

BOUNDING REFLECTION LENGTH IN AN AFFINE COXETER GROUP

JON MCCAMMOND AND T. KYLE PETERSEN

ABSTRACT. In any Coxeter group, the conjugates of elements in the standard minimal generating set are called reflections and the minimal number of reflections needed to factor a particular element is called its reflection length. In this article we prove that the reflection length function on an affine Coxeter group has a uniform upper bound. More precisely we prove that the reflection length function on an affine Coxeter group that naturally acts faithfully and cocompactly on \mathbb{R}^n is bounded above by $2n$ and we also show that this bound is optimal. Conjecturally, spherical and affine Coxeter groups are the only Coxeter groups with a uniform bound on reflection length.

Every Coxeter group W has two natural generating sets: the finite set S used in its standard finite presentation and the set R of reflections formed by collecting all conjugates of the elements in S . The first generating set leads to the standard length function $\ell_S: W \rightarrow \mathbb{N}$ and the second is used to define the reflection length function $\ell_R: W \rightarrow \mathbb{N}$. When W is finite, both length functions are uniformly bounded for trivial reasons and are fairly well understood.¹ For infinite Coxeter groups the function ℓ_S is always unbounded because there are only finitely many group elements of a given length as a consequence of the fact that S is finite. Our main result is that ℓ_R remains bounded for affine Coxeter groups and we provide an explicit optimal upper bound.

Theorem A (Explicit affine upper bounds). *If W is an affine Coxeter group that naturally acts faithfully and cocompactly on \mathbb{R}^n then every element of W has reflection length at most $2n$ and there exist elements in W with reflection length equal to $2n$.*

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¹For finite W these generating sets and length functions exhibit an interesting duality: the maximum value of ℓ_S is $|R|$ and the maximum value of ℓ_R is $|S|$. See [1] for further details and for additional illustrations of this phenomenon.

The article is structured as follows. The first two sections recall basic facts, the third establishes a key technical result and the fourth contains the proof of our main result. The final section contains a conjecture about infinite non-affine Coxeter groups.

1. REFLECTION LENGTH

We assume the reader is familiar with the basic theory of reflection groups (as described, for example, in [4]) and we generally follow the standard notational conventions.

Definition 1.1 (Reflection length). Let W be a Coxeter group with standard generating set S . A *reflection* in W is any conjugate of an element of S and we use R to denote the set of all reflections in W . In other words, $R = \{wsw^{-1} \mid s \in S, w \in W\}$. We should note that unless W is a finite group, R is an infinite set. For any element $w \in W$, its *reflection length* $\ell_R(w)$ is the minimal number of reflections whose product is w . Thus $w = r_1 r_2 \cdots r_k$ with $r_i \in R$ means $\ell_R(w) \leq k$. Alternatively, $\ell_R(w)$ can be defined as the combinatorial distance from the vertex labeled by the identity to the vertex labeled by w in the Cayley graph of W with respect to R .

Since combinatorial distance defines a metric on the vertex set of any graph and Cayley graphs are homogeneous in the sense that there is a vertex-transitive group action, metric properties of the distance function translate into properties of ℓ_R . Symmetry and the triangle inequality, for example, imply that $\ell_R(w) = \ell_R(w^{-1})$, and $\ell_R(uv) \leq \ell_R(u) + \ell_R(v)$, respectively.

It is sufficient to investigate reflection length in irreducible Coxeter groups because of the following elementary result.

Proposition 1.2 (Reducible Coxeter groups). *When W is a reducible Coxeter group, its standard generating set S has a nontrivial partition $S = S_1 \sqcup S_2$ in which every element in S_1 commutes with every element in S_2 . In this context, $W = W_1 \times W_2$ where W_i denotes the parabolic subgroup generated by S_i , the reflections R in W can be partitioned as $R = R_1 \sqcup R_2$ where R_i denotes the reflections in W_i and when $w \in W$ is written in the form $w = w_1 w_2$ with $w_i \in W_i$, we have $\ell_R(w) = \ell_{R_1}(w_1) + \ell_{R_2}(w_2)$.*

2. AFFINE COXETER GROUPS

Next we review the construction of an affine Coxeter group from a crystallographic root system.

Definition 2.1 (Affine Coxeter groups). Recall that a crystallographic root system Φ is a finite collection of vectors that span a real Euclidean space V satisfying a few elementary properties and that an affine Coxeter group W can be constructed from Φ as follows. For each $\alpha \in \Phi$ and $i \in \mathbb{Z}$ let $H_{\alpha,i}$ denote the (affine) *hyperplane* of solutions to the equation $\langle x, \alpha \rangle = i$ where the brackets denote the standard inner product on V . The unique nontrivial isometry of V that fixes $H_{\alpha,i}$ pointwise is a *reflection* that we call $r_{\alpha,i}$. The collection $R = \{r_{\alpha,i} \mid \alpha \in \Phi, i \in \mathbb{Z}\}$ generates the affine Coxeter group W and R is its set of reflections in the sense of Definition 1.1. A standard minimal generating set S can be obtained by restricting to those reflections that reflect across the facets of a certain polytope in V .

Remark 2.2 (No finite factors). Every irreducible affine Coxeter group can be constructed from its crystallographic root system as described in Definition 2.1 but the construction also works equally well when the root system is reducible. It is not, however, sufficient to construct arbitrary reducible affine Coxeter groups because it always constructs affine Coxeter groups with affine irreducible components (i.e. no finite irreducible components). The affine Coxeter groups constructible from a root system in this way can also be characterized as those that naturally act faithfully and cocompactly on some Euclidean space.

For each affine Coxeter group W constructed from a root system Φ there is a finite Coxeter group W_0 related to W in two distinct ways.

Definition 2.3 (Subgroups and quotients). We abbreviate $r_{\alpha,0}$ and $H_{\alpha,0}$ as r_α and H_α , respectively. The hyperplanes H_α are precisely the ones that contain the origin and the reflections $R_0 = \{r_\alpha \mid \alpha \in \Phi\}$ generate a finite Coxeter group W_0 that contains all elements of W that fix the origin. This embeds W_0 as a subgroup of W . There is a group homomorphism $p: W \rightarrow W_0$ defined by sending each generating reflection $r_{\alpha,i}$ in W to r_α in W_0 .

Because the map $p: W \rightarrow W_0$ sends R to R_0 , it sends reflection factorizations to reflection factorizations, proving the following.

Proposition 2.4 (Lengths and quotients). *If the map $p: W \rightarrow W_0$ sends u to u_0 then $\ell_R(u) \geq \ell_{R_0}(u_0)$.*

Finite Coxeter groups such as W_0 are also known as spherical Coxeter groups because they act by isometries on spheres and we digress for a moment to recall a few of their key properties. In a finite Coxeter group, reflection length has a geometric interpretation that yields a spherical version of Theorem A as an immediate corollary. The following properties were observed by Carter in [2].

Proposition 2.5 (Spherical reflection length). *The reflection length of an element w in a finite Coxeter group W_0 is equal to the codimension of the subspace of vectors that w fixes in the standard orthogonal representation of W_0 . In addition, a reflection factorization $w = r_1 r_2 \cdots r_m$ is of minimum length if and only if the vectors normal to the hyperplanes of these reflections are linearly independent.*

Corollary 2.6 (Spherical upper bounds). *Let W_0 be a finite Coxeter group whose standard representation acts on \mathbb{R}^n by orthogonal transformations. For all $w \in W_0$, $\ell_R(w) \leq n$ and for elements that only fix the origin, $\ell_R(w) = n$. More concretely, multiplying the elements in its standard minimal generating set S_0 produces an element w with $\ell_R(w) = n$.*

Proposition 2.5 also has consequences for some elements in W .

Corollary 2.7 (Linearly independent roots). *If $w = r_1 r_2 \cdots r_m$ is a reflection factorization of $w \in W$ in which the roots of the reflections are linearly independent, then $\ell_R(w) = m$.*

Proof. The factorization shows $\ell_R(w) \leq m$. For the lower bound note that each reflection $r_i \in R$ is r_{α_i, c_i} for some $\alpha_i \in \Phi$ and $c_i \in \mathbb{Z}$, and by hypothesis the roots α_i are linearly independent. If we let $w_0 = p(w)$ then by Propositions 2.4 and 2.5 we have $\ell_R(w) \geq \ell_{R_0}(w_0) = m$. \square

Although we never use the following result established by Solomon in [6], we mention it because it highlights how reflection length captures fundamental aspects of the behavior of finite Coxeter groups.

Proposition 2.8 (Solomon's factorization formula). *For each finite Coxeter group W_0 , the polynomial recording the distribution of reflection lengths factors completely over the integers. In particular,*

$$f(x) = \sum_{w \in W_0} x^{\ell_R(w)} = \prod_{i=1}^n (1 + e_i x),$$

where the e_i are the exponents of W_0 .

We now return to the structure of the affine Coxeter group W constructed from a root system Φ .

Definition 2.9 (Coroots). For each root $\alpha \in \Phi$, there is a corresponding coroot $\alpha^\vee = c\alpha$ with $c = \frac{2}{\langle \alpha, \alpha \rangle}$. The collection of all coroots is denoted Φ^\vee and the integral linear combinations of vectors in Φ^\vee is a lattice $L = \mathbb{Z}\Phi^\vee \cong \mathbb{Z}^n$ called the *coroot lattice*. The isomorphism with \mathbb{Z}^n is a result of the existence of a subset $\Delta^\vee \subset \Phi^\vee$ of linearly independent vectors that form a basis for L .

The coroot lattice describes the translations in W .

Definition 2.10 (Translations). A *translation* is a map that shifts each point by the same vector λ and we let t_λ denote the map sending each point $x \in V$ to $x + \lambda$. For each $\alpha \in \Phi$ consider the product $r_{\alpha,1}r_\alpha$. Reflecting through parallel hyperplanes produces a translation in the α direction and the exact translation is t_{α^\vee} . Because $t_\mu t_\nu = t_{\mu+\nu}$, there is a translation of the form t_λ in W for each $\lambda \in L$ and the set $T = \{t_\lambda \mid \lambda \in L\}$ forms an abelian subgroup of W . In fact, these are the only translations that are contained in W .

The subgroup T is also the kernel of the map $p: W \rightarrow W_0$ and W can be viewed as a semidirect product $W = W_0 \ltimes T$.

Proposition 2.11 (Normal forms). *For each element $w \in W$ there is a unique factorization $w = t_\lambda w_0$ where t_λ is a translation with $\lambda \in L$ and w_0 is an element in W_0 .*

Proof. If such an expression exists then w_0 is the image of w under the map $p: W \rightarrow W_0$ and λ is the image of the origin under w (keeping in mind that elements of W act on V in function notation so that composition is from right to left). This proves uniqueness. For existence define $w_0 = p(w) \in W_0 \subset W$ and consider the element ww_0^{-1} . Since it is in the kernel of p , it is translation in the form t_λ for some $\lambda \in L$. \square

3. TRANSLATION DIMENSION

In this section we introduce the notion of the dimension of a vector in the coroot lattice. As above, W is an affine Coxeter group, constructed from a root system Φ , acting on a vector space V .

Definition 3.1 (Real dimension). We call a subspace of V spanned by a collection of coroots in Φ^\vee a *real coroot subspace* and we say a vector λ has *real dimension k* when λ is contained in a k -dimensional coroot subspace but it is not contained in any real coroot subspace of strictly smaller dimension. Since it is always possible find a coroot basis for each coroot subspace, real dimension k means that $\lambda = c_1\alpha_1^\vee + c_2\alpha_2^\vee + \cdots + c_k\alpha_k^\vee$ for some $c_i \in \mathbb{R}$ and $\alpha_i^\vee \in \Phi^\vee$ but that no such expression exists with fewer summands.

Remark 3.2 (Non-unique subspaces). There may be more than one k -dimensional coroot subspace containing a vector $\lambda \in L$ of real dimension k because coroot subspaces are not closed under intersection. Consider the D_n root system $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$. Every root has norm 2 so $\alpha = \alpha^\vee$ and $\Phi = \Phi^\vee$. The translation t_λ with $\lambda = 2e_1 \in L$ is 2-dimensional because $2e_1$ is in the span of

$\{e_1 + e_j, e_1 - e_j\}$ for any $j > 1$ but as j varies we get distinct 2-planes containing λ . They intersect along the e_1 -axis but this line is not a real coroot subspace.

The following definition is very similar.

Definition 3.3 (Integral dimension). We say a vector $\lambda \in L$ has *integral dimension* k when λ can be expressed in the form $\lambda = c_1\alpha_1^\vee + c_2\alpha_2^\vee + \cdots + c_k\alpha_k^\vee$ for some $c_i \in \mathbb{Z}$ and $\alpha_i^\vee \in \Phi^\vee$ but no such expression exists with fewer summands. The restriction to integral coefficients means that only vectors in L have a dimension in this sense. When λ has integral dimension k we say λ is *k-dimensional* as is the corresponding translation t_λ . Note that the real dimension of λ is a lower bound on its integral dimension.

Proposition 3.4 (Dimensions). *If W is an affine Coxeter group constructed from a root system Φ and naturally acts faithfully and cocompactly on $V = \mathbb{R}^n$, then every vector in its coroot lattice has integral dimension at most n and n -dimensional vectors do exist.*

Proof. The first assertion is a consequence of the fact that $L \cong \mathbb{Z}^n$ is a lattice with a \mathbb{Z} -basis in Φ^\vee . For the second assertion note that any vector $\lambda \in L$ that does not lie in the union of the finite number of proper subspaces through the origin that are spanned by coroots has real dimension n and thus integral dimension at least n . \square

The integral dimension of a vector $\lambda \in L$ bounds how hard it is to move the origin to λ using reflections. To prove this assertion we need an elementary result about factorizations.

Lemma 3.5 (Rewriting factorizations). *Let W be a Coxeter group with reflections R and let $w = r_1r_2 \cdots r_m$ be a reflection factorization. For any selection $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ of positions there is a length m reflection factorization of w whose first k reflections are $r_{i_1}r_{i_2} \cdots r_{i_k}$ and another length m reflection factorization of w where these are the last k reflections in the factorization.*

Proof. Because reflections are closed under conjugation, for any reflections r and r' there exist reflections r'' and r''' such that $rr' = r''r$ and $r'r = rr'''$. Iterating these rewriting operations allows us to move the selected reflections into the desired positions without altering the length of the factorization. \square

In preparation for the next result recall that $t_{\alpha^\vee} = r_{\alpha,1}r_\alpha$ and note that because the hyperplanes $H_{\alpha,i}$ are evenly spaced, the product $r_{\alpha,i+1}r_{\alpha,i}$ is also t_{α^\vee} for every $i \in \mathbb{Z}$. More generally $r_{\alpha,i+j}r_{\alpha,i} = t_{j\alpha^\vee}$.

Proposition 3.6 (Moving points). *If $\lambda \in L$ has integral dimension k then $\ell_R(t_\lambda) \leq 2k$ and there is an element $u \in W$ with $\ell_R(u) \leq k$ that sends the origin to λ .*

Proof. By the definition of integral dimension there is an equation of the form $\lambda = c_1\alpha_1^\vee + c_2\alpha_2^\vee + \cdots + c_k\alpha_k^\vee$ and the formulas $t_{\mu+\nu} = t_\mu t_\nu$ and $t_{j\alpha^\vee} = r_{\alpha,j}r_\alpha$ show that t_λ has a length $2k$ reflection factorization of the form $t_\lambda = (r_{\alpha_1,c_1}r_{\alpha_1})(r_{\alpha_2,c_2}r_{\alpha_2}) \cdots (r_{\alpha_k,c_k}r_{\alpha_k})$. Next, by Lemma 3.5 there is another length $2k$ reflection factorization of t_λ where the final k reflections are $r_{\alpha_1}r_{\alpha_2} \cdots r_{\alpha_k}$. Since all of these reflections fix the origin, the product of the first k reflections is an element u that sends the origin to λ . \square

Note that just as there can be distinct minimal expressions for λ (Remark 3.2), there are often distinct elements of reflection length k that send the origin to λ .

Theorem 3.7 (Equivalent definitions). *For each $\lambda \in L$ the real dimension of λ , the integral dimension of λ and the minimal reflection length of an element sending the origin to λ are equal.*

Proof. Let k_r , k_d and k_m be three numbers at issue in the order listed. Proposition 3.6 shows that $k_d \geq k_m$. Next, let u be an element sending the origin to λ . Because a reflection $r_{\alpha,i}$ move points in the α^\vee direction, λ is in the span of the coroots associated to the reflections in a minimal length reflection factorization of u . Thus $k_m \geq k_r$. And finally, let V' be a k_r -dimensional coroot subspace of V containing λ . The set $\Phi' = \Phi \cap V'$ satisfies the requirements to be a root system and $L' = L \cap V'$ is its coroot lattice. Because λ lies in the lattice L' , $k_r \geq k_d$. The combination $k_d \geq k_m \geq k_r \geq k_d$ shows all three are equal. \square

4. BOUNDING REFLECTION LENGTH

We are now ready to prove our main result.

Proposition 4.1 (Bounds). *If W is a affine Coxeter group and $w \in W$ has the form $w = t_\lambda w_0$ where t_λ is a k -dimensional translation and $w_0 \in W_0$ is an element fixing the origin then $k \leq \ell_R(w) \leq k + n$. In particular, every element $w \in W$ has $\ell_R(w) \leq 2n$.*

Proof. The lower bound, $\ell_R(w) \geq k$, follows from Theorem 3.7. By Proposition 3.6 and Theorem 3.7 there is an element u with $\ell_R(u) = k$ that sends the origin to λ . Because the element $v_0 = u^{-1}w$ fixes the origin, it is in W_0 and $\ell_R(v_0) \leq n$ by Corollary 2.6. Thus $\ell_R(w) = \ell_R(uv_0) \leq \ell_R(u) + \ell_R(v_0) \leq k + n$. For the final assertion note that

every element can be written in this form (Proposition 2.11) and every translation is k -dimensional for some $k \leq n$ (Proposition 3.4). \square

Based on the proof of this proposition, one might conjecture that each $w = t_\lambda w_0$ with k -dimensional λ has another factorization $w = uv_0$ with $v_0 \in W_0$ and $\ell_R(w) = \ell_R(u) + \ell_R(v_0)$. This is not always the case.

Example 4.2 (Exact bounds). Let α_{ij} denote the vector $e_i - e_j$, let $\Phi = \{\alpha_{ij} \mid 1 \leq i, j \leq 4\}$ be the A_3 root system with coroot lattice L , and consider the elements $w_{ijk} = r_{\alpha_{12},i} r_{\alpha_{23},j} r_{\alpha_{34},k}$. By Corollary 2.7 $\ell_R(w_{ijk}) = 3$. All of these elements are sent under $p: W \rightarrow W_0$ to $w_{000} = r_{\alpha_{12}} r_{\alpha_{23}} r_{\alpha_{34}}$ which cyclically permutes the four coordinates moving the point (x, y, z, w) to (w, x, y, z) . The minimum length reflection factorizations of w_{000} in W_0 are well-known and they are encoded by maximal chains in the lattice NC_4 of non-crossing partitions on four elements [5]. Every reflection in R_0 occurs in some factorization of w_{000} but there exists a pair of reflections, $r_{\alpha_{13}}$ and $r_{\alpha_{24}}$, representing a ‘‘crossing’’ partition that cannot both occur in the same factorization. By varying i, j and k , the elements w_{ijk} produce all elements of the form $t_\lambda w_{000}$ with $\lambda \in L$. Thus for a careful choice of values, the vector λ is in the span of α_{13} and α_{24} and this is the unique minimal dimensional coroot subspace containing λ . If u is a product of two reflections sending such a λ to the origin, then the coroots involved in this product must be α_{13} and α_{24} . As a consequence, there is no length 3 reflection factorization of the form $w_{ijk} = uv_0$ since the projection of such a factorization to W_0 would be a length 3 reflection factorization of w_{000} containing both $r_{\alpha_{13}}$ and $r_{\alpha_{24}}$ which is known not to exist.

Proposition 4.3 (Optimality). *If W is a affine Coxeter group and $w = t_\lambda$ is a k -dimensional translation then $\ell_R(w) = 2k$. In particular, n -dimensional translations have reflection length $2n$.*

Proof. Being a translation in W , $w = t_\lambda$ for some $\lambda \in L$ and by definition of dimension $\lambda = c_1 \alpha_1^\vee + c_2 \alpha_2^\vee + \cdots + c_k \alpha_k^\vee$ for $\alpha_i^\vee \in \Phi^\vee$ and $c_i \in \mathbb{Z}$. By Proposition 3.6 $\ell_R(w) \leq 2k$. To show that $2k$ is also a lower bound suppose that $w = r_1 r_2 \cdots r_m$ with each $r_i = r_{\alpha_i, c_i} \in R$. Because the reflection r_i moves points in the α_i^\vee direction and λ has real dimension k (Theorem 3.7), the coroots α_i^\vee must span a subspace of dimension at least k . Next, we may assume that the first k reflections have linearly independent coroots since it is possible to use Lemma 3.5 to move any k reflections with linearly independent coroots to the front. This alters the reflection factorization of w but leaves the total length unchanged. Let $u = r_1 r_2 \cdots r_k$ and let $v = r_{k+1} \cdots r_m$ so that $w = uv$, let $u_0 = p(u)$ and $v_0 = p(v)$ where p is the homomorphism $p: W \rightarrow W_0$

and note that $\ell_{R_0}(u_0) = k$ by Proposition 2.5. Since w is a translation, $p(w)$ is the identity and $v_0 = u_0^{-1}$. Finally, by Proposition 2.4 we have $\ell_R(v) \geq \ell_{R_0}(v_0) = \ell_{R_0}(u_0^{-1}) = \ell_{R_0}(u_0) = k$, which means that any reflection factorization of v has length at least k . This implies that $m \geq 2k$ and as a consequence every reflection factorization of w has length at least $2k$. \square

Theorem A (Explicit affine upper bounds). *If W is an affine Coxeter group that naturally acts faithfully and cocompactly on \mathbb{R}^n then every element of W has reflection length at most $2n$ and there exist elements in W with reflection length equal to $2n$.*

Proof. Proposition 4.1 shows that $2n$ is an upper bound. Propositions 3.4 and 4.3 show that it is optimal. \square

Remark 4.4 (Finite factors). The cocompactness assumption in Theorem A essentially means that W does not have any finite irreducible factors (Remark 2.2). When finite irreducible factors are present, the optimal upper bound can be lowered accordingly. In particular, if $W = W_f \times W_a$ where W_f is the product of the finite irreducible factors, W_a is the product of the affine irreducible factors and $\mathbb{R}^n = \mathbb{R}^{n_f} \oplus \mathbb{R}^{n_a}$ is the orthogonal decomposition preserved by this splitting then by Proposition 1.2, Corollary 2.6 and Theorem A the optimal upper bound for the reflection length function on W is $n_f + 2n_a = 2n - n_f$.

We should note that we have not found an elementary way to compute $\ell_R(w)$ exactly for a generic w in an affine Coxeter group W . Techniques for easily computing $\ell_R(w)$ would be useful. For example, they would enable one to investigate whether the polynomial $f_\lambda(x) = \sum_{w \in W_0} x^{\ell_R(t_\lambda w)}$ for $\lambda \in L$ has properties similar to $f_0(x)$, the Solomon polynomial discussed in Proposition 2.8. All we can say at the moment (as a consequence of Proposition 4.1) is that when λ is k -dimensional, the polynomial $f_\lambda(x)$ has degree at most $k + n$ and is divisible by t^k .

5. AN UNBOUNDED EXAMPLE

When W is a Coxeter group that is neither finite nor affine we conjecture the following.

Conjecture 5.1 (No upper bounds). *The reflection length function on a Coxeter group W has a uniform upper bound if and only if W is of spherical or affine type.*

We are not currently able to prove this conjecture but we can show that it holds in at least one special case. Let W be the free Coxeter group on three generators, i.e. the group generated by three involutions

and with no other relations. Using a criterion of Dyer [3] we can show that the n -th power of the product of the three standard generators has reflection length $n + 2$. In particular, the reflection length function is unbounded on W and Conjecture 5.1 holds in this case. We note that W can be viewed as a reflection group acting on the hyperbolic plane generated by the reflections in the sides of an ideal triangle. Proving that all hyperbolic triangle Coxeter groups have unbounded reflection length would be first step towards proving Conjecture 5.1.

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