Primitive orthogonal idempotents for R-trivial monoids

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Abstract

We show that the notions of R-trivial monoid and weakly ordered monoid are equivalent. We use this fact to construct a complete system of orthogonal idempotents for all R-trivial monoids.

1 Introduction

Recently, Denton ([5], [6]) gave a construction for a complete system of orthogonal idempotents for the 0-*Hecke algebra* of type A, the first since the question was posed by Norton [9] in 1979. A complete system of orthogonal idempotents for *left regular bands* was found by Brown [3] and Saliola [10]. Finding such collections is an important problem in representation theory because they give a decomposition of the algebra into projective indecomposable modules: If $\{e_J\}_{J\in\mathfrak{J}}$ is such a collection for an algebra A, then $A = \oplus_{J\in\mathfrak{J}} Ae_J$ for indecomposable modules Ae_J. They also allow for the explicit computation of the quiver, the Cartan invariants, and the Wedderburn decomposition of the algebra (see [2], [1]).

Schocker [11] constructed a class of monoids, called *weakly ordered monoids*, to generalize 0-Hecke monoids and left regular bands, with the broader aim of finding a complete system of orthogonal idempotents for the corresponding monoid algebras.

A key step in being able to do so is recognizing that the notions of weakly ordered monoid and R-*trivial monoid* are one and the same. In Section 2, we fill out an outline of a proof that Steinberg [12] pointed out to us that connects the two concepts. In Section 3, we use the equivalence to provide a construction of a complete system of orthogonal idempotents for the resulting monoid algebras.

It should be noted that Denton, Hivert, Schilling and Thiéry [7] give a construction of a complete system of orthogonal idempotents for J*-trivial monoids*, which are a subclass of R-trivial monoids. Left regular bands, for example, are R-trivial but not necessarily J-trivial. In this paper, we give a uniform construction for all R-trivial monoids.

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2 Weakly ordered monoids and R-trivial monoids

Given any monoid T, that is, a set with an associative multiplication and an identity element, we can define a preorder $\leq$ on T as follows: Given $u, v \in T$, $u \leq v$ if there exists $w \in T$ such that $uw = v$. We write $u < v$ if $u \leq v$ but $u \neq v$. Unless stated otherwise, the monoids throughout the paper are endowed with this "weak" order.

Definition 2.1. *Let $(\mathcal{L}, \preceq)$ be an upper semilattice. A* **weakly ordered monoid** *W is a finite monoid with two maps $C, D : W \to \mathcal{L}$ such that*

1. *C is a surjection of monoids.*
2. *If $u, v \in W$ are such that $uv \leq u$, then $C(v) \preceq D(u)$.*
3. *If $u, v \in W$ are such that $C(v) \preceq D(u)$, then $uv = u$.*

Remark 2.2. *In Schocker's paper, he actually calls these weakly ordered semigroups. However our understanding is that monoids include a unity and semigroups do not. So throughout the paper we call these weakly ordered monoids.*

Definition 2.3. *We say that the monoid S is* ***R*-trivial** *if, for all $x, y \in S$, $xS = yS$ implies $x = y$.*

We restrict our discussion to *finite* R-trivial monoids.

Example 2.4. A monoid W is called a **left regular band** if $x^2 = x$ and $xyx = xy$ for all $x, y \in W$. Left-regular bands are R-trivial. Indeed, if $xW = yW$, then there exist $u, v \in W$ such that $xu = y$ and $x = yv$. But then, since $uv = uvu$,

$$x = yv = xuv = xuvu = yvu = xu = y.$$

Left regular bands are also weakly ordered monoids, see Shocker [11], Eg. 2.4 and Brown [3], Appendix B.

Example 2.5. Let G be a Coxeter group with simple generators $\{s_i : i \in I\}$ and relations:

- $s_i^2 = 1$,
- $\underbrace{s_i s_j s_i s_j \ldots}_{m_{ij}} = \underbrace{s_j s_i s_j s_i \ldots}_{m_{ij}}$ for positive integers m_{ij}.

Then the 0-Hecke monoid $H_n^G(0)$ has generators $\{T_i : i \in I\}$ and relations:

- $T_i^2 = T_i$,
- $\underbrace{T_i T_j T_i T_j \ldots}_{m_{ij}} = \underbrace{T_j T_i T_j T_i \ldots}_{m_{ij}}$ for positive integers m_{ij}.

Of particular interest is the case when G is the symmetric group $\mathfrak{S}_n$. Norton [9] gave a decomposition of the monoid algebra $\mathbb{C}H_n^{\mathfrak{S}_n}(0)$ into left ideals and classified its irreducible representations. She was not able to construct a complete system of orthogonal idempotents for the algebra. Denton [5] gave the first construction of a set of orthogonal idempotents for $\mathbb{C}H_n^{\mathfrak{S}_n}(0)$.

The weakly ordered monoid $H_n^{\mathfrak{S}_n}(0)$ has maps C and D onto the lattice of subsets of $\{1, \ldots, n-1\}$. The map C is the *content set* of an element, that is, if $x = T_{i_1} T_{i_2} \ldots T_{i_k}$, then $C(x)$ is the set containing $i_1, i_2, \ldots, i_k$. The map D is the subset of right descents of an element, that is, $xT_i = x$ if and only if $i \in D(x)$.

Example 2.6. Consider the 3×3–matrices over $\mathbb{Z}$ given by

$$g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let S be the monoid with identity generated by g_1 and g_2, that is, $S = \{1,\ g_1,\ g_2,\ g_1g_2,\ g_2g_1\}$. S is both an R-trivial monoid and a weakly ordered monoid. For example, we can take $\mathcal{L}$ be to be usual lattice of subsets of $\{1,2\}$, with $C : S \to \mathcal{L}$ given by

$$C(1) = \emptyset,\ C(g_1) = \{1\},\ C(g_2) = \{2\},\ C(g_1g_2) = C(g_2g_1) = \{1,2\},$$

and $D : S \to \mathcal{L}$ given by

$$D(1) = \emptyset,\ D(g_1) = \{1\},\ D(g_2) = D(g_1g_2) = \{2\},\ D(g_2g_1) = \{1,2\}.$$

S, however, is neither a left regular band, since g_1g_2 is not idempotent, nor isomorphic to the 0-Hecke monoid $\mathbb{C}H_3^{\mathfrak{S}_3}(0)$ on 2 generators, since the latter has 6 elements.

The fact that the above examples are both weakly ordered monoids and R-trivial monoids is no coincidence: these two notions are equivalent.

Proposition 2.7. *A monoid S is R-trivial if and only if the preorder $\leq$ defined above is a partial order.*

Proof. Suppose S is an R-trivial monoid and $x, y \in S$ are such that $x \leq y$ and $y \leq x$. Then there exist $u, v \in S$ such that $xu = y$ and $yv = x$. So $y \in xS$ and $x \in yS$, implying that $yS \subseteq xS$ and $xS \subseteq yS$. That is, $xS = yS$. Since S is R-trivial, $x = y$.

On the other hand, suppose that the given preorder is a partial order, and that $xS = yS$ for some $x, y \in S$. Since $x = x \cdot 1 \in xS = yS$, we have that $x = yu$ for some $u \in S$. So $y \leq x$. Similarly, $y \in xS$ implies that $x \leq y$. The antisymmetry of $\leq$ implies then that $x = y$. So S is R-trivial. □

Corollary 2.8. *A weakly ordered monoid is an R-trivial monoid.*

Proof. Let W be a weakly ordered monoid. Lemma 2.1 in [11] shows that the defining conditions of a weakly ordered monoid imply that the preorder on W is a partial order. The result now follows from Proposition 2.7. □

Let S be a finite R-trivial monoid. We will show that S is a weakly ordered monoid using an argument outlined by Steinberg [12].

We must establish the existence of an upper semi-lattice $\mathcal{L}$ and two maps C and D from S to $\mathcal{L}$ that satisfy the conditions of Definition 2.1. We gather here the definitions of $\mathcal{L}$, C and D:

1. $\mathcal{L}$ is the set of left ideals Se generated by idempotents e in S, ordered by reverse inclusion;

2. $C : S \to \mathcal{L}$ is defined as $C(x) = Sx^\omega$, where x^ω is the idempotent power of x (see Corollary 2.10);

3. $D : S \to \mathcal{L}$ is defined as $D(u) = C(e)$, where e is a maximal element in the set $\{s \in S : us = u\}$ (with respect to the preorder $\leq$).

The remainder of this section is dedicated to showing that these objects are well-defined and that they satisfy the conditions of Definition 2.1.

We begin by using Proposition 2.7 to show that the submonoid generated by any x in S stabilizes at a particular power of x.

Lemma 2.9. *For each $x \in S$, there exists a positive integer $\omega = \omega(x)$ such that $x^\omega x = x^\omega$.*

Proof. If $x = 1$, we may take ω to be 1. If $x \neq 1$, consider the set of positive integers $N = \{n : x^n = x^k \text{ for some } 0 \leq k < n\}$. Since the set $\{1, x, x^2, x^3, \ldots\}$ is a subset of S and S is finite, the set N is nonempty. Let $m + 1$ be the smallest member of N. Since $x \neq 1$, $m + 1 \geq 2$. The minimality of $m + 1$ tells us that $1, x, x^2, \ldots, x^m$ are distinct: Given our order, $1 < x < x^2 < \cdots < x^m$. If $x^{m+1} = x^k$, where $k < m$, then we would have that $x^m \leq x^k$ because $x^m x = x^{m+1} = x^k$, and $x^k \leq x^m$ because $x^k x^{m-k} = x^m$. But then Propostion 2.7 tells us that $x^k = x^m$, contradicting $x^k < x^m$. So x^{m+1} must be x^m and we may take ω to equal m. ☐

Consequently, every element in an R-trivial monoid has some power that is idempotent.

Corollary 2.10. *For each $x \in S$, there exists a positive integer $\omega = \omega(x)$ such that $\left(x^\omega\right)^2 = x^\omega$.*

Remark 2.11. *In what follows, if $x \in \mathbb{C}S$ and there exists an N such that $x^{N+1} = x^N$, we sometimes abuse notation by writing x^N instead of x^ω.*

The next technical lemma sets the groundwork needed to define the lattice $\mathcal{L}$ and the maps $C, D : S \to \mathcal{L}$.

Lemma 2.12. *Let S be a finite R-trivial monoid. For all x and y in S,*

1. $(xy)^\omega x = (xy)^\omega$
2. $(xy)^\omega x^\omega = (xy)^\omega$
3. $(x^\omega y^\omega)^\omega x^\omega = (x^\omega y^\omega)^\omega$
4. $(x^\omega y^\omega)^\omega = (x^\omega y^\omega)^\omega (xy)$
5. $(x^\omega y^\omega)^\omega = (x^\omega y^\omega)^\omega (xy)^\omega$

Proof. (1) Since $(xy)^\omega x \in (xy)^\omega S$, it follows that $(xy)^\omega xS \subseteq (xy)^\omega S$. To show the reverse inclusion, note that $(xy)^\omega = (xy)^\omega(xy) = \big((xy)^\omega x\big)y \in (xy)^\omega xS$, where the first equality follows from Lemma 2.9. So $(xy)^\omega S \subseteq (xy)^\omega xS$. Thus $(xy)^\omega xS = (xy)^\omega S$. Since S is an R-trivial monoid, the desired result follows.

(2) This follows from applying (1) repeatedly.

(3) Let $u = x^\omega$ and $v = y^\omega$. Now, by (1), $(uv)^\omega u = (uv)^\omega$.

(4) We compute:

$$\begin{aligned}
(x^\omega y^\omega)^\omega &= (x^\omega y^\omega)^{\omega-1} x^\omega y^\omega \\
&= (x^\omega y^\omega)^{\omega-1} x^\omega y^\omega y && \text{(by Lemma 2.9)} \\
&= (x^\omega y^\omega)^\omega y \\
&= (x^\omega y^\omega)^\omega x^\omega y && \text{(by (3))} \\
&= (x^\omega y^\omega)^\omega x^\omega x y && \text{(by Lemma 2.9)} \\
&= (x^\omega y^\omega)^\omega x y && \text{(by (3))}
\end{aligned}$$

(5) This follows by repeatedly applying part (4). □

We are now ready to construct a lattice corresponding to the R-trivial monoid S. Define

$$\mathcal{L} = \{Se : \ e \in S \text{ such that } e^2 = e\}.$$

That is, $\mathcal{L}$ is the set of left ideals generated by idempotents. Define a partial order on $\mathcal{L}$ by

$$Se \preceq Sf \iff Se \supseteq Sf.$$

Proposition 2.13. *If e and f are idempotents in S, then $S(ef)^\omega$ is the least upper bound of Se and Sf in $\mathcal{L}$.*

Proof. First, let us show that $S(ef)^\omega$ is an upper bound for Se and Sf. Since, by Lemma 2.12 (1), $(ef)^\omega = (ef)^\omega e$, we have that $(ef)^\omega \in Se$. Hence $S(ef)^\omega \subseteq Se$ and $S(ef)^\omega \succeq Se$. Moreover, $(ef)^\omega = \Big((ef)^{\omega-1}e\Big)f \in Sf$. So

$S(ef)^\omega \subseteq Sf$ and $S(ef)^\omega \succeq Sf$. So $S(ef)^\omega$ is an upper bound for Se and Sf.

Next, let us show that $S(ef)^\omega$ is the least upper bound for Se and Sf. Suppose g is an idempotent in S such that Sg is an upper bound for Se and Sf. That is, $Sg \subseteq Se$ and $Sg \subseteq Sf$. Since $Sg \subseteq Se$, $g = te$ for some $t \in S$. But then $ge = (te)e = te^2 = te = g$. Similarly, $Sg \subseteq Sf$ implies that $gf = g$. So $g(ef) = (ge)f = gf = g$ and it follows that

$$g = g(ef) = \Big(g(ef)\Big)(ef) = g(ef)^2 = \Big(g(ef)\Big)(ef)^2 = g(ef)^3 = \cdots = g(ef)^\omega.$$

Consequently, $g \in S(ef)^\omega$, $Sg \subseteq S(ef)^\omega$, and $Sg \succeq S(ef)^\omega$. So $S(ef)^\omega$ is the least upper bound of Se and Sf. $\square$

Hence, we may define the join of two elements Se and Sf in $\mathcal{L}$ by

$$Se \vee Sf = S(ef)^\omega.$$

That is, $\mathcal{L}$ is an upper semilattice with respect to this join operation.

Define a map $C : S \to \mathcal{L}$ by $C(x) = Sx^\omega$.

Proposition 2.14. *C is a surjective monoid morphism.*

Proof. Let $x, y \in S$. By Lemma 2.12 (5), we know that $(x^\omega y^\omega)^\omega = (x^\omega y^\omega)^\omega (xy)^\omega$. Hence, $(x^\omega y^\omega)^\omega \in S(xy)^\omega$ and $S(x^\omega y^\omega)^\omega \subseteq S(xy)^\omega$.

To show the reverse inclusion, we begin by noting that, by Lemma 2.12 (2), $(xy)^\omega = (xy)^\omega x^\omega$. So $(xy)^\omega \in Sx^\omega$ and $S(xy)^\omega \subseteq Sx^\omega$. That is, $S(xy)^\omega \succeq Sx^\omega$. Also, by using Lemmas 2.9 and 2.12 (1), we have

$$\begin{aligned}
(xy)^\omega &= (xy)^\omega (xy) \\
&= \Big((xy)^\omega x\Big) y = (xy)^\omega y \\
&= \Big((xy)^\omega (xy)\Big) y \\
&= \Big((xy)^\omega x\Big) y^2 = (xy)^\omega y^2 \\
&= \cdots \\
&= \Big((xy)^\omega x\Big) y^\omega = (xy)^\omega y^\omega.
\end{aligned}$$

So $(xy)^\omega \in Sy^\omega$, which implies that $S(xy)^\omega \subseteq Sy^\omega$ and $S(xy)^\omega \succeq Sy^\omega$. In particular, $S(xy)^\omega$ is an upper bound for both Sx^ω and Sy^ω. So $S(xy)^\omega \succeq Sx^\omega \vee Sy^\omega = S(x^\omega y^\omega)^\omega$, that is, $S(xy)^\omega \subseteq S(x^\omega y^\omega)^\omega$.

Thus $C(xy) = S(xy)^\omega = S(x^\omega y^\omega)^\omega = Sx^\omega \vee Sy^\omega = C(x) \vee C(y)$, and C is a monoid morphism. Finally, we know that every element of $\mathcal{L}$ is of the form Se for some idempotent e in S. But then $C(e) = Se^\omega = Se$; that is, C is a surjective morphism. □

Given $x \in S$, we defined $C(x)$ to be the left ideal of S generated by x^ω. Here is an alternate characterization of $C(x)$:

Proposition 2.15. *Given $x \in S$, $C(x) = \{a \in S \,:\, ax = a\}$.*

Proof. Take an arbitrary element in Sx^ω, say tx^ω. Since $\big(tx^\omega\big)x = t\big(x^\omega x\big) = tx^\omega$ by Lemma 2.9, we see that $tx^\omega \in \{a \in S \,:\, ax = a\}$. On the other hand, take $b \in \{a \in S \,:\, ax = a\}$. Then

$$bx^\omega = (bx)x^{\omega-1} = bx^{\omega-1} = (bx)x^{\omega-2} = bx^{\omega-2} = \cdots = bx = b.$$

So $b \in Sx^\omega$. □

We now define a map $D : S \to \mathcal{L}$. Given $u \in S$, let $D(u) = C(e)$, where e is a maximal element in the set $\{s \in S \,:\, us = u\}$.

To check whether the map D is well-defined, let e and f be two distinct maximal elements in $\{s \in S \,:\, us = u\}$. Since $e \leq ef$ and $u(ef) = (ue)f = uf = u$, by the maximality of e, $e = ef$. Similarly, since $f \leq fe$ and $u(fe) = u$, the maximality of f implies $f = fe$. But then, by Proposition 2.14,

$$C(e) = C(ef) = C(e) \vee C(f) = C(f) \vee C(e) = C(fe) = C(f).$$

Note that the maximality of e and $ue^2 = u$ also implies that $e = e^2$, that is, e is idempotent.

The next proposition shows that the C and D maps on S interact in precisely the manner given in conditions 2 and 3 in Definition 2.1 of a weakly ordered monoid. The following lemma will help us prove this proposition.

Lemma 2.16. *Let $x, y \in S$. If $x \leq y$, then $C(x) \preceq C(y)$.*

Proof. Take $s \in C(y)$. Then $sy = s$. Since $x \leq y$, there exists $t \in S$ such that $y = xt$. So $sxt = s$ implying $sx \leq s$. But $s \leq sx$. Since, by Proposition 2.7, the order on S is a partial order, $sx = s$. That is, $s \in C(x)$. Hence $C(y) \subseteq C(x)$, that is, $C(x) \preceq C(y)$. □

Proposition 2.17. *Let $u, v \in S$.*

(i) If $uv \leq u$, then $C(v) \preceq D(u)$.

(ii) If $C(v) \preceq D(u)$, then $uv = u$.

Proof. (i) Since $u \leq uv$, by Proposition 2.7, $u = uv$. Hence v lies in the set $\{s \in S \ : \ us = u\}$. Let e be a maximal element in this set such that $v \leq e$. Then, by Lemma 2.16, $C(v) \preceq C(e) = D(u)$.

(ii) By definition, $D(u) = C(e)$, where e is a maximal element of $\{s \in S \ : \ us = u\}$. So if $C(v) \preceq D(u)$, then $C(v) \preceq C(e)$. Hence $C(e) \subseteq C(v)$. Since $ue = u$, u lies in $C(e)$. So u is also a member of $C(v)$; that is, $uv = u$. □

Propositions 2.14 and 2.17 tell us that an R-trivial monoid is a weakly ordered monoid. Combining this with Corollary 2.8, we have the following result.

Theorem 2.18. *A monoid W is a weakly ordered monoid if and only if it is an R-trivial monoid.*

3 Constructing idempotents

We begin this section with a small technical lemma about R-trivial monoids. The proof is rather trivial, but we use it often enough in proofs to justify stating it at the onset.

Lemma 3.1. *Suppose W is an R-trivial monoid. If $x, y, z \in W$ are such that $xyz = x$, then $xy = x$.*

Consequently, if $x, y_1, y_2, \ldots, y_m \in W$ are such that $xy_1y_2 \ldots y_m = x$, then $xy_i = x$ for all $1 \leq i \leq m$.

Proof. If $xyz = x$ then $xyW = xW$. Therefore $xy = x$ by the definition of W being R-trivial. The second statement immediately follows from the first. □

Definition 3.2. *Let A be an algebra. Let $\Lambda = \{e_J : J \in \mathcal{I}\}$ be a set of nonzero elements of A. We say that Λ is a* **complete system of orthogonal idempotents for** *A if:*

1. *e_J is idempotent, that is, $e_J^2 = e_J$ for all $J \in \mathcal{I}$;*

2. *e_J is orthogonal to e_K, that is, $e_J e_K = 0$ for $J, K \in \mathcal{I}$ with $J \neq K$; and*

3. *the collection Λ is a maximal set of nonzero elements with properties 1 and 2.*

Remark 3.3. *A collection of nonzero elements that satisfies 1, 2 and 3 in the above defintion will also satisfy the following two conditions:*

- *e_J is primitive for all $J \in \mathcal{I}$, that is, if $e_J = x + y$, where x and y are idempotent and $xy = yx = 0$, then either x or y is zero; and*
- $\displaystyle\sum_{J \in \mathcal{I}} e_J = 1.$

To see primitive, just note that if e_J can be written as $x + y$, then we could replace e_J in Λ with x and y, contradicting the maximality of Λ. To see the second condition, we just note that if $\sum_J e_J \neq 1$, then $1 - \sum_J e_J$ is idempotent and orthogonal to all other e_J. Combining this element with Λ would again contradict the maximality of Λ.

Let W denote a weakly ordered monoid with C and D being the associated "content" and "descent" maps from W to an upper semi-lattice $\mathcal{L}$. We let $\mathcal{G}$ denote a set of generators of W. The main goal of this paper is to build a method for finding a complete system of orthogonal idempotents for the monoid algebra $\mathbb{C}W$. In particular, this solves the problem posed by Norton about the 0-Hecke algebra for the symmetric group.

For each $J \in \mathcal{L}$, we define a **Norton element** $A_J T_J$. Let us begin by defining T_J.

For each $J \in \mathcal{L}$, let

$$T_J = \Big(\prod_{\substack{g \in \mathcal{G} \\ C(g) \preceq J}} g^\omega \Big)^\omega \in W.$$

Remark 3.4. *A different ordering of the set $\mathcal{G}$ of generators may produce different T_J's; so we fix an (arbitrarily chosen) order.*

We now define the A_J in the "Norton element" $A_J T_J$. First we let

$$B_J = \prod_{\substack{g \in \mathcal{G} \\ C(g) \not\preceq J}} (1 - g^\omega) \in \mathbb{C}W.$$

We would like to raise B_J to a high enough power to make it idempotent. However, $B_J \notin W$, so B_J^ω may not be well defined. The following lemma and corollary resolve this problem.

Definition 3.5. *Given $x \in \mathbb{C}W$ if $x = \sum_{w \in W} c_w w$, then the **coefficient** of w in x is c_w. We say w is a **term** of x if the coefficient of w in x is nonzero.*

Lemma 3.6. *Let $b \in W$ and suppose $bx^\omega = b$ for some $x \in \mathcal{G}$ with $C(x) \not\leq J$.*

1. *Then the coefficient of b in bB_J is zero; and*
2. *if c is a term of bB_J, then $c > b$.*

Proof. Let $\mathcal{D} = \{x^\omega : x \in \mathcal{G}, C(x) \not\leq J, bx^\omega = b\}$. By assumption $\mathcal{D}$ is not empty. Let $g_1, g_2, \ldots, g_m$ be the generators which appear in the definition of B_J. Then

$$B_J = \sum_{i_1 < i_2 < \cdots < i_k} (-1)^k g_{i_1}^\omega g_{i_2}^\omega \cdots g_{i_k}^\omega.$$

It follows from Lemma 3.1 that the coefficient of b in bB_J is counting the terms in B_J where each of $g_{i_1}, \ldots, g_{i_k}$ come from $\mathcal{D}$, weighted with sign $(-1)^k$. If $|\mathcal{D}| = n \geq 1$ then this is $1 - n + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n = 0$.

The second statement follows from the first and the definition of order, as every term c of bB_J must be of the form $c = bm$ for some term m appearing in B_J, and hence $c \geq b$. $\square$

Lemma 3.7. *For every $J \in \mathcal{L}$, there exists an integer N such that $y^\omega B_J^N = 0$ for all $y \in \mathcal{G}$ with $C(y) \not\leq J$.*

Proof. Let $N = \ell + 1$, where ℓ is the length of the longest chain of elements in the poset $(W, \leq)$.

Suppose $y^\omega B_J^N \neq 0$. Let c_N be a term of B_J^N. Then c_N is a term of $c_{N-1}B_J$ for some term c_{N-1} in $y^\omega B_J^{N-1}$. Since $y^\omega y^\omega = y^\omega$, Lemma 3.6 (1) implies that y^ω is not a term of $y^\omega B_J^k$ for any $k \geq 1$. Hence, $c_{N-1} = y^\omega g_1^\omega \cdots g_m^\omega$ for some $m \geq 1$ and $g_i \in \mathcal{G}$ with $C(g_i) \not\leq J$. In particular, $c_{N-1} g_m^\omega = c_{N-1}$, and so, by Lemma 3.6 (2), $c_N > c_{N-1}$.

Repeated application of this argument produces a decreasing chain

$$c_N > c_{N-1} > c_{N-2} > \cdots > c_1$$

of N elements in W, contradicting the fact that the length of the longest chain of elements in $(W, \leq)$ is $\ell < N$. $\square$

Corollary 3.8. *For every $J \in \mathcal{L}$ there exists an N such that $B_J^{N+1} = B_J^N$.*

Proof. By Lemma 3.7, $(B-1)B^N = 0$ for a sufficiently large N since every element of $B-1$ is of the form y^ω where $y \in \mathcal{G}$ and $C(y) \not\preceq J$. □

This now allows us to define $A_J = B_J^\omega$.

Lemma 3.9. *Let $J \in \mathcal{L}$. Then:*

1. *$T_J x = T_J$ for all x such that $C(x) \preceq J$;*

2. *$y^\omega A_J = 0$ for all y such that $C(y) \not\preceq J$ and $y \in \mathcal{G}$.*

Proof. Since $J = C(T_J)$, $C(x) \preceq J$ implies $C(x) \supseteq C(T_J)$. We also know that $T_J \in C(T_J)$ because T_J is idempotent. So $T_J \in C(x)$, that is, $T_J x = T_J$.

The second part follows from Lemma 3.7 since $A = B^N$. □

Remark 3.10. *Although T_J and A_J are idempotents individually, their product, the "Norton element" z_J, need not be. For example, take the 0-Hecke algebra $H_6(0)$ corresponding to the symmetric group $\mathfrak{S}_6$. Let J be the subset $\{1,4,5\}$ of $\{1,2,3,4,5\}$. Then $T_J = T_1T_4T_5T_4$, $A_J = (1-T_2)(1-T_3)(1-T_2)$ and z_J is their product. No power of z_J is idempotent.*

Recall that $z_J = A_J T_J$.

Lemma 3.11. *The coefficient of T_J in z_J is 1. All other terms y in z_J have $C(y) \succ J$.*

Proof. The coefficient of the identity element 1 in A_J is 1. Each term of $A_J T_J$ is of the form aT_J for a term a of A_J. If $a \neq 1$, then $C(a) \not\preceq J$ so $C(aT_J) = C(a) \vee C(T_J) \succ C(T_J) = J$. Hence the coefficient of T_J in $A_J T_J$ is 1 and all other terms have content greater than J. □

Lemma 3.12. *If $J \not\preceq K$ then $z_J z_K = 0$.*

Proof. Since $J \not\preceq K$, there exists a $g \in \mathcal{G}$ with $C(g) \preceq J$ but $C(g) \not\preceq K$. Expanding the product

$$z_J z_K = A_J T_J A_K T_K.$$

We will show $T_J A_K = 0$. By Lemma 3.9 (1), $T_J g^\omega = T_J$ and by Lemma 3.9 (2), $(1-g^\omega)A_J = A_J$ or $g^\omega A_J = 0$. Hence $T_J A_J = T_J g^\omega A_J = 0$. □

Definition 3.13. *Let $J \in \mathcal{L}$. Let*

$$P_J := \sum_{n,m \geq 0} (1 - z_J)^{n+m} z_J^2 = \sum_{k \geq 0} (k+1)(1 - z_J)^k z_J^2.$$

(Lemma 3.15 shows there are only finitely many terms in this summation.)

Remark 3.14. *A monoid S is called J-trivial if $SxS = SyS$ implies $x = y$ for all $x, y \in S$. When S is J-trivial it suffices to define*

$$P_K = \sum_{n \geq 0} (1 - z_K)^n z_K.$$

The next result shows that the sum in the definition of P_J contains only finitely many summands, and so P_J is a well-defined element of $\mathbb{C}W$ for each $J \in \mathcal{L}$.

Lemma 3.15. *For all $J \in \mathcal{L}$, there exists an N such that $(1 - z_J)^N z_J^2 = 0$.*

Proof. To simplify the notation, let us temporarily set $T = T_J$, $A = A_J$ and $z = z_J = AT$. We first note that for any integer $k \geq 0$,

$$\begin{aligned}(1-z)^k z^2 &= z(1-z)^k z \\ &= AT(1-AT)^k AT \\ &= A(T(1-A)T)^k AT.\end{aligned}$$

We will show that $(T(1-A)T)^N A = 0$ for $N > \ell$, where ℓ is the length of the longest chain in the poset $(W, \leq)$.

Let us write $1 - A = \sum_{a \in W} c_a a$ where each term has $c_a \neq 0$ only if $a = g_1^\omega \cdots g_k^\omega$ with $C(g_i) \not\leq J$ for all i. Therefore

$$T(1-A)T = \sum_{a \in W} c_a TaT = \sum_{\substack{a \in W \\ TaT = Ta}} c_a Ta \; + \sum_{\substack{a \in W \\ TaT \neq Ta}} c_a TaT.$$

Note that $c_1 = 0$ since 1 is not a term of $(1 - A)$. If $TaT = Ta$, then we have

$$TaT \cdot (T(1-A)T) = Ta(1-A)T = Ta - TaAT = Ta$$

since $aA = 0$ by Lemma 3.9. Thus,

$$(T(1-A)T)^N = \left(\sum_{\substack{a_1 \in W \\ Ta_1 T = Ta_1}} c_{a_1} Ta_1 + \sum_{\substack{a_1 \in W \\ Ta_1 T \neq Ta_1}} c_{a_1} Ta_1 T \right) (T(1-A)T)^{N-1}$$

$$= \sum_{\substack{a_1 \in W \\ Ta_1 T = Ta_1}} c_{a_1} Ta_1 + \left(\sum_{\substack{a_1 \in W \\ Ta_1 T \neq Ta_1}} c_{a_1} Ta_1 T \right) (T(1-A)T)^{N-1}.$$

Next, rewrite the second summand above using the same argument:

$$\left(\sum_{\substack{a_1 \in W \\ Ta_1 T \neq Ta_1}} c_{a_1} Ta_1 T \right) (T(1-A)T)^{N-1}$$

$$= \left(\sum_{\substack{a_1 \in W \\ Ta_1 T \neq Ta_1}} c_{a_1} Ta_1 T \right) \left(\sum_{a_2 \in W} c_{a_2} Ta_2 T \right) (T(1-A)T)^{N-2}$$

$$= \left(\sum_{\substack{a_1, a_2 \in W \\ Ta_1 T \neq Ta_1}} c_{a_1} c_{a_2} Ta_1 Ta_2 T \right) (T(1-A)T)^{N-2}$$

$$= \sum_{\substack{Ta_1 T \neq Ta_1 \\ Ta_1 Ta_2 T = Ta_1 Ta_2}} c_{a_1} c_{a_2} Ta_1 Ta_2$$

$$+ \left(\sum_{\substack{Ta_1 T \neq Ta_1 \\ Ta_1 Ta_2 T \neq Ta_1 Ta_2}} c_{a_1} c_{a_2} Ta_1 Ta_2 T \right) (T(1-A)T)^{N-2}.$$

Continuing in this way, we can write $(T(1-A)T)^N$ in the form

$$(T(1-A)T)^N = \left(\sum c_{a_1} Ta_1 + \cdots + \sum c_{a_1} \cdots c_{a_N} Ta_1 \cdots Ta_N \right)$$

$$+ \sum_{\substack{Ta_1 \cdots Ta_i T \neq Ta_1 \cdots Ta_i \\ 1 \leq i \leq N}} c_{a_1} \cdots c_{a_N} Ta_1 \cdots Ta_N T.$$

By Lemma 3.9, we have $a_i A = 0$ for all terms a_i in $1 - A$, and so

$$(T(1-A)T)^N \cdot A = \left(\sum_{\substack{Ta_1 \cdots Ta_i T \neq Ta_1 \cdots Ta_i \\ 1 \leq i \leq N}} c_{a_1} \cdots c_{a_N} Ta_1 \cdots Ta_N T \right) A.$$

This summation is 0 as it ranges over an empty set: indeed, if it is not empty, we would have an increasing chain of length $N > \ell$, namely

$$Ta_1 \; < \; Ta_1 Ta_2 \; < \; Ta_1 Ta_2 Ta_3 \; < \; \cdots \; < \; Ta_1 Ta_2 \cdots Ta_N,$$

Therefore, $(T(1-A)T)^N A = 0$. ☐

Lemma 3.16. *The coefficient of T_J in P_J is 1 and all other terms y of P_J have $C(y) \succ J$.*

Proof. If $n + m > 0$ then

$$A_J T_J A_J T_J (1 - A_J T_J)^{n+m} = A_J T_J A_J (T_J - T_J A_J T_J)^{n+m}.$$

Each term x in $(T_J - T_J A_J T_J)^{n+m}$ has $C(x) \succ J$, so no T_J appears in $z_J^2 (1 - z_J)^{n+m}$. The coefficient of T_J in z_J is 1, by Lemma 3.11. Hence T_J appears in $z_J^2 (1 - z_J)^0$ with coefficient 1. By Lemma 3.11, since all of the terms $y \neq T_J$ of z_J have $C(y) \succ J$ and P_J is a polynomial in z_J, all other terms w of P_J must have $C(w) \succ J$. ☐

Lemma 3.17. *As polynomials in x,*

$$x \sum_{n=0}^{N} (1-x)^n = 1 - (1-x)^{N+1},$$

for any nonnegative integer N.

Proof. Induct on N. ☐

Proposition 3.18. *For each $J \in \mathcal{L}$, the element P_J is idempotent.*

Proof. Let $J \in \mathcal{L}$ be fixed and let N be such that $(1-z_J)^N z_J^2 = 0$. Let us temporarily denote z_J by z. We can use Lemma 3.17 to rewrite P_J as

$$\begin{aligned}
P_J &= \sum_{n,m \geq 0} z^2 (1-z)^{n+m} \\
&= \sum_{n=0}^{N} \sum_{m=0}^{N-n} z^2 (1-z)^{n+m} \\
&= \sum_{n=0}^{N} (1-z)^n \left(z^2 \sum_{m=0}^{N-n} (1-z)^m \right) \\
&= \sum_{n=0}^{N} (1-z)^n \left(z - z(1-z)^{N-n+1} \right) \\
&= z \left(\sum_{n=0}^{N} (1-z)^n \right) - (N+1) z (1-z)^{N+1} \\
&= 1 - (1-z)^{N+1} - (N+1) z (1-z)^{N+1}.
\end{aligned}$$

This implies that $z^2 P_J = z^2$ since $z^2(1-z)^{N+1} = 0$, and so

$$P_J^2 = \left(\sum_{n=0}^{N} \sum_{m=0}^{N-n} (1-z)^{n+m} z^2 \right) P_J = \sum_{n=0}^{N} \sum_{m=0}^{N-n} (1-z)^{n+m} z^2 = P_J. \qquad \square$$

Lemma 3.19. *For all $J, K \in \mathcal{L}$, with $J \not\preceq K$, $P_J P_K = 0$.*

Proof. This is implied by Lemma 3.12 and the fact that P_J is a polynomial in z_J with no constant term. $\square$

Definition 3.20. *For each $J \in \mathcal{L}$, let $e_J := P_J \left(1 - \sum_{K \succ J} e_K \right)$.*

Lemma 3.21. *T_J occurs in e_J with coefficient 1. All other terms y of e_J have $C(y) \succ J$. In particular, $e_J \neq 0$.*

Proof. We proceed by induction. If J is maximal, then $e_J = P_J$, so the statement is implied by Lemma 3.16.

Now suppose the statement is true for all $M \succ J$. Then $e_J = P_J(1 - \sum_{M \succ J} e_M)$. By induction, all terms x of e_M have $C(x) \succeq M \succ J$. So terms y from $P_J e_M$ have $C(y) \succeq M \succ J$. The only other terms are those from P_J, for which the statement was proved in Lemma 3.16. $\square$

Lemma 3.22. *$e_K P_J = 0$ for $K \not\preceq J$.*

Proof. The proof is by a downward induction on the semilattice. If K is maximal, then $e_K = P_K$, so by Lemma 3.19, $e_K P_J = P_K P_J = 0$.

Now suppose that for every $L \succ K$, $e_L P_J = 0$ for $L \not\preceq J$, and we will show that $e_K P_J = 0$ for $K \not\preceq J$. We expand $e_K P_J$:

$$e_K P_J = P_K \left(1 - \sum_{L \succ K} e_L\right) P_J = P_K P_J - \sum_{L \succ K} P_K e_L P_J.$$

Since $K \not\preceq J$, we have $P_K P_J = 0$ by Lemma 3.19, and $e_L P_J = 0$ by induction, since $L \succ K$ and $K \not\preceq J$ implies $L \not\preceq J$. ☐

Corollary 3.23. *e_J is idempotent.*

Proof. We expand $e_J e_J$:

$$\begin{aligned} e_J e_J &= P_J \left(1 - \sum_{M \succ J} e_M\right) P_J \left(1 - \sum_{M \succ J} e_M\right) \\ &= P_J \left(P_J - \sum_{M \succ J} e_M P_J\right) \left(1 - \sum_{M \succ J} e_M\right) \\ &= P_J^2 \left(1 - \sum_{M \succ J} e_M\right) && \text{(by Lemma 3.22)} \\ &= P_J \left(1 - \sum_{M \succ J} e_M\right) && \text{(by Lemma 3.18)} \\ &= e_J \end{aligned}$$

☐

Lemma 3.24. *$e_J e_K = 0$ for $J \neq K$.*

Proof. The proof is by downward induction on the lattice $\mathcal{L}$. For a maximal element $M \in \mathcal{L}$, $e_M = P_M$, so $e_M e_K = P_M P_K (1 - \sum e_L) = 0$ by Lemma 3.19. Now suppose that for all $M \succ J$, $e_M e_K = 0$ for $M \neq K$ and we will show that $e_J e_K = 0$ for $J \neq K$. We expand $e_J e_K$:

$$e_J e_K = P_J (1 - \sum_{L \succ J} e_L) e_K = P_J (e_K - \sum_{L \succ J} e_L e_K) \tag{1}$$

If $K \not\succ J$, then $\sum_{L \succ J} e_L e_K = 0$ by our induction hypothesis, so $P_J(e_K - \sum_{L \succ J} e_L e_K) = P_J e_K = P_J P_K(1 - \sum_{M \succ K} e_M) = 0$ by Lemma 3.19.

If $K \succ J$, then $\sum_{L \succ J} e_L e_K = e_K$ since e_K is idempotent and $e_L e_K = 0$ for $L \neq K$ by the inductive hypothesis. Therefore $e_K - \sum_{L \succ J} e_L e_K = 0$ and hence the right hand side of (1) is zero. □

Theorem 3.25. *The set $\{e_J : J \in \mathcal{L}\}$ is a complete collection of orthogonal idempotents for $\mathbb{C}W$.*

Proof. From [11], we know that the maximal number of such idempotents is the cardinality of $\mathcal{L}$. The rest of the claim is just Lemma 3.21, Corollary 3.23 and Lemma 3.24. □

Appendix: Two examples

We show by example how to use the above construction to create orthogonal idempotents for R-trivial monoids.

Idempotents of the free left regular band on two generators

Let W be the left regular band freely generated by two elements a, b. Then $W = \{1, a, b, ab, ba\}$. All elements of W are idempotent. Also $aba = ab$ and $bab = ba$. The lattice $\mathcal{L}$ has four elements: $\emptyset := W, \mathfrak{a} := Wa, \mathfrak{b} := Wb$ and $\mathfrak{ab} := Wab = Wba$, where $\emptyset \prec \mathfrak{a} \prec \mathfrak{ab}$ and $\emptyset \prec \mathfrak{b} \prec \mathfrak{ab}$, but $\mathfrak{a}$ and $\mathfrak{b}$ have no relation.

When $J = \emptyset$, neither of the generators satisfies $C(g) \preceq J$, so $T_\emptyset = 1 \in W$. $B_\emptyset = (1-a)(1-b)$. Also

$$\begin{aligned}
B_\emptyset^2 &= (1-a)(1-b)(1-a)(1-b) \\
&= (1-a-b+ab)(1-a)(1-b) \\
&= (1-a-b+ab)(1-b) \\
&= (1-a-b+ab) \\
&= B_\emptyset.
\end{aligned}$$

Therefore $A_\emptyset = B_\emptyset = 1-a-b+ab$, so $z_\emptyset = 1-a-b+ab$. Therefore $z_\emptyset$ is idempotent, so $P_\emptyset = 1-a-b+ab$.

When $J = \mathfrak{a}$, then $C(a) \preceq \mathfrak{a}$ and $C(b) \not\preceq \mathfrak{a}$, so $T_\mathfrak{a} = a$ and $B_\mathfrak{a} = 1-b = A_\mathfrak{a}$ since $1-b$ is idempotent. Therefore $z_\mathfrak{a} = (1-b)a = a-ba$. $z_\mathfrak{a}^2 = a-ab$ and one can check that $z_\mathfrak{a}^3 = z_\mathfrak{a}^2$, so $P_\mathfrak{a} = z_\mathfrak{a}^2(1+(1-z_\mathfrak{a})+(1-z_\mathfrak{a})^2+\dots) = z_\mathfrak{a}^2 = a-ab$. One can check that $P_\mathfrak{a}$ is idempotent.

Similarly, $P_\mathfrak{b} = b - ba$.

When $J = \mathfrak{ab}$, $C(a), C(b) \preceq \mathfrak{ab}$, so $T_{\mathfrak{ab}} = ab$ and $A_{\mathfrak{ab}} = 1$. $z_{\mathfrak{ab}} = ab$ is idempotent, so $P_{\mathfrak{ab}} = ab$. Since $\mathfrak{ab}$ is maximal, $e_{\mathfrak{ab}} = ab$.

Since $P_\mathfrak{a} e_{\mathfrak{ab}} = (a-ab)ab = ab - ab = 0$, $e_\mathfrak{a} = P_\mathfrak{a}(1-e_{\mathfrak{ab}}) = P_\mathfrak{a} = a - ab$.

Similarly, $e_\mathfrak{b} = b - ba$.

$P_\emptyset e_\mathfrak{a} = (1-a-b+ab)(a-ab) = 0$. Similarly, $P_\emptyset e_\mathfrak{b} = 0$. However, $P_\emptyset e_{\mathfrak{ab}} = (1-a-b+ab)ab = ab - ab - ba + ab = ab - ba$. So we let $e_\emptyset = P_\emptyset(1-e_{\mathfrak{ab}}) = P_\emptyset - P_\emptyset e_{\mathfrak{ab}} = 1-a-b+ab-ab+ba = 1-a-b+ba$.

One can check that $\{e_\emptyset, e_\mathfrak{a}, e_\mathfrak{b}, e_{\mathfrak{ab}}\}$ is a collection of mutually orthogonal idempotents.

Idempotents of $H^{\mathfrak{S}_5}(0)$

As mentioned above, $H^{\mathfrak{S}_5}(0)$ has generators T_1, T_2, T_3, T_4. In this case, the corresponding lattice is the lattice of subsets of $\{1,2,3,4\}$. $H^{\mathfrak{S}_5}(0)$ is actually a J-trivial monoid, so we can use the simplified formula from Remark 3.14. We use the shorthand notation $T(i_1, \dots, i_k)$ to denote the element $T_{i_1} \dots T_{i_k}$.

If $J = \{1,2,3,4\}$, then $T_J = T(1,2,3,4)^\omega = T(1,2,3,4,1,2,3,1,2,1)$. Also $A_J = 1$, so $z_J = A_J T_J = T_J$. Also, $P_J = z_J$, and since J is maximal, $e_J = P_J$, so

$$e_{\{1,2,3,4\}} = T(1,2,3,4,1,2,3,1,2,1).$$

If $J = \{1,2,3\}$, then $T_J = T(1,2,3,1,2,1)$ and $A_J = 1 - T(4)$. Then $z_J = (1-T(4))T(1,2,3,1,2,1) = T(1,2,3,1,2,1) - T(4,1,2,3,1,2,1)$. One can check that $z_J^2 = z_J$, so $P_J = z_J$. Also, one can check that P_J is orthogonal to $e_{\{1,2,3,4\}}$. So $e_J = P_J$. Therefore

$$e_{\{1,2,3\}} = T(1,2,3,1,2,1) - T(4,1,2,3,1,2,1).$$

Similarly,

$$e_{\{2,3,4\}} = T(2,3,4,2,3,2) - T(1,2,3,4,2,3,2).$$

Now let $J = \{1,2,4\}$. $T_J = T(1,2,1,4)$ and $A_J = (1-T(3))$. Letting $z_J = A_J T_J$, one can check that $z_J(1-z_J)^2 = 0$, so $P_J = z_J(1+(1-z_J))$.

Again P_J is orthogonal to $e_{\{1,2,3,4\}}$, so $e_J = P_J$. Therefore

$e_{\{1,2,4\}} = -T(1,2,3,4,2,3,1,2,1) + T(1,2,3,4,3,1,2,1) - T(3,4,1,2,1) + T(4,1,2,1)$.

Similarly,

$e_{\{1,3,4\}} = -T(1,2,3,4,1,2,3,2,1) + T(1,2,3,4,2,3,2,1) - T(2,3,4,3,1) + T(3,4,3,1)$.

When $J = \{1,2\}$, $T_J = T(1,2,1)$ and $A_J = (1-T(3))(1-T(4))(1-T(3))$. Then z_J is already idempotent, so $P_J = z_J$. One can check that P_J is already orthogonal to $e_{\{1,2,3,4\}}, e_{\{1,2,3\}}, e_{\{1,2,4\}}$. Therefore,

$e_{\{1,2\}} = -T(3,4,3,1,2,1) + T(3,4,1,2,1) + T(4,3,1,2,1) - T(3,1,2,1) - T(4,1,2,1) + T(1,2,1)$.

Similarly,

$e_{\{3,4\}} = -T(1,2,3,4,3,1) + T(1,2,3,4,3) + T(2,3,4,3,1) - T(3,4,3,1) - T(2,3,4,3) + T(3,4,3)$.

If $J = \{1,3\}$, $T_J = T_1T_3$ and $A_J = (1-T_2)(1-T_4)$. One can check that $z_J(1-z_J)^2 = 0$, and $P_J = z_J(1+1-z_J)$ is idempotent. P_J is orthogonal to $e_{\{1,2,3,4\}}$ and $e_{\{1,2,3\}}$, but not orthogonal to $e_{\{1,2,4\}}$. So we define $e_{\{1,3\}} = P_{\{1,3\}}(1-e_{\{1,2,4\}})$. Then

$e_{\{1,3\}} = T(1,2,3,4,1,2,3,2,1) - T(1,2,3,4,1,2,3,1) - T(1,2,3,4,2,3,2,1) + T(1,2,3,4,2,3,1) - T(2,3,4,1,2,3,2,1) + T(2,3,4,1,2,3,1) + T(4,1,2,3,1,2,1) - T(1,2,3,1,2,1) + T(3,4,1,2,3,2,1) - T(3,4,1,2,3,1) - T(4,1,2,3,2,1) + T(1,2,3,2,1) + T(4,2,3,1) - T(2,3,1) - T(4,3,1) + T(3,1)$.

Similarly,

$e_{\{2,4\}} = T(1,2,3,4,2,3,1,2,1) - T(1,2,3,4,2,3,1,2) - T(1,2,3,4,3,1,2,1) + T(1,2,3,4,3,1,2) + T(1,2,3,4,2,3,2) - T(1,2,3,4,3,2) - T(2,3,4,2,3,1,2,1) + T(2,3,4,2,3,1,2) + T(2,3,4,3,1,2,1) - T(2,3,4,3,1,2) + T(3,4,1,2) - T(4,1,2) - T(2,3,4,2,3,2) + T(2,3,4,3,2) - T(3,4,2) + T(4,2)$.

We continue in this way, constructing all of the idempotents for the algebra. For the sake of completeness, the other idempotents are:

$e_{\{2,3\}} = -T(1,2,3,4,1,2,3,1,2,1)+T(1,2,3,4,1,2,3,1,2)+T(2,3,4,1,2,3,1,2,1)-T(2,3,4,1,2,3,1,2)+T(4,1,2,3,2)-T(1,2,3,2)-T(4,2,3,2)+T(2,3,2);$

$e_{\{1,4\}} = -T(1,2,3,4,1,2,3,1,2,1)+T(1,2,3,4,1,2,3,2,1)+T(1,2,3,4,2,3,1,2,1)-T(1,2,3,4,3,1,2,1)-T(1,2,3,4,2,3,2,1)+T(1,2,3,4,3,2,1)-T(2,3,4,2,1)+T(2,3,4,1)+T(3,4,2,1)-T(4,2,1)-T(3,4,1)+T(4,1);$

$e_{\{4\}} = T(1,2,3,4,1,2,1)-T(1,2,3,4,1,2)-T(1,2,3,4,2,1)+T(1,2,3,4,1)+T(1,2,3,4,2)-T(1,2,3,4)-T(2,3,4,1,2,1)+T(2,3,4,1,2)+T(3,4,1,2,1)-T(4,1,2,1)-T(3,4,1,2)+T(4,1,2)+T(2,3,4,2,1)-T(2,3,4,1)-T(3,4,2,1)+T(4,2,1)+T(3,4,1)-T(4,1)-T(2,3,4,2)+T(2,3,4)+T(3,4,2)-T(4,2)-T(3,4)+T(4);$

$e_{\{3\}} = -T(1,2,3,4,1,2,3,2)+T(1,2,3,4,1,2,3)+T(1,2,3,4,2,3,2)-T(1,2,3,4,2,3)+T(2,3,4,1,2,3,2)-T(2,3,4,1,2,3)+T(4,1,2,3,1)-T(1,2,3,1)-T(3,4,1,2,3,2)+T(3,4,1,2,3)-T(4,1,2,3)+T(1,2,3)-T(4,2,3,1)+T(2,3,1)+T(4,3,1)-T(3,1)-T(2,3,4,2,3,2)+T(2,3,4,2,3)+T(3,4,2,3,2)-T(3,4,2,3)+T(4,2,3)-T(2,3)-T(4,3)+T(3);$

$e_{\{2\}} = -T(3,4,1,2,3,1,2,1)+T(3,4,1,2,3,1,2)+T(4,1,2,3,1,2,1)-T(1,2,3,1,2,1)-T(4,1,2,3,1,2)+T(1,2,3,1,2)+T(3,4,2,3,1,2,1)-T(3,4,2,3,1,2)-T(4,2,3,1,2,1)+T(2,3,1,2,1)+T(4,2,3,1,2)-T(2,3,1,2)+T(3,4,3,1,2)-T(3,4,1,2)-T(4,3,1,2)+T(3,1,2)+T(4,1,2)-T(1,2)-T(3,4,3,2)+T(3,4,2)+T(4,3,2)-T(3,2)-T(4,2)+T(2);$

$e_{\{1\}} = T(2,3,4,2,3,2,1)-T(2,3,4,2,3,1)-T(2,3,4,3,2,1)+T(2,3,4,2,1)+T(2,3,4,3,1)-T(2,3,4,1)-T(3,4,2,3,2,1)+T(3,4,2,3,1)+T(4,2,3,2,1)-T(2,3,2,1)-T(4,2,3,1)+T(2,3,1)+T(3,4,3,2,1)-T(3,4,2,1)-T(4,3,2,1)+T(3,2,1)+T(4,2,1)-T(2,1)-T(3,4,3,1)+T(3,4,1)+T(4,3,1)-T(3,1)-T(4,1)+T(1).$

Finally, $e_\emptyset$ is just the signed sum of all elements, with sign determined by Coxeter length.

One can check (ideally not by hand!) that $\{e_J : J \subset \{1,2,3,4\}\}$ is a complete system of orthogonal idempotents.

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