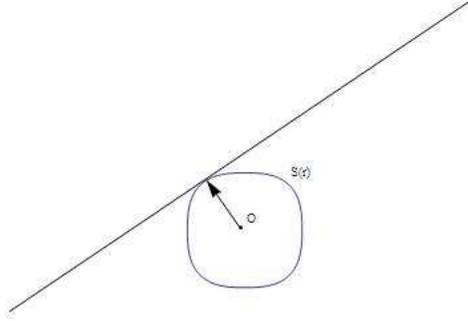
$$Hess(F) \star \bar{\xi} d\bar{\xi}|_{S\mathbb{R}^n} = 0 \quad (2.15)$$

in which $\bar{\xi} d\bar{\xi} := (\bar{\xi}_i d\bar{\xi}_j)_{n \times n}$ is a matrix and \star is the Frobenius inner product of matrices.

Next, the canonical symplectic form $\tilde{\omega}$ on $T^*S_F^{n-1}$, $\tilde{\omega} = \omega|_{T^*S_F^{n-1}}$ in which ω is the canonical symplectic form on the cotangent bundle $T^*\mathbb{R}^n$. Thus, from (2.12) and (2.13), one can obtain that

$$\tilde{\varphi}_F^*\tilde{\omega} = Hess(F) \star d\bar{\eta} \wedge d\bar{\xi}|_{T^*S_F^{n-1}}, \quad (2.16)$$

here $d\bar{\xi} \wedge d\bar{\eta}$ is a matrix with 2-form entries and \star is the Frobenius inner product of matrices.


 FIGURE 2.2. “ p -norm distance” r

Therefore, by plugging (2.6) into (2.16) and using (2.15), we obtain

$$\begin{aligned}
 p^* \tilde{\varphi}_F^* \tilde{\omega} &= \text{Hess}(F) \star d\bar{\eta} \wedge d\bar{\xi}|_{TS^{n-1}} \\
 &= \text{Hess}(F) \star d(x - dF(\xi)(x)\bar{\xi}) \wedge d\bar{\xi}|_{S\mathbb{R}^n} \\
 &= \text{Hess}(F) \star dx \wedge d\bar{\xi}|_{S\mathbb{R}^n} - d(dF(\xi)(x)) \wedge \text{Hess}(F) \star \bar{\xi}d\bar{\xi}|_{S\mathbb{R}^n} \quad (2.17) \\
 &= \text{Hess}(F) \star dx \wedge d\bar{\xi}|_{S\mathbb{R}^n} \\
 &= \bar{\omega},
 \end{aligned}$$

which by Theorem 2.3 implies the claim. \square

At the end to this section, we make a remark on the symplectic structure on $\overline{Gr_1(\mathbb{R}^n)}$.

Remark 2.5. From (1.10) we see the symplectic structure $\bar{\omega}$ on $T\mathbb{R}^n$ relies on the Minkowski metric F , then we know, by the above construction, the symplectic structure on $\overline{Gr_1(\mathbb{R}^n)}$ depends on the Minkowski metric F as well. Let us see the following example of Minkowski plane with p -norm as a Minkowski metric.

Example 2.6. Given a Minkowski plane by $(\mathbb{R}^2, \|\cdot\|_p)$, $1 < p < \infty$, where $\|(\alpha, \beta)\|_p = (|\alpha|^p + |\beta|^p)^{1/p}$ and the dual norm is $\|\cdot\|_{\frac{p}{p-1}}$ we can obtain the symplectic form ω on $\overline{Gr_1(\mathbb{R}^2)}$, the space of affine lines in $(\mathbb{R}^2, \|\cdot\|_p)$, by following the general construction above.

By (1.10) and Theorem 2.3, we have

$$p^* \omega_0 = (p-1)\alpha^{p-2} dx \wedge d\alpha + (p-1)\beta^{p-2} dy \wedge d\beta, \quad (2.18)$$

for $((x, y), (\alpha, \beta)) \in S\mathbb{R}^2$.

Since $\overline{Gr_1(\mathbb{R}^2)}$ is a 2-dimensional manifold, we can parametrize affine lines in $\overline{Gr_1(\mathbb{R}^2)}$ with two variables in a natural way. For any straight line l passing through (x, y) with direction (α, β) of unit p -norm, let $(-\Theta, \Omega)$ be the unit vector in p -norm such that l is tangent to the Minkowski sphere $S(r)$ of radius r at $(-r\Theta, r\Omega)$, here we can call r the “ p -norm distance” of l to the origin, see Figure 2.2 on page 7. Thus we can denote the line by $l(r, \Theta)$.

We have the following theorem about the symplectic structure on $\overline{Gr_1(\mathbb{R}^2)}$ by the above parametrization.

Theorem 2.7. *The symplectic structure on $\overline{Gr_1(\mathbb{R}^2)}$ is*

$$\omega_0 = \frac{(p-1)^2 \Theta^{p(p-2)} \Omega^{p^2-3p+1}}{\|(\Theta, \Omega)\|_{p(p-1)}^{(p-1)(2p-1)}} dr \wedge d\Theta. \quad (2.19)$$

Proof. For a line l passing through (x, y) with direction (α, β) of unit p -norm, the “ p -norm distance”

$$r = -\Theta^{p-1}x + \Omega^{p-1}y \quad (2.20)$$

and

$$(-\Theta, \Omega) = \left(-\frac{\beta^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}}, \frac{\alpha^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}} \right). \quad (2.21)$$

In order to express ω_0 in terms of r and Θ , at first we use (2.20) and (2.21) to compute

$$\begin{aligned} dr \wedge d\Theta &= (-\Theta^{p-1}dx + \Omega^{p-1}dy) \wedge d\Theta \\ &= -\Theta^{p-1}dx \wedge d\left(\frac{\beta^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}}\right) + \Omega^{p-1}dy \wedge d\left(\frac{\alpha^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}}\right) \\ &= -\Theta^{p-1}dx \wedge d\left(\frac{1}{\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{\frac{1}{p}}}\right) + \Omega^{p-1}dy \wedge d\left(\frac{1}{\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{\frac{1}{p}}}\right) \\ &= -\Theta^{p-1}dx \wedge \left(-\frac{1}{p}\right)\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{p}{p-1} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \frac{\beta d\alpha - \alpha d\beta}{\beta^2} \\ &\quad + \Omega^{p-1}dy \wedge \left(-\frac{1}{p}\right)\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{p}{p-1} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \frac{\beta d\alpha - \alpha d\beta}{\beta^2} \\ &= \left(\frac{1}{p-1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \left(\Theta^{p-1}dx \wedge \left(\frac{1}{\beta}d\alpha - \frac{\alpha}{\beta^2}d\beta\right) \right. \\ &\quad \left. - \Omega^{p-1}dy \wedge \left(\frac{1}{\beta}d\alpha - \frac{\alpha}{\beta^2}d\beta\right)\right) \\ &= \left(\frac{1}{p-1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \left(\Theta^{p-1}\left(\frac{1}{\beta} + \frac{\alpha^p}{\beta^{p+1}}\right)dx \wedge d\alpha \right. \\ &\quad \left. - \Omega^{p-1}\left(-\frac{1}{\beta} \frac{\beta^{p-1}}{\alpha^{p-1}} - \frac{\alpha}{\beta^2}\right)dy \wedge d\beta\right) \\ &= \left(\frac{1}{p-1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \left(\frac{\Theta^{p-1}}{\beta^{p+1}}dx \wedge d\alpha + \frac{\Omega^{p-1}}{\alpha^{p-1}\beta^2}dy \wedge d\beta\right) \\ &= \frac{1}{(p-1)^2} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \left(\frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}}(p-1)\alpha^{p-2}dx \wedge d\alpha \right. \\ &\quad \left. + \frac{\Omega^{p-1}}{\alpha^{p-1}\beta^p}(p-1)\beta^{p-2}dy \wedge d\beta\right). \end{aligned} \quad (2.22)$$

Indeed,

$$\frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}} = \frac{\Omega^{p-1}}{\alpha^{p-1}\beta^p} \quad (2.23)$$

since

$$\frac{\Theta^{p-1}}{\Omega^{p-1}} = \frac{\beta}{\alpha} \quad (2.24)$$

by (2.21).

Therefore, using (2.23), (2.24) and $\|(\Theta, \Omega)\|_p = 1$, we have

$$\begin{aligned}
 dr \wedge d\Theta &= \frac{1}{(p-1)^2} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}} \\
 &\quad \left((p-1)\alpha^{p-2}dx \wedge d\alpha + (p-1)\beta^{p-2}dy \wedge d\beta\right) \\
 &= \frac{1}{(p-1)^2} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}} \omega_0 \\
 &= \frac{1}{(p-1)^2} \frac{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \Theta^{p-1}}{\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{\frac{p+1}{p}} \beta^{p+1}\alpha^{p-2}} \omega_0 \\
 &= \frac{1}{(p-1)^2} \frac{\frac{\Omega}{\Theta} \Theta^{p-1}}{\left(\left(\frac{\Omega}{\Theta}\right)^{p+1} \left(\frac{\Theta^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p+1} \left(\frac{\Omega^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p-2}\right)} \omega_0 \quad (2.25) \\
 &= \frac{1}{(p-1)^2} \frac{\Omega \Theta^{2p-1}}{\left(\frac{\Theta^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p+1} \left(\frac{\Omega^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p-2}} \omega_0 \\
 &= \frac{1}{(p-1)^2} \frac{\Omega \Theta^{2p-1} \|(\Theta^{p-1}, \Omega^{p-1})\|_p^{2p-1}}{\Theta^{(p-1)(p+1)} \Omega^{(p-1)(p-2)}} \omega_0 \\
 &= \frac{1}{(p-1)^2} \frac{\|(\Theta^{p-1}, \Omega^{p-1})\|_p^{2p-1}}{\Theta^{p(p-2)} \Omega^{p^2-3p+1}} \omega_0 \\
 &= \frac{\|(\Theta, \Omega)\|_p^{(p-1)(2p-1)}}{(p-1)^2 \Theta^{p(p-2)} \Omega^{p^2-3p+1}} \omega_0,
 \end{aligned}$$

Thus we have shown

$$dr \wedge d\Theta = \frac{\|(\Theta, \Omega)\|_p^{(p-1)(2p-1)}}{(p-1)^2 \Theta^{p(p-2)} \Omega^{p^2-3p+1}} \omega_0, \quad (2.26)$$

which implies (2.19) in the claim. \square

So from (2.18) and (2.19) we see the symplectic structure on $\overline{Gr_1(\mathbb{R}^2)}$ is determined by the Minkowski metric $\|\cdot\|_p$ on \mathbb{R}^2 .

3. INTEGRAL GEOMETRY ON LENGTH IN MINKOWSKI SPACE

The length of a straight line segment in (\mathbb{R}^2, F) can be obtained by integrating the canonical contact form α introduced in Section 1.2. For any $x, y \in \mathbb{R}^2$, let \overrightarrow{xy} be the vector from x to y , and

$$c(t) := \left(x + \frac{t}{F(\overrightarrow{xy})}(y-x), dF\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right)\right), \quad t \in [0, F(\overrightarrow{xy})] \quad (3.1)$$

be a straight line segment in $T^*\mathbb{R}^2$. By the positive homogeneity of F , one can get the useful fact that

$$dF\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right)\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right) = F\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right) = 1. \quad (3.2)$$

Therefore,

$$\int_c \alpha = \int_0^{F(\overrightarrow{xy})} dF\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right)\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right) dt = F(\overrightarrow{xy}) = \mathbf{L}(\overrightarrow{xy}), \quad (3.3)$$

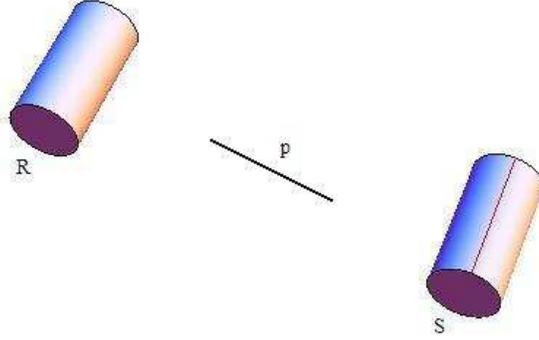
where $\mathbf{L}(\overrightarrow{xy})$ is the length of \overrightarrow{xy} .

Here let us introduce a general definition in integral geometry first.

Definition 3.1. A Crofton measure ϕ for a degree k measure Φ on (\mathbb{R}^n, F) is a measure on $\overline{Gr_{n-k}(\mathbb{R}^n)}$ (Definition 1.3), such that it satisfies the Crofton-type formula

$$\Phi(M) = \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(M \cap P) \Phi(P) \quad (3.4)$$

for any compact convex subset M (\mathbb{R}^n, F).

FIGURE 3.1. From $S\mathbb{R}^2$ to $\overline{Gr_1(\mathbb{R}^2)}$

Furthermore, we have the following

Proposition 3.2. *The Crofton measure on $\overline{Gr_1(\mathbb{R}^2)}$ for the length is $|\omega_0|$.*

Our treatment of applying Stokes' theorem here is primarily based on [4].

Proof. From Section 2, we know $\overline{Gr_1(\mathbb{R}^2)} \xrightarrow{\psi} TS_F$ which is a cylinder, and it has a symplectic form ω_0 as TS_F embedded in $T^*\mathbb{R}^2$.

Let $S := \left\{ l \in \overline{Gr_1(\mathbb{R}^2)} : l \cap \overset{\circ}{xy} \neq \emptyset \right\}$, C_x and C_y be the family of oriented lines passing through x and y respectively, then $C_x \cap C_y = \{l_{xy}^+, l_{xy}^-\}$ that are the two oriented lines connecting x and y , and $\partial S = C_x \cup C_y$.

Let $R := \left\{ \xi \in S\mathbb{R}^2 \subset T\mathbb{R}^2 : l((x + t(y-x), dF(\xi)) \cap \overset{\circ}{xy}) \neq \emptyset \right\}$, where $l((x + t(y-x), dF(\xi)))$ is the line passing through $x + t(y-x)$ with direction ξ , then $p(R) = S$, where p is the natural projection from $S\mathbb{R}^2$ to $\overline{Gr_1(\mathbb{R}^2)}$.

Additionally, let $C'_x = \{\xi_x : \xi_x \in S_x\mathbb{R}^2\}$, $C'_y = \{\xi_y : \xi_y \in S_y\mathbb{R}^2\}$, $l'_{xy}{}^+ = \frac{\overline{xy}}{F(\overline{xy})} \in S_x\mathbb{R}^2$, and $l'_{xy}{}^- = -\frac{\overline{xy}}{F(\overline{xy})} \in S_y\mathbb{R}^2$, then p maps C'_x , C'_y , $l'_{xy}{}^+$ and $l'_{xy}{}^-$ to C_x , C_y , l_{xy}^+ and l_{xy}^- respectively, see Figure 3.1 on page 10.

Applying Stokes' theorem to the two regions individually, using the fact that $\int_{C'} \alpha = 0$ because of the fixed base points for any $C' \subset C'_x$ or C'_y , and combining with (3.3), we obtain

$$\begin{aligned}
 \int_S |\omega_0| &= \int_{p(R)} |\omega_0| = \int_R |p^* \omega_0| \\
 &= \int_R |\omega| \\
 &= 2 \int_{l'_{xy}{}^+ \cup l'_{xy}{}^-} \alpha \\
 &= 4\mathbf{L}(\overline{xy}).
 \end{aligned} \tag{3.5}$$

Therefore, for any rectifiable curve γ in (\mathbb{R}^2, F) , the length of γ ,

$$\mathbf{L}(\gamma) = \frac{1}{4} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\omega_0|, \tag{3.6}$$

which is the desired claim. \square

Remark 3.3. The proof above can be applied to \mathbb{R}^2 with projective Finsler metric, in which geodesics are straight lines. Furthermore, for \mathbb{R}^n with a projective Finsler metric F , we choose a plane $P \subset \mathbb{R}^n$ containing \overline{xy} for any $x, y \in \mathbb{R}^n$, then

$$\mathbf{L}(\overline{xy}) = \frac{1}{4} \int_{l \in Gr_1(P)} \#(\overline{xy} \cap l) |\omega_0|. \quad (3.7)$$

4. VOLUME OF HYPERSURFACES

A standard definition of Holmes-Thompson volume in Minkowski space (\mathbb{R}^n, F) is given and its importance in Finsler geometry and integral geometry is illustrated in [3].

The Holmes-Thompson volumes are defined as follows.

Definition 4.1. Let N be a k -dimensional manifold and

$$D^*N := \{\xi_x \in T^*N : F^*(\xi_x) \leq 1\}, \quad (4.1)$$

where F^* is the dual norm in (1.7), be the codisc bundle of N , then the k -th Holmes-Thompson volume is defined as

$$vol_k(N) := \frac{1}{\epsilon_k} \int_{D^*N} |\omega^k|, \quad (4.2)$$

where ϵ_k is the Euclidean volume of k -dimensional Euclidean ball and ω is the canonical symplectic form on the cotangent bundle of N .

Let $\Lambda \in \overline{Gr_k(\mathbb{R}^n)}$ for some $k \leq n$, ω_0 and $\hat{\omega}_0$ are the natural symplectic forms on $\overline{Gr_1(\mathbb{R}^n)}$ and $\overline{Gr_1(\Lambda)}$ constructed in the way described in Section 2. The relation between ω_0 and $\hat{\omega}_0$ is shown in the following

Lemma 4.2. $i^*\omega_0 = \omega$ for $i : \overline{Gr_1(\Lambda)} \hookrightarrow \overline{Gr_1(\mathbb{R}^n)}$.

Proof. First consider the diagram

$$S^*\Lambda \xrightarrow{\varphi_{F^*}} S\Lambda \xrightarrow{\hat{i}} S\mathbb{R}^n \xrightarrow{\varphi_F} S^*\mathbb{R}^n. \quad (4.3)$$

We have a canonical contact form $\hat{\alpha}_\xi(X) := \xi(\hat{\pi}_0^*X)$ for $X \in T_\xi S^*\Lambda$ on $S^*\Lambda$ in diagram (4.3), where $\hat{\pi}_0 : S^*\Lambda \rightarrow \Lambda$ is the natural projection, and define $\hat{\omega} := d\hat{\alpha}$ on $S^*\Lambda$.

Let $j = \varphi_F \circ \hat{i} \circ \varphi_{F^*}$, then for any $X \in T_\xi S^*\Lambda$,

$$(j^*\alpha)_\xi(X) = \alpha_{j(\xi)}(j_*X) = j(\xi)(\pi_*j_*X) = \xi(\hat{\pi}_0^*X) = \hat{\alpha}_\xi(X) \quad (4.4)$$

in which α and ω on $S^*\mathbb{R}^n$ are introduced in Section 1.2, then (4.4) implies

$$j^*\alpha = \hat{\alpha}, \quad (4.5)$$

and furthermore we have

$$j^*\omega = \hat{\omega} \quad (4.6)$$

by differentiating (4.5).

Next, let \hat{p} be the projection taking $\bar{\xi}_x \in S\Lambda$ to the line passing x with the direction $\bar{\xi}_x$, and similarly for p which is described in (2.1). Consider the diagram

$$\begin{array}{ccccccc} S^*\Lambda & \xrightarrow{\varphi_{F^*}} & S\Lambda & \xrightarrow{\hat{i}} & S\mathbb{R}^n & \xrightarrow{\varphi_F} & S^*\mathbb{R}^n \\ & & \downarrow \hat{p} & & \downarrow p & & \\ & & \overline{Gr_1(\Lambda)} & \xrightarrow{i} & \overline{Gr_1(\mathbb{R}^n)} & & \end{array} \quad (4.7)$$

obtained by combining diagram (2.1) and (4.4). By the definitions of the maps in (4.7), we know the diagram is commutative. By Theorem 2.3, we have $p^*\omega_0 = \varphi_F^*\omega$ and $\hat{p}^*\hat{\omega}_0 = \varphi_F^*\hat{\omega}$. Combining with (4.6) and the commutativity of the diagram (4.7), we obtain the desired claim $i^*\omega_0 = \hat{\omega}_0$. \square

Suppose N is a hypersurface in (\mathbb{R}^n, F) , then we have the following

Proposition 4.3. $vol_{n-1}(N) = \frac{1}{2\epsilon_{n-1}} \int_{l \in \overline{Gr_1(\mathbb{R}^n)}} \#(N \cap l) |\omega_0^{n-1}|$, where ω_0 is the symplectic form on $\overline{Gr_1(\mathbb{R}^n)}$.

This idea of intrinsic proof is given by Dr. Joseph H. G. Fu.

Proof. It suffices to prove the claim in the case when N is affine. Without loss of generality, assume $N \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ is compact and convex with smooth boundary.

Consider the following diagram

$$S^*N \xrightarrow{i} S^*\mathbb{R}^{n-1} \xrightarrow{\cong} S\mathbb{R}^{n-1} \xrightarrow{i} S\mathbb{R}^n \xrightarrow{\cong} S^*\mathbb{R}^n \xrightarrow{\pi} \overline{Gr_1(\mathbb{R}^n)}, \quad (4.8)$$

where i and k are embeddings, and $\pi := p \circ \varphi_F^{-1} = p \circ \varphi_{F^*}$ is a projection from diagram (2.1).

As N is a $(n-1)$ -dimensional manifold, the canonical contact form $\hat{\alpha}$ on S^*N is defined as $\hat{\alpha}_\xi(X) := \xi(\hat{\pi}_0^*X)$ for $X \in T_\xi S^*N$, where $\hat{\pi}_0 : S^*N \rightarrow N$ is the projection.

Let $j = \varphi_F \circ i \circ \varphi_{F^*}$, then

$$(j^*\alpha)_{\hat{i}(\xi)}(\hat{i}_*X) = ((\varphi_F \circ i \circ \varphi_{F^*})^*\alpha)_{\hat{i}(\xi)}(\hat{i}_*X) = \xi(\pi_0^*X) = \hat{\alpha}_\xi(X). \quad (4.9)$$

for any $X \in T_\xi S^*N$, which implies $(\hat{i} \circ j)^*\alpha = \hat{i}^*j^*\alpha = \hat{\alpha}$, and then $(i \circ j)^*\omega = \hat{\omega}$ where $\hat{\omega} := d\hat{\alpha}$ and ω is introduced in Section 1.2.

Applying Stokes' theorem, we have

$$\begin{aligned} \int_{D^*N} \hat{\omega}^{n-1} &= \int_{\partial(D^*N)} \hat{\alpha} \wedge \hat{\omega}^{n-2} = \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2} + \int_{\hat{\pi}_0^{-1}(\partial N)} \hat{\alpha} \wedge \hat{\omega}^{n-2} \\ &= \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2} \end{aligned} \quad (4.10)$$

since the degree of $\hat{\alpha} \wedge \hat{\omega}^{n-2}$ on the component measuring perturbations of base points is bigger than the dimension of the base manifold, and

$$\begin{aligned} \int_{S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \omega^{n-1} &= \int_{\partial(S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N))} \alpha \wedge \omega^{n-2} \\ &= \int_{S^*N} \hat{i}^*j^*\alpha \wedge \hat{i}^*j^*\omega^{n-2} + \int_{\hat{\pi}_0^{-1}(\partial N)} \hat{i}^*j^*\alpha \wedge \hat{i}^*j^*\omega^{n-2} \\ &= \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2}, \end{aligned} \quad (4.11)$$

where

$$S_+^*\mathbb{R}^n = \{\xi \in S^*\mathbb{R}^n : \xi(v_0) \geq 0, v_0 \text{ satisfies } dF(v_0)(v) = 0 \text{ for all } v \in S\mathbb{R}^{n-1}\}. \quad (4.12)$$

Therefore,

$$\int_{D^*N} \hat{\omega}^{n-1} = \int_{S_+^*\mathbb{R}^n \cap \pi_2^{-1}(N)} \omega^{n-1}. \quad (4.13)$$

Now let us consider the ‘‘upper’’ half space of geodesics in (\mathbb{R}^n, F) ,

$$\overline{Gr_1^+(\mathbb{R}^n)} := \{l(x, \eta) : dF(\eta)(\eta_0) \geq 0, \eta_0 \text{ satisfies } dF(\eta_0)(v) = 0 \text{ for all } v \in S\mathbb{R}^{n-1}\}. \quad (4.14)$$

Since $\pi^*\omega_0 = \omega$, then we get

$$\begin{aligned} \int_{S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \omega^{n-1} &= \int_{S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \pi^*\omega_0^{n-1} \\ &= \int_{\pi^{-1}(l) \in S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \#(N \cap l) \omega_0^{n-1} \\ &= \int_{l \in \overline{Gr_1^+(\mathbb{R}^n)}} \#(N \cap l) \omega_0^{n-1}. \end{aligned} \quad (4.15)$$

Combining with (4.13), we obtain

$$\begin{aligned} \text{vol}_{n-1}(N) &= \frac{1}{\epsilon_{n-1}} \int_{D^*N} \hat{\omega}^{n-1} \\ &= \frac{1}{\epsilon_{n-1}} \int_{l \in \overline{Gr_1^+(\mathbb{R}^n)}} \#(N \cap l) \omega_0^{n-1} \\ &= \frac{1}{2\epsilon_{n-1}} \int_{l \in \overline{Gr_1(\mathbb{R}^n)}} \#(N \cap l) |\omega_0^{n-1}|, \end{aligned} \quad (4.16)$$

that finishes the proof. \square

5. k -TH HOLMES-THOMPSON VOLUME AND CROFTON MEASURES

Let us introduce a general fact first. Busemann constructed all projective metrics F for projective Finsler space (\mathbb{R}^n, F) , and it was also proved in [12] by Schneider using spherical harmonics.

Theorem 5.1. (*Busemann*) *Suppose F is a projective metric on \mathbb{R}^n , then $F(x, v) = \int_{\xi \in S^{n-1}} |\langle \xi, v \rangle| f(\xi, \langle \xi, x \rangle) \Omega_0$ for any $(x, v) \in T\mathbb{R}^n$, where Ω_0 is the Euclidean volume form on S^{n-1} and f is some continuous function on $S^{n-1} \times \mathbb{R}$.*

In fact, for the case that (\mathbb{R}^n, F) is Minkowski, we can use a theorem on surjectivity of cosine transform,

$$\mathcal{C}(f)(\cdot) = \int_{\xi \in S^{n-1}} |\langle \xi, \cdot \rangle| f(\xi) \Omega_0, \quad (5.1)$$

of even functions from Chapter 3 of [9],

Theorem 5.2. *For any even $C^{2[(n+3)/2]}$ function g on S^{n-1} , $n \geq 2$, where $[\cdot]$ is the greatest integer function, there is an even function f on S^{n-1} such that $\mathcal{C}(f) = g$.*

From it we directly obtain that there exists an even function f on S^{n-1} , such that

$$L(\overline{xy}) = \frac{1}{4} \int_{\xi \in S^{n-1}} |\langle \xi, \overline{xy} \rangle| f(\xi) \Omega_0. \quad (5.2)$$

On the other hand, for any $v = \overline{xy}$, $x, y \in (\mathbb{R}^2, F)$, by Proposition 3.2 we know

$$F(x, v) = \frac{1}{\omega_{n-1}} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\overline{xy} \cap l) |\omega_0|. \quad (5.3)$$

In fact, there is a relation between Ω_0 and ω_0 . Considering the following double fibration

$$\overline{Gr_1(\mathbb{R}^n)} \xleftarrow{\pi_1} \mathcal{I} \xrightarrow{\pi_2} \overline{Gr_{n-1}(\mathbb{R}^n)}, \quad (5.4)$$

where $\mathcal{I} = \left\{ (l, H) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-1}(\mathbb{R}^n)} : l \subset H \right\}$, we have

Proposition 5.3. *$GT(f\Omega_0 \wedge dr) = \omega_0$, where GT is the Gelfand transform for the double fibration (5.4) and r is the Euclidean distance of a hyperplane H to the origin.*

Proof. For any x, y in any 2-plane $\Pi \subset \mathbb{R}^n$, we know that the length of \overline{xy} ,

$$L(\overline{xy}) = \frac{1}{4} \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |\omega_0|_{\overline{Gr_1(\Pi)}}. \quad (5.5)$$

where $\overline{Gr_1(\Pi)} := \{l \in \overline{Gr_1(\mathbb{R}^n)} : l \subset \Pi\}$.

Let $I = \{l \in \overline{Gr_1(\Pi)} \subset \overline{Gr_1(\mathbb{R}^n)} : \overline{xy} \cap l \neq \emptyset\}$ and $G_H := \pi_1(\pi_2^{-1}(H))$ for $H \in \overline{Gr_{n-1}(\mathbb{R}^n)}$. By the fundamental theorem of Gelfand transform, Theorem 1.5,

$$\int_{H \in \overline{Gr_{n-1}(\mathbb{R}^n)}} \#(I \cap G_H) |f\Omega_0 \wedge dr| = \int_I |GT(f\Omega_0 \wedge dr)|. \quad (5.6)$$

Therefore,

$$\begin{aligned} \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |GT(f\Omega_0 \wedge dr)| &= \int_I |GT(f\Omega_0 \wedge dr)| \\ &= \int_{H \in \overline{Gr_{n-1}(\mathbb{R}^n)}} \#(I \cap G_H) |f\Omega_0 \wedge dr| \\ &= \int_{\xi \in S^{n-1}} |\langle \xi, \overline{xy} \rangle| f(\xi) \Omega_0 \end{aligned} \quad (5.7)$$

since $\overline{Gr_{n-1}(\mathbb{R}^n)} \cong S^{n-1} \times \mathbb{R}$. By (5.2) and (5.5) we thus obtain

$$\int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |GT(f\Omega_0 \wedge dr)| = \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |\omega_0|, \quad (5.8)$$

which implies $|GT(f\Omega_0 \wedge dr)|_{\overline{Gr_1(\Pi)}} = |\omega_0|_{\overline{Gr_1(\Pi)}}$ for any plane $\Pi \subset \mathbb{R}^n$ by the injectivity of cosine transform (5.1). (In Chapter 3 of [9] Groemer shows by using condensed harmonic expansion and Parseval's equation, that $\mathcal{C}(f_1) = \mathcal{C}(f_2)$ iff $f_1^+ = f_2^+$, where $f_1^+(v) = \frac{f_1(v) + f_1(-v)}{2}$ and similarly for f_2^+ , for any bounded integrable functions f_1 and f_2 on S^{n-1} .)

Now define a basis for $T_l \overline{Gr_1(\mathbb{R}^n)}$, the tangent space of $\overline{Gr_1(\mathbb{R}^n)}$ at $l \in \overline{Gr_1(\mathbb{R}^n)}$.

Note that $\overline{Gr_1(\mathbb{R}^n)} \stackrel{\psi}{\cong} TS_F^{n-1}$ from Section 2. Let $\{e_i : i = 1, \dots, n\}$ be the basis for \mathbb{R}^n , and curve $\overline{\gamma}_i$ with $\overline{\gamma}_i(t) = l + te_i$ for $i = 1, \dots, n-1$, where $l \in \overline{Gr_1(\mathbb{R}^n)}$, and then define $\overline{e}_i := \overline{\gamma}'_i(0)$ for $i = 1, \dots, n-1$. Let $l(x, \xi)$ be a line in $\overline{Gr_1(\mathbb{R}^n)}$ passing through x with direction ξ and $r_i(t)(\xi)$ for be the rotation about origin with the direction from e_n towards e_i for time t , then let $v_i(t)$ be the parallel transport from $\psi(l(x, \xi))$ along on $r_i(t)(\xi)$ on S_F^{n-1} , and then define curves $\overline{\gamma}_i(t) = \psi^{-1}(v_i(t))$ for $i = 1, \dots, n-1$, thus we can define $\overline{e}_i := \overline{\gamma}'_i(0)$. Then $\{\overline{e}_i, \overline{e}_j : i, j = 1, \dots, n-1\}$ is a basis for $T_l \overline{Gr_1(\mathbb{R}^n)}$.

Here we have four cases to discuss.

First of all, one can obtain the fact

$$GT(f\Omega_0 \wedge dr)(\overline{e}_i, \overline{e}_i) = \omega_0(\overline{e}_i, \overline{e}_i) \quad (5.9)$$

by choosing a plane Π_i with the tangent space of $\overline{Gr_1(\Pi_i)}$ spanned by \overline{e}_i and \overline{e}_i for $i = 1, \dots, n-1$.

On the other hand, in the double fibration (5.4), $\pi_2|_{\pi_1^{-1}(L_{ij})}$, in which L_{ij} is be the lines in $\overline{Gr_1(\mathbb{R}^n)}$ obtained by translation along \overline{e}_i or \overline{e}_j for $i, j = 1, \dots, n-1$, is not a submersion from $\pi_1^{-1}(L_{ij})$ to $\overline{Gr_{n-1}(\mathbb{R}^n)}$. Precisely, choose \tilde{e}_i and \tilde{e}_j in $T_{(l, H)} \mathcal{I}$, $l \subset H$ such that $d\pi_1(\tilde{e}_i) = \overline{e}_i$ and $d\pi_1(\tilde{e}_j) = \overline{e}_j$, moreover, $d\pi_2(\tilde{e}_i)$ and $d\pi_2(\tilde{e}_j)$ are linearly dependent in $T_H \overline{Gr_{n-1}(\mathbb{R}^n)}$. Therefore $\pi_{1*} \pi_2^*(f\Omega_0 \wedge dr)_l(\tilde{e}_i, \tilde{e}_j) =$

$\int_{\pi_1^{-1}(l)} \pi_2^*(f\Omega_0 \wedge dr)(\bar{e}_i, \bar{e}_j) = 0$ for $i, j = 1, \dots, n-1$, and obviously $\omega_0(\bar{e}_i, \bar{e}_j) = 0$, thus

$$GT(f\Omega_0 \wedge dr)(\bar{e}_i, \bar{e}_j) = \omega_0(\bar{e}_i, \bar{e}_j) = 0 \quad (5.10)$$

for $i, j = 1, \dots, n-1$.

For the case of \bar{e}_i and \bar{e}_j , $i \neq j$, $i = 1, \dots, n-1$. Let \bar{L}_{ij} be the lines in $\overline{Gr_1(\mathbb{R}^n)}$ obtained by translation along \bar{e}_i or rotation along \bar{e}_j . Again, $\pi_2|_{\bar{L}_{ij}}$ in (5.4) is not a submersion from $\pi_1^{-1}(\bar{L}_{ij})$ to $\overline{Gr_{n-1}(\mathbb{R}^n)}$ either, and it also can be explained precisely as the above case, therefore $\pi_{1*}\pi_2^*(f\Omega_0 \wedge dr)(\bar{e}_i, \bar{e}_j) = 0$ for $i, j = 1, \dots, n-1$, and obviously $\omega_0(\bar{e}_i, \bar{e}_j) = 0$, thus

$$GT(f\Omega_0 \wedge dr)(\bar{e}_i, \bar{e}_j) = \omega_0(\bar{e}_i, \bar{e}_j) = 0 \quad (5.11)$$

for $i \neq j$, $i, j = 1, \dots, n-1$.

Similarly for the last case of \bar{e}_i and \bar{e}_j , $i, j = 1, \dots, n-1$,

$$GT(f\Omega_0 \wedge dr)(\bar{e}_i, \bar{e}_j) = \omega_0(\bar{e}_i, \bar{e}_j) = 0. \quad (5.12)$$

So we have $GT(f\Omega_0 \wedge dr) = \omega_0$ on $\overline{Gr_1(\mathbb{R}^n)}$. \square

One can use the diagonal intersection map and Gelfand transform by following [6] to construct Crofton measure for the k -th Holmes-Thompson volume.

Let $\Omega_{n-1} := f\Omega_0 \wedge dr$ and define a map

$$\begin{aligned} \pi : \overline{Gr_{n-1}(\mathbb{R}^n)}^k \setminus \Delta_k &\rightarrow \overline{Gr_{n-k}(\mathbb{R}^n)} \\ \pi((H_1, \dots, H_k)) &= H_1 \cap \dots \cap H_k, \end{aligned} \quad (5.13)$$

where $\Delta_k = \{(H_1, \dots, H_k) : \dim(H_1 \cap \dots \cap H_k) > n-k\}$ and then let $\Omega_{n-k} := \pi_*\Omega_{n-1}^k$.

Now consider the following double fibration,

$$\overline{Gr_1(\mathbb{R}^n)} \xleftarrow{\pi_{1,k}} \mathcal{I}_k \xrightarrow{\pi_{2,k}} \overline{Gr_{n-k}(\mathbb{R}^n)}, \quad (5.14)$$

where $\mathcal{I}_k = \{(l, S) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-k}(\mathbb{R}^n)} : l \subset S\}$. Then we have the following proposition about the Gelfand transform on (5.14)

Proposition 5.4. $GT(\Omega_{n-k}) = \omega_0^k$ for $1 \leq k \leq n-1$.

Proof. Let

$$\mathcal{H} := \left\{ (l, (H_1, H_2, \dots, H_k)) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-1}(\mathbb{R}^n)}^k : l \subset H_1 \cap \dots \cap H_k \right\} \quad (5.15)$$

and consider the following diagram

$$\begin{array}{ccccc} \overline{Gr_1(\mathbb{R}^n)} & \xleftarrow{\pi_{1,k}} & \mathcal{I}_k & \xrightarrow{\pi_{2,k}} & \overline{Gr_{n-k}(\mathbb{R}^n)} \\ & \nwarrow \tilde{\pi}_1 & \uparrow \tilde{\pi} & & \uparrow \pi \\ & & \mathcal{H} & \xrightarrow{\tilde{\pi}_2} & \overline{Gr_{n-1}(\mathbb{R}^n)}^k, \end{array} \quad (5.16)$$

in which $\tilde{\pi} : \mathcal{H} \rightarrow \mathcal{I}_k$ is defined by $\tilde{\pi}((l, (H_1, H_2, \dots, H_k))) = (l, H_1 \cap H_2 \cap \dots \cap H_k)$.

Note that

$$\pi_{1*}\pi_{2*}^*\Omega_{n-1} = \omega_0, \quad (5.17)$$

by Proposition 5.3.

For the lower part of the diagram (5.16),

$$\overline{Gr_1(\mathbb{R}^n)} \xleftarrow{\tilde{\pi}_1} \mathcal{H} \xrightarrow{\tilde{\pi}_2} \overline{Gr_1(\mathbb{R}^n)}^k, \quad (5.18)$$

By manipulating the map $\tilde{\pi}_2 = \underbrace{\pi_2 \times \cdots \times \pi_2}_k$, the product of k copies of the map π_2 , applying Fubini theorem for (5.17) and using the fact that $\tilde{\pi}_1 \times \tilde{\pi}_2 : \mathcal{H} \rightarrow \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_1(\mathbb{R}^n)}^k$ is an immersion, one can infer $\tilde{\pi}_1^* \tilde{\pi}_2^* \Omega_{n-1}^k = \omega_0^k$.

Thus, by the commutativity of the diagram (5.16) we obtain $\pi_{1,k*} \pi_{2,k}^* \Omega_{n-k} = \omega_0^k$. \square

In order to study the k -th Holmes-Thompson volume, one can restrict on some $k+1$ -dimensional flat subspace. So fix $S \in \overline{Gr_{k+1}(\mathbb{R}^n)}$ and then define a map by intersection

$$\begin{aligned} \pi_S : \overline{Gr_{n-k}(\mathbb{R}^n)} \setminus \Delta(S) &\rightarrow \overline{Gr_1(S)} \\ \pi_S(H^{n-k}) &= H^{n-k} \cap S \end{aligned} \quad (5.19)$$

for $H^{n-k} \in \overline{Gr_{n-k}(\mathbb{R}^n)} \setminus \Delta(S)$, where

$$\Delta(S) := \left\{ H^{n-k} \in \overline{Gr_{n-k}(\mathbb{R}^n)} : \dim(H^{n-k} \cap S) > 0 \right\}. \quad (5.20)$$

Then we have the following proposition

Proposition 5.5. $(\pi_S)_* \Omega_{n-k} = \omega_0^k|_{\overline{Gr_1(S)}}$, for $1 \leq k \leq n-1$.

Proof. From Proposition 5.4, we know that $\pi_{1,k*} \pi_{2,k}^* \Omega_{n-k} = \omega_0^k$ for the double fibration $\overline{Gr_{n-k}(\mathbb{R}^n)} \xleftarrow{\pi_{2,k}} \mathcal{I}_k \xrightarrow{\pi_{1,k}} \overline{Gr_1(\mathbb{R}^n)}$.

Therefore, one can obtain by the definition of the intersection map (5.19)

$$(\pi_S)_* \Omega_{n-k} = \pi_{1,k*} \pi_{2,k}^* \Omega_{n-k}|_{\overline{Gr_1(S)}} = \omega_0^k|_{\overline{Gr_1(S)}}. \quad (5.21)$$

\square

Finally, one can obtain the following theorem about Holmes-Thompson volumes.

Theorem 5.6. (Alvarez) *Suppose N is a k -dimensional submanifold in (\mathbb{R}^n, F) . Then $\text{vol}_k(N) = \frac{1}{2\epsilon_k} \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N \cap P) |\Omega_{n-k}|$ for $1 \leq k \leq n-1$.*

Proof. By Proposition 4.3, the claim is true for hypersurface case.

It is sufficient to show the claim for the case when $N \subset S$ for some $S \in \overline{Gr_{k+1}(\mathbb{R}^n)}$. We obtain by Proposition 4.3 and Proposition 5.5,

$$\begin{aligned} \text{vol}_k(N) &= \frac{1}{2\epsilon_k} \int_{l \in \overline{Gr_1(S)}} \#(N \cap l) |\omega_0^k| \\ &= \frac{1}{2\epsilon_k} \int_{l \in \overline{Gr_1(S)}} \#(N \cap l) |(\pi_S)_* \Omega_{n-k}| \\ &= \frac{1}{2\epsilon_k} \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N \cap P) |\Omega_{n-k}|. \end{aligned} \quad (5.22)$$

as desired. \square

6. LENGTH AND RELATED

The classic Crofton formula is

$$\text{Length}(\gamma) = \frac{1}{4} \int_0^\infty \int_0^{2\pi} \#(\gamma \cap l(r, \theta)) d\theta dr \quad (6.1)$$

for any rectifiable curve in Euclidean plane, where θ is the angle of the normal of the oriented line l to the x -axis and r is its distance to the origin. Let us denote the affine 1-Grassmannians (lines) in \mathbb{R}^2 by $\overline{Gr_1(\mathbb{R}^2)}$.

As for Minkowski plane, it is a normed two dimensional space with a norm $F(\cdot) = \|\cdot\|$, in which the unit disk is convex and F has some smoothness.

Two of the key tools used to obtain the Crofton formula for Minkowski plane are the cosine transform and Gelfand transform. Let us explain them one by one first and see their connection next. A fact from spherical harmonics about cosine transform is there is some even function on S^1 such that

$$F(\cdot) = \frac{1}{4} \int_{S^1} |\langle \xi, \cdot \rangle| g(\xi) d\xi, \quad (6.2)$$

if F is an even C^4 function on S^1 . A good reference for this is [9]. As for Gelfand transform, it is the transform of differential forms and densities on double fibrations, for instance, $\mathbb{R}^2 \xleftarrow{\pi_1} \mathcal{I} \xrightarrow{\pi_2} \overline{Gr_1(\mathbb{R}^2)}$, where $\mathcal{I} := \{(x, l) \in \mathbb{R}^2 \times \overline{Gr_1(\mathbb{R}^2)} : x \in l\}$ is the incidence relations and π_1 and π_2 are projections. A formula one can take as an example of the fundamental theorem of Gelfand transform is the following

$$\int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\Omega|, \quad (6.3)$$

where $\Omega := g(\theta) d\theta \wedge dr$. But we give a direct proof here.

Proof. First, consider the case of $\Omega = d\theta \wedge dr$. For any $v \in T_x \gamma$, since there is some $v' \in T_{x'} \mathcal{I}$, such that $\pi_{1*}(v') = v$, then

$$\begin{aligned} (\pi_{1*} \pi_2^* |\Omega|)_x(v) &= (\int_{\pi_1^{-1}(x)} \pi_2^* |\Omega|)_x(v) \\ &= \int_{x' \in \pi_1^{-1}(x)} (\pi_2^* |\Omega|)_{x'}(v') \\ &= \int_{S^1} (\pi_2^* |d\theta \wedge dr|)_{x'}(v') \\ &= \int_{S^1} |dr(\pi_{2*}(v'))| d\theta \\ &= \int_{S^1} |\langle v, \theta \rangle| d\theta \\ &= 4|v|. \end{aligned} \quad (6.4)$$

So $\int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = 4Length(\gamma) = \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\Omega|$ by the classic Crofton formula.

When $\Omega = f(\theta) d\theta \wedge dr$, we just need to replace $d\theta$ by $g(\theta) d\theta$ in the equalities in the first case. \square

Moreover, from the above proof and (6.2), for any curve $\gamma(t) : [a, b] \rightarrow \mathbb{R}^2$ differentiable almost everywhere in the Minkowski space,

$$\int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = \int_a^b (\pi_{1*} \pi_2^* |\Omega|)(\gamma'(t)) dt = \int_a^b 4F(\gamma'(t)) dt = 4Length(\gamma), \quad (6.5)$$

so then by (6.3) we know

$$Length(\gamma) = \frac{1}{4} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |g(\theta) d\theta \wedge dr| \quad (6.6)$$

for Minkowski plane.

The Holmes-Thompson Area $HT^2(U)$ of a measurable set U in a Minkowski plane is defined as $HT^2(U) := \frac{1}{\pi} \int_{D^*U} |\omega_0|^2$, where ω_0 is the natural symplectic form on the cotangent bundle of \mathbb{R}^2 and $D^*U := \{(x, \xi) \in T^*\mathbb{R}^2 : F^*(\xi) \leq 1\}$. To study it from the perspective of integral geometry, we need to introduce a symplectic form ω on the space of affine lines $\overline{Gr_1(\mathbb{R}^2)}$, that one can see [1].

7. HT AREA AND RELATED

Now let's see the Crofton formula for Minkowski plane, which is $Length(\gamma) = \frac{1}{4} \int_{\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\omega|$. To prove this, it is sufficient to show that it holds for any straight line segment

$$L : [0, \|p_2 - p_1\|] \rightarrow \mathbb{R}^2, L(t) = p_1 + \frac{p_2 - p_1}{\|p_2 - p_1\|} t, \quad (7.1)$$

starting at p_1 and ending at p_2 in \mathbb{R}^2 . First, using the diffeomorphism between the circle bundle and co-circle bundle, which is

$$\begin{aligned} \varphi_F : S\mathbb{R}^2 &\rightarrow S^*\mathbb{R}^2 \\ \varphi_F(x, \xi) &= (x, dF_\xi), \end{aligned} \quad (7.2)$$

we can obtain a fact that

$$\begin{aligned} \int_{L \times \left\{ \frac{p_2 - p_1}{\|p_2 - p_1\|} \right\}} \varphi_F^* \alpha_0 &= \int_{\varphi_F(L \times \left\{ \frac{p_2 - p_1}{\|p_2 - p_1\|} \right\})} \alpha_0 \\ &= \int_0^{\|p_2 - p_1\|} \alpha_0 dF_{\frac{p_2 - p_1}{\|p_2 - p_1\|}} \left(\left(\frac{p_2 - p_1}{\|p_2 - p_1\|}, 0 \right) \right) dt \\ &= \int_0^{\|p_2 - p_1\|} dF_{\frac{p_2 - p_1}{\|p_2 - p_1\|}} \left(\frac{p_2 - p_1}{\|p_2 - p_1\|} \right) dt, \end{aligned} \quad (7.3)$$

where α_0 is the tautological one-form, precisely $\alpha_{0\xi}(X) := \xi(\pi_{0*} X)$ for any $X \in T_\xi T^*\mathbb{R}^2$, and $d\alpha_0 = \omega_0$. Applying the basic equality that $dF_\xi(\xi) = 1$, which is derived from the positive homogeneity of F , for all $\xi \in S\mathbb{R}^2$, the above quantity becomes $\int_0^{\|p_2 - p_1\|} 1 dt$, which equals to $\|p_2 - p_1\|$.

Let $R := \{\xi_x \in S^*\mathbb{R}^2 : x \in \overline{p_1 p_2}\}$ and $T = \left\{ l \in \overline{Gr_1(\mathbb{R}^2)} : l \cap \overline{p_1 p_2} \neq \emptyset \right\}$, and p' is the projection (composition) from $S^*\mathbb{R}^2$ to $\overline{Gr_1(\mathbb{R}^2)}$.

Apply the above fact and $p'^*\omega = \omega_0$,

$$\begin{aligned} \int_T |\omega| &= \int_{p'(R)} |\omega| = \int_R |p'^*\omega| = \int_R |\omega_0| \\ &= \left| \int_{R^+} \omega_0 \right| + \left| \int_{R^-} \omega_0 \right| \\ &= \left| \int_{\partial R^+} \alpha_0 \right| + \left| \int_{\partial R^-} \alpha_0 \right| \\ &= 4\|p_2 - p_1\|. \end{aligned} \quad (7.4)$$

Thus we have shown the Crofton formula for Minkowski plane.

Furthermore, combining with (6.6), we have

$$\frac{1}{4} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\Omega| = \frac{1}{4} \int_{\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\omega|, \quad (7.5)$$

where $\Omega = g(\theta) d\theta \wedge dr$. Then, by the injectivity of cosine transform in [9], $|\Omega| = |\omega|$.

To obtain the HT area, one can define a map

$$\begin{aligned} \pi : \overline{Gr_1(\mathbb{R}^2)} \times \overline{Gr_1(\mathbb{R}^2)} \setminus \tilde{\Delta} &\rightarrow \mathbb{R}^2 \\ \pi(l, l') &= l \cap l', \end{aligned} \quad (7.6)$$

where $\tilde{\Delta} := \{(l, l') : l \parallel l' \text{ or } l = l'\}$, extended from Alvarez's construction of taking intersections, [?]. The following theorem can be obtained.

Theorem 7.1. $HT^2(U) = \frac{1}{2\pi} \int_{x \in \mathbb{R}^2} \chi(x \cap U) |\pi_* \Omega^2|$ for any bounded measurable subset U of a Minkowski plane.

Proof. On one hand,

$$\frac{1}{\pi} \int_{D^*U} \omega_0^2 = \frac{1}{\pi} \int_{\partial D^*U} \omega_0^2 = \frac{1}{\pi} \int_{S^*U} \alpha_0 \wedge \omega_0. \quad (7.7)$$

On the other hand, let $\mathcal{T}_U := \left\{ ((l, l') \in \overline{Gr_1(\mathbb{R}^2)} \times \overline{Gr_1(\mathbb{R}^2)} : l \cap l' \in U) \right\}$,

$$\frac{1}{\pi} \int_{x \in \mathbb{R}^2} \chi(x \cap U) \pi_* \Omega^2 = \frac{1}{\pi} \int_U \pi_* \omega^2 = \frac{1}{\pi} \int_{\mathcal{T}_U} \omega^2. \quad (7.8)$$

Let $\mathbb{T}^*U := \{(\xi_x, \xi'_x) : \xi_x, \xi'_x \in S_x^*U\}$, then

$$(p' \times p')^{-1}(\mathcal{T}_U) = \mathbb{T}^*U \setminus \{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}. \quad (7.9)$$

Therefore

$$\begin{aligned} \frac{1}{\pi} \int_{\mathcal{T}_U} \omega^2 &= \frac{1}{\pi} \int_{\mathbb{T}^*U \setminus \{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}} p'^* \omega^2 \\ &= \frac{1}{\pi} \int_{\mathbb{T}^*U \setminus \{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}} \omega_0^2 \\ &= \frac{2}{\pi} \int_{\{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}} \alpha_0 \wedge \omega_0 \\ &= \frac{2}{\pi} \int_{S^*U} \alpha_0 \wedge \omega_0. \end{aligned} \quad (7.10)$$

So the claim follows from (7.7), (7.8) and (7.10). \square

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