A REFINED AGLER DECOMPOSITION AND GEOMETRIC APPLICATIONS

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Abstract. We prove a refined Agler decomposition for bounded analytic functions on the bidisk and show how it can be used to reprove an interesting result of Guo et al. related to extending holomorphic functions without increasing their norm. In addition, we give a new treatment of Heath and Suffridge's characterization of holomorphic retracts on the polydisk.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ the unit bidisk.

[Agler, 1988] proved that a holomorphic function $f : \mathbb{D}^2 \to \mathbb{D}$ satisfies a decomposition (later called an *Agler decomposition*) of the form

$$1 - f(z)\overline{f(\zeta)} = (1 - z_1\bar{\zeta}_1)K_1(z, \zeta) + (1 - z_2\bar{\zeta}_2)K_2(z, \zeta)$$

where K_1, K_2 are positive semi-definite kernel functions. A kernel function $K : \Omega \times \Omega \to \mathbb{C}$ is *positive semi-definite* if for every finite subset $F \subset \Omega$ the matrix

$$(K(z, \zeta))_{z,\zeta \in F}$$

is positive semi-definite. (In this article, Ω will be either $\mathbb{D}^2$ or $\mathbb{D}$.) The Agler decomposition generalizes the Pick interpolation theorem from one-variable complex analysis, which implies that for any $f : \mathbb{D} \to \mathbb{D}$, holomorphic,

$$\frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}}$$

is a positive semi-definite kernel.

In recent years, more refined versions of the Agler decomposition have been found for rational inner functions. See [Cole and Wermer, 1999], [Geronimo and Woerdeman, 2004], [Knese, 2008], or [Knese, 2010]. (Unrelated to rational inner functions, in specific, but still relevant are [Ball et al., 2005] and [Lata et al., 2009]). It has not been clear which

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of the "refined" aspects of these decompositions for rational inner functions would extend to more general bounded analytic functions (and which would actually be useful). The following theorem represents an offering in this direction. Our hope is that others may find it useful without having to learn any of the underlying theory required to prove it.

Theorem 1.1. *Let $f : \mathbb{D}^2 \to \mathbb{D}$ be holomorphic. Then, there exist positive semidefinite kernels K_1, K_2, and holomorphic kernels L_1, L_2 such that*

$$1 - f(z)\overline{f(\zeta)} = (1 - z_1\bar{\zeta}_1)K_1(z, \zeta) + (1 - z_2\bar{\zeta}_2)K_2(z, \zeta)$$

and

$$f(z) - f(\zeta) = (z_1 - \zeta_1)L_1(z, \zeta) + (z_2 - \zeta_2)L_2(z, \zeta),$$

where $(z, \zeta) = ((z_1, z_2), (\zeta_1, \zeta_2))$. In addition, the following (pointwise) inequalities hold

$$|L_j(z, \zeta)|^2 \leq K_j(z, z)K_j(\zeta, \zeta)$$

for $j = 1, 2$.

Notice $L_j(z, z) = \frac{\partial f}{\partial z_j}(z)$. So, estimates on the positive semi-definite kernels in this decomposition provide estimates on the derivatives of f.

The analogous inequalities in one variable are

$$\left|\frac{f(z) - f(\zeta)}{z - \zeta}\right|^2 \leq \left|\frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}}\right|^2 \leq \frac{1 - |f(z)|^2}{1 - |z|^2}\frac{1 - |f(\zeta)|^2}{1 - |\zeta|^2}$$

which are consequences of the Schwarz-Pick lemma.

As an application of this theorem, we are able to reprove a useful theorem of [Guo et al., 2008] related to norm preserving extensions of holomorphic functions on the polydisk and holomorphic retracts of the polydisk. When working in $\mathbb{D}^{n+1}$ we will typically denote points by (z, w) where $z \in \mathbb{D}^n$ and $w \in \mathbb{D}$.

Theorem 1.2 ([Guo et al., 2008])**.** *Let $V \subset \mathbb{D}^{n+1}$, and suppose $w|_V$ has a nontrivial norm 1 holomorphic extension to $\mathbb{D}^n$. Then, V is a subset of the graph of a holomorphic function of z.*

Here *nontrivial norm 1 extension* refers to a function on $\mathbb{D}^{n+1}$ other than w which agrees with w on V and whose modulus has supremum norm at most 1. Guo et al.'s proof involved an interesting use of the one-variable Denjoy-Wolff theorem. Guo et al. used this result to continue some of the work initiated in the paper [Agler and McCarthy, 2003]. Additionally, they reproved Heath and Suffridge's characterization of holomorphic retracts of the polydisk, which we now define.

Definition 1.3. A subset $V \subset \mathbb{D}^n$ is a *holomorphic retract* if there exists a holomorphic function (*a retraction*) $\rho : \mathbb{D}^n \to \mathbb{D}^n$ such that

$$\rho \circ \rho = \rho \text{ and } \rho(\mathbb{D}^n) = V$$

Heath and Suffridge characterized all holomorphic retracts of the polydisk as graphs.

Theorem 1.4 ([Heath and Suffridge, 1981])**.** *Suppose $V \subset \mathbb{D}^n$ is a holomorphic retract. Then, after applying an automorphism of $\mathbb{D}^n$, V can be put into the form*

$$\{(z, f(z)) : z \in \mathbb{D}^k\}$$

where $f : \mathbb{D}^k \to \mathbb{D}^{n-k}$ is holomorphic.

The proof of Heath and Suffridge involves an impressive and technical study of properties of Taylor series of retracts. Guo et al. gave a new proof by rehashing some of their proof of Theorem 1.2. Although it is something of an aside, we think it is worth it to show a slightly different approach in Section 4. While our proofs are different from both Heath and Suffridge and Guo et al., the general roadmap of our approach owes a great deal to Guo et al.

2. Proof of Theorem 1.1

Let us first explain the result for rational inner functions and then use an approximation argument to prove it for all analytic functions bounded by one on $\mathbb{D}^2$.

As shown in [Rudin, 1969] (Theorem 5.2.5), every rational inner function on $\mathbb{D}^2$ can be represented as

$$f(z) = \frac{\tilde{p}(z_1, z_2)}{p(z_1, z_2)}$$

where $p \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{D}^2$, $\tilde{p}(z_1, z_2) = z_1^n z_2^m \overline{p(1/\bar{z}_1, 1/\bar{z}_2)}$ for appropriate powers n, m, and $\tilde{p}$ and p have no common factor. Necessarily p and $\tilde{p}$ have bidegree at most (n, m) (i.e. degree at most n in z_1 and m in z_2).

[Geronimo and Woerdeman, 2004] proved a detailed version of a two-variable Christoffel-Darboux formula (see their Proposition 2.3.3 and also [Cole and Wermer, 1999] and [Knese, 2008]), which can be stated as follows: there exist polynomials $A_1, \dots, A_n \in \mathbb{C}[z_1, z_2]$ of bidegree at most $(n-1, m)$ and polynomials $B_1, \dots, B_m \in \mathbb{C}[z_1, z_2]$ of bidegree at most $(n, m-1)$ such that

$$p(z)\overline{p(\zeta)} - \tilde{p}(z)\overline{\tilde{p}(\zeta)} = (1 - z_1\bar{\zeta}_1)\sum_{j=1}^{n} A_j(z)\overline{A_j(\zeta)} + (1 - z_2\bar{\zeta}_2)\sum_{j=1}^{m} B_j(z)\overline{B_j(\zeta)} \tag{2.1}$$

Let $\tilde{A}_j(z) := z_1^{n-1}z_2^{m}\overline{A_j(1/\bar{z}_1, 1/\bar{z}_2)}$, $\tilde{B}_j(z) := z_1^{n}z_2^{m-1}\overline{B_j(1/\bar{z}_1, 1/\bar{z}_2)}$. If we perform a similar reflection operation to (2.1) (i.e. replace (z, ζ) with $((1/\bar{z}_1, 1/\bar{z}_2), (1/\bar{\zeta}_1, 1/\bar{\zeta}_2))$, take complex conjugates and multiply through by $z_1^n z_2^m \bar{\zeta}_1^n \bar{\zeta}_2^m$) we get

$$p(z)\overline{p(\zeta)} - \tilde{p}(z)\overline{\tilde{p}(\zeta)} = (1 - z_1\bar{\zeta}_1)\sum_{j=1}^{n} \tilde{A}_j(z)\overline{\tilde{A}_j(\zeta)} + (1 - z_2\bar{\zeta}_2)\sum_{j=1}^{m} \tilde{B}_j(z)\overline{\tilde{B}_j(\zeta)} \tag{2.2}$$

If we average (2.1) and (2.2) and rewrite using vector notation, we get

$$p(z)\overline{p(\zeta)} - \tilde{p}(z)\overline{\tilde{p}(\zeta)} = (1 - z_1\bar{\zeta}_1)\langle A(z), A(\zeta)\rangle + (1 - z_2\bar{\zeta}_2)\langle B(z), B(\zeta)\rangle \tag{2.3}$$

where

$$A = \frac{1}{\sqrt{2}}[A_1, \dots, A_n, \tilde{A}_1, \dots, \tilde{A}_n]^t$$

$$B = \frac{1}{\sqrt{2}}[B_1, \dots, B_m, \tilde{B}_1, \dots, \tilde{B}_m]^t$$

and $\langle v, w\rangle = w^* v$ denotes the standard complex euclidean inner product (with dimension taken from context).

If we reflect (2.3) in z alone we get

$$\tilde{p}(z)p(\zeta) - p(z)\tilde{p}(\zeta) = (z_1 - \zeta_1)\tilde{A}(z)\cdot A(\zeta) + (z_2 - \zeta_2)\tilde{B}(z)\cdot B(\zeta) \tag{2.4}$$

where "$\cdot$" denotes the dot product: $v \cdot w = w^t v$.

Now, if we divide (2.3) by $p(z)\overline{p(\zeta)}$ and divide (2.4) by $p(z)p(\zeta)$, we get equations of the form

$$1 - f(z)\overline{f(\zeta)} = \sum_{j=1}^{2}(1 - z_j\bar{\zeta}_j)K_j(z, \zeta)$$

$$f(z) - f(\zeta) = \sum_{j=1}^{2}(z_j - \zeta_j)L_j(z, \zeta)$$

where K_1, K_2 are positive semidefinite kernels given explicitly by

$$K_1(z, \zeta) = \frac{\langle A(z), A(\zeta)\rangle}{p(z)\overline{p(\zeta)}} \qquad K_2(z, \zeta) = \frac{\langle B(z), B(\zeta)\rangle}{p(z)\overline{p(\zeta)}}$$

and L_1, L_2 are holomorphic kernels given explicitly by

$$L_1(z,\zeta) = \frac{\tilde{A}(z)\cdot A(\zeta)}{p(z)p(\zeta)} \qquad L_2(z,\zeta) = \frac{\tilde{B}(z)\cdot B(\zeta)}{p(z)p(\zeta)}.$$

The inequality

$$|L_j(z,\zeta)|^2 \leq K_j(z,z)K_j(\zeta,\zeta)$$

follows from Cauchy-Schwarz and the fact that $|A| = |\tilde{A}|$ and $|B| = |\tilde{B}|$.

This proves the theorem for rational inner functions.

It is proven in [Rudin, 1969] (Theorem 5.5.1) that holomorphic functions $f : \mathbb{D}^2 \to \mathbb{D}$ can be approximated locally uniformly by rational inner functions. So, let $\{f^{(i)}\}_i$ be a sequence of rational inner functions converging locally uniformly to f with corresponding $K_1^{(i)}, K_2^{(i)}, L_i^{(i)}, L_2^{(i)}$ satisfying the above formulas/inequalities. Because of the inequalities

$$|L_j^{(i)}(z,\zeta)|^2, |K_j^{(i)}(z,\zeta)|^2 \leq K_j^{(i)}(z,z)K_j^{(i)}(\zeta,\zeta)$$
$$\leq \frac{1}{(1-|z_1|^2)(1-|z_2|^2)(1-|\zeta_1|^2)(1-|\zeta_2|^2)}$$

the kernel functions are locally bounded and hence form a normal family. We can select subsequences so that $K_1^{(i)} \to K_1$, $K_2^{(i)} \to K_2$, $L_1^{(i)} \to L_1$, $L_2^{(i)} \to L_2$ locally uniformly. Positive semi-definiteness and pointwise inequalities are preserved under this limit and therefore the statement of the theorem holds.

3. Guo et al.'s extension theorem

As an application we prove Theorem 1.2 in the following slightly more detailed form. Except for uniqueness, this is contained in [Guo et al., 2008].

Theorem 3.1. *Let $V \subset \mathbb{D}^{n+1}$ be a set with more than one w-value and let πV be the projection of V onto the first n coordinates. Suppose $w|_V$ has a nontrivial norm 1 holomorphic extension F. Then, there is a unique holomorphic $f : \mathbb{D}^n \to \mathbb{D}$ such that $F(z, f(z)) = f(z)$ and $V = \{(z, f(z)) : z \in \pi V\}$.*

In this section we generally follow the convention of denoting points in $\mathbb{D}^{n+1}$ by (z, w) with $z \in \mathbb{D}^n$ and $w \in \mathbb{D}$.

Lemma 3.2. *If $f : \mathbb{D}^{n+1} \to \mathbb{D}$ is holomorphic, ϕ is an automorphism of $\mathbb{D}$, and there exists a $z_0 \in \mathbb{D}^n$ such that $f(z_0, w) = \phi(w)$ for all w, then $f(z, w) = \phi(w)$ for all (z, w).*

Proof. We may assume $\phi(w) = w$ and $z_0 = (0, \ldots, 0)$. Then,

$$G(z, w) = \frac{f(z, w) - w}{1 - \bar{w} f(z, w)}$$

is holomorphic in z, $G(0, w) = 0$, and $|G| \leq 1$.

Write $|z|_\infty$ for the maximum modulus of the components of z. By the Schwarz lemma,

$$|G(z, w)|^2 \leq |z|_\infty^2$$

and

$$1 - |z|_\infty^2 \leq 1 - |G(z, w)|^2 = \frac{(1 - |w|^2)(1 - |f(z, w)|^2)}{|1 - \bar{w} f(z, w)|^2} \leq \frac{1 - |w|^2}{|1 - \bar{w} f(z, w)|^2}$$

and so

$$|w - f(z, w)|^2 \leq |1 - \bar{w} f(z, w)|^2 \leq \frac{1 - |w|^2}{1 - |z|_\infty^2}.$$

Then, by the maximum principle

$$\sup_{w \in r\mathbb{D}} |w - f(z, w)|^2 \leq \frac{1 - r^2}{1 - |z|_\infty^2}$$

which implies $f(z, w) \equiv w$ after letting $r \nearrow 1$. ∎

Lemma 3.3. *Let $F : \mathbb{D}^{n+1} \to \mathbb{D}$ be holomorphic and suppose $F(z_0, w_0) = w_0$ at some point. Necessarily,*

$$|\frac{\partial F}{\partial w}(z_0, w_0)| \leq 1. \tag{3.1}$$

If equality holds in (3.1), then $F(z, w) = \phi(w)$ for some automorphism ϕ. If strict inequality holds in (3.1), then there exists a unique holomorphic function $f : \Omega \to \mathbb{D}$ defined in a neighborhood Ω of z_0 such that $f(z_0) = w_0$ and $F(z, f(z)) = f(z)$ where defined.

Proof. By the Schwarz lemma

$$\frac{1 - |F(z_0, w_0)|^2}{1 - |w_0|^2} = 1 \geq |\frac{\partial F}{\partial w}(z_0, w_0)|.$$

If equality occurs then $F(z_0, w)$ is an automorphism of $\mathbb{D}$ and by the previous lemma $F(z, w) = \phi(w)$ identically.

If equality does not hold, then setting $G(z, w) = F(z, w) - w$ we see that $\frac{\partial G}{\partial w}(z_0, w_0) = \frac{\partial F}{\partial w}(z_0, w_0) - 1 \neq 0$. By the implicit function theorem, there exists a function of z in a neighborhood of z_0 such that $G(z, f(z)) = 0$; i.e. $F(z, f(z)) = f(z)$.

To see that f is unique, we note that if $F(z_1, w_1) = w_1$, there cannot be a different $w_2 \neq w_1$ such that $F(z_1, w_2) = w_2$, for then $F(z_1, w) \equiv w$ and hence $F(z, w) \equiv w$. By assumption this cannot occur, so any point (z_1, w_1) satisfying $F(z_1, w_1) = w_1$ is uniquely determined by the z component. In particular, $F(z, f(z)) = f(z)$ cannot hold for two different choices of $f : \Omega \to \mathbb{D}$. $\square$

The final lemma is the most important and it utilizes the main theorem.

Lemma 3.4. *Let $F : \mathbb{D}^{n+1} \to \mathbb{D}$ be holomorphic. Suppose $F(z_0, w_0) = w_0$, $F(z_1, w_1) = w_1$, $F(z_2, w_2) \neq w_2$, where $w_0 \neq w_1$. Then there exists a unique $f : \mathbb{D}^n \to \mathbb{D}$ such that $F(z, f(z)) = f(z)$. In particular, if $F(z_3, w_3) = w_3$, then $f(z_3) = w_3$.*

Proof. By the three assumptions, F cannot be an automorphism as a function of w. So, we are in the second case of the previous lemma and there locally (say on a domain $\Omega \subset \mathbb{D}^n$) exists a unique $f : \Omega \to \mathbb{D}$ satisfying $F(z, f(z)) = f(z)$. We need to extend f to all of $\mathbb{D}^n$.

We will show f can be extended one variable at a time. Letting $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) = (\zeta_1, \zeta') \in \Omega$, we plan to show f can be extended to $\mathbb{D} \times \{\zeta'\}$ in such a way that the identity $F(z, f(z)) = f(z)$ is preserved. By Lemma 3.3, the identity will then extend to a unique function on an open neighborhood of $\mathbb{D} \times \{\zeta'\}$. So, given any other point $\eta = (\eta_1, \dots, \eta_n)$, we will be able to successively extend f to $(\eta_1, \zeta_2, \dots, \zeta_n), (\eta_1, \eta_2, \zeta_3, \dots), \dots, (\eta_1, \dots, \eta_n)$. This will imply f is holomorphic on all of $\mathbb{D}^n$ and $F(z, f(z)) = f(z)$.

For this argument we will use t for the first coordinate of z and write $z = (t, z')$ (we are avoiding "z_1" since we have used this in a different way in the lemma statement).

Let $g(t) = f(t, \zeta')$ and $G(t, w) = F(t, \zeta', w)$. Now g is holomorphic in some neighborhood of ζ_1 and $G(t, g(t)) = g(t)$ holds in said neighborhood. If g is constant, then clearly g extends to be holomorphic on $\mathbb{D}$ and $G(t, g(t)) = g(t)$ holds on all of $\mathbb{D}$. So, suppose g is nonconstant. Perturb ζ_1 if necessary to make $g'(\zeta_1) \neq 0$, and let ∂_t, ∂_w denote the partial derivatives with respect to t, w, respectively.

Then, $\partial_t G(t, g(t)) + \partial_w G(t, g(t)) \partial_t g(t) = \partial_t g(t)$, so $\partial_t G(t, g(t)) = \partial_t g(t)(1 - \partial_w G(t, g(t)))$. Now, $\partial_t G(\zeta_1, g(\zeta_1)) \neq 0$ by the previous lemma (i.e. $\partial_w G(t, g(t))$ cannot equal 1, since this would imply G is an automorphism as a function of w) and since $\partial_t g(\zeta_1) \neq 0$.

We apply the main theorem to $G(t, w)$. Theorem 1.1 implies

$$1 - G(t, w)\overline{G(\tau, \eta)} = (1 - t\bar{\tau})K_1 + (1 - w\bar{\eta})K_2$$

with K_1, K_2 positive semi-definite, where K_1, K_2 should be evaluated at $((t, w), (\tau, \eta))$.

Substituting $w = g(t), \eta = g(\tau)$ for t, τ in a neighborhood of ζ_1, and writing $v(t,\tau) = ((t, g(t)), (\tau, g(\tau)))$ for short, we get

$$1 - g(t)\overline{g(\tau)} = (1 - t\bar{\tau})K_1(v(t,\tau)) + (1 - g(t)\overline{g(\tau)})K_2(v(t,\tau))$$

or

$$(1 - g(t)\overline{g(\tau)})(1 - K_2(v(t,\tau))) = (1 - t\bar{\tau})K_1(v(t,\tau)) \tag{3.2}$$

We cannot have $K_2(v(\zeta_1, \zeta_1)) = 1$ for then $K_1(v(\zeta_1, \zeta_1)) = 0$, which by the main theorem implies a contradiction. Specifically,

$$|\partial_t G(\zeta_1, g(\zeta_1))| = |L_1(v(\zeta_1, \zeta_1))| \leq |K_1(v(\zeta_1, \zeta_1))| = 0$$

which is not the case as $\partial_t G(\zeta_1, g(\zeta_1)) \neq 0$.

So, $|K_2(v(t,\tau))| < 1$ for t, τ in some open set around ζ_1. By (3.2), for such t, τ

$$\frac{1 - g(t)\overline{g(\tau)}}{1 - t\bar{\tau}} = \frac{K_1(v(t,\tau))}{1 - K_2(v(t,\tau))} = K_1(v(t,\tau)) \sum_{j=0}^{\infty} K_2(v(t,\tau))^j$$

is positive semi-definite. By the Pick interpolation theorem, g extends to be holomorphic on all of $\mathbb{D}$. (See [Agler and McCarthy, 2002] for the Pick interpolation theorem from this point of view.) Also, $G(t, g(t)) = g(t)$ then automatically holds on all of $\mathbb{D}$ by analyticity. This completes the proof. □

Theorem 3.1 is just a rephrasal of this lemma.

4. Retracts and Theorem 1.4

We prove the following refinement of Theorem 1.4 (which can also be found in [Guo et al., 2008]).

Theorem 4.1. *Suppose $V \subset \mathbb{D}^n$ is a holomorphic retract. Then, after applying an automorphism of $\mathbb{D}^n$, V can be put into the form*

$$\{(z, e(z), f(z)) : z \in \mathbb{D}^k\}$$

where $e : \mathbb{D}^k \to \mathbb{D}^m$ is a coordinate function in each component and $f : \mathbb{D}^k \to \mathbb{D}^{n-m-k}$ is holomorphic with no components equal to an automorphism as a function of a single variable.

An example for e might be $e(z_1, z_2) = (z_1, z_1, z_1, z_2, z_2)$.

Lemma 4.2. *Suppose $V \subset \mathbb{D}^{n+1}$ is a holomorphic retract, with retraction $\rho(z, w) = (\rho_1, \ldots, \rho_n, \rho_{n+1}) = (\rho', \rho_{n+1})$ where we assume ρ_{n+1} is not an automorphism as a function of one variable. Then, there exists*

$f : \mathbb{D}^n \to \mathbb{D}$, holomorphic, such that $(z, w) \mapsto (\rho'(z, f(z)), f(z))$ is a retraction of V,

$$V = \{(z, f(z)) : z \in \pi V\} \tag{4.1}$$

and πV is a retract with retraction $z \mapsto \rho'(z, f(z))$.

Proof. If ρ_{n+1} is constant, there is nothing to prove, so assume otherwise. Then, $\rho_{n+1}(z, w) = w$ for two distinct values of w (and necessarily different values of z) since $\rho_{n+1}(\rho', \rho_{n+1}) = \rho_{n+1}$. Therefore, Theorem 3.1 applies. There exists $f : \mathbb{D}^n \to \mathbb{D}$ holomorphic satisfying

$$\rho_{n+1}(z, f(z)) = f(z) \tag{4.2}$$

and

$$\{(z, w) : \rho_{n+1}(z, w) = w\} = \{(z, f(z)) : z \in \mathbb{D}^n\} \supset V. \tag{4.3}$$

This proves (4.1).

As $\rho(z, f(z)) \in V$ we see that by (4.3) and (4.2), $f(\rho'(z, f(z))) = \rho_{n+1}(z, f(z)) = f(z)$, which shows $(z, w) \mapsto (\rho'(z, f(z)), f(z))$ agrees with the map $(z, w) \mapsto \rho(z, f(z))$. This is a retraction since its composition with itself is

$$\begin{aligned} \rho(\rho'(z, f(z)), f(\rho'(z, f(z)))) &= \rho(\rho'(z, f(z)), \rho_{n+1}(z, f(z))) \\ &= \rho(\rho(z, f(z)) = \rho(z, f(z)) \end{aligned}$$

as desired.

We need to show that the range of $(z, w) \mapsto \rho(z, f(z))$ contains V (it certainly is contained in V). If $(z, w) \in V$, then $w = f(z)$ and $\rho(z, f(z)) = (z, f(z)) = (z, w)$. So, this map is a retraction *of* V.

Finally, we must show πV is a retract with retraction $z \mapsto \rho'(z, f(z))$. This map is indeed a retraction since $(z, w) \mapsto (\rho'(z, f(z)), f(z))$ is, and the first n components necessarily trace out πV. □

Lemma 4.3. *Let $\rho = (\rho_1, \dots, \rho_n) : \mathbb{D}^n \to \mathbb{D}^n$ be a retraction of V. If $\rho_1(z_1, \dots, z_n)$ is an automorphism as a function of z_1, then $\rho_1(z) \equiv z_1$. If ρ_1 is an automorphism as a function of z_2 then $\rho_2(z) \equiv z_2$ and $\rho_1(z) \equiv \phi(z_2)$ for some ϕ.*

Proof. If ρ_1 is an automorphism, say ϕ, as a function of z_1, then $\phi \circ \phi = \phi$, which means $\phi = \mathrm{id}$. This means $\rho_1(z) = z_1$. If ρ_1 is an automorphism, say ϕ, as a function of z_2, then $\phi(\rho_2(z)) = \rho_1(\rho(z)) = \rho_1(z) = \phi(z_2)$. This implies $\rho_2(z) = z_2$. □

Lemma 4.4. *Let $\rho = (\rho_1, \dots, \rho_n) : \mathbb{D}^n \to \mathbb{D}^n$ be a retraction of V with all components equal to an automorphism as a function of a single* *variable. After conjugating by automorphisms of $\mathbb{D}^n$ we may put ρ into the form*

$$\rho(z, w) = (z, e(z))$$

where $z \in \mathbb{D}^k$, $w \in \mathbb{D}^{n-k}$, $e : \mathbb{D}^k \to \mathbb{D}^{n-k}$, where each component of e is a coordinate function.

Proof. Let us just illustrate. If $\rho_2(z) = z_2$ and $\rho_1(z) = \phi(z_2)$ for some one variable automorphism ϕ, we can conjugate by the automorphism of $\mathbb{D}^n$ given by $\psi(z) = (\phi^{-1}(z_1), z_2, \dots, z_n)$ to get

$$\psi \circ \rho \circ \psi^{-1}(z) = (z_2, z_2, \rho_3 \circ \psi^{-1}(z), \dots, \rho_n \circ \psi^{-1}(z))$$

The lemma then follows from the previous lemma after reordering and conjugating by analogous automorphisms as necessary. □

Proof of Theorem 4.1. There is no harm in assuming V is not a Cartesian product of a point and a retract (this is equivalent to assuming our retractions do not possess a constant component).

We proceed by induction. Let $n = 1$ and let $\rho : \mathbb{D} \to V$ be a retraction. One variable retractions are either constant (which by assumption is ruled out) or equal to the identity (by the Schwarz-Pick lemma, a self-map of the disk with two fixed points equals the identity).

Suppose the theorem holds for n dimensional retracts. Let $\rho : \mathbb{D}^{n+1} \to V$ be a retraction onto V. If all components of ρ are automorphisms (in a single variable) then we are finished by Lemma 4.4. So, we assume some component is not an automorphism in a single variable and relabel to make ρ_{n+1} such a component. By Lemma 4.2, we can replace ρ with a retraction r of the form

$$r(z, w) = (r'(z), f(z))$$

where $z \in \mathbb{D}^n$ and $w \in \mathbb{D}$ and the projection of V onto the first n coordinates, denoted πV, is a retract with retraction $r'(z)$. By induction (after possibly applying automorphisms of $\mathbb{D}^n$) we may put πV into the form

$$\pi V = \{(z, e(z), g(z)) : z \in \mathbb{D}^k\}$$

where $e : \mathbb{D}^k \to \mathbb{D}^m$ consists of coordinate functions, $g : \mathbb{D}^k \to \mathbb{D}^{n-m-k}$ is holomorphic with no components equal to an automorphism.

Then,

$$V = \{(z, e(z), g(z), f(z, e(z), g(z))) : z \in \mathbb{D}^k\}$$

which is of the desired form. □

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