INVARIANT DISTRIBUTIONS ON PROJECTIVE SPACES OVER LOCAL FIELDS

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ABSTRACT. Let Γ be an $\widetilde{A}_n$ subgroup of $\mathrm{PGL}_{n+1}(\mathbb{K})$, with $n \geq 2$, where $\mathbb{K}$ is a local field with residue field of order q and let $\mathbb{P}^n_{\mathbb{K}}$ be projective n-space over $\mathbb{K}$. The module of coinvariants $H_0(\Gamma; C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z}))$ is shown to be finite. Consequently there is no nonzero Γ-invariant $\mathbb{Z}$-valued distribution on $\mathbb{P}^n_{\mathbb{K}}$.

1. Introduction

Let $\mathbb{K}$ be a nonarchimedean local field with residue field k of order q and uniformizer π. Denote by $\mathbb{P}^n_{\mathbb{K}}$ the set of one dimensional subspaces of the vector space $\mathbb{K}^{n+1}$, i.e. the set of points in projective n-space over $\mathbb{K}$. Then $\mathbb{P}^n_{\mathbb{K}}$ is a compact totally disconnected space with the quotient topology inherited from $\mathbb{K}^{n+1}$, and there is a continuous action of $G = \mathrm{PGL}_{n+1}(\mathbb{K})$ on $\mathbb{P}^n_{\mathbb{K}}$.

Let Γ be a lattice subgroup of G. The abelian group $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})$ of continuous integer-valued functions on $\mathbb{P}^n_{\mathbb{K}}$ has the structure of a Γ-module and the module of coinvariants $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma = H_0(\Gamma; C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z}))$ is a finitely generated group. Now suppose that Γ is an $\widetilde{A}_n$ group [3, 4], i.e. Γ acts freely and transitively on the vertex set of the Bruhat-Tits building of G, which has type $\widetilde{A}_n$. A free group is an $\widetilde{A}_1$ group since it acts freely and transitively on the vertex set of a tree, which is a building of type $\widetilde{A}_1$. For $n \geq 2$, the $\widetilde{A}_n$ groups are unlike free groups. This article proves the following.

Theorem 1.1. *If Γ is an $\widetilde{A}_n$ subgroup of $\mathrm{PGL}_{n+1}(\mathbb{K})$, where $n \geq 2$, then $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$ is a finite group.*

The proof depends upon the fact that Γ has Kazhdan's property (T). A *distribution* on $\mathbb{P}^n_{\mathbb{K}}$ is a finitely additive $\mathbb{Z}$-valued measure μ defined on the clopen subsets of $\mathbb{P}^n_{\mathbb{K}}$.

Corollary 1.2. *If Γ is an $\widetilde{A}_n$ subgroup of $\mathrm{PGL}_{n+1}(\mathbb{K})$, where $n \geq 2$, then there is no nonzero Γ-invariant $\mathbb{Z}$-valued distribution on $\mathbb{P}^n_{\mathbb{K}}$.*

This contrasts strongly with the main result of [8] concerning boundary distributions associated with finite graphs. A torsion free lattice subgroup Γ of $\mathrm{PGL}_2(\mathbb{K})$ is a free group, of rank r say. It was shown in [8] that in this case the group of Γ-invariant $\mathbb{Z}$-valued distributions on $\mathbb{P}^1_{\mathbb{K}}$ is isomorphic to $\mathbb{Z}^r$. In particular, there are many such distributions.

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2. Background

2.1. **The Bruhat-Tits building.** If $\mathbb{K}$ is a local field, with discrete valuation $v : \mathbb{K}^\times \to \mathbb{Z}$, let $\mathcal{O} = \{x \in \mathbb{K} : v(x) \geq 0\}$ and let $\pi \in \mathbb{K}$ satisfy $v(\pi) = 1$. A *lattice* L is an $\mathcal{O}$-submodule of $\mathbb{K}^{n+1}$ of rank $n+1$. In other words $L = \mathcal{O}e_1 + \mathcal{O}e_2 + \cdots + \mathcal{O}e_{n+1}$, for some basis $\{e_1, e_2, \ldots, e_{n+1}\}$ of $\mathbb{K}^{n+1}$. Two lattices L_1 and L_2 are *equivalent* if $L_1 = \alpha L_2$ for some $\alpha \in \mathbb{K}^\times$. The Bruhat-Tits building of $\mathrm{PGL}_{n+1}(\mathbb{K})$ is a two dimensional simplicial complex Δ whose vertices are equivalence classes of lattices in $\mathbb{K}^{n+1}$ [9]. Two lattice classes $[L_0], [L_1]$ are *adjacent* if, for suitable representatives L_1, L_2, we have $L_0 \subset L_1 \subset \pi^{-1}L_0$. A *simplex* is a set of pairwise adjacent lattice classes. The maximal simplices (*chambers*) are the sets $\{[L_0], [L_1], \ldots, [L_n]\}$ where $L_0 \subset L_1 \subset \cdots \subset L_n \subset \pi^{-1}L_0$. These inclusions determine a canonical ordering of the vertices in a chamber, up to cyclic permutation. Each vertex v of Δ has a *type* $\tau(v) \in \mathbb{Z}/(n+1)\mathbb{Z}$, and each chamber of Δ has exactly one vertex of each type. If the Haar measure on $\mathbb{K}^{n+1}$ is normalized so that $\mathcal{O}^{n+1}$ has measure 1 then the type map may be defined by $\tau([L]) = \log_q(\mathrm{vol}(L)) + (n+1)\mathbb{Z}$. The cyclic ordering of the vertices of a chamber coincides with the natural ordering given by the vertex types (Figure 1). Let E^1 denote the set of directed edges $e = (x, y)$ of Δ such that $\tau(y) = \tau(x) + 1$. Write $o(e) = x$ and $t(e) = y$. The subgraph of the 1-skeleton of Δ with edge set E^1 is studied in [5, 7].

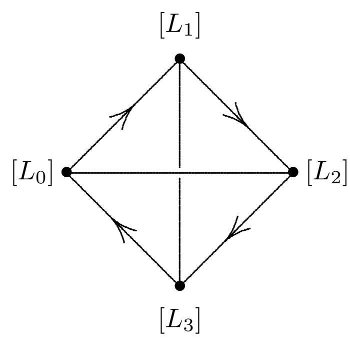

Figure 1. $\widetilde{A}_3$ case: cyclic ordering of the vertices of a chamber

Lemma 2.1. *Let C be a chamber of Δ. Then C contains $n+1$ directed edges $e \in E^1$.*

Proof. By [9, Chapter 9.2], there is a basis $(e_1, \ldots, e_{n+1})$ of $\mathbb{K}^{n+1}$ such that the vertices of C are the classes of the lattices

$$\begin{aligned}
L_0 &= \pi\mathcal{O}e_1 + \pi\mathcal{O}e_2 + \pi\mathcal{O}e_3 + \cdots + \pi\mathcal{O}e_{n+1}\\
L_1 &= \mathcal{O}e_1 + \pi\mathcal{O}e_2 + \pi\mathcal{O}e_3 + \cdots + \pi\mathcal{O}e_{n+1}\\
L_2 &= \mathcal{O}e_1 + \mathcal{O}e_2 + \pi\mathcal{O}e_3 + \cdots + \pi\mathcal{O}e_{n+1}\\
&\cdots\cdots\\
L_n &= \mathcal{O}e_1 + \mathcal{O}e_2 + \mathcal{O}e_3 + \cdots + \pi\mathcal{O}e_{n+1}.
\end{aligned}$$

Define $L_{n+1} = L_0$. Then the edges C which lie in E^1 are $([L_k], [L_{k+1}])$, where $0 \leq k \leq n$. $\square$

The building Δ is of type $\widetilde{A}_n$ and the action of $\mathrm{GL}_{n+1}(\mathbb{K})$ on the set of lattices induces an action of $\mathrm{PGL}_{n+1}(\mathbb{K})$ on Δ which is transitive on the vertex set. The action of $\mathrm{PGL}_{n+1}(\mathbb{K})$ on Δ is *type rotating* in the sense that, for each $g \in \mathrm{PGL}_{n+1}(\mathbb{K})$, there exists $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ such that $\tau(gv) = \tau(v) + i$ for all vertices $v \in \Delta$.

Fix a vertex $v_0 \in \Delta$ of type 0, and let $\Pi(v_0)$ be the set of vertices adjacent to v_0. Then $\Pi(v_0)$ has a natural incidence structure: if $u, v \in \Pi(v_0)$ are distinct, then u and v are *incident* if u, v and v_0 lie in a common chamber of Δ. If v_0 is the lattice class $[L_0]$, then $\Pi(v_0)$ consists of the classes $[L]$ where $L_0 \subset L \subset \pi^{-1}L_0$, and one can associate to $[L] \in \Pi(v_0)$ the subspace $v = L/L_0$ of $\pi^{-1}L_0/L_0 \cong k^{n+1}$. Thus we may identify $\Pi(v_0)$ with the flag complex of subspaces of the vector space k^{n+1}. Under this identification, a vertex $v \in \Pi(v_0)$ has type $\tau(v) = \dim(v) + \mathbb{Z}/(n+1)\mathbb{Z}$ where $\dim(v)$ is the dimension of v over k. A chamber C of Δ which contains v_0 has vertices $v_0, v_1, \dots, v_n$ where $(0) = v_0 \subset v_1 \subset \cdots \subset v_n \subset k^{n+1}$ is a complete flag. For brevity, write $C = \{v_0 \subset v_1 \subset \cdots \subset v_n\}$.

Definition 2.2. If $e = ([L_0], [L_1]) \in E^1$, where $L_0 \subset L_1 \subset \pi^{-1}L_0$ and $\tau([L_1]) = \tau([L_0]) + 1$, then define $\Omega(e)$ to be the set of lines $\ell \in \mathbb{P}^n_{\mathbb{K}}$ such that $L_1 = L_0 + (\ell \cap \pi^{-1}L_0)$. The sets $\Omega(e)$, $e \in E^1$, form a basis for the topology on $P^n_{\mathbb{K}}$ (c.f. [10, Ch.II.1.1], [1, 1.6]).

Lemma 2.3. *If $e \in E^1$, then $\Omega(e)$ may be expressed as a disjoint union of q^n sets*

$$\Omega(e) = \bigsqcup_{\substack{o(e')=t(e) \\ \Omega(e') \subset \Omega(e)}} \Omega(e') . \tag{1}$$

Proof. Let $e = ([L_0], [L_1]) \in E^1$, where $L_0 \subset L_1 \subset \pi^{-1}L_0$ and $\tau([L_1]) = \tau([L_0]) + 1$. If $\ell \in \Omega(e)$ then $L_1 = L_0 + (\ell \cap \pi^{-1}L_0)$. Choose $e' = ([L_1], [L_2])$ where $L_2 = L_0 + (\ell \cap \pi^{-2}L_0)$. Now $L_0 \subset L_1 \subset L_2 \subset \pi^{-1}L_1$ and L_2/L_1 is a 1-dimensional subspace of $\pi^{-1}L_1/L_1 \cong k^{n+1}$. Moreover, L_2/L_1 is not incident with the n-dimensional subspace $\pi^{-1}L_0/L_1$ of $\pi^{-1}L_1/L_1 \cong k^{n+1}$. There are precisely q^n such 1-dimensional subspaces of k^{n+1}, each of which corresponds to an edge $e' \in E^1$. $\square$

Lemma 2.4. *If ξ is a fixed vertex of Δ, then $\mathbb{P}^n_{\mathbb{K}}$ may be expressed as a disjoint union*

$$\mathbb{P}^n_{\mathbb{K}} = \bigsqcup_{o(e)=\xi} \Omega(e) . \tag{2}$$

Proof. Let $\xi = [L_0]$, where L_0 is a lattice. If $\ell \in \mathbb{P}^n_{\mathbb{K}}$, define the lattice $L_1 = L_0 + (\ell \cap \pi^{-1}L_0)$. Then $L_0 \subset L_1 \subset \pi^{-1}L_0$ and $\tau([L_1]) = \tau([L_0]) + 1$, since L_0 is maximal in L_1. Thus the edge $e = ([L_0], [L_1])$ lies in E^1, and $\ell \in \Omega(e)$. $\square$

Lemma 2.5. *Let C be a chamber of Δ and denote the directed edges of $C \cap E^1$ by $e_0, e_1, \dots, e_n$. Then $\mathbb{P}^n_{\mathbb{K}}$ may be expressed as a disjoint union*

$$\mathbb{P}^n_{\mathbb{K}} = \bigsqcup_{i=0}^{n} \Omega(e_i) . \tag{3}$$

Proof. Let C have vertex set $\{[L_0], [L_1], \dots, [L_n]\}$ where $L_0 \subset L_1 \subset \cdots \subset L_n \subset \pi^{-1}L_0$. Let $\ell = \mathbb{K}a \in \mathbb{P}^n_{\mathbb{K}}$, where $a \in \mathbb{K}^{n+1}$ is scaled so that $a \in \pi^{-1}L_0 - L_0$. Then $a \in L_{i+1} - L_i$ for some i, where $L_{i+1}/L_i \cong k$ and $L_{n+1} = \pi^{-1}L_0$. Thus $\ell \in \Omega(e_i)$. $\square$

2.2. $\widetilde{A}_n$ **groups.** From now on let $\Pi = \Pi(v_0)$, the set of neighbours of the fixed vertex $v_0 \in \Delta$. Thus Π is isomorphic to the flag complex of subspaces of k^{n+1} and a chamber C of Δ which contains v_0 is a complete flag $\{v_0 \subset v_1 \subset \cdots \subset v_n\}$. For $1 \le r \le n$, let $\Pi_r = \{u \in \Pi(v_0) : \dim u = r\}$.

Now suppose that Γ is an $\widetilde{A}_n$ group i.e. Γ acts freely and transitively on the vertex set of Δ [3, 4]. Then for each $v \in \Pi(v_0)$, there is a unique element $g_v \in \Gamma$ such that $g_v v_0 = v$. If $v \in \Pi(v_0)$, then $g_v^{-1} v_0$ also lies in $\Pi(v_0)$, and $\lambda(v) = g_v^{-1} v_0$ defines an involution $\lambda : \Pi(v_0) \to \Pi(v_0)$ such that $g_{\lambda(v)} = g_v^{-1}$. Let $\mathcal{T} = \{(u, v, w) \in \Pi(v_0)^3 : g_u g_v g_w = 1\}$. If $(u, v, w) \in \mathcal{T}$ then w is uniquely determined by (u, v) and there is a bijective correspondence between triples $(u, v, w) \in \mathcal{T}$ and directed triangles $(v_0, \lambda(u), v)$ of Δ containing v_0. By [6, Proposition 2.2], the abstract group Γ has a presentation with generating set $\{g_v : v \in \Pi(v_0)\}$ and relations

$$g_u g_{\lambda(u)} = 1, \quad u \in \Pi(v_0); \tag{4a}$$

$$g_u g_v g_w = 1, \quad (u, v, w) \in \mathcal{T}. \tag{4b}$$

If $u \in \Pi(v_0)$ and then $\tau(g_u v_0) = \tau(u) = \tau(u) + \tau(v_0)$. Hence $\tau(g_u x) = \tau(u) + \tau(x)$ for each vertex x of Δ, since g_u is type rotating. In particular, if $u, v \in \Pi(v_0)$ then

$$\tau(g_u g_v v_0) = \tau(u) + \tau(v). \tag{5}$$

It follows from (5) that

$$\tau(\lambda(u)) = -\tau(u)$$

for each $u \in \Pi$. Also, if $(u, v, w) \in \mathcal{T}$, then

$$\tau(u) + \tau(v) + \tau(w) = 0.$$

Let $C = \{v_0 \subset v_1 \subset \cdots \subset v_n\}$ be a chamber of Δ containing v_0. Since the vertices v_{i-1} and v_i are adjacent, so are the vertices $v_0 = g_{v_{i-1}}^{-1} v_{i-1}$ and $g_{v_{i-1}}^{-1} g_{v_i} v_0 = g_{v_{i-1}}^{-1} v_i$. Also $\tau(g_{v_{i-1}}^{-1} g_{v_i} v_0) = \tau(v_i) - \tau(v_{i-1}) = 1$. Therefore $g_{v_{i-1}}^{-1} g_{v_i} = g_{a_i}$ where $a_i \in \Pi_1$, $v_{n+1} = v_0$ and $g_{v_0} = 1$. Thus $g_{a_1} g_{a_2} \dots g_{a_k} = g_{v_k}$ $(1 \le k \le n)$ and $g_{a_1} g_{a_2} \dots g_{a_{n+1}} = 1$.

The $(n+1)$-tuple $\sigma(C) = (a_1, a_2, \dots, a_{n+1}) \in \Pi_1^{n+1}$ is uniquely determined by the chamber C containing v_0. Denote by $\mathfrak{S}$ the set of all $(n+1)$-tuples $\sigma(C)$ associated with such chambers C. If $u \in \Pi(v_0)$ with $\dim(u) = k$, then u is a vertex of a chamber C containing v_0. Therefore

$$g_u = g_{a_1} g_{a_2} \dots g_{a_k}, \quad \text{where} \quad a_i \in \Pi_1, 1 \le i \le k. \tag{6}$$

In particular, the set $\{g_a : a \in \Pi_1\}$ generates Γ. Since $g_{\lambda(u)} = g_u^{-1}$, we have

$$g_{\lambda(u)} = g_{a_{i+1}} \dots g_{a_{n+1}}. \tag{7}$$

Note that the expression (6) for g_u is not unique, but depends on the choice of the chamber C containing u and v_0. An edge in E^1 has the form $(x, g_a x)$ where $a \in \Pi_1$.

Lemma 2.6. *The $\widetilde{A}_n$ group Γ has a presentation with generating set $\{g_a : a \in \Pi_1\}$ and relations*

$$g_{a_1} g_{a_2} \dots g_{a_{n+1}} = 1, \qquad (a_1, a_2, \dots, a_{n+1}) \in \mathfrak{S}. \tag{8}$$

Proof. It is enough to show that the relations (4) follow from the relations (8). Let $(u, v, w) \in \mathcal{T}$ with $\dim(u) = i, \dim v = j$ and $\dim w = k$, where $i + j + k \equiv 0 \mod (n+1)$. Choose a chamber $C = \{v_0 \subset v_1 \subset \cdots \subset v_n\}$ containing $\{v_0, g_u v_0, g_u g_v v_0\}$. Let $(a_1, a_2, \dots, a_{n+1}) = \sigma(C) \in \Pi_1^{n+1}$ be the element of $\mathfrak{S}$ determined by C. Then $g_u v_0$ is the vertex of C of type i, so $g_u = g_{a_1} g_{a_2} \dots g_{a_i}$.

Suppose that $j < n+1-i$. Then $g_u g_v v_0$ is the vertex of C of type $i+j$ and $g_u g_v = g_{a_1} g_{a_2} \dots g_{a_{i+j}}$. Thus $g_v = g_{a_{i+1}} \dots g_{a_{i+j}}$ and $g_w = g_{a_{i+j+1}} \dots g_{a_{n+1}}$. Therefore

$$g_u g_v g_w = g_{a_1} g_{a_2} \dots g_{a_{n+1}}.$$

Suppose that $j > n+1-i$. Then $g_u g_v v_0$ has type $i+j-n-1$ and

$$g_u g_v = g_{a_1} g_{a_2} \dots g_{a_{i+j-n-1}} = g_{a_1} g_{a_2} \dots g_{a_{n+1}} g_{a_1} \dots g_{a_{i+j-n-1}}.$$

Thus $g_v = g_{a_{i+1}} \dots g_{a_{n+1}} g_{a_1} \dots g_{a_{i+j-n-1}}$ and $g_w = g_{a_{i+j-n}} \dots g_{a_{n+1}}$. Therefore

$$g_u g_v g_w = (g_{a_1} g_{a_2} \dots g_{a_{n+1}})^2.$$

In each case the relations (4b) follow from the relations (8). The same is true for the relations (4a), by equation (7). $\square$

3. The coinvariants

If Γ is an $\widetilde{A}_n$ group acting on Δ, then Γ acts on $\mathbb{P}^n_{\mathbb{K}}$, and the abelian group $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})$ has the structure of a Γ-module, with $(g \cdot f)(\ell) = f(g^{-1}\ell)$, $g \in \Gamma$, $\ell \in \mathbb{P}^n_{\mathbb{K}}$. The module of coinvariants, $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$, is the quotient of $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})$ by the submodule generated by $\{g \cdot f - f : g \in \Gamma, f \in C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})\}$. If $f \in C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})$ then let $[f]$ denote its class in $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$. Also, let $\mathbf{1}$ denote the constant function defined by $\mathbf{1}(\ell) = 1$ for $\ell \in \mathbb{P}^n_{\mathbb{K}}$, and let $\varepsilon = [\mathbf{1}]$.

If $e \in E^1$, let χ_e be the characteristic function of $\Omega(e)$. For each $g \in \Gamma$, the functions χ_e and $g \cdot \chi_e = \chi_{ge}$ project to the same element in $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$. Any edge $e \in E^1$ is in the Γ-orbit of some edge $(v_0, g_a v_0)$, where $a \in \Pi_1$ is uniquely determined by e. Therefore it makes sense to denote by $[a]$ the class of χ_e in $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$.

Lemma 3.1. *The group $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$ is finitely generated, with generating set $\{[a] : a \in \Pi_1\}$.*

Proof. Every clopen set V in $\mathbb{P}^n_{\mathbb{K}}$ may be expressed as a finite disjoint union of sets of the form $\Omega(e)$, $e \in E^1$. Any function $f \in C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})$ is bounded, by compactness of $\mathbb{P}^n_{\mathbb{K}}$, and so takes finitely many values $n_i \in \mathbb{Z}$. Therefore f may be expressed as a finite sum $f = \sum_j n_j \chi_{e_j}$, with $e_j \in E^1$. The result follows, since $\{[\chi_e] : e \in E^1\} = \{[a] : a \in \Pi_1\}$. $\square$

Suppose that $e, e' \in E^1$ with $o(e') = t(e) = x$, so that $o(e) = g_{\lambda(a)} x$ and $t(e') = g_b x$ for (unique) $a, b \in \Pi_1$. Then, by the proof of Lemma 2.3, $\Omega(e') \subset \Omega(e)$ if and only if $b \cap \lambda(a) = (0)$.

Equations (1) and (2) imply the following relations in $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$.

$$\varepsilon = \sum_{a \in \Pi_1} [a]; \tag{9a}$$

$$[a] = \sum_{\substack{b \in \Pi_1 \\ b \cap \lambda(a) = (0)}} [b], \quad a \in \Pi_1. \tag{9b}$$

It is easy to see that $|\Pi_1| = \frac{q^{n+1}-1}{q-1}$. If $a \in \Pi_1$, then $\lambda(a) \in \Pi_n$ and so the number of elements $b \in \Pi_1$ which are incident with $\lambda(a)$ is $\frac{q^n-1}{q-1}$. Thus there exist q^n elements $b \in \Pi_1$ such that $b \cap \lambda(a) = (0)$. In other words, the right side of (9b) contains q^n terms. As a first step towards proving that $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$ is finite, we show that the element $\varepsilon = [\mathbf{1}]$ has finite order.

Lemma 3.2. *In the group $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$, $(q^n - 1)\varepsilon = 0$.*

Proof. By (9a) and (9b),

$$\varepsilon = \sum_{a \in \Pi_1} [a] = \sum_{a \in \Pi_1} \left(\sum_{\substack{b \in \Pi_1 \\ b \cap \lambda(a) = (0)}} [b] \right) = \sum_{b \in \Pi_1} q^n [b] = q^n \varepsilon.$$

□

We can now prove Theorem 1.1. It follows from (3) that if $(a_1, a_2, \dots, a_{n+1}) \in \mathfrak{S}$ then

$$\sum_{i=1}^{n+1} [a_i] = \varepsilon. \tag{10}$$

Therefore, by Lemmas 2.6 and 3.1, there is a homomorphism θ from Γ onto the abelian group $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma / \langle \varepsilon \rangle$ defined by $\theta(g_a) = [a] + \langle \varepsilon \rangle$, for $a \in \Pi_1$.

The $\widetilde{A}_n$ group Γ has Kazhdan's property (T) [2, Theorems 1.6.1 and 1.7.1]. It follows that $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma / \langle \varepsilon \rangle$ is finite [2, Corollary 1.3.5]. Therefore $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$ is also finite, since $\langle \varepsilon \rangle$ is finite, by Lemma 3.2. □

Distributions. A *distribution* on $\mathbb{P}^n_{\mathbb{K}}$ is a finitely additive $\mathbb{Z}$-valued measure μ defined on the clopen subsets of $\mathbb{P}^n_{\mathbb{K}}$ [1, 1.4]. By integration, a distribution may be regarded as a $\mathbb{Z}$-linear function on the group $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})$. Therefore a Γ-invariant distribution defines a homomorphism $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma \to \mathbb{Z}$. This homomorphism is necessarily trivial, since $C(\mathbb{P}^n_{\mathbb{K}}, \mathbb{Z})_\Gamma$ is finite. This proves Corollary 1.2.

References

[1] G. Alon and E. de Shalit, On the cohomology of Drinfel'd's p-adic symmetric domain, *Israel J. Math.* **129** (2002), 1–20.

[2] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)*, Cambridge University Press, Cambridge, 2008.

[3] D. I. Cartwright, Groups acting simply transitively on the vertices of a building of type $\tilde{A}_n$, *Groups of Lie type and their Geometries*, W. M. Kantor and L. Di Martino, editors, 43–76. Cambridge University Press, 1995.

[4] D. I. Cartwright and T. Steger, A family of $\widetilde{A}_n$ groups, *Israel J. Math.* **103** (1998), 125–140.

[5] D. I. Cartwright, P. Solé and A. Żuk, Ramanujan geometries of type $\widetilde{A}_n$, *Discrete Math.* **269** (2003), 35–43.

[6] D. I. Cartwright, A. M. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, I, *Geom. Ded.* **47** (1993), 143–166.

[7] A. Lubotzky, B. Samuels and U. Vishne, Ramanujan complexes of type $\widetilde{A}_d$, *Israel J. Math.* **149** (2005), 267–299.

[8] G. Robertson, Invariant boundary distributions associated with finite graphs, *J. Combin. Theory Ser. A*, **115** (2008), 1272–1278.

[9] M. Ronan, *Lectures on Buildings*, University of Chicago Press, 2009.

[10] J.-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980.

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