NOTES ON PROJECTIVE NORMALITY OF REDUCIBLE CURVES

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Abstract. We give some results on quadratic normality of reducible curves canonically embedded and partially extend this study to their projective normality.

Introduction

Let C be a smooth curve of genus g over an algebraically closed field k. The canonical bundle ω_C induces an embedding of C in $\mathbb{P}^{g-1}$ if and only if C is not hyperelliptic; we indicate the power $\omega_C^{\otimes n}$ by ω_C^n for any $n \in \mathbb{N}$. One says that C is *projectively normal* if the maps

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(k)) \to H^0(C, \omega_C^k) \tag{1}$$

are surjective for every $k \geq 1$. In other words, C is projectively normal if and only if the hypersurfaces of degree k in $\mathbb{P}^{g-1}$ cut a complete linear series on C for any k. If $k = 1$ and the map (1) is surjective, we say that C is *linearly normal*, which means that the curve is embedded via a complete linear series. If ω_C is ample, then an equivalent formulation states that C is projectively normal if the maps

$$\mathrm{Sym}^k H^0(C, \omega_C) \to H^0(C, \omega_C^k) \tag{2}$$

are surjective for every $k \geq 1$, because the surjectivity of all these maps when ω_C is ample implies the very ampleness of ω_C.

If C is a smooth, non-hyperelliptic curve, Castelnuovo and Noether proved that its canonical model is projectively normal (see [ACGH]). When we deal with singular curves, though, the problem becomes harder: for integral curves, in [KM09] the authors generalize Castelnuovo's approach proving that linear normality is equivalent to projective normality. For reducible curves yet not much is known: properties of the canonical map for Gorenstein curves, i.e. the map induced by the dualising sheaf, are investigated in [CFHR99], whereas in [F04] the author gives a sufficient condition for line bundles on non-reduced curves to be *normally generated* (see 1.9). The projective normality of reducible curves is studied in [S91]; more in general,

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since the problem of studying projective normality reduces to the study of multiplication maps, we refer to [B01] and [F04] for these items.

In this paper we investigate the projective normality of reducible curves restricting the problem to suitable subcurves. The first step is to study the *quadratic normality*, i.e. the surjectivity of the maps in (1) for $k = 2$. Let X be a connected, reduced and Gorenstein projective curve of genus g with ω_X very ample. Assume that X has planar singularities at the points lying on at least two irreducible components. Our main result about quadratic normality is the following theorem.

Theorem 1. *Let X be a curve as above, and set $X = A \cup B$ with A, B connected subcurves being smooth at $D := A \cap B$. If $A \neq \emptyset$ and the map*

$$\mu_{\omega_A,\omega_X|_A} : H^0(A, \omega_A) \otimes H^0(A, \omega_X|_A) \to H^0(X, \omega_A \otimes \omega_X|_A)$$

is surjective, then X is quadratically normal.

We also study certain multiplication maps in order to establish sufficient conditions that imply the surjectivity of the map in (2) for some k (*k-normal generation*) assuming to know the surjectivity for $(k-1)$ (see Proposition 2.9).

We divided the paper in two sections: in the first one we show our results about multiplication maps of reducible curves and apply them to the study of quadratic normality and of k-normal generation given the $(k-1)$-normal generation of the canonical bundle. In the second section we show some applications of our results to interesting cases and give some examples.

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1. Quadratic normality

For any reduced projective curve X and any line bundles M, N on X let

$$\mu_{M,N} : H^0(X, M) \otimes H^0(X, N) \longrightarrow H^0(X, M \otimes N); \tag{3}$$

denote the multiplication map. Set $\mu_M = \mu_{M,M}$. Given the dualizing sheaf ω_X on X, we are interested in studying the surjectivity of the map μ_{ω_X}. In particular, when we assume that X is canonically embedded this is equivalent to saying that X is *quadratically normal*. We have

Proposition 1.1. *Let X be a connected reduced curve of genus g with planar singularities and ω_X very ample. Assume that $X = A \cup B$, with A, B connected and smooth at $D := A \cap B$. If*

(i) $\mu_{\omega_A,\omega_X|_A}$ *is surjective,*
(ii) $\mu_{\omega_X|_B}$ *is surjective,*

then μ_{ω_X} *is surjective.*

In order to prove the proposition, we need some background material. We are going to keep the notation used in the statement of Proposition 1.1. Let $D := A \cap B$ be the scheme-theoretic intersection. We will view D also as a subscheme of A and B. Since both A and B are smooth at each point of the support of D, that we denote by $supp(D)$, the scheme D is a Cartier divisor of both A and B; more in general, this is true if X has only planar singularities at each point of $supp(D)$, because in this case a local equation of B in an ambient germ of a smooth surface gives a local equation of D as a subscheme of A.

Remark 1.2. According to the notation above, we have that

(i) It is well known that a curve with planar singularities is Gorenstein.
(ii) Since X is Gorenstein and locally planar at the points of $supp(D)$, then A and B are Gorenstein as well, so that ω_A and ω_B are both line bundles on A and B.
(iii) Since X is locally planar at the points of $supp(D)$, the adjunction formula gives $\omega_X|_A = \omega_A(D)$ and $\omega_X|_B = \omega_B(D)$. Thus $\deg(\omega_X|_A) = 2g_A - 2 + \delta$ and $\deg(\omega_X|_B) = 2g_B - 2 + \delta$, where of course g_A, g_B are the arithmetic genera of A and B, and $\delta = \deg(D)$.

Lemma 1.3. *Let Z be a reduced, Gorenstein and connected projective curve. Let E be an effective Cartier divisor on Z such that $E \neq 0$. Then $h^0(\mathcal{I}_E) = 0$ and $h^1(\omega_Z(E)) = 0$.*

Proof. Since Z is connected, $h^0(\mathcal{O}_Z) = 1$. Since E is effective and non-empty, we get $h^0(\mathcal{I}_E) = 0$. We apply the duality for locally Cohen-Macaulay schemes, i.e. we apply to the scheme $X := Z$ and the sheaf $F := \omega_Z(E)$ the case $r = p = 1$ of the theorem at page 1 of [AK70]. We get $h^1(\omega_Z(E)) = \dim(Ext^0(\omega_Z(E), \omega_Z))$, i.e. $h^1(\omega_Z(E)) = h^0(Hom(\omega_Z(E), \omega_Z))$. Since ω_Z is assumed to be locally free, we get $h^1(\omega_Z(E)) = h^0(Hom(\mathcal{O}_Z(E), \mathcal{O}_Z)) = 0$. □

Lemma 1.4. *Let X be a connected reduced curve of genus g with planar singularities and ω_X very ample. Assume that $X = A \cup B$, with A, B connected and smooth at $D := A \cap B$. For any subcurve Z of X we consider the map*

$$\rho_Z : H^0(X, \omega_X) \longrightarrow H^0(Z, \omega_X|_Z).$$

Then ρ_A and ρ_B are surjective.

Proof. To fix ideas we work on $Z = A$; let us consider the exact sequence:

$$0 \to \mathcal{I}_A \otimes \omega_X \to \omega_X \to \omega_X|_A \to 0.$$

We claim that $\mathcal{I}_A \otimes \omega_X = \omega_B$. To prove this, we notice that since X has only planar singularities, it can be embedded in a smooth surface S, where X, A and B are Cartier divisors. Thus D is a Cartier divisor of A and of B (but seldom of X). By the adjunction formula we have that

$$\omega_X = \omega_S(A+B)|_X,$$

then

$$\omega_B = \omega_S(B)|_B = \omega_S(A+B-A)|_B = (\omega_S(A+B-A)|_X)|_B$$
$$= (\omega_S(A+B)|_X \otimes \mathcal{I}_A)|_B = (\omega_X \otimes \mathcal{I}_A)|_B.$$

So the claim is proved and the previous sequence becomes

$$0 \to \omega_B \to \omega_X \to \omega_X|_A \to 0.$$

The corresponding long exact sequence in cohomology is

$$0 \to H^0(\omega_B) \to H^0(\omega_X) \to H^0(\omega_X|_A) \to H^1(\omega_B) \to H^1(\omega_X) \to H^1(\omega_X|_A) \to \cdots$$

Since $\omega_X|_A = \omega_A(D)$, by lemma 1.3 we have that $\dim H^1(\omega_X|_A) = 0$. Moreover, being both B and X connected, we have that $\dim H^1(\omega_B) = 1$ and $\dim H^1(\omega_X) = 1$, so the map $H^0(\omega_X) \to H^0(\omega_X|_A)$ is surjective. □

We are now able to prove proposition 1.1:

Proof of proposition 1.1. Let us consider the composition

$$H^0(\omega_X) \otimes H^0(\omega_X) \xrightarrow{\mu_{\omega_X}} H^0(\omega_X^2) \xrightarrow{\rho_B^2} H^0({\omega_X}^2|_B); \tag{4}$$

In order to show that μ_{ω_X} is surjective, it suffices, by a basic argument of linear algebra, to prove that

(a) $\rho_B^2 \circ \mu_{\omega_X}$ is surjective,
(b) $\mathrm{Ker}\rho_B^2 \subseteq \mathrm{Im}\mu_{\omega_X}$.

So let us show (a): we have a commutative diagram

$$\begin{array}{ccc} H^0(\omega_X) \otimes H^0(\omega_X) & \xrightarrow{\rho_B^2 \circ \mu_{\omega_X}} & H^0({\omega_X}^2|_B) \\ \Big\downarrow{\scriptstyle \rho_B \otimes \rho_B} & \nearrow{\scriptstyle \mu_{\omega_B(D)}} & \\ H^0(\omega_X|_B) \otimes H^0(\omega_X|_B) & & \end{array} \tag{5}$$

where the map $\rho_B \otimes \rho_B$ is surjective by lemma 1.4 and $\mu_{\omega_B(D)}$ is surjective by assumption (ii). So, by the commutativity of the diagram we get (a).

In order to prove (b), we notice that

$$\mathrm{Ker}\rho_{\mathrm{B}}^2 = H^0(X, \mathcal{I}_B \otimes \omega_X^2),$$

and take

$$\mu := \mu_{\omega_X}|_{H^0(X, \mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X)}.$$

So we have the following commutative diagram:

$$
\begin{array}{ccc}
H^0(\mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X) & \xrightarrow{\mu} & H^0(\mathcal{I}_B \otimes {\omega_X}^2) \\
\downarrow{\scriptstyle id\otimes\rho_A} & & \downarrow{\scriptstyle \cong} \\
H^0(\omega_A) \otimes H^0(\omega_X|_A) & \xrightarrow{\mu_{\omega_A,\omega_X|_A}} & H^0(\omega_A \otimes \omega_X|_A)
\end{array}
\tag{6}
$$

The map $id \otimes \rho_A$ is surjective by lemma 1.4, while $\mu_{\omega_A,\omega_X|_A}$ is surjective by assumption (i). Hence μ is surjective. Since μ is a restriction of μ_{ω_X}, we get $\mathrm{Ker}\rho_B^2 \subseteq \mathrm{Im}\mu_{\omega_X}$. □

Definition 1.5. Fix an integer $m > 0$; let X be a reduced and Gorenstein projective curve. We say that X is *m-connected* (resp. *numerically m-connected*) if for any decomposition $X = U \cup V$ with U, V subcurves without common irreducible components, the scheme $U \cap V$ has degree at least m (resp. $\deg \omega_X|_U - \deg \omega_U \geq m$ and $\deg \omega_X|_V - \deg \omega_V \geq m$).

Remark 1.6. If every point of X lying on at least two irreducible components of X is a planar singularity of X, then X is m-connected if and only if it is numerically m-connected (see [CFHR99], Remark 3.2).

Notation 1.7. Given a reduced curve X, we will denote by $X_{\mathrm{mult}} \subset X$ the set of points of X lying on at least two irreducible components of X and by X_{sm} the open set of smooth points of X.

Lemma 1.8. *Let X be a connected, reduced and Gorenstein curve of genus g with ω_X very ample. Assume that X has planar singularities at the points of X_{mult}. Then X is 3-connected.*

Proof. Let us fix any decomposition $X = U \cup V$ of X, with U, V subcurves and $\dim(U \cap V) = 0$. Set $D := U \cap V$. Since X has planar singularities at the points of *supp*(D), D is a Cartier divisor of U. To prove the lemma it is sufficient to show the inequality $\deg(D) \geq 3$. Assume $\deg(D) \leq 2$. Since ω_X is globally generated, X is 2-connected (see [Ca81], Theorem D). Assume, then, $\deg D = 2$. Remark 1.2 gives $\omega_X|_U \cong \omega_U(D)$. Since X is 2-connected and $\deg D = 2$, we easily see that U is connected. By lemma 1.3 we get that $\dim H^1(\omega_U(D)) = 0$. Thus Riemann-Roch gives

$$\dim H^0(\omega_U(D)) = \dim H^0(\omega_U) + 1.$$

Since D is a Cartier divisor of U, we get $\mathcal{I}_D \otimes \omega_U(D) \cong \omega_U$. Thus

$$\dim H^0(\mathcal{I}_D \otimes \omega_X|_U) = \dim H^0(\omega_X|_U) - 1,$$

hence the restriction to D of the morphism induced by $|\omega_X|$ is not very ample, contradiction. □

Definition 1.9. One says that a line bundle L on a curve X is *normally generated* if the maps

$$H^0(X,L)^k \to H^0(X,L^k)$$

are surjective for any $k \geq 1$.

Now we need to recall Theorem B in [F04].

Theorem 1.10 (Franciosi)**.** *Let C be a connected reduced curve and let $\mathcal{H}$ be an invertible sheaf on C such that*

$$\deg \mathcal{H}|_Z \geq 2p_a(Z)+1 \text{ for all subcurves } Z \subseteq C.$$

Then $\mathcal{H}$ is normally generated on C.

We are now able to prove the following lemma.

Lemma 1.11. *Let $X = A \cup B$, with $A, B \neq \emptyset$ and assume that X is Gorenstein, with planar singularities at the points of X_{mult}. Let ω_X be very ample. Then $\omega_X|_A$ and $\omega_X|_B$ are normally generated.*

Proof. Let us prove the conclusions for B. By Theorem 1.10 it is sufficient to prove that $\deg \omega_X|_Z \geq 2p_a(Z)+1$ for every subcurve $Z \subseteq B$. Since $A \neq \emptyset$, we have that $Z \subsetneq X$. But since ω_X is very ample, by lemma 1.8 we have that X is 3-connected, hence the conclusions. □

We are now ready to prove Theorem 1:

Proof of theorem 1. We recall that X is a connected, reduced and Gorenstein projective curve of genus g with ω_X very ample. By hypothesis we assume that X has planar singularities at the points of X_{mult}, and that $X = A \cup B$ with A, B connected subcurves being smooth at $D := A \cap B$. Since $\mu_{\omega_A, \omega_X|_A}$ is surjective, by proposition 1.1 it suffices to show that (ii) holds. But this is true by lemma 1.11. □

In what follows we will investigate when condition (i) of proposition 1.1 holds. If X is any curve, we denote by X_{sm} its smooth locus. We recall a result from [B01]; before doing this, let us introduce some notation: if L is a line bundle on a curve C globally generated and such that $\dim H^0(C,L) = r$, it induces a morphism

$$h_L : C \to \mathbb{P}^{r-1}.$$

Lemma 1.12 (Ballico)**.** *Let C be an integral projective curve with $C \neq \mathbb{P}^1$ and $R \in \text{Pic}C$, R globally generated and such that h_R is birational onto its image. Then the multiplication map*

$$\mu_{\omega_C,R} : H^0(C,\omega_C) \otimes H^0(C,R) \to H^0(C,\omega_C \otimes R)$$

is surjective.

More in general we have the following result.

Theorem 1.13. *Let A be a reduced, connected and Gorenstein projective curve such that ω_A is very ample and the map μ_{ω_A} is surjective. Let $E \subset A_{\mathrm{sm}}$ be an effective divisor on A such that $\deg E \geq 2$. Then $\mu_{\omega_A,\omega_A(E)}$ is surjective.*

Proof. Since A is connected, lemma 1.3 gives $H^1(\omega_A(D)) = 0$ for every effective and nonzero Cartier divisor D on A. Thus

$$\dim H^0(\omega_A(D)) = g_A + \deg D - 1$$

for every such D. We use induction on $e := \deg E$.

(a) Let us first assume $e = 2$. We check that $\omega_A(E)$ is globally generated. Set $E = p_1 + p_2$, where p_1, p_2 are smooth points for A. Since ω_A is globally generated, then $\omega_A(E)$ is globally generated outside $\{p_1, p_2\}$. We just proved that

$$\dim H^0(\omega_A(p_i)) = \dim H^0(\omega_A(p_1 + p_2)) - 1.$$

Thus there is at least one section of $\omega_A(E)$ that doesn't vanish at p_i, with $i = 1, 2$. Hence $\omega_A(E)$ is globally generated. The divisor E induces two inclusions $j : \omega_A \hookrightarrow \omega_A(E)$ and $j' : {\omega_A}^2 \hookrightarrow {\omega_A}^2(E)$, which in turn induce the linear maps $j_* : H^0(\omega_A) \longrightarrow H^0(\omega_A(E))$ and $j'_* : H^0({\omega_A}^2) \longrightarrow H^0({\omega_A}^2(E))$ which have respectively corank 1 and 2. Consider the following diagram:

$$\begin{array}{ccc} H^0(\omega_A) \otimes H^0(\omega_A) & \xrightarrow{id \otimes j_*} & H^0(\omega_A) \otimes H^0(\omega_A(E)) \\ \downarrow \mu_{\omega_A,\omega_A} & & \downarrow \mu_{\omega_A,\omega_A(E)} \\ H^0({\omega_A}^2) & \xrightarrow{j'_*} & H^0({\omega_A}^2(E)) \end{array} \tag{7}$$

Since by hypothesis μ_{ω_A,ω_A} is surjective and

$$\dim H^0({\omega_A}^2(E)) = \dim H^0({\omega_A}^2) + 2,$$

then $j'_*(Im(\mu_{\omega_A,\omega_A}))$ is the codimension 2 linear subspace $\Gamma := H^0(\mathcal{I}_E \otimes \omega_A(E))$ of $H^0({\omega_A}^2(E))$. Since the subspace $j'_*(Im(\mu_{\omega_A,\omega_A}))$ is contained in $Im(\mu_{\omega_A,\omega_A(E)})$, in order to get the conclusions for $e = 2$ it suffices to prove the existence of two elements of $Im(\mu_{\omega_A,\omega_A(E)})$ which together with a basis of $j'_*(Im(\mu_{\omega_A,\omega_A}))$, i.e. of Γ, are linearly independent. Since $\omega_A(E)$ is globally generated, there exists $\alpha \in H^0(\omega_A(E))$ not vanishing at p_1 and p_2. Since ω_A is globally generated, there is $\beta \in H^0(\omega_A)$ not vanishing at p_1 and p_2 as well. Since ω_A is very ample, there is $\gamma \in H^0(\omega_A)$ vanishing at p_1 but not at p_2, or, in the case when $p_1 = p_2$, vanishing at p_1 with order exactly 1. Now the section $\sigma := \mu_{\omega_A,\omega_A(E)}(\gamma \otimes \alpha)$ doesn't belong to Γ; indeed, if $p_1 \neq p_2$, σ doesn't vanish at p_2, and if $p_1 = p_2$, it vanishes at p_1 with order exactly 1. Since the section $\mu_{\omega_A,\omega_A(E)}(\beta \otimes \alpha)$ does not vanish at p_1, it is not contained in the linear span of Γ and σ. Thus

$$\dim Im(\mu_{\omega_A,\omega_A(E)}) \geq dim\Gamma + 2.$$

Thus $\mu_{\omega_A,\omega_A(E)}$ is surjective in the case $e = 2$.

(b) Let now $e \geq 3$. We use induction on e. We fix a point p contained in the support of the divisor E, and set $F := E - p$. We check that $\omega_A(E)$ is globally generated, By inductive hypothesis the line bundle $\omega_A(F)$ is globally generated, hence so is $\omega_A(E)$ outside p. Since $\dim H^1(\omega_A(F)) = 0$, Riemann-Roch gives $\dim H^0(\omega_A(E)) > \dim H^0(\omega_A(F))$. Thus $\omega_A(F)$ has a section not vanishing at p. Hence $\omega_A(E)$ is globally generated. We define two inclusions: $\iota : \omega_A(F) \hookrightarrow \omega_A(E)$ and $\iota' : {\omega_A}^2(F) \hookrightarrow {\omega_A}^2(E)$, which induce the linear maps $\iota_* : H^0(\omega_A(F)) \longrightarrow H^0(\omega_A(E))$ and $\iota'_* : H^0({\omega_A}^2(F)) \longrightarrow H^0({\omega_A}^2(E))$, both having corank 1. We consider the diagram

$$\begin{array}{ccc} H^0(\omega_A) \otimes H^0(\omega_A(F)) & \xrightarrow{id \otimes u_*} & H^0(\omega_A) \otimes H^0(\omega_A(E)) \\ \downarrow{\scriptstyle \mu_{\omega_A,\omega_A(F)}} & & \downarrow{\scriptstyle \mu_{\omega_A,\omega_A(E)}} \\ H^0({\omega_A}^2(F)) & \xrightarrow{u'_*} & H^0({\omega_A}^2(E)) \end{array} \tag{8}$$

By the inductive hypothesis the map $\mu_{\omega_A,\omega_A(F)}$ is surjective. Thus the linear subspace $u'_*(Im(\mu_{\omega_A,\omega_A(F)}))$ has codimension 1 in $H^0({\omega_A}^2(E))$. Fix $\eta \in H^0(\omega_A)$ not vanishing at p and $\tau \in H^0(\omega_A(E))$ not vanishing at p. Since $\mu_{\omega_A,\omega_A(E)}(\eta \otimes \tau)$ does not vanish at p, it doesn't belong to $u'_*(Im(\mu_{\omega_A,\omega_A(F)}))$. Thus $\mu_{\omega_A,\omega_A(E)}$ is surjective. □

2. k-NORMALITY IN HIGHER DEGREE

We are now interested in studying the surjectivity of higher order maps, i.e. of

$$Sym^k(H^0(\omega_X)) \longrightarrow H^0(\omega_X^k)$$

when $k \geq 3$, but since $Sym^k(H^0(\omega_X))$ is a quotient of $H^0(\omega_X)^{\otimes k}$, we can equivalently study the surjectivity of

$$H^0(\omega_X)^{\otimes k} \longrightarrow H^0(\omega_X^k).$$

We observe that by applying part (b) in the proof of theorem 1.13 we get the following:

Proposition 2.1. *Let A be a reduced, connected and Gorenstein curve such that ω_A is globally generated. Fix a globally generated $R \in \mathrm{Pic}A$ such that $H^1(R) = 0$ and $\mu_{\omega_A,R}$ is surjective. Let $D \subset A_{\mathrm{sm}}$ be any effective divisor. Then $\mu_{\omega_A,R(D)}$ is surjective.*

As a corollary of theorem 1.13, we get the following result.

Corollary 2.2. *Let A be a reduced, connected and Gorenstein projective curve such that ω_A is very ample and μ_{ω_A} is surjective. Let $E \subset A_{\mathrm{sm}}$ be an effective divisor such that $\deg E \geq 2$. Then the maps $\mu_{\omega_A,\omega_A^k(kE)}$ are surjective for all $k \geq 2$.*

We are now going to give some definitions in order to state a result;

Definition 2.3. A *simple* $(r-1)$-secant is a configuration of $r-1$ smooth points $p_1, \dots, p_{r-1}$ on a curve $X \subset \mathbb{P}^N$, spanning a $\mathbb{P}^{r-2}$ and such that $X \cap \mathbb{P}^{r-2} = \{p_1, \dots, p_{r-1}\}$ as schemes.

Definition 2.4. Let R be a globally generated line bundle on a curve X, inducing a map $h_R : X \longrightarrow \mathbb{P}^r$, $r := \dim H^0(R) - 1$, which is birational onto the image. A *good* $(r-1)$-secant of R is a set $S := \{p_1, \dots, p_{r-1}\}$ such that $\dim H^0(R(-\sum_{i=1}^{r-1} p_i)) = 2$, $R(-\sum_{i=1}^{r-1} p_i)$ is still globally generated, and h_R is an embedding at each p_i.

We recall the following result from [B01]

Lemma 2.5 (Ballico)**.** *Let X be a one-dimensional projective locally Cohen-Macaulay scheme with $\dim H^0(\mathcal{O}_X) = 1$ and $R \in \mathrm{Pic}X$ globally generated and such that $\dim H^0(R) = 2$. Then the multiplication map*

$$\mu_{\omega_X, R} : H^0(\omega_X) \otimes H^0(R) \longrightarrow H^0(\omega_X \otimes R)$$

is surjective.

Lemma 2.6. *Let A be a connected, projective curve, $L, M \in \mathrm{Pic}A$, M globally generated, and such that $\dim H^0(M) = 2$ and $\dim H^1(L \otimes M^\vee) = 0$. Then $\mu_{L,M}$ is surjective.*

Proof. Obvious by the base point free pencil trick. □

Proposition 2.7. *Let A be a connected, Gorenstein curve with ω_A globally generated, $R \in \mathrm{Pic}A$ with R globally generated, with h_R birational onto its image and with a good $(r-1)$-secant, where $r := h^0(R) - 1$. Then the maps μ_{ω_A, R^k} are surjective for all $k \geq 1$.*

Proof. Fix a good $(r-1)$-secant set $S = \{q_1, \dots, q_{r-1}\}$. Thus the linear span $\langle h_R(q_1), \dots, h_R(q_{r-1})\rangle$ has dimension $r-2$, $h_R(A) \cap \langle h_R(q_1), \dots, h_R(q_{r-1})\rangle = \{h_R(q_1), \dots, h_R(q_{r-1})\}$ as schemes and

$$h_R^{-1}(\{h_R(q_1), \dots, h_R(q_{r-1})\}) = \{q_1, \dots, q_{r-1}\}.$$

Set $M := R(-S)$. We start by examining the case $k = 1$. Since ω_A is globally generated, we have $A \neq \mathbb{P}^1$. Since the map h_R induced by R is birational onto its image, we have $r \geq 2$. The first condition on the good $(r-1)$-secant points gives $h^0(M) = 2$. The last two conditions give that M is globally generated. Since $h^0(R) = h^0(M) + r - 1$, we also get $h^0(M(q_1)) = h^0(M) + 1$. Thus there is $\eta \in H^0(M(q_1))$ such that $\eta(q_1) \neq 0$. The factorization shown in the following diagram

$$\begin{array}{ccc} H^0(\omega_A)\otimes H^0(M) & \longrightarrow\!\!\!\!\rightarrow & H^0(\omega_A\otimes M) \\ \downarrow & & \downarrow j \\ H^0(\omega_A)\otimes H^0(M(q_1)) & \xrightarrow{\varphi} & H^0(\omega_A\otimes M(q_1)) \end{array}$$

shows that the image of φ contains a copy of $H^0(\omega_A \otimes M)$ as a hyperplane. Since q_1 is not a base point for M and ω_A is globally generated, there is $\sigma \in H^0(\omega_A) \otimes H^0(M(q_1))$ that doesn't vanish on q_1. Hence the image of σ via φ doesn't vanish on q_1, and we get the surjectivity of φ. Repeating this argument for all the points $q_1, \dots, q_{r-1}$ adding them one by one we get that $\mu_{\omega_A,R}$ is surjective.

Now we assume $k \geq 2$ and use induction on k. The inductive assumption gives the surjectivity of the map $H^0(\omega_A) \otimes H^0(R^{k-1}) \longrightarrow H^0(\omega_A \otimes R^{k-1})$. We use the following commutative diagram:

$$\begin{array}{ccc} H^0(\omega_A)\otimes H^0(R^{k-1})\otimes H^0(R^k) & \xrightarrow{\psi} & H^0(\omega_A\otimes R^{k-1})\otimes H^0(R) \\ \downarrow & & \downarrow \phi \\ H^0(\omega_A)\otimes H^0(R^k) & \xrightarrow{\mu} & H^0(\omega_A\otimes R^k) \end{array}$$

It suffices to prove that ϕ is surjective, indeed, if it is, then $\phi \circ \psi$ is surjective, hence μ must be surjective. We proved that M is globally generated and $\dim H^0(M) = 2$. Moreover we notice that

$$\omega_A \otimes R^{k-1} \otimes M^\vee = \omega_A \otimes R^{k-2}(S).$$

Since $k \geq 2$ and $S \neq \emptyset$, we have that $\dim H^1(\omega_A \otimes R^{k-2}(S) = 0$. The base point free pencil trick applied to $\omega_A \otimes R^{k-1}$ and M gives the surjectivity of $\mu_{\omega_A \otimes R^{k-1},M}$. By Riemann-Roch theorem we get that

$$\dim H^0(\omega_A \otimes R^k) = \dim H^0(\omega_A \otimes R^{k-1} \otimes M) + \sharp S.$$

Arguing as in case $k = 1$ we get that the map μ_{ω_A,R^k} is surjective. □

Definition 2.8. We say that a line bundle L on a curve X is *k-normally generated* if the map

$$H^0(\omega_X)^{\otimes k} \longrightarrow H^0(\omega_X^k)$$

is surjective.

For instance "quadratically normal" means "linearly normal" plus "2-normally generated".

Proposition 2.9. *Let X be a connected, reduced, Gorenstein projective curve with planar singularities and ω_X very ample. Assume that $X = A \cup B$, with A, B connected and smooth at $D := A \cap B$. Fix $k \geq 3$; if*

(i) ω_X *is* $(k-1)$*-normally generated,*
(ii) $\mu_{\omega_A,\omega_X^j|_A}$ *is surjective for* $1 \le j \le k$,
(iii) $\omega_X|_B$ *is* j*-normally generated for* $1 \le j \le k$,

then ω_X *is* k*-normally generated.*

Proof. The proof is similar to the one of proposition 1.1; we just change notation slightly, denoting the multiplication maps in an easier way. We notice that in order to prove that the map

$$H^0(\omega_X)^{\otimes k} \xrightarrow{\mu_k} H^0(\omega_X^k)$$

is surjective, by factorizing we get

$$H^0(\omega_X) \otimes H^0(\omega_X)^{\otimes k-1} \xrightarrow{\mu \otimes \mu_{k-1}} H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) \xrightarrow{\widetilde{\mu}} H^0(\omega_X^k),$$

so it suffices to see that the map $\widetilde{\mu}$ is surjective. We consider the diagram

$$\begin{array}{ccc} H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) & \xrightarrow{\widetilde{\mu}} & H^0(\omega_X^k) \\ \downarrow \eta & & \downarrow \psi \\ H^0(\omega_X|_B) \otimes H^0(\omega_X^{k-1}|_B) & \xrightarrow{\phi} & H^0(\omega_X^k|_B) \end{array} \tag{9}$$

where the map $\widetilde{\mu} = \mu_{\omega_X,\omega_X^{k-1}}$. We know that ϕ is surjective by (iii), and if

(a) $\psi \circ \widetilde{\mu}$ is surjective,
(b) $\mathrm{Ker}\psi \subseteq \mathrm{Im}\widetilde{\mu}$,

then by linear algebra we get that $\widetilde{\mu}$ is surjective. In order to prove (a), by (9) we equivalently show that the map $\phi \circ \eta$ is surjective. We claim that η is surjective. Indeed, since ω_X is locally free we have the exact sequence

$$0 \to \mathcal{I}_B \otimes \omega_X \to \omega_X \to \omega_X|_B \to 0.$$

If we tensor by ω_X^{k-2}, we get

$$0 \to \mathcal{I}_B \otimes \omega_X^{k-1} \to \omega_X^{k-1} \to \omega_X|_B \otimes \omega_X^{k-2} \to 0,$$

which is equivalent to

$$0 \to \omega_A \otimes \omega_X^{k-2} \to \omega_X^{k-1} \to \omega_X^{k-1}|_B \to 0,$$

The corresponding long exact sequence in cohomology is

$$0 \to H^0(\omega_A \otimes \omega_X^{k-2}) \to H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1}|_B) \to H^1(\omega_A \otimes \omega_X^{k-2}) \to$$
$$\to H^1(\omega_X^{k-1}) \to H^1(\omega_X^{k-1}|_B) \to \cdots$$

Now we consider $H^1(\omega_A \otimes \omega_X^{k-2})$; we have that $\omega_A \otimes \omega_X^{k-2} = \omega_A \otimes \omega_X^{k-2}|_A = \omega_A \otimes \omega_A^{k-2}((k-2)D)$, hence by lemma 1.3 we obtain that $H^1(\omega_A \otimes \omega_A^{k-2}((k-2)D)) = 0$, therefore the map

$$H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1}|_B)$$

is surjective, and we get (a).

Now we want to prove (b). We notice that

$$\mathrm{Ker}\psi = H^0(\mathcal{I}_B \otimes \omega_X^k)$$

and set

$$\mu := \widetilde{\mu}|_{H^0(X,\mathcal{I}_B\otimes\omega_X)\otimes H^0(\omega_X^{k-1})}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X^{k-1}) & \xrightarrow{\mu} & H^0(\mathcal{I}_B \otimes {\omega_X}^k) \\ \downarrow \gamma & & \downarrow \cong \\ H^0(\omega_A) \otimes H^0(\omega_X^{k-1}|_A) & \xrightarrow{\mu_{\omega_A,\omega_X^{k-1}|_A}} & H^0(\omega_A \otimes \omega_X^{k-1}|_A) \end{array} \tag{10}$$

Now we have that $\mathcal{I}_B \otimes \omega_X \cong \omega_A$ and applying the previous argument to A rather than to B, we obtain that

$$H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1}|_A)$$

is surjective, hence so is γ in (10). Applying hypothesis (ii) we have that μ is surjective, hence as in the proof of 1.1, we get that $\mathrm{Ker}\psi = \mathrm{Im}\mu \subseteq \mathrm{Im}\widetilde{\mu}$. □

We notice that when k grows, the hypothesis in proposition 2.9 can be simplified:

Proposition 2.10. *Let X be a connected, reduced, Gorenstein projective curve of genus g, with ω_X globally generated. Fix $k \geq 4$ and assume that ω_X is $(k-1)$-normally generated. Then ω_X is k-normally generated.*

Proof. As in the proof of 2.9, looking at the factorization

$$H^0(\omega_X) \otimes H^0(\omega_X)^{\otimes k-1} \xrightarrow{\mu\otimes\mu_{k-1}} H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) \xrightarrow{\mu_{\omega_X,\omega_X^{k-1}}} H^0(\omega_X^k),$$

by hypothesis it suffices to prove that $\mu_{\omega_X,\omega_X^{k-1}}$ is surjective. We use Proposition 8 in [F07] in the following way: we take $\mathcal{F} := \omega_X$ and $\mathcal{H} := \omega_X^{k-1}$, so we have that $H^0(\mathcal{F})$ is globally generated. Moreover we have that

$$H^1(\mathcal{H} \otimes \mathcal{F}^{-1}) = H^1(\omega_X^{k-2}) = 0$$

if $k \geq 4$, so we get that the $\mu_{\omega_X,\omega_X^{k-1}}$ is surjective. □

3. Applications

In the sequel we are going to study some cases where we can apply our results.

Lemma 3.1. *Let Z be a connected and Gorenstein curve such that ω_Z is globally generated. Let $D \subset Z_{\mathrm{sm}}$ be an effective Cartier divisor such that $\deg(D) \geq 2$. Then $\omega_Z(D)$ is globally generated.*

Proof. Since $\omega_Z(D)$ is a line bundle, it is globally generated if and only if for every $q \in Z$ there is $s \in H^0(\omega_Z(D))$ such that $s(q) \neq 0$. Since ω_Z is assumed to be globally generated and D is effective, the sheaf $\omega_Z(D)$ is globally generated outside the finitely many points appearing in $supp(D)$. Fix $p \in supp(D)$ and set $D' := \mathcal{I}_p \otimes D$. Since $p \in X_{sm}$, D' is a Cartier divisor of degree $\deg(D) - 1$. Moreover, since $p \in supp(D)$, D' is effective. Thus Lemma 1.3 gives $h^1(\omega_Z(D')) = 0$. Riemann-Roch gives $h^0(\omega_Z(D)) = h^0(\omega_Z(D')) + 1$. Thus there is $s \in H^0(\omega_Z(D))$ such that $s(p) \neq 0$. □

Corollary 3.2. *Let X be a connected reduced curve with two irreducible non-rational components C_1, C_2 meeting at planar singularities for X and both smooth at $C_1 \cap C_2$; assume that ω_X is very ample. Then X is canonically embedded is projectively normal.*

Proof. First of all we have to prove that X is quadratically normal, so let us use the set-up of proposition 1.1, and set $A = C_1$, $B = C_2$. We look at hypothesis (i) and (ii) of the theorem; hypothesis (i) is verified by applying 1.12 to C_1. Indeed in our situation $R = \omega_X|_{C_1}$, i.e. $R = \omega_{C_1}(D)$ where D is the divisor on C_1 and C_2 corresponding to $C_1 \cap C_2$. Hence by lemma 3.1 we have that R is globally generated and birational onto the image, and we get (i). Concerning (ii), it suffices to apply 1.11, and then by 1.1 we obtain that X is quadratically generated. Now we want to study the 3-normal generation of X. So we look at the hypothesis of 2.9: we know that ω_X is quadratically normal, and of course (iii) holds by lemma 1.11. So it remains to prove (ii): but this is a consequence of corollary 2.2, indeed we have that μ_{ω_A} is surjective since A is irreducible and hence projectively normal, moreover, being ω_X very ample, $A \cdot B \geq 3$. Now when $k \geq 4$ we just apply 2.10 and get the conclusions. □

Remark 3.3. We observe that in the case of nodal connected curves with two non-rational irreducible components, the corollary above says that if the two components C_1 and C_2 meet at least at 3 points, then $X = C_1 \cup C_2$ canonically embedded is projectively normal. The corollary leaves out the curves having at least one $\mathbb{P}^1$ as a component, and in particular binary curves (i.e. a curve X is binary if it is composed of two $\mathbb{P}^1$'s meeting at $g+1$ points where g is the genus of X), but for the latter special class of curves we can use [S91] (see 3.6) and easily get projective normality. Concerning the class of curves $X = C_1 \cup C_2$ with $C_1 \neq \mathbb{P}^1$ and $C_2 = \mathbb{P}^1$, we get the projective normality by applying the same proof as in corollary 3.2, once we denote by A the component C_1. Indeed the hypothesis $C_1 \neq \mathbb{P}^1$ is used only when we apply 1.12 to A.

We can generalize the previous result:

Corollary 3.4. *Let X be a connected reduced Gorenstein curve with ω_X very ample and with planar singularities. Assume that $X = A \cup B$ with $A \neq \mathbb{P}^1$ irreducible and let B be a connected curve. Let A and B be smooth at $A \cap B$. Then ω_X is k-normally generated for any $k \geq 2$.*

Proof. The proof is straightforward once we notice that we can apply 1.12 to A and by Theorem 1 we get quadratic normality of X; for $k = 3$ we apply 2.9 since both 1.11 for B and 1.12 for A hold, and when $k \geq 4$ we apply 2.10. □

Corollary 3.5. *Let X be a connected reduced Gorenstein curve with ω_X very ample and with planar singularities. Assume that $X = A \cup B$ with A as in theorem 1.13 and let B be a connected curve. Let A and B be smooth at $A \cap B$. Then X canonically embedded is projectively normal.*

Proof. The proof is as in corollary 3.4, we just apply theorem 1.13 to A. □

We give now an example; before doing this, we recall an important result from [S91]:

Theorem 3.6 (Schreyer)**.** *Let $X \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g. If X has a simple $(g-2)$-secant, then X is projectively normal.*

Schreyer's theorem can be used in the most general setting once one is able to verify the existence of a simple $(g-2)$-secant. In [S91]pp.86 gave an example of a reducible canonically embedded curve admitting no simple $(g-2)$-secant. In the following example we show that our theorem applies to that case.

Example 3.7. Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$, with X_i smooth of genus g_i and such that the components intersect in 6 distinct points $p_{ij} = X_i \cap X_j$ that are ordinary nodes for X. Then X has genus $g = g_1 + g_2 + g_3 + g_4 + 3$. We have that ω_X is a very ample line bundle; if $g_i = 0$ for every i we have a graph curve, and it is projectively normal, as we see in [BE91]. Hence we can assume $g_i \neq 0$ for some i, say $g_1 > 0$. Set $A := X_1$, $B := X_2 \cup X_3 \cup X_4$. Since $A \neq \mathbb{P}^1$ we can apply 1.12 and get that the multiplication map $\mu_{\omega_A, \omega_X|_A}$ is surjective. Sincethe conditions on the degree of $\omega_X|_B$ in 1.10 are satisfied, the map $\mu_{\omega_X|_B}$ is surjective and we can apply proposition 1.1 and get that X is quadratically normal.

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